Spec. Matrices 2019; 7:92–94 DE GRUYTER

Comment Open Access

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Determinant of binary circulant matrices

https://doi.org/110.1515/spma-2019-0008
Received April 22, 2018; accepted August 8, 2019

Abstract: This article gives a closed-form expression for the determinant of binary circulant matrices.

Keywords: Greatest common divisor, Binary circulant matrix, Circulant matrix determinants, Permutations, Toeplitz determinant

MSC: 11A05, 15B05, 15A15, 05A05

1 Introduction

Let C be an adjacency matrix of a full directed cycle (or a permutation matrix of cyclic shift corresponding to n letters) which is expressed as

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}.$$

Let the matrix I denotes the identity matrix of dimension n, and the vector $\mathfrak{e}$ denote the $n \times 1$ vector with all ones. Let (m, n) denote the GCD (Greatest common divisor) of integers m and n. Any circulant matrix can be represented as $A = \sum_{k=0}^{n-1} a_k C^k$. The matrix C has eigenvalues $e^{\frac{2\pi i t}{n}}$ with an eigenvector u_t, having entry $u_t(r, 1) = (e^{\frac{2\pi i t r}{n}})$ for $0 \le t < n$ and $0 \le r < n$.
For a permutation matrix P, let $sign(P) = 0$ if it represents even permutation and $sign(P) = 1$ if it represents odd permutation.

In [1] the authors claim that when a circulant matrix has the first row $[a, a, \cdots a, b, b, \cdots b]$ (with a repeated $n-k$ times and b repeated k times) then the determinant is 0 when $n \equiv p \mod k$ and $(p, k) \neq 1$, otherwise the determinant is $k((n-k)(a/k) + b)(a-b)^{n-1}$. Here this claim is proved positively.

2 Results

Theorem 2.1. *The binary circulant matrix having the first row $[a, a, \cdots a, b, b, \cdots b]$ (with a repeated $n-k$ times and b repeated k times) has determinant zero when $(n-k, n) \neq 1$ and $((n-k)a + kb)(a-b)^{n-1}$ otherwise.*

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Proof. The binary circulant matrix having the first row $[a, a, \cdots a, b, b, \cdots b]$ (with k repeated b s and $n-k$ repeated a s) is of the form

$$A = a \sum_{t=0}^{n-k-1} C^t + b \sum_{t=n-k}^{n-1} C^t \tag{1}$$

Let us consider two sums

$$S_1 = \sum_{t=0}^{n-k-1} C^t$$

and

$$S_2 = \sum_{t=n-k}^{n-1} C^t.$$

We have

$$S_1(C-I) = C^{n-k} - I, \tag{2}$$

and

$$S_2(C-I) = C^n - C^{n-k} \tag{3}$$
$$= I - C^{n-k}. \tag{4}$$

From equations (2) and (4) we have

$$A(C-I) = (a-b)(C^{n-k} - I). \tag{5}$$

When we multiply by an eigenvector (C and A share same eigenvectors), which is not $\mathfrak{e}$, denoted by u_t (for $1 \le t < n$) with eigenvalue $e^{\frac{2\pi i t}{n}}$ on both sides of equation (5), we get

$$Au_t = \frac{(a-b)(e^{\frac{2\pi i(n-k)t}{n}} - 1)}{e^{\frac{2\pi i t}{n}} - 1} u_t. \tag{6}$$

When $(n-k, n) > 1$, we have a zero eigenvalue from equation 6. Hence $\det(A) = 0$. Note that we have $A\mathfrak{e} = ((n-k)a + kb)\mathfrak{e}$.

So the expression for the determinant when $(n-k, n) = 1$ is given by,

$$\det(A) = ((n-k)a + kb) \prod_{t=1}^{n-1} \frac{(a-b)(e^{\frac{2\pi i(n-k)t}{n}} - 1)}{e^{\frac{2\pi i t}{n}} - 1} \tag{7}$$
$$= ((n-k)a + kb)(a-b)^{n-1}. \tag{8}$$

The equation (8) is obtained by noting that when $n-k$ is relatively prime to n, the sets $\{(n-k)t \mod n : 1 \le t < n\}$ and $\{t : 1 \le t < n\}$ are equal. □

Corollary 2.1.1. *When a binary circulant matrix is multiplied by a permutation matrix, represented as $B = AP$, where A has first row $[a, a, \cdots a, b, b, \cdots b]$ (with a repeated $n-k$ times and b repeated k times) then determinant of B is*

$$(-1)^{sign(P)}((n-k)a + kb)(a-b)^{n-1},$$

when $(n-k, n) = 1$ and zero otherwise.

Proof. Note that $\det(B) = \det(A)\det(P)$. If P is even permutation, then it can be represented as product of even number of flips. With flip having submatrix as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and each flip has determinant -1. So if P is odd permutation $\det(B) = -\det(A)$ else $\det(B) = \det(A)$. □

References

[1] C. Kravvaritis, Determinant evaluations for binary circulant matrices, Spec. Matrices 2 (2014), 187-199.