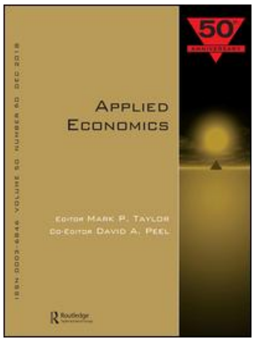

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Rational functions: an alternative approach to asset pricing

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ABSTRACT
This paper shows that asset prices are linear polynomials of various underlying explanatory factors and asset returns being ratios of these polynomials, are rational functions that do not add linearly when averaging. Hence, average returns should be modeled based on stock prices. However, continuous returns may be treated as approximately linear across time and modeled directly. Our new Rational Function (RF) models, empirically outperform the traditional asset pricing models like the Capital Asset Pricing Model (CAPM) and the Fama–French three and five-factor models for both average and continuous returns. Moreover, the RF theory also provides a model to estimate the asset volumes. The average change in asset volumes together with average returns provide the estimates for average change in market values of assets. Thus, the RF model approach can be used to select assets that provide either highest returns for profit maximization or highest change in market values for wealth maximization for given levels of risk.

KEYWORDS
Asset pricing; average returns; rational function model; CAPM; Fama French 3 and 5 factor models; asset volumes

JEL CLASSIFICATION
G11; G12

I. Introduction

Existing asset pricing models are based on a linear relation between expected returns of financial assets and relevant explanatory factors. This includes both theoretically derived models, such as the Capital Asset Pricing Model (CAPM: Sharpe 1964; Lintner 1965) and Arbitrage Pricing Theory (Ross 1976) as well as more recent empirically motivated models beginning with the Fama and French (1993) three-factor model and continuing with other factor based models (see, for example, Aharoni et al. 2013; Novy-Marx 2013; Fama and French 2015). However, these asset pricing models often fail to provide an accurate description of expected returns when compared to actual values. Indeed, the development of the empirical asset pricing literature arose from the inability of CAPM to explain the cross-sectional differences between the stock returns (starting with the value-growth anomaly of Basu 1977 and the size anomaly of Banz 1981). This paper seeks to contribute to the literature by focusing on the basic definition of asset returns and the price-quantity framework that defines any free market. We argue that since an asset return is a quotient of two consecutive asset prices that are themselves polynomials of various factors, the return is a rational function and hence does not add linearly within a portfolio. This motivates one to consider an alternative expected return model that improves our understanding of asset price movements.

Our work adds to the capital asset pricing literature by explaining the discrepancies between the estimations of existing asset pricing models and actual average return values by questioning a basic assumption that lies behind these models. We question whether stock returns add linearly across portfolios or across time and whether they are linearly related to index returns. We address this through a theoretical re-examination of the stock market behaviour by considering first an individual security and then the market portfolio. We attempt to explain asset pricing through the Rational Function (RF) model, which asserts that returns do not add linearly when averaging. This means that we should not model returns directly using linear regression techniques but rather model prices and then compute average returns from the estimated prices.

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Our findings show that the RF model not only outperforms the traditional linear models like the CAPM and the Fama and French 3 and 5 Factor models but follows the actual average returns so closely that it seems that it has finally solved the CAPM enigma. Thus, the CAPM anomalies were mainly arising due to averaging errors. Further, apart from the averaging concept, the RF theory also relies on the theoretical link between price and quantity (called trading volume in the stock market) – the two fundamental forces of any free market. This logic is very much in the spirit of earlier endeavours to base asset pricing fundamentals on price–volume relationships (Lo and Wang 2000, 2006; Wang 2002). Accordingly, the RF theory leads to a model for estimating asset volumes as well. We have compared the performance of our RF model for volumes with that of the Lo and Wang 2 Factor (LW2F) model and again, the RF model estimates for average change in volumes are found to be more accurate. These findings bring very important implications for the investment managers and the investors as the RF model introduces a powerful technique to study financial assets that can help in making better investment decisions.

The remainder of the paper is organized as follows: Section 2 reviews the popular asset pricing models for estimating returns and volumes, while Section 3 introduces the RF model by establishing its theoretical base and attributes. Section 4 conducts an empirical analysis to compare the performance of the abovementioned models against that of the RF model. Section 5 highlights the practical implications of the RF model while Section 6 concludes the paper.

II. Literature review

According to the Sharpe–Lintner CAPM equation, the relation between the expected return of asset i, denoted as $E(R_i)$, and the market beta $\beta_{i,m}$ is linear and given by:

$$E(R_i) = R_f + \beta_{i,m}\left[E(R_m) - R_f\right] \tag{1}$$

Asset i represents either an individual stock or a portfolio, while R_f and $E(R_m)$ denote the risk-free rate of return and the expected market return, respectively. However, several empirical studies have consistently found that the CAPM average returns are lower than the actual average returns for lower risk assets, whereas they are higher than the actual average returns for higher risk assets (see, for example, Jensen 1968; Blume 1970; Fama and French 1993, 1996, 2004). This led to the articulation of the Joint Hypothesis Problem by Fama (1970), which attributed the discrepancies between the actual and the CAPM average returns to a flawed asset pricing model and/or market inefficiencies.[1]

Fama and French (1993) provided empirical evidence that a single factor encapsulating risk does not adequately explain cross-sectional differences in stock returns. This finding motivated a substantial research agenda on asset pricing that ran alongside a debate regarding the fundamental relationship between risk and return (Baker and Haugen 2012). Fama and French (1993, 1996, 2004) introduced a three-factor model (FF3F) that included a size and value premium to explain variations in stock returns. These two additional factors were included due to their supportive empirical evidence, even though their underlying theoretical rationales remain unclear. Nevertheless, a considerable amount of subsequent research has been devoted to investigating the explanations for these factors (recent examples include Erdos et al. 2011; Dempsey 2013; Elgammal and McMillan 2014; Elgammal et al. 2016; Bao et al. 2017).

Various empirical studies examining cross-sectional variations in stock returns have reported patterns unexplained by CAPM, commonly referred to as anomalies. Based on these anomalies, various investment styles have been developed. These investment styles are based on size (Banz 1981; Fama and French 1993), value/growth (Basu 1977; Fama and French 1992, 1993), momentum and reversals (Jegadeesh and Titman

[1]A related issue concerns the view that the CAPM is an ex ante concept, whereas the tests of the CAPM are conducted ex post. Roll (1977) concludes that the CAPM tests are invalid because they use inefficient benchmark portfolios. Both researchers and practitioners have an innate belief that ex ante risk matters and an ex ante risk premium exists, even if, ex post, such a belief is not empirically validated (Roll and Ross 1994; Diacogiannis and Feldman 2013).

1993; O' Keeffe and Gallagher 2017), liquidity (Haugen and Baker 1996; Datar, Naik, and Radcliffe 1998), profitability and investment (Fama and French 2015). However, these models lack a robust economic explanation for their precise nature. Recently, Fama and French (2015) introduced a five-factor model (FF5F) that adds profitability and investment factors to their FF3F model. This model in turn builds on the work of Hou et al. (2015) and Novy-Marx (2013). However, Fama and French (2015) concede that, although, the five-factor model (FF5F) may outperform the three-factor model (FF3F), neither of these models provide a complete description of the expected returns. Indeed, Fama and French (2015) admit that their and the other models have difficulty explaining the behaviour of small stocks.

Given the above gap, this paper introduces a Rational Function (RF) model which argues that asset returns do not add linearly in a portfolio because returns are rational functions (i.e. ratios) of asset prices which themselves are polynomials of various relevant variables. These relevant variables include asset volumes, preceding asset price, time trends and other market factors in addition to the market return. For identifying the relevant variables, the RF theory draws from the price-volume framework in a free market environment which then logically mirrors the price model into a volume model as well. The RF model for volumes addresses another gap in the literature that being the lack of attention in modeling volumes as compared to modeling returns. A few others (Lo and Wang 2000, 2006; Wang 2002) have also linked both price and volume to economic fundamentals for asset pricing. Lo and Wang (2000) have introduced a two factor (LW2F) model that estimates asset volumes by the volumes of two portfolios, one being the market portfolio and the other being the hedging portfolio. The former is an equal-weighted index while the latter is a share-weighted index. Though Lo and Wang (2000) have estimated asset returns also using their two-portfolio model, they found its performance could not beat that of the FF3F model. Hence, we consider the LW2F for comparing the performance of the RF volume model only and did not include it for comparison with the RF returns models. Thus, this paper ends by comparing the RF returns models with the CAPM, the FF3F and the FF5F models (which remain the dominant models used in the literature by academics as well as practitioners) and by comparing the RF volume model with the LW2F model.

III. Rational function model (RF): theoretical development

We begin by revisiting the basics of price determination for a publicly traded asset, viz., the demand and supply framework as advocated by Lo and Wang (2000), Wang (2002) etc. Following the laws of increasing marginal cost for the supply curve and decreasing marginal utility for the demand curve (jointly, the laws of diminishing returns), we have an exponential supply curve and a logarithmic demand curve. Here, a fixed quantity of issued stocks limits the supply while the demand for stocks is limited by the amount of money available for investment in the economy. These principles were first stated by Cournot (1838) currently available in his textbook published in 1897, and further developed by Marshall (1890). Mathematically, we express the exponential supply curve of stock '*i*' on day '*t*' as follows:

$$p_{s,t} = a_s + b_s\left[exp\left(v_{s,t}\right)\right] \tag{2}$$

where $p_{s,t}$ is the stock price that corresponds to the stock trading volume $v_{s,t}$, a_s is the minimum price at which the supply curve starts and b_s is the linear regression coefficient.

The law of decreasing marginal utility is shaped by the 'willingness' and the 'ability' of investors to buy stocks. According to this law, the elasticity of demand is best expressed by a decreasing logarithmic demand curve where the decrease in utility (or price-willingness) for increasing quantity is sharp at first and hereafter gradually flattens out with further increase in quantity or volume. This has been captured in Equation (3) where the logarithmic demand curve for stock *i* on day *t* is given as follows:

$$p_{d,t} = a_d - b_d\left[ln\left(v_{d,t}\right)\right] \tag{3}$$

where $p_{d,t}$ is the price that corresponds to volume $v_{d,t}$, a_d is the minimum price at which the demand curve starts and b_d is the linear regression coefficient. Investors allocate their limited wealth to obtain the greatest satisfaction or utility (See, Samuelson and Nordhaus 2001). At the point of market equilibrium, Equations (2) and (3) intersect each other, and we have:

$$p_{s,t} = p_{d,t} = p_{i,t};\ and\ v_{s,t} = v_{d,t} = v_{i,t} \tag{4}$$

Taking an average of Equations (2) and (3) at the point of equilibrium and substituting Equation (4) leads us to Equation (5):

$$p_{i,t} = (a_s + a_d)/2 + (b_s/2)\left[exp\ \left(v_{i,t}\right)\right] - (b_d/2)[ln\ (v_{i,t})] \tag{5}$$

Equation (5) can be further generalized by representing the constants and the slope coefficients by single terms, as follows:

$$p_{i,t} = a_i + b_{i1}\left[exp\ \left(v_{i,t}\right)\right] - b_{i2}[ln\ (v_{i,t})] \tag{6}$$

Here, a_i is a constant, while b_{i1} and b_{i2} are the coefficients of the exponential and logarithmic values of the stock volume. This equation, thus, expresses the basic relationship between the equilibrium price and the equilibrium volume of a stock 'i'. Next, we consider the situation where numerous stocks are traded simultaneously, which collectively influence the demand and supply of any given stock through various common economic, business or technological factors. In such a case, the prices and volumes of multiple stocks may be correlated. Therefore, we expand our analysis by including a market portfolio m, the price of which on day 't' is given by:

$$p_{m,t} = \sum\nolimits_{j=1}^{w} q_j p_{j,t} \tag{7}$$

where $j = 1$ to w, including i, thus representing all stocks contained in combination m, while q_j denotes the corresponding weight of the stock j in combination m. Thus,

$$\sum\nolimits_{j=1}^{w} q_j = 1 \tag{8}$$

From Equation (7) we can see that $p_{m,t}$ is a linear aggregate of $p_{i,t}$ and other stock prices on day 't'. However, the price of some stocks may be correlated with that of stock i and so $p_{i,t}$ would vary in response to a change in the price of any one of these stocks. By representing the relationships between $p_{i,t}$ and those of other stocks (whether linear, non-linear or uncorrelated) into linear functions that describe them most closely and then adding these resulting simultaneous equations, we get a linear relationship between $p_{i,t}$ and $p_{m,t}$. Considering that both $p_{i,t}$ and $p_{i,t-1}$ are correlated with $p_{m,t}$ and $p_{m,t-1}$ on days 't' and '$t-1$' respectively, their ratio or the 'rate of change of price' of stock 'i' (i.e. $p_{i,t}/p_{i,t-1}$) would also be correlated with the 'rate of change of price' of the market combination m (i.e. $p_{m,t}/p_{m,t-1}$).[2] This linear relationship between $p_{i,t}/p_{i,t-1}$ and $p_{m,t}/p_{m,t-1}$ can be expressed in a generalized form as follows:

$$\left(p_{i,t}/p_{i,t-1}\right) = q_i + r_i\left(p_{m,t}/p_{m,t-1}\right) \tag{9}$$

Here, q_i is the constant and r_i is the slope. Although Equation (9) looks like a generalized form of the CAPM, its objective is different, in the sense that it aims to estimate asset prices and not returns. Prior to our work, there exists earlier research, which linked capital asset pricing models to the change in the prices. Hagerman and Kim (1976) develop a capital asset-pricing model that relates risk and return under conditions of changing price levels. Similarly, Long (1974) has shown a connection between commodity prices and equilibrium stock prices.

Taking the price equivalent for $p_{i,t}$ and adding the term $[q_i\ p_{i,t-1} + r_i\ \{(p_{m,t}/p_{m,t-1})\ p_{i,t-1}\}]$ obtained from Equation (9) with Equation (6) and generalizing, we get:

$$p_{i,t} = k_i + s_{i1}\left\{\left(p_{m,t}/p_{m,t-1}\right)p_{i,t-1}\right\}+)] + s_{i2}p_{i,t-1} + s_{i3}exp\ \left(v_{i,t}\right) - s_{i4}ln\ \left(v_{i,t}\right) \tag{10}$$

Equation (10) outlines the relation between the price of an individual stock 'i', the prices of the market combination 'm', the preceding price of

[2]Let us consider a hypothetical market where there are only three stocks trading – stocks 1, 2 and 3. Then, if their daily prices are denoted by the variables $p_{1,t}$, $p_{2,t}$ and $p_{3,t}$, respectively, and if $p_{1,t}$ is correlated with $p_{2,t}$ and uncorrelated with $p_{3,t}$, we may express these relationships in linear functions that describe them most closely as follows: $p_{2,t} = k' + b'\ (p_{1,t})$…(i); $p_{3,t} = k'' + 0\ (p_{1,t})$ …(ii); and finally $p_{1,t} = 0 + (p_{1,t})$ …(iii). Assuming the market combination 'm' to be a simple average of the three stock prices and averaging the Equations (i), (ii) and (iii), we get: $(p_{1,t}+p_{2,t}+p_{3,t})/3 = p_{m,t} = (k'+k'')/3 + \{(1 + b')/3\}(p_{1,t})$ …(iv), where $p_{m,t}$ is the price of the market combination 'm'. The Equation (iv) indicates a linear relationship between $p_{m,t}$ and $p_{1,t}$ which may be generalized as $p_{i,t} = c_i + d_i\ p_{m,t}$…(v). This logic also holds for day '$t-1$' when $p_{i,t-1} = c_i + d_i\ p_{m,t-1}$…(vi). From equations (v) and (vi), we may deduce that the ratios $(p_{i,t}/p_{i,t-1})$ and $(p_{m,t}/p_{m,t-1})$ are also correlated which can be expressed as $(p_{i,t}/p_{i,t-1}) = q_i + r_i\ (p_{m,t}/p_{m,t-1})$…(vii).

the stock '*i*' and the volume of stock '*i*' under economic equilibrium in the market.

Besides the asset volume and market return, an asset price may be affected by other market factors, such as firm size, book to market ratios, etc. (Fama and French 1993, 1996, 2004, 2015) as well as time and seasonal trends (Gallant, Rossi, and Tauchen 1992; Chen, Firth, and Rui 2001; Pisedtasalasai and Gunasekarage 2007). The presence of the time trends in asset pricing can be explained by the seasonality of sales or the regularity of certain anticipated events, such as earnings or dividend announcements. Accordingly, a time trend component can be expressed by the log function of the chronological order of the observation within the data sample (Pisedtasalasai and Gunasekarage 2007). Incorporating other market factors and the time trend in Equation (10), we get the following resulting equation defining the stock price $p_{i,t}$:

$$\begin{aligned} p_{i,t} = {} & \alpha_i + \beta_{i1}\left\{\left(p_{m,t}/p_{m,t-1}\right)p_{i,t-1}\right\} \\ & + \beta_{i2}p_{i,t-1} + \beta_{i3}exp\left(v_{i,t}\right) \\ & + \beta_{i4}ln\left(v_{i,t}\right) + \beta_{i5}ln\left(t_t\right) + \beta_{i6}MF_{i,t} \end{aligned} \tag{11}$$

Here, $MF_{i,t}$ represents other market factors and $ln(t_t)$ is the time-trend component. Equation (11) reflects a conceptual framework for estimating the price of stock '*i*' using firm, market and time parameters. From Equation (11), we can clearly infer that the stock return $R_{i,t}$ is not a linear polynomial function of the return of the market combination $R_{m,t}$ but rather a quotient of two polynomials. This follows simply from the definition of return which requires it to be a ratio of the change in the price on day '*t*' to the price on day '*t-1'*. Thus, we have:

$$R_{i,t} = \frac{\begin{array}{c}\left[\left\{\alpha_i + \beta_{i1}\left\{\left(p_{m,t}/p_{m,t-1}\right)p_{i,t-1}\right\} + \beta_{i2}p_{i,t-1} + \beta_{i3}exp\left(v_{i,t}\right) + \beta_{i4}ln\left(v_{i,t}\right) + \beta_{i5}ln\left(t_t\right) + \beta_{i6}MF_{i,t}\right\} - \right. \\ \left. \alpha_i + \beta_{i1}\left\{\left(p_{m,t-1}/p_{m,t-2}\right)p_{i,t-2}\right\} + \beta_{i2}p_{i,t-2} + \beta_{i3}exp\left(v_{i,t-1}\right) + \beta_{i4}ln\left(v_{i,t-1}\right) + \beta_{i5}ln\left(t_{t-1}\right) + \beta_{i6}MF_{i,t-1}\right\}\right]\end{array}}{\left[\alpha_i + \beta_{i1}\left\{\left(p_{m,t-1}/p_{m,t-2}\right)p_{i,t-2}\right\} + \beta_{i2}p_{i,t-2} + \beta_{i3}exp\left(v_{i,t-1}\right) + \beta_{i4}ln\left(v_{i,t-1}\right) + \beta_{i5}ln\left(t_{t-1}\right) + \beta_{i6}MF_{i,t-1}\right]} \tag{12}$$

Equation (12) shows that the relationship between stock return $R_{i,t}$ and market return $R_{m,t}$ is not linear, as suggested by the CAPM, but non-linear because $R_{i,t}$ is a rational function of two consecutive readings of the index return, the preceding asset price, asset volume, time trend and other relevant market factors.

Further, like Equation (11), it can be shown that the asset volume $v_{i,t}$ is given by:

$$\begin{aligned} v_{i,t} = {} & \gamma_i + \delta_{i1}\left\{\left(v_{m,t}/v_{m,t-1}\right)v_{i,t-1}\right\} \\ & + \delta_{i2}v_{i,t-1} + \delta_{i3}exp\left(p_{i,t}\right) + \delta_{i4}ln\left(p_{i,t}\right) \\ & + \delta_{i5}ln\left(t_t\right) + \delta_{i6}MF_{i,t} \end{aligned} \tag{13}$$

Here $v_{m,t}$ is the volume of market combination *m* on day '*t*' and like $p_{m,t}$ is a weighted aggregate of the volumes of all the stocks trading in the market. This shows that both price and volume of an asset can be estimated using the RF theory.

Klassen and McLaughlin (1996) used cumulative abnormal returns (CARs) obtained during an event period and the number of shares issued by firms multiplied by its preceding share price to arrive at the 'market value' of various environmental events like award and crisis announcements. We have built on this concept by replacing CARs by average returns and the preceding firm value (i.e. number of issued shares multiplied by preceding share price) by average change in volumes, to estimate average change in market value of an asset during a time period. From this, it can be deduced that investors who just wish to maximize profits should focus on maximizing return $R_{i,t}$ but investors who wish to maximize wealth and not just profits (like corporates) should aim at maximizing the change in market value of their portfolio. The percentage change in market value $\Delta MV_{i,t}$ is given by the arithmetic product of the percentage terms of the changed price and changed volume minus 1 and is given as follows:

$$\Delta MV_{i,t} = \left[\left(1 + R_{i,t}\right)\left(1 + V_{i,t}\right) - 1\right] \tag{14}$$

Here, for empirical application of Equations (12) and (14), we take $R_{i,t} = ln\ (p_{i,t}/p_{i,t-1})$ and $V_{i,t} = ln\ (v_{i,t}/v_{i,t-1})$. However, we must clarify here that Equations (11) and (13) are conceptual models that need to be refined through empirical validation. In this study, we have considered the variables identified in Equations (11) and (13) and examined their behaviour in an empirical context, thus deriving the final RF models from both theoretical rationales and empirical evidence. In developing the final empirical RF model for returns, we distinguish between two types of asset returns – average asset returns and continuous asset returns. This is because average asset returns exhibit greater non-linearity due to the effect of averaging over multiple time intervals and hence these two types of asset returns need to be modeled differently. The next section presents the empirically adjusted RF models for both average and continuous returns and tests their performance in comparison with the existing models like the CAPM, the FF3F and the FF5F. It also presents the empirically adjusted RF model for average change in volumes and compares it with the LW2F. We have not considered continuous change in volumes for single time intervals because volumes are much more volatile than returns and studying them on a continual basis would be of little practical value.

IV. Empirical models and methodology

We test the validity of the RF Models against the Capital Asset Pricing Model (CAPM), the Fama–French 3 Factor model (FF3F) and the Fama–French 5 Factor model (FF5F). Our tests utilize twenty-one monthly and daily samples from three international markets (the USA, Australia and India) from May 2003 to April 2013. This ten-year time frame provides us with sufficient data for the study, noting, in particular, the unavailability of reliable data on trading volumes for periods much earlier than this, especially for the Indian market.

Empirical models

The CAPM regression equation is:

$$R_{i,t} = R_{f,t} + \beta_{i,m}\left(R_{m,t} - R_{f,t}\right) + e_{it} \tag{15}$$

while the FF3F model regression equation is:

$$R_{i,t} = R_{f,t} + \beta_{i,m}\left(R_{m,t} - R_{f,t}\right) + \beta_{i,s}SMB_t + \beta_{i,h}HML_t + e_{it} \tag{16a}$$

The FF5F model, which we consider only for the US market, is:

$$R_{i,t} = R_{f,t} + \beta_{i,m}\left(R_{m,t} - R_{f,t}\right) + \beta_{i,s}SMB_t + \beta_{i,h}HML_t + \beta_{i,r}RMW_t + \beta_{i,c}CMA_t + e_{it} \tag{16b}$$

Here, the return of asset 'i' on day 't' is denoted by $R_{i,t}$, while $R_{f,t}$ and $R_{m,t}$ denote the risk-free rate of return and the market return on day 't' respectively. SMB_t (for Small minus Big) is the difference between the returns of the diversified portfolios of small and big size stocks. HML_t (High minus Low) is the difference between the returns of the diversified portfolios of high and low BE/ME (ratio of book equity to market equity) stocks. RMW_t (Robust minus Weak) is the difference between the returns of the diversified portfolios of robust and weak profitability. CMA_t (Conservative minus Aggressive) is the difference between the returns of the diversified portfolios of low and high investment firms while e_{it} is a zero-mean residual term. The data for SMB_t, HML_t, RMW_t and CMA_t are obtained from Kenneth French's data library.[3]

As already mentioned, we examine the asset returns in two different formats – average returns and continuous returns. Our preliminary empirical tests indicate that non-linearity due to the rational function nature is quite pronounced in average returns, whether they are averaged across portfolios or across time. Citing other empirical studies, Fama and MacBeth (1973) have reported evidence of stochastic non-linearities in average returns from period to period. Similarly, Fama and French (2004) have reported evidence that the relationship between average return and market beta was somewhat non-linear. However, since continuous returns are computed as time series of daily or monthly returns based on single time intervals they do not exhibit the non-linear attributes of rational functions. As a result, the continuous returns behave 'approximately'

[3]We thank Kenneth French for making the data available at: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

linearly across time. This argument is consistent with the evidence in the literature that continuous returns can be estimated using linear models (please see, Brown and Warner 1985; Chen and Epstein 2002). Hence, continuous returns have been modelled directly and not indirectly from prices as we do with average returns (see, Reinganum 1982; Pandey et al. 1998).

To examine the average returns, the conceptual RFM Equation (11) is refined empirically to simplify it without loss of information. As discussed above, the RFM average return is not computed linearly across time but is calculated as a ratio of two consecutive average prices obtained for intervals (*t* to *t* + *n-1*) and (*t* + *1* to *t* + *n*), respectively. Our preliminary estimation of the asset prices shows that the preceding asset price and the index return can explain asset prices by themselves (with the coefficient of determination, R^2 above 93.5%). Thus, our final empirical models for estimating asset prices are as given below:[4]

$$RF_{1a}: \ p_{i,t} = \alpha_i + \beta_i\left[\left\{\left(p_{m,t}/p_{m,t-1}\right)p_{i,t-1}\right\}\right] + e_{it} \tag{17a}$$

$$RF_{1b}: \ p_{i,t} = \beta_i\left[\left\{\left(p_{m,t}/p_{m,t-1}\right) \ p_{i,t-1}\right\}\right] + e_{it} \tag{17b}$$

The intercept α_i is included in Equation (17a) while it is assumed to be zero for Equation (17b). In Equation (17a), the intercept α_i, which is a risk-free component of asset prices, is included for the sake of consistency with the CAPM, FF3F and FF5F models. However, we include a zero-intercept version of the model, as this would be the lowest price that an asset could trade at. The zero-intercept model is also consistent with the empirical values of the average risk-free rate of return $R_{f,t}$ that have been negligible or zero in the literature (e.g. Mehra and Prescott 1985). According to the RF theory, the risk factor identified for the asset prices is the arithmetic product of the market return and the preceding asset price $[\{(p_{m,t}/p_{m,t-1})p_{i,t-1}\}]$ as shown in Equations (17a) and (17b). The RF theory thus indicates that there is no direct linear relationship between the average returns and the risk factor, since the average returns are rational functions. However, the assets can be separately sorted based on the risk factors as identified by the CAPM, the FF3F or the FF5F model (or any other model for that matter) and the average returns of the portfolios thus formed could then be estimated using the RF models RF_{1a} and RF_{1b}.

After estimating average returns, we have also estimated average change in asset volumes ($V_{i,t}$) as discussed earlier in Equation (14). For this, just like average returns, we have computed average change in asset volumes for both LW2F and the RF volume model as a ratio of two consecutive average volumes obtained for intervals (*t* to *t* + *n-1*) and (*t* + *1* to *t* + *n*) respectively. The empirical model for estimating the asset volumes is a logical variation of Equation (17b), where we have substituted the price variables with the volume variables as shown below:

$$RF_{1c}: \ v_{i,t} = \delta_i\left[\left\{\left(v_{m,t}/v_{m,t-1}\right) \ v_{i,t-1}\right\}\right] + e_{it} \tag{17c}$$

Here, we have considered only the zero intercept model for estimating asset volumes, because there is no economic compulsion for a positive risk-free level of trading volume for an asset and in reality the asset volumes do dip to zero levels during various trading cycles.

Next, we have compared the RF_{1c} model with the LW2F model. From Lo and Wang (2000) we have taken the LW2F regression Equation as:

$$v_{i,t} = \delta_{i,1}v_{m',t} + \delta_{i,1}v_{h,t} + e_{it} \tag{18}$$

Here, $v_{m',t}$ and $v_{h,t}$ are the volumes of the equally weighted and share-weighted indices, respectively. The volume of the equally weighted portfolio $v_{m',t}$ is a simple average of the constituent stock volumes of the index on day '*t*'. For $v_{h,t}$, Lo and Wang (2000) had computed the share-weight as the ratio of outstanding shares of a stock to the total number of outstanding shares of all stocks in the index. They then multiplied the share-weights with the stock volume and summed

[4]All results for the preliminary estimation of the asset prices and specification tests between different versions of the model are available upon request.

up all these share-weighted volumes to arrive at the volume of the hedging portfolio $v_{h,t}$ on day 't'. However, as these share weights remain constant for extended time-periods as the number of outstanding shares is constant over considerable time-periods, they tend to become meaningless for daily data being studied for small time windows like we have considered here in this study. Hence, instead of using number of outstanding shares, we have computed the share-weights as ratios of traded volume of a stock to the total traded volume of all the stocks in the index. Further, Lo and Wang (2000) had imposed the constraints that [$\delta_{i,1} + \delta_{i,2} = 1$] and [$\Sigma\ \delta_{i,1} = J$ for $i = 1$ to J] but they had tested the unconstrained version of the model as well and found its results comparable to the constrained version. Hence, we have considered the unconstrained version of the LW2F model here for greater versatility.

The continuous returns behave approximately linearly across time as they are computed from data based on single time intervals and hence do not require averaging. Thus, continuous returns can be modeled directly from the relevant firm-specific and market-specific factors. Our empirical results, detailed below, indeed show that the linear models (CAPM, FF3F and FF5F) are found to provide reasonable estimates (e.g. above 70% correlation with the actual values) for the continuous returns, although the FF3F and FF5F estimates are marginally more accurate than the CAPM estimates. However, the accuracy of the FF3F and FF5F models for continuous asset returns can be further improved by including three additional factors based on the RF theory, namely, the change in market volume, the time factor and the preceding asset return. In keeping with Equation (11) we replace the term $MF_{i,t}$ (for other market factors) with SMB_t, HML_t, RMW_t and CMA_t, enabling us to combine the Fama-French models with the RF theory to give the following equations for continuous asset returns:

$$RF_{2a}:\ R_{i,t} - R_{f,t} = \beta_{i,m}(R_{m,t} - R_{f,t}) + \beta_{i,s}SMB_t + \beta_{i,h}HML_t + \beta_{i,v}(V_{m,t}) + \beta_{i,o}(t_t)^2 + \beta_{i,l}(R_{i,t-1}) + e_{it} \tag{19a}$$

$$RF_{2b}:\ R_{i,t} - R_{f,t} = \beta_{i,m}(R_{m,t} - R_{f,t}) + \beta_{i,s}SMB_t + \beta_{i,h}HML_t + \beta_{i,r}RMW_t + \beta_{i,c}CMA_t + \beta_{i,v}(V_{m,t}) + \beta_{i,o}(t_t)^2 + \beta_{i,l}(R_{i,t-1}) + e_{it} \tag{19b}$$

Equations (19a) and (19b) are modified versions of the FF3F and FF5F, respectively, by including the additional variables relevant to the RF theory as noted above. Here, $V_{m,t} = \ln(v_{m,t}/v_{m,t-1})$ and we have used it instead of change in asset volume $V_{i,t}$ because the latter is dependent on the former as shown in Equation (13). The time factor used for modeling continuous returns is based on the works of Pisedtasalasai and Gunasekarage (2007) and Chen et al. (2001), who show that the time series of trading volumes exhibit quadratic time trends. The continuous returns being single-interval time series like the trading volume data and being connected to the volumes through supply-demand framework, quadratic variations of time ranks have been used in estimating these continuous returns as well, which find support through our empirical results.

Thus, we study asset returns in two different formats as can be seen from Equations (17a) and (17b) for RF_1 models used for estimating average returns and Equations (19a) and (19b) for RF_2 models used for estimating continuous returns. We also study the average change in asset volumes using Equation (17c). Further, we compute and report two different kinds of actual average returns, denoted actual$_1$ and actual$_2$, for the sake of comparing the performances of the various models. Here, actual$_1$ average returns are computed from the ratio of average prices while

[5]For computing the RFM average returns, the time series of the estimated prices of the RF_{1a} and RF_{1b} models are averaged from intervals t to $t + n-1$ and from $t + 1$ to $t + n$ respectively, and then divided to obtain their ratios. Thus, the actual$_1$ average returns, the RF_{1a} average returns and the RF_{1b} average returns are computed from the ratios of average portfolio prices. However, the actual$_2$ average returns and the average returns estimated by the CAPM, FF3F and FF5F models are computed by following the standard practice by directly averaging the time series of the continuous returns for each portfolio, from $t + 1$ to $t + n$ intervals. The study of average returns is useful for plotting the multiple period risk-return profile of each asset.

[6]Different time series of n day-to-day or month-to-month continuous returns over a period of $t + 1$ to $t + n$ are computed for different asset portfolios. This format examines the time series of continuous asset returns across increasing time as computed from the same time series of actual portfolio prices that are used for the average returns format. The continuous returns are studied for both actual as well as estimated values as per the CAPM, the FF3F, the FF5F and the RFM equations. This format is useful in describing the single period contemporaneous asset returns across increasing risk and time.

actual$_2$ average returns are computed by directly averaging the time series of continuous returns.[5,6] The actual average change in asset volume has been computed like the actual$_1$ average returns.

Methodology

To test the RF models against the extant models, various sample portfolios have been obtained for three different markets – the USA, Australia and India. These markets have been selected to represent financial markets in different developing stages. Further, our data set has different time windows to demonstrate the empirical validity of the RF theory. The details of these samples are provided in Table 1. Of the 18 USA samples, the first 11 samples (S1 to S11) are constructed out of the constituent stocks of three indices of various sizes. These include Dow Jones Industrial Average (30 stocks) as on 30 April 2013; 395 stocks constituting the Barron's 400 (B400) index as on 1 August 2013; and 500 stocks constituting the S&P 500 index as on 1 August 2013. The monthly samples have been obtained for a ten-year period from May 2003 to April 2013, while the daily samples are from 12 December 2012 to 30 April 2013. Further, we have considered another seven samples of the USA stocks, S12 to S18 representing the Fama and French portfolios of monthly data constructed by sorting on different financial parameters for the above ten years period. This diversity in portfolio design has been used to demonstrate the robustness of the RF Model to various selection parameters. As both small and large sample analyses of stocks are needed to test out any new asset-pricing model, this paper has considered both.

Two more portfolios were constructed from the 100-pooled components of the S&P ASX 50 and the S&P ASX Mid-cap 50 indices. The Australian samples have been used to study their daily returns for 95 and 120-day time-windows contained within 12 April 2013 to 30 September 2013. Finally, one more sample of 30 stocks constituting the Bombay Stock Exchange (BSE) Sensex as on 1 January 2005 has been collected from the Indian capital market, with monthly returns from January 2002 to November 2009. The samples selected for this study, thus, represent a diversified view of the US, Australian and Indian markets over various time windows and the findings may be taken to be free from any selection bias.[7]

The FF5F model has been tested only for the US market due to data unavailability for the RMW_t and the CMA_t factors for Australia and India. Kenneth French's data library reports that the daily values of $R_{f,t}$ were negligible for Australia over our sample period and thus the $R_{f,t}$ values are taken to be zero for studying the Australian samples S19 and S20. The Australian FF3F factors SMB_t and HML_t were obtained by computing the difference in portfolio returns formed by sorting the 100 companies based on market capitalization and the BE/ME ratio, over the two quarters during April 2013 to September 2013.

The average change in asset volumes has been studied only for samples S1 to S11 and S19 to S21 where we could collect the volume data for the various stocks being considered. We could not do the volume analysis for the Fama–French portfolios S12 to S18 as their volume data are not available.

For the samples S1 to S11 (USA) and S19 to S21 (Australia and India), the stocks in each of these samples were sorted according to risk. The risk factor has been assessed in two ways, first, using the returns variance (for samples S1 to S8 and S19 to S21) and second, idiosyncratic volatility (for samples S9 to S11). More specifically, the returns variance has been measured by the variance of the stock returns through a rolling period of the last 12 observations. Similarly, idiosyncratic volatility has been measured by the variance of the residual e_{it} through a rolling time frame of the last 12 observations, where the residual e_{it} is obtained from the difference in actual stock returns and the estimated return from the market model:

[7]The data for the samples S1 to S11 (USA) and S19 and S20 (Australia) were collected from Norgate Investor Services databases. The Australian Size and BE/ME data were collected from the Australian Financial Review. The data for the samples S12 to S18 (USA), and the Fama–French factors for USA and India were collected from Kenneth R. French's data library. The Indian stock market data were collected from the Prowess Database provided by the Centre for Monitoring Indian Economy (CMIE).

Table 1. Portfolios description.

S. No.	Portfolios	Market	Data from	to	Type of Returns	Number of time intervals	Sorting Factor(s)	Market Proxy	Names
1	30 components of DJIA as on 30 April 2013	USA	30 May 2003	30 April 2013	Monthly	120	Returns Variance	DJIA	S1
2	30 components of DJIA as on 30 April 2013	USA	30 May 2003	30 April 2013	Monthly	120	Returns Variance	S&P 500	S2
3	30 components of DJIA as on 30 April 2013	USA	30 June 2005	30 April 2013	Monthly	95	Returns Variance	DJIA	S3
4	30 components of DJIA as on 30 April 2013	USA	30 June 2005	30 April 2013	Monthly	95	Returns Variance	S&P 500	S4
5	30 components of DJIA as on 30 April 2013	USA	12 December 2012	30 April 2013	Daily	95	Returns Variance	DJIA	S5
6	30 components of DJIA as on 30 April 2013	USA	12 December 2012	30 April 2013	Daily	95	Returns Variance	S&P 500	S6
7	396 components of B400 as on 1 August 2013	USA	12 December 2012	30 April 2013	Daily	95	Returns Variance	S&P 500	S7
8	500 components of S&P 500 as on 1 August 2013	USA	12 December 2012	30 April 2013	Daily	95	Returns Variance	S&P 500	S8
9	30 components of DJIA as on 30 April 2013	USA	30 May 2003	30 April 2013	Monthly	120	Idiosyncratic Volatility	S&P 500	S9
10	396 components of B400 as on 1 August 2013	USA	12 December 2012	30 April 2013	Daily	95	Idiosyncratic Volatility	S&P 500	S10
11	500 components of S&P 500 as on 1 August 2013	USA	12 December 2012	30 April 2013	Daily	95	Idiosyncratic Volatility	S&P 500	S11
12	Fama-French 5 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Industry	S&P 500	S12
13	Fama-French 6 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Size & Investment	S&P 500	S13
14	Fama-French 6 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Size & Long term reversals	S&P 500	S14
15	Fama-French 6 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Size and Momentum	S&P 500	S15
16	Fama-French 6 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Size and Operating profits	S&P 500	S16
17	Fama-French 6 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Size & Short term reversals	S&P 500	S17
18	Fama-French 6 Portfolios of All USA stocks	USA	30 May 2003	30 April 2013	Monthly	120	Size and BE/ME ratio	S&P 500	S18
19	100 components of S&P ASX 50 and S&P ASX Mid-Cap 50 as on 15 May 2013	Australia	20 May 2013	30 September 2013	Daily	95	Returns Variance	ASX All Ordinaries	S19
20	100 components of S&P ASX 50 and S&P ASX Mid-Cap 50 as on 15 May 2013	Australia	12 April 2013	30 September 2013	Daily	120	Returns Variance	ASX All Ordinaries	S20
21	30 components of BSE Sensex as on 1 January 2005	India	31 January 2002	30 November 2009	Monthly	95	Returns Variance	BSE Sensex	S21

$$R_{i,t} = \beta_{i,m} R_{m,t} + e_{it} \tag{20}$$

Furthermore, the stocks for the samples S1 to S11 and S19 to S21 were grouped into five sub-portfolios P1 to P5, according to risk. Each sub-portfolio contains 6 stocks for the DJIA, 79 stocks for the B400, 100 stocks for the S&P500, 20 stocks for the Australian samples and 6 stocks for the BSE Sensex. P1 consists of the lowest risk stocks while P5 contains the highest risk stocks. We also examine the full sample portfolio (P-full) consisting of all stocks. After sorting, actual stock prices were reconstructed from actual stock returns using a common base number (e.g. 100) to avoid any sudden or abrupt change in price after each sorting.

The seven samples S12 to S18 are the Fama–French samples of all actively traded US stocks. Sample S12 consists of all actively traded US stocks sorted into five industry-based sub-portfolios: consumer, manufacturing, hi-technology, health and others as P1 to P5, respectively. We have studied the value-weighted returns of these five sub-portfolios as well as the aggregate of these sub-portfolios, P-full. The value-weighted returns are converted to time series of prices from a base price of 100. Samples S13 to S18 consist of six sub-portfolios, P1 to P6, of value-weighted returns sorted on the basis of two-variables. First, the stocks have been sorted into two size portfolios and then each split further into three portfolios based on another financial parameter like investment, long-term reversal (based on prior 13–60 returns), momentum (based on prior 2–12 returns), operating profit, short-term reversal (based on prior 1–1 returns) and value. The breakpoint for the size variable is the median NYSE market equity, whereas the breakpoints for the other variables are the 30th and the 70th NYSE percentiles. As before, we converted the time series of value-weighted returns to time series of prices from a base of 100 for further analysis.

The above samples were analysed using the CAPM, FF3F, FF5F (for the US samples only), RF_1 (17a & 17b) and RF_2 (18a & 18b) models for both average and continuous returns. To obtain homogenous estimates from the RF_{1a} model across the sub-portfolios, P1 to P5, the intercept α_i is computed for the full sample (P-full) and used for estimating the prices for P1 to P5, for the samples S1 to S12 and S19 to S21. For the samples S13 to S18 (where we do not consider the aggregate portfolio), the intercept α_i is the arithmetic average of the intercepts of all the six sub-portfolios. In this way, we have a common intercept value for all the sub-portfolios in a given sample so that we can compare their performance. The empirical accuracy of estimated results from the CAPM, FF3F, FF5F and the RF models have been compared with each other based on their correlations with the actual returns as well as their sum of squared errors (SSE).[8] We have also carried out the volume analyses for the fourteen samples S1 to S11 and S19 to S21 using RF_1 (17c) and LW2F models and compared their results.

Empirical results and discussion

Tables 2, 3 and 4 report the slopes estimated by RF_{1a}, RF_{1b} and RF_{1c} models for the different samples. As can be seen, the values of β_i for the RF_{1a} and RF_{1b} models are stable. In fact, for the RF_1 models, the values of β_i are all positive and are all very close to 1.00. This maybe is because empirically, the change in the asset price $p_{i,t}$ is nearly equal to the change in market price $p_{m,t}$.[9] All the β_i coefficients have very high t-values for both RF_{1a} and RF_{1b} models in Tables 2 and 3. This implies that the arithmetic product of market return and the preceding asset price is the most important factor in estimating the asset prices and hence average asset returns. Another noteworthy finding from Table 2 is that the t-statistics of the intercepts α_i for the RF_{1a} model are all insignificant

[8] Obviously, our analysis has generated a large number of results that cannot be reasonably reported here. Thus, we report only the important results [i.e. the regression results of Equations (17a), (17b), (17c) and (19b); the results of correlation analysis and the SSE values for average returns and average change in volumes; results of correlation analysis for continuous returns; and finally the improvements in accuracy of estimation of continuous returns due to Equation (18b)]. Other results are available upon request.

[9] We also estimate the CAPM and the FF3F for all samples and the FF5F for the US data. However, we do not report these results due to space limitation. All results are available upon request. The estimated values of CAPM's $\beta_{i,m}$ increase with risk for portfolios sorted by risk, however, the values of $\beta_{i,m}$ for other portfolios do not follow any systematic pattern. The values of $\beta_{i,m}$, estimated by FF3F, FF5F, RF_{2a} and RF_{2b} models are generally increasing across risk-sorted portfolios.

Table 2. Slope coefficients and t-stats of the RFM Equation (17a – with intercept): RF_{1a}: $p_{i,t} = \alpha_i + \beta_i[\{(p_{m,t}/ p_{m,t-1})\ p_{i,t-1}\}] + e_{it}$.

	Slope coefficients across P1 to P-full				Slope coefficients across P1 to P-full		
Sample Portfolios	Sub portfolios	α_i	β_i	Sample Portfolios	Sub portfolios	α_i	β_i
S1	P1		1.00 (49.902)	S10	P1		0.99 (135.59)
	P2		1.00 (110.85)		P2		0.98 (89.093)
	P3		0.99 (101.55)		P3		0.98 (75.988)
	P4		0.97 (62.040)		P4		0.96 (58.225)
	P5		0.98 (61.915)		P5		0.96 (44.426)
	P-full	0.35 (0.46)	0.99 (170.84)		P-full	1.92 (1.60)	0.98 (81.112)
S2	P1		0.99 (39.986)	S11	P1		1.00 (144.28)
	P2		0.99 (97.915)		P2		1.00 (149.26)
	P3		0.99 (97.491)		P3		0.99 (145.45)
	P4		0.98 (65.704)		P4		0.98 (82.446)
	P5		0.99 (65.846)		P5		0.96 (50.963)
	P-full	0.11 (0.14)	0.99 (153.73)		P-full	0.46 (0.66)	0.99 (145.11)
S3	P1		1.00 (45.441)	S12	P1		1.00 (151.91)
	P2		1.00 (97.870)		P2		0.99 (102.36)
	P3		0.99 (93.150)		P3		0.99 (121.94)
	P4		0.98 (59.949)		P4		1.00 (72.912)
	P5		0.99 (51.117)		P5		0.99 (113.85)
	P-full	−0.45 (−0.69)	1.00 (176.79)		P-full	0.03 (0.11)	1.00 (431.53)
S4	P1		0.99 (35.812)	S13	P1		0.99 (84.023)
	P2		0.99 (85.160)		P2		0.99 (97.351)
	P3		0.99 (90.832)		P3	0.79 (0.51)	0.98 (67.908)
	P4		0.98 (62.910)		P4		1.00 (176.53)
	P5		1.00 (53.596)		P5		1.00 (300.28)
	P-full	−0.29 (−0.35)	1.00 (139.63)		P-full		0.99 (176.97)
S5	P1		0.99 (81.714)	S14	P1		0.98 (58.832)
	P2		1.01 (77.688)		P2		0.99 (102.56)
	P3		0.98 (99.304)		P3	1.32 (0.78)	0.99 (93.675)
	P4		1.01 (81.851)		P4		0.99 (70.892)
	P5		0.97 (45.708)		P5		1.00 (236.99)
	P-full	−0.14 (−0.30)	1.00 (221.37)		P-full		0.99 (106.76)
S6	P1		0.98 (61.889)	S15	P1		0.97 (74.616)
	P2		1.00 (86.040)		P2		0.99 (105.38)
	P3		0.97 (93.221)		P3	1.81 (1.04)	1.00 (74.499)
	P4		1.00 (83.234)		P4		0.97 (66.420)
	P5		0.97 (47.608)		P5		0.99 (263.86)
	P-full	0.28 (0.44)	0.99 (161.76)		P-full		1.00 (102.37)
S7	P1		0.99 (119.27)	S16	P1		0.98 (65.503)
	P2		0.98 (97.872)		P2		0.99 (101.19)
	P3		0.97 (69.931)		P3	0.83 (0.71)	0.99 (92.163)
	P4		0.97 (59.854)		P4		1.00 (118.36)
	P5		0.96 (42.219)		P5		1.00 (276.27)
	P-full	1.92 (1.58)	0.98 (80.612)		P-full		0.99 (252.70)
S8	P1		1.00 (128.44)	S17	P1		0.98 (66.220)
	P2		0.99 (152.71)		P2		0.99 (92.274)
	P3		0.99 (106.11)		P3		0.98 (72.263)
	P4		0.98 (83.984)		P4	1.29 (0.71)	0.99 (88.608)
	P5		0.97 (50.929)		P5		1.00 (245.88)
	P-full	0.46 (0.66)	0.99 (145.04)		P-full		0.99 (116.70)
S9	P1		0.96 (55.165)	S18	P1		0.99 (74.678)
	P2		1.00 (123.48)		P2		0.98 (91.969)
	P3		0.96 (62.214)		P3		0.98 (74.880)
	P4		0.99 (80.148)		P4	0.87 (0.70)	0.99 (209.34)
	P5		0.97 (52.576)		P5		0.99 (196.31)
	P-full	0.20 (0.23)	0.99 (149.50)		P-full		1.01 (92.183)
				S19	P1		0.98 (50.475)
					P2		0.99 (113.25)
					P3		1.00 (66.014)
					P4		1.00 (107.45)
					P5		1.00 (60.653)
					P-full	−1.06 (−1.31)	1.00 (142.21)
				S20	P1		1.01 (67.880)
					P2		0.99 (108.21)
					P3		1.00 (81.172)
					P4		0.99 (91.064)
					P5		0.99 (59.589)
					P-full	−1.05 (−1.22)	1.00 (135.29)
				S21	P1		0.96 (79.087)
					P2		0.99 (94.814)
					P3		1.00 (115.54)
					P4		1.00 (85.058)
					P5		0.99 (69.505)
					P-full	0.09 (0.21)	0.99 (227.75)

Table 3. Slope coefficients and t-stats of the RFM Equation (17b – without intercept): RF_{1b}: $p_{i,t} = \beta_i\ [\{(p_{m,t}/\ p_{m,t-1})\ p_{i,t-1}\}] + e_{it.}$

Sample Portfolios	Slope coefficients across P1 to P-full		Sample Portfolios	Slope coefficients across P1 to P-full		Sample Portfolios	Slope coefficients across P1 to P-full		Sample Portfolios	Slope coefficients across P1 to P-full		Sample Portfolios	Slope coefficients across P1 to P-full	
	Sub portfolios	β_i		Sub portfolios	β_i		Sub portfolios	β_i		Sub portfolios	β_i		Sub portfolios	β_i
S1	P1	1.00 (471.85)	S6	P1	0.99 (1654.1)	S11	P1	1.00 (5012.7)	S16	P1	1.00 (382.21)	S21	P1	0.98 (213.95)
	P2	1.00 (538.63)		P2	1.00 (2586.9)		P2	1.00 (5289.7)		P2	1.00 (492.61)		P2	0.99 (262.05)
	P3	1.00 (605.99)		P3	1.00 (2773.9)		P3	1.00 (4573.9)		P3	1.00 (420.93)		P3	1.00 (278.81)
	P4	0.99 (391.94)		P4	1.00 (2306.4)		P4	1.00 (3075.3)		P4	1.00 (717.52)		P4	0.99 (248.63)
	P5	0.99 (287.34)		P5	1.00 (1573.8)		P5	0.99 (2320.0)		P5	1.00 (1639.3)		P5	0.99 (167.51)
	P-full	0.99 (1238.0)		P-full	1.00 (4918.6)		P-full	1.00 (5365.0)		P-full	1.00 (1339.5)		P-full	0.99 (612.95)
S2	P1	1.00 (381.42)	S7	P1	0.99 (4468.4)	S12	P1	1.00 (680.51)	S17	P1	1.00 (318.36)			
	P2	1.00 (474.89)		P2	1.00 (3246.3)		P2	1.00 (479.14)		P2	1.00 (464.61)			
	P3	1.00 (582.88)		P3	1.00 (2850.2)		P3	1.00 (598.82)		P3	1.00 (426.21)			
	P4	0.99 (416.24)		P4	1.00 (2092.7)		P4	1.00 (402.58)		P4	0.99 (424.75)			
	P5	0.99 (303.81)		P5	1.00 (1605.3)		P5	0.99 (530.94)		P5	1.00 (1326.5)			
	P-full	0.99 (1121.8)		P-full	1.00 (2995.6)		P-full	1.00 (2532.6)		P-full	1.00 (654.61)			
S3	P1	0.99 (396.20)	S8	P1	1.00 (3932.7)	S13	P1	1.00 (369.70)	S18	P1	1.00 (421.14)			
	P2	1.00 (467.18)		P2	1.00 (5533.1)		P2	1.00 (481.73)		P2	1.00 (456.74)			
	P3	1.00 (533.59)		P3	1.00 (4015.8)		P3	1.00 (424.10)		P3	1.00 (375.93)			
	P4	0.99 (328.83)		P4	1.00 (3073.3)		P4	1.00 (1058.1)		P4	1.00 (1138.8)			
	P5	0.99 (228.41)		P5	0.99 (2063.0)		P5	1.00 (1768.1)		P5	1.00 (1152.5)			
	P-full	0.99 (1164.5)		P-full	1.00 (5353.9)		P-full	1.00 (1083.3)		P-full	1.00 (538.96)			
S4	P1	0.99 (315.91)	S9	P1	0.99 (476.39)	S14	P1	1.00 (308.03)	S19	P1	0.99 (2549.3)			
	P2	1.00 (405.38)		P2	1.00 (560.68)		P2	1.00 (477.37)		P2	1.00 (3271.4)			
	P3	1.00 (520.12)		P3	0.99 (502.39)		P3	1.00 (450.33)		P3	0.99 (2547.1)			
	P4	0.99 (344.51)		P4	0.99 (401.12)		P4	1.00 (496.31)		P4	1.00 (2147.1)			
	P5	0.99 (239.13)		P5	0.99 (288.13)		P5	1.00 (1439.6)		P5	0.99 (1129.9)			
	P-full	1.00 (919.87)		P-full	0.99 (1069.1)		P-full	1.00 (615.43)		P-full	0.99 (3994.0)			
S5	P1	0.99 (2178.2)	S10	P1	0.99 (5081.4)	S15	P1	1.00 (290.60)	S20	P1	1.00 (2996.2)			
	P2	1.00 (2329.3)		P2	0.99 (3312.8)		P2	1.00 (493.12)		P2	1.00 (3388.7)			
	P3	1.00 (2958.7)		P3	1.00 (2536.8)		P3	1.00 (391.74)		P3	0.99 (2801.2)			
	P4	1.00 (2280.6)		P4	0.99 (2140.7)		P4	0.99 (352.85)		P4	1.00 (1972.0)			
	P5	0.99 (1518.1)		P5	1.00 (1701.1)		P5	1.00 (1326.8)		P5	1.00 (1155.0)			
	P-full	1.00 (6797.1)		P-full	1.00 (3011.5)		P-full	1.00 (580.35)		P-full	1.00 (3972.8)			

and the intercepts themselves have quite small values (roughly within a range of −0.9% to +1.9% of the average asset prices). This supports the view that the zero intercept RF_{1b} model is both accurate and sufficient in determining the estimates of asset prices. This is consistent with other empirical evidence, whereby the risk-free rate of return $R_{f,t}$ has been found to be negligible or zero (see, Fama and French 2015). Moreover, we believe that a common intercept α_i could give erroneous results for long time series data since it would be large compared to the initial prices and small compared to the later prices.

From Table 4 we can see the results of regressing RF_{1c} Equation (17c) for estimating asset volumes where the values of δ_i coefficients are all positive but increase or decrease randomly across the sub-portfolios that have been formed according to increasing risk as well as other financial parameters. Unlike the price slopes which are all nearly equal to 1, the volume slopes lie between a range of 0.45 to 1.27. However, like the price slopes, the volume slopes also have high t-statistics indicating that $\{(v_{m,t}/v_{m,t-1})\ v_{i,t-1}\}$ is the most important as well as a sufficient factor in estimating asset volumes.

The results of regressing the RF_{2b} Equation (19b) are reported in Table 5. The regression results indicate that the $\beta_{i,m}$ values are generally significant and increasing with risk (lying in a range of 0.5 to 1.5) which show that the market return is very important in explaining continuous asset returns . Again, while the t-statistics for $\beta_{i,m}$ maybe the largest, it was noted that the t-statistics of the RF factors - change in index volume ($\beta_{i,v}$), time trend ($\beta_{i,o}$) and preceding asset return ($\beta_{i,l}$) are not dissimilar to those of the FF3F and FF5F factors. This indicates that the RF factors are also important contributors in the assessment of continuous returns.

Next, we compare the estimated average returns from different models with the actual average returns to test the accuracy of each model. As mentioned earlier, two types of actual average returns have been computed - actual_1 and actual_2, where the former are computed from the ratios of two consecutive average prices, while the latter have been computed by directly averaging the time series of continuous

Table 4. Slope coefficients and t-stats of the RFM Equation (17c – volume estimation): RF_{1c}: $v_{i,t} = \delta_i\ [\{(v_{m,t}/\ v_{m,t-1})\ v_{i,t-1}\}] + e_{it}$.

	Slope coefficients across P1 to P-full			Slope coefficients across P1 to P-full			Slope coefficients across P1 to P-full	
Sample Portfolios	Sub portfolios	δ_i	Sample Portfolios	Sub portfolios	δ_i	Sample Portfolios	Sub portfolios	δ_i
S1	P1	1.11 (42.54)	S6	P1	1.03 (56.93)	S11	P1	1.02 (31.70)
	P2	0.86 (20.81)		P2	1.02 (48.11)		P2	1.22 (45.12)
	P3	1.27 (69.49)		P3	0.97 (49.80)		P3	0.89 (57.38)
	P4	0.96 (42.12)		P4	0.96 (55.16)		P4	0.89 (54.59)
	P5	0.88 (28.43)		P5	0.92 (51.70)		P5	0.90 (59.24)
	P-full	1.15 (55.20)		P-full	0.96 (73.89)		P-full	0.91 (71.38)
S2	P1	1.13 (43.84)	S7	P1	1.13 (47.02)	S19	P1	1.00 (20.36)
	P2	0.90 (22.75)		P2	1.07 (35.13)		P2	0.80 (13.45)
	P3	1.27 (72.35)		P3	1.19 (36.63)		P3	0.72 (10.31)
	P4	1.00 (52.11)		P4	1.11 (22.35)		P4	0.84 (18.32)
	P5	0.89 (25.30)		P5	0.88 (99.53)		P5	0.88 (18.92)
	P-full	1.16 (66.36)		P-full	0.90 (77.65)		P-full	1.00 (20.36)
S3	P1	0.99 (30.46)	S8	P1	1.19 (42.46)	S20	P1	0.77 (21.82)
	P2	0.87 (18.62)		P2	1.03 (15.72)		P2	0.99 (23.69)
	P3	1.24 (51.65)		P3	1.00 (81.88)		P3	0.86 (15.51)
	P4	0.92 (37.90)		P4	0.93 (45.24)		P4	0.92 (30.57)
	P5	0.83 (25.16)		P5	0.91 (70.27)		P5	0.96 (29.22)
	P-full	1.00 (29.57)		P-full	0.92 (52.99)		P-full	0.77 (21.82)
S4	P1	1.05 (45.33)	S9	P1	1.05 (43.86)	S21	P1	1.27 (9.77)
	P2	0.90 (20.43)		P2	0.92 (38.56)		P2	0.45 (6.05)
	P3	1.25 (59.40)		P3	0.92 (43.62)		P3	0.73 (15.00)
	P4	0.97 (49.49)		P4	0.97 (29.33)		P4	0.66 (10.45)
	P5	0.91 (34.30)		P5	0.76 (23.37)		P5	0.53 (33.72)
	P-full	1.03 (34.71)		P-full	0.95 (44.26)		P-full	0.83 (8.94)
S5	P1	1.02 (49.55)	S10	P1	1.18 (82.83)			
	P2	1.04 (46.18)		P2	1.05 (65.96)			
	P3	0.96 (44.20)		P3	1.03 (62.53)			
	P4	0.96 (47.72)		P4	0.92 (61.92)			
	P5	0.93 (53.14)		P5	0.93 (42.65)			
	P-full	0.96 (66.35)		P-full	0.93 (43.56)			

Table 5. Slope coefficients and their t-stats for the Combined RFM Equation (18b): $RF_{2b} : R_{i,t} - R_{f,t} = \beta_{i,m} (R_{m,t} - R_{f,t}) + \beta_{i,s} SMB_t + \beta_{i,h} HML_t + \beta_{i,r} RMW_t + \beta_{i,c} CMA_t + \beta_{i,v}[ln(v_{m,t}/v_{m,t-1})] + \beta_{i,o} (t_t)^2 + \beta_{i,l}(R_{i,t-1}) + e_{it.}$

		Slope coefficients across P1 to P-full							
Sample Portfolios	Sub portfolios	$\beta_{i,m}$	$\beta_{i,s}$	$\beta_{i,h}$	$\beta_{i,r}$	$\beta_{i,c}$	$\beta_{i,v}$	$\beta_{i,o}$	$\beta_{i,l}$
S1	P1	0.82 (13.88)	−0.19 (−2.19)	−0.09 (−1.01)	0.28 (1.919)	0.15 (1.007)	−0.010 (−2.04)	−4.23 (−0.11)	0.02 (0.331)
	P2	0.77 (13.82)	0.03 (0.352)	−0.06 (−0.74)	0.17 (1.285)	0.18 (1.215)	−0.005 (−1.26)	7.86 (2.193)	−0.03 (−0.51)
	P3	0.99 (17.52)	0.07 (0.866)	0.11 (1.216)	−0.2 (−1.78)	−0.08 (−0.54)	0.0054 (1.136)	0.00 (0.931)	0.05 (1.140)
	P4	1.14 (14.12)	0.16 (1.338)	−0.01 (−0.05)	−0.38 (−1.89)	−0.37 (−1.71)	−0.009 (−1.31)	1.58 (0.312)	0.0001 (0.003)
	P5	1.42 (14.18)	0.46 (2.995)	0.56 (3.525)	−0.25 (−1.03)	−0.28 (−1.04)	−0.002 (−0.26)	−1.33 (−2.12)	−0.001 (−0.03)
	P-full	1.02 (43.50)	0.10 (2.881)	0.08 (2.114)	−0.11 (−2.02)	−0.04 (−0.68)	−0.003 (−1.97)	3.35 (0.226)	−0.001 (−0.09)
S2	P1	0.77 (11.68)	−0.30 (−2.99)	−0.05 (−0.49)	0.37 (2.219)	0.29 (1.654)	−0.011 (−1.81)	−6.06 (−0.14)	0.0012 (0.021)
	P2	0.78 (13.65)	−0.08 (−1.02)	−0.04 (−0.45)	0.34 (2.341)	0.33 (2.193)	−0.005 (−0.96)	7.55 (2.092)	−0.071 (−1.32)
	P3	0.98 (16.75)	−0.07 (−0.81)	0.12 (1.319)	−0.04 (−0.28)	0.13 (0.853)	0.0067 (1.218)	2.35 (0.637)	0.019 (0.480)
	P4	1.18 (14.39)	−0.04 (−0.34)	0.01 (0.084)	−0.14 (−0.66)	−0.06 (−0.29)	−0.000 (−0.04)	−5.10 (−9.98)	−0.06 (−1.21)
	P5	1.44 (14.30)	0.21 (1.358)	0.56 (3.513)	0.06 (0.248)	0.11 (0.406)	0.0090 (0.949)	−1.58 (−2.49)	−0.02 (−0.48)
	P-full	1.03 (36.05)	−0.06 (−1.44)	0.09 (2.250)	0.089 (1.218)	0.20 (2.600)	0.0004 (0.180)	−7.76 (−0.43)	−0.04 (−1.99)
S3	P1	0.80 (12.92)	−0.24 (−2.34)	−0.07 (−0.73)	0.31 (1.894)	0.16 (0.947)	−0.014 (−2.48)	1.49 (0.206)	0.01 (0.108)
	P2	0.77 (13.00)	0.08 (0.769)	−0.10 (−0.98)	0.18 (1.196)	0.25 (1.482)	−0.006 (−1.18)	1.62 (2.313)	−0.04 (−0.64)
	P3	1.01 (16.54)	0.09 (0.921)	0.07 (0.712)	−0.24 (−1.49)	−0.02 (−0.12)	0.0056 (0.980)	4.79 (0.673)	0.04 (0.929)
	P4	1.19 (13.37)	0.012 (0.086)	−0.02 (−0.15)	−0.44 (−1.90)	−0.20 (−0.81)	−0.010 (−1.24)	−1.31 (−0.12)	−0.01 (−0.14)
	P5	1.42 (14.34)	0.622 (3.752)	0.96 (5.382)	−0.06 (−0.26)	−0.88 (−3.13)	0.0084 (0.927)	−1.87 (−1.64)	3.42 (0.000)
	P-full	1.02 (46.13)	0.083 (2.249)	0.14 (3.622)	−0.06 (−1.10)	−0.08 (−1.37)	−0.003 (−1.48)	1.28 (0.499)	−0.01 (−0.49)
S4	P1	0.7541 (10.80)	−0.37 (−3.08)	0.003 (0.023)	0.41 (2.152)	0.26 (1.257)	−0.013 (−1.97)	1.29 (0.157)	−0.01 (−0.09)
	P2	0.77 (12.63)	−0.06 (−0.63)	−0.04 (−0.44)	0.35 (2.144)	0.37 (2.110)	−0.003 (−0.59)	1.55 (2.150)	−0.07 (−1.19)
	P3	0.98 (16.09)	−0.08 (−0.84)	0.12 (1.134)	−0.01 (−0.10)	0.16 (0.890)	0.0044 (0.726)	3.02 (0.415)	0.017 (0.388)
	P4	1.22 (14.00)	−0.23 (−1.59)	0.04 (0.284)	−0.16 (−0.69)	0.05 (0.208)	0.0005 (0.068)	−4.57 (−0.45)	−0.06 (−1.28)
	P5	1.38 (13.33)	0.36 (1.989)	1.03 (5.504)	0.23 (0.823)	−0.58 (−1.91)	0.0150 (1.473)	−2.20 (−1.81)	−0.02 (−0.38)
	P-full	1.01 (33.10)	−0.11 (−2.06)	0.21 (3.884)	0.15 (1.824)	0.10 (1.129)	0.0010 (0.349)	−4.42 (−0.12)	−0.04 (−1.92)
S5	P1	0.884 (11.73)	−0.09 (−0.78)	−0.46 (−2.67)	0.02 (0.078)	0.49 (2.293)	0.0028 (1.345)	−1.52 (−1.17)	−0.18 (−2.86)
	P2	0.982 (12.19)	0.008 (0.069)	0.27 (1.512)	−0.07 (−0.25)	−0.10 (−0.45)	−0.001 (−0.67)	1.47 (1.052)	0.03 (0.431)
	P3	1.01 (15.84)	−0.03 (−0.32)	−0.09 (−0.64)	−0.12 (−0.55)	0.36 (1.989)	−0.001 (−0.72)	3.53 (0.318)	0.08 (1.550)
	P4	1.02 (13.69)	0.173 (1.466)	0.08 (0.473)	−0.37 (−1.43)	−0.67 (−3.13)	−0.002 (−1.05)	2.65 (2.028)	−0.07 (−1.39)
	P5	0.99 (9.009)	0.025 (0.144)	0.94 (3.757)	−0.18 (−0.47)	−0.94 (−3.00)	0.0023 (0.788)	1.117 (0.586)	−0.12 (−1.82)
	P-full	0.98 (39.19)	0.008 (0.222)	0.14 (2.576)	−0.15 (−1.73)	−0.18 (−2.61)	−5.174 (−0.07)	8.201 (1.880)	−0.04 (−1.76)
S6	P1	0.73 (9.194)	−0.16 (−1.16)	−0.56 (−2.79)	0.06 (0.215)	0.74 (2.872)	0.0036 (1.218)	−1.43 (−0.95)	−0.16 (−2.18)
	P2	0.94 (14.51)	−0.15 (−1.33)	0.17 (1.047)	0.21 (0.840)	0.29 (1.389)	−0.000 (−0.31)	1.06 (0.854)	0.05 (0.989)
	P3	0.95 (18.35)	−0.18 (−2.00)	−0.22 (−1.67)	0.13 (0.653)	0.75 (4.514)	0.0017 (0.884)	3.95 (0.039)	0.09 (2.128)
	P4	0.89 (11.91)	0.04 (0.347)	−0.015 (−0.08)	−0.25 (−0.85)	−0.35 (−1.45)	−0.003 (−1.25)	2.48 (1.730)	−0.02 (−0.42)
	P5	0.87 (8.327)	−0.09 (−0.49)	0.84 (3.227)	−0.02 (−0.05)	−0.61 (−1.83)	0.0011 (0.303)	1.02 (0.514)	−0.10 (−1.50)
	P-full	0.88 (30.65)	−0.11 (−2.37)	0.04 (0.573)	0.02 (0.212)	0.15 (1.645)	0.0002 (0.277)	6.08 (1.106)	−0.00 (−0.12)

(*Continued*)

Table 5. (Continued).

Sample Portfolios	Sub portfolios	Slope coefficients across P1 to P-full							
		$\beta_{i,m}$	$\beta_{i,s}$	$\beta_{i,h}$	$\beta_{i,r}$	$\beta_{i,c}$	$\beta_{i,v}$	$\beta_{i,o}$	$\beta_{i,l}$
S7	P1	0.89 (29.54)	0.26 (5.022)	−0.36 (−4.85)	−0.16 (−1.39)	0.27 (2.852)	0.0007 (0.687)	−4.22 (−0.74)	0.0131 (0.516)
	P2	0.98 (25.19)	0.57 (8.412)	−0.35 (−3.66)	−0.12(−0.82)	0.17 (1.398)	−0.001 (−0.79)	3.47 (0.472)	0.0397 (1.426)
	P3	0.96 (23.35)	0.68 (9.517)	−0.40 (−3.94)	−0.44 (−2.76)	0.01 (0.108)	−0.000 (−0.37)	1.78 (0.230)	−0.005 (−0.19)
	P4	1.03 (21.24)	0.91 (10.75)	−0.29 (−2.46)	−0.39(−2.12)	−0.15 (−0.96)	−0.004 (−2.32)	−2.04 (−0.22)	0.0280 (0.957)
	P5	1.06 (16.20)	1.20 (10.59)	−0.24 (−1.51)	−0.25 (−0.84)	0.28 (1.370)	−0.006 (−2.81)	8.81 (0.715)	−0.021 (−0.61)
	P-full	0.98 (31.76)	0.72 (13.46)	−0.33 (−4.32)	−0.26 (−2.21)	0.12 (1.240)	−0.002 (−2.13)	1.45 (0.249)	0.0139 (0.665)
S8	P1	0.85 (26.21)	0.03 (0.693)	−0.17 (−2.19)	0.01 (0.147)	0.49 (4.737)	0.0013 (1.112)	1.33 (2.152)	−0.011 (−0.37)
	P2	0.96 (32.85)	0.16 (3.253)	−0.10 (−1.43)	0.12 (1.131)	0.35 (3.793)	3.25 (0.029)	−7.40 (−0.13)	0.0164 (0.694)
	P3	1.05 (27.92)	0.26 (3.984)	−0.03 (−0.35)	−0.11 (−0.76)	0.04 (0.360)	−0.000 (−0.03)	1.60 (0.225)	0.0192 (0.737)
	P4	1.1 (30.83)	0.3 (5.315)	0.146 (1.580)	−0.07 (−0.48)	−0.15 (−0.96)	−0.002 (−1.48)	−4.14 (−0.59)	0.0254 (1.099)
	P5	1.05 (20.40)	0.5 (6.506)	0.616 (4.755)	0.08 (0.402)	−0.39 (−2.36)	−0.006 (−3.17)	−5.67 (−0.05)	−0.017 (−0.56)
	P-full	1.01 (43.27)	0.27 (6.848)	0.087 (1.497)	0.01 (0.095)	0.08 (1.038)	−0.001 (−1.57)	1.85 (0.419)	0.0069 (0.408)
S9	P1	0.94 (15.58)	−0.11 (−1.26)	−0.03 (−0.39)	0.43 (2.764)	0.21 (1.239)	−0.000 (−0.15)	−4.27 (−1.12)	−0.095 (−2.01)
	P2	0.95 (19.05)	−0.21 (−2.79)	0.02 (0.353)	0.34 (2.692)	0.32 (2.336)	−0.003 (−0.71)	5.58 (1.748)	−0.021 (−0.52)
	P3	0.86 (13.92)	−0.06(−0.69)	0.06 (0.719)	0.11 (0.746)	0.25 (1.463)	−0.003 (−0.55)	−3.59 (−0.92)	−0.017 (−0.35)
	P4	1.02 (12.17)	−0.12 (−0.96)	−0.0 (−0.08)	−0.25 (−1.20)	0.27 (1.177)	−0.011 (−1.48)	8.48 (0.164)	−0.019 (−0.37)
	P5	1.41 (12.54)	0.07 (0.422)	0.49 (2.811)	−0.01 (−0.05)	−0.06 (−0.21)	0.03 (2.462)	−1.65 (−0.23)	−0.021 (−0.43)
	P-full	1.05 (34.16)	−0.08 (−1.83)	0.07 (1.668)	0.09 (1.200)	0.22 (2.607)	0.0025 (0.868)	−3.53 (−0.18)	−0.045 (−2.03)
S10	P1	0.92 (33.36)	0.25 (5.241)	−0.34 (−5.01)	−0.20 (−1.89)	0.25 (2.895)	−0.000 (−0.11)	−2.09 (−0.40)	0.0058 (0.261)
	P2	0.97 (26.06)	0.56 (8.764)	−0.37 (−4.00)	−0.26 (−1.83)	0.14 (1.252)	−0.000 (−0.65)	4.30 (0.061)	0.0089 (0.336)
	P3	0.98 (21.75)	0.77 (9.840)	−0.34 (−3.00)	−0.30 (−1.73)	0.00 (0.040)	0.0003 (0.207)	1.32 (1.547)	0.0039 (0.132)
	P4	1.01 (19.50)	0.85 (9.451)	−0.22 (−1.73)	−0.27 (−1.35)	−0.15 (−0.90)	−0.005 (−2.83)	−1.29 (−1.33)	0.0326 (1.011)
	P5	1.04 (16.59)	1.17 (10.72)	−0.35 (−2.23)	−0.23 (−0.95)	0.37 (1.858)	−0.006 (−2.68)	9.35 (0.792)	−0.011 (−0.33)
	P-full	0.9 (31.85)	0.72 (13.41)	−0.32 (−4.23)	−0.25 (−2.11)	0.13 (1.294)	−0.002 (−2.20)	1.37 (0.236)	0.0147 (0.701)
S11	P1	0.91 (30.31)	0.06 (1.312)	−0.13 (−1.74)	0.01 (0.081)	0.37 (3.934)	0.0004 (0.412)	7.22 (1.272)	−0.000 (−0.01)
	P2	0.98 (31.37)	0.15 (2.901)	−0.08 (−1.06)	0.016 (0.137)	0.28 (2.800)	0.0003 (0.325)	1.64 (0.277)	0.0017 (0.073)
	P3	1.04 (31.50)	0.21 (3.744)	0.02 (0.274)	−0.05 (−0.35)	0.06 (0.581)	−0.000 (−0.27)	6.19 (0.986)	0.0233 (0.995)
	P4	1.12 (32.40)	0.44 (7.368)	0.07 (0.827)	−0.08 (−0.62)	0.02 (0.242)	−0.001 (−1.24)	−3.81 (−0.58)	0.0110 (0.508)
	P5	1.00 (18.85)	0.51 (5.568)	0.54 (4.087)	0.13 (0.635)	−0.36 (−2.15)	−0.005 (−2.87)	−1.04 (−0.10)	−0.021 (−0.65)
	P-full	1.01 (43.19)	0.28 (6.834)	0.08 (1.412)	0.01 (0.051)	0.08 (1.040)	−0.001 (−1.59)	1.97 (0.446)	0.0065 (0.386)
S12	P1	0.82 (21.99)	0.28 (4.962)	−0.14 (−2.41)	0.27 (2.820)	0.26 (2.533)	0.0025 (0.713)	6.77 (2.809)	0.0034 (0.106)
	P2	1.00 (14.27)	0.10 (0.952)	0.033 (0.304)	0.12 (0.667)	−0.43 (−2.24)	−0.000 (−0.03)	3.61 (0.813)	0.0337 (0.738)
	P3	1.06 (24.79)	0.38 (5.813)	−0.50 (−7.61)	−0.12 (−1.17)	−0.07 (−0.60)	−0.006 (−1.58)	2.81 (1.052)	0.0101 (0.372)
	P4	0.66 (9.246)	−0.12 (−1.14)	−0.14 (−1.29)	−0.32 (−1.76)	0.15 (0.759)	−0.000 (−0.05)	1.23 (2.675)	−0.057 (−0.91)
	P5	1.15 (23.69)	0.031 (0.418)	0.55 (7.206)	−0.17 (−1.35)	0.05 (0.366)	0.0071 (1.556)	−2.49 (−0.81)	−0.047 (−1.73)
	P-full	0.94 (88.96)	0.13 (7.992)	−0.07 (−4.36)	−0.03 (−1.30)	0.001 (0.035)	7.89 (0.078)	4.37 (6.511)	0.0005 (0.065)
S13	P1	1.05 (53.81)	1.11 (36.16)	0.04 (1.450)	−0.22 (−4.45)	0.40 (7.467)	0.0044 (2.348)	1.57 (1.260)	0.0134 (1.350)
	P2	0.91 (48.89)	0.96 (32.89)	0.13 (4.491)	−0.01 (−0.32)	0.05 (1.078)	−0.001 (−1.04)	3.87 (3.248)	−0.024 (−2.18)
	P3	1.00 (54.52)	1.03 (35.96)	−0.06 (−1.95)	−0.25 (−5.42)	−0.49 (−9.69)	−0.000 (−0.28)	9.77 (0.839)	0.0179 (1.807)
	P4	0.94 (47.94)	0.06 (2.029)	−0.11 (−3.40)	−0.04 (−0.91)	0.55 (10.36)	−0.004 (−2.35)	2.42 (1.934)	0.0154 (1.005)
	P5	0.96 (49.95)	−0.004 (−0.15)	−0.03 (−1.02)	−0.01 (−0.39)	0.10 (2.001)	0.0020 (1.126)	3.45 (2.831)	−0.021 (−1.39)
	P6	1.00 (43.99)	0.13 (3.801)	−0.01 (−0.21)	−0.02 (−0.42)	−0.52 (−8.45)	0.0004 (0.188)	3.41 (2.370)	0.0187 (1.169)

(*Continued*)

Table 5. (Continued).

Sample Portfolios	Sub portfolios	Slope coefficients across P1 to P-full							
		$\beta_{i,m}$	$\beta_{i,s}$	$\beta_{i,h}$	$\beta_{i,r}$	$\beta_{i,c}$	$\beta_{i,v}$	$\beta_{i,o}$	$\beta_{i,l}$
S14	P1	1.13 (32.08)	1.21 (22.07)	0.17 (3.150)	−0.33 (−3.63)	0.22 (2.314)	0.0034 (1.028)	−2.15 (−0.09)	0.0001 (0.009)
	P2	0.92 (32.73)	0.94 (21.39)	0.12 (2.875)	−0.05 (−0.69)	0.03 (0.403)	0.0015 (0.589)	4.54 (2.538)	−0.042 (−2.55)
	P3	1.0 (29.22)	0.97 (17.10)	−0.006 (−0.11)	−0.00 (−0.03)	−0.49 (−4.99)	−0.000 (−0.19)	4.19 (1.823)	0.0489 (2.528)
	P4	1.04 (20.44)	0.17 (2.156)	0.16 (2.020)	−0.38 (−2.95)	0.50 (3.621)	0.0056 (1.168)	2.70 (0.084)	−0.105 (−3.34)
	P5	0.92 (46.94)	0.02 (0.835)	0.16 (5.292)	0.02 (0.446)	−0.01 (−0.22)	0.0008 (0.444)	4.68 (3.762)	−0.022 (−1.49)
	P6	0.9 (22.67)	0.07 (1.032)	−0.26 (−3.84)	0.008 (0.078)	−0.47 (−3.96)	−0.002 (−0.62)	3.16 (1.143)	0.0537 (1.721)
S15	P1	1.18 (13.78)	1.09 (8.195)	0.31 (2.283)	−0.47 (−2.14)	−0.37 (−1.59)	0.0019 (0.239)	2.2 (0.423)	0.0856 (2.449)
	P2	0.96 (36.51)	0.94 (22.84)	0.12 (2.809)	−0.08 (−1.22)	−0.16 (−2.22)	−0.000 (−0.31)	3.94 (2.348)	0.0170 (1.136)
	P3	0.99 (20.41)	1.0 (13.99)	−0.16 (−2.08)	−0.11 (−0.91)	−0.06 (−0.48)	−0.002 (−0.55)	3.56 (1.154)	−0.005 (−0.20)
	P4	1.2 (13.90)	−0.0 (−0.43)	0.38 (2.610)	−0.45 (−1.87)	−0.50 (−1.96)	0.0231 (2.667)	2.27 (0.388)	−0.000 (−0.01)
	P5	0.95 (35.06)	0.07 (1.693)	0.055 (1.298)	0.07 (1.062)	0.05 (0.798)	−0.000 (−0.18)	5.1 (2.971)	−0.020 (−1.00)
	P6	0.94 (16.50)	0.31 (3.576)	−0.25 (−2.76)	0.05 (0.372)	−0.09 (−0.60)	−0.006 (−1.24)	1.93 (0.538)	0.0497 (1.258)
S16	P1	1.0 (61.70)	1.07 (42.27)	−0.12 (−4.83)	−0.63 (−15.0)	−0.02 (−0.36)	−0.001 (−1.11)	1.2 (1.267)	0.0048 (0.579)
	P2	0.93 (47.41)	0.96 (31.17)	0.16 (5.123)	0.07 (1.392)	−0.07 (−1.21)	0.0007 (0.403)	3.11 (2.481)	−0.007 (−0.67)
	P3	1.01 (45.20)	1.11 (31.73)	0.11 (3.213)	0.32 (5.495)	−0.16 (−2.56)	0.0011 (0.516)	2.21 (1.546)	0.0305 (2.448)
	P4	0.9 (36.36)	0.09 (2.381)	0.07 (1.688)	−0.82 (−11.9)	−0.15 (−2.04)	0.0036 (1.437)	3.2 (1.936)	0.0267 (1.591)
	P5	1.03 (54.36)	0.04 (1.384)	0.08 (2.848)	−0.01 (−0.24)	−0.06 (−1.21)	−0.001 (−1.03)	2.75 (2.299)	0.0124 (0.926)
	P6	0.94 (59.62)	0.05 (2.263)	−0.16 (−6.40)	0.25 (6.258)	0.02 (0.415)	0.0005 (0.383)	3.72 (3.723)	−0.020 (−1.52)
S17	P1	1.2 (24.88)	1.11 (14.61)	0.003 (0.041)	−0.48 (−3.82)	−0.23 (−1.72)	0.0080 (1.719)	2.1 (0.680)	0.0233 (1.084)
	P2	0.99 (42.60)	0.96 (26.44)	0.12 (3.432)	−0.12 (−2.04)	−0.24 (−3.77)	−0.000 (−0.23)	3.69 (2.507)	0.0068 (0.534)
	P3	0.94 (21.82)	0.9 (14.33)	0.02 (0.222)	−0.16 (−1.47)	−0.09 (−0.78)	−0.005 (−1.40)	6.70 (0.245)	0.0520 (2.162)
	P4	1.2 (17.72)	0.13 (1.178)	0.10 (0.915)	−0.33 (−1.84)	−0.33 (−1.70)	0.0089 (1.334)	−3.08 (−0.69)	−0.003 (−0.08)
	P5	0.95 (37.59)	0.11 (2.928)	−0.04 (−1.11)	0.07 (1.196)	−0.02 (−0.28)	0.0014 (0.589)	4.91 (3.052)	−0.023 (−1.18)
	P6	0.92 (17.33)	0.07 (0.875)	−0.02 (−0.19)	0.008 (0.059)	−0.16 (−1.13)	−0.004 (−0.98)	5.65 (1.665)	0.0489 (1.257)
S18	P1	0.99 (53.56)	1.08 (37.56)	−0.35 (−12.4)	−0.36 (−7.61)	−0.21 (−4.25)	−0.001 (−0.62)	2.89 (2.481)	0.0136 (1.356)
	P2	0.9 (54.67)	1.0 (38.07)	0.08 (3.004)	−0.05 (−1.21)	−0.05 (−1.26)	−0.000 (−0.06)	2.33 (2.139)	−0.001 (−0.11)
	P3	1.01 (53.86)	0.99 (33.69)	0.54 (18.30)	0.02 (0.351)	0.13 (2.693)	0.0017 (0.999)	1.08 (0.906)	−0.004 (−0.51)
	P4	0.95 (53.86)	0.06 (2.198)	−0.34 (−12.4)	0.07 (0.833)	0.02 (0.584)	0.0006 (0.377)	4.20 (3.747)	−0.003 (−0.23)
	P5	1.03 (39.04)	0.02 (0.514)	0.17 (4.188)	0.03 (0.418)	−0.01 (−0.14)	0.0006 (0.267)	1.13 (0.681)	0.0003 (0.021)
	P6	0.96 (37.35)	0.13 (3.401)	0.79 (19.44)	−0.30 (−4.50)	−0.35 (−5.03)	−0.001 (−0.55)	4.17 (2.552)	0.0176 (1.154)

returns. Therefore, the RF_{1a} and RF_{1b} average returns are computed like $actual_1$ average returns while the CAPM, FF3F and FF5F average returns have been computed like $actual_2$ average returns. We may mention here that the $actual_1$ average returns are more accurate measurements of the empirical average returns. This is because the asset returns are non-linear rational functions (i.e., ratios) of asset prices. This may be illustrated by the fact that $[(b/a) + (c/b) \neq [(b + c)/(a + b)]$, where a, b and c are the prices of an asset for three consecutive time intervals. Hence, direct averages like $actual_2$ average returns that follow the left-hand side of the above equation are not correct measures of the ratios of average prices that are shown on the right-hand side of the above equation. This non-linear rational function characteristic of the asset returns becomes more prominent the more they are averaged out across portfolios or across time. However, we have considered both $actual_1$ and $actual_2$ types of average returns for the sake of comparison among the results.

Thus, we have obtained both actual and estimated average returns for the sub-portfolios P1 to P5 and P-full (i.e., P6 for samples S13 to S18).[10] Tables 6 and 7 show that the estimates of RF_1 models for average returns consistently have the highest correlations with the actual average returns, whether $actual_1$ or $actual_2$, with all above 90% (We would like to clarify that for longer span data the correlations between $actual_2$ average returns and the RFM estimates would gradually decrease since the former are not correct measures of average asset returns). The correlations for the CAPM and FF3F models are not only much smaller than those for the RF_1 models, but are quite unpredictable taking different signs in different samples. The correlations of the FF5F estimates are higher than those of both the CAPM and the FF3F estimates but are still consistently lower than those of the RF_1 estimates. Moreover, the correlations of the FF5F model with actual average returns are negative for samples S1, S2 and S12 and only 3.23% with $actual_1$ and 7.16% with $actual_2$ for the sample S8. The highest correlation of FF5F estimates with $actual_1$ average returns is 93.81% (93.58% with $actual_2$) for S16, whereas all the correlations of the RF_1 models with actual average returns are above 90% and consistently higher than those of FF5F. Moreover, the RF_1 models have the lowest Sum of Squared Errors (SSE) between the actual and the estimated average returns. This demonstrates that the average returns estimated by the RF_1 models are the most accurate.

Figure 1 shows the plots for the $actual_1$ average returns and their estimates from the CAPM, FF3F, FF5F and RF_{1b} model.[11] These plots highlight the greater accuracy of the RFM and the non-linear behaviour of actual returns across the market beta. In fact, the negative correlations with actual values that can be observed for the CAPM, FF3F and FF5F models arise as these models impose a linear relation. Thus, even though the CAPM, the FF3F and the FF5F models might have identified the important and relevant factors influencing the asset returns, they are unable to capture the complete empirical reality for average asset returns. These results show that the average asset returns are basically non-linear in nature across market risk and other financial attributes. Thus, it follows that though the index price might be the most important variable in computing asset returns, by itself it cannot provide the important improvements brought in by the non-linearity of the rational function treatment and other factors used in the RF theory.

These findings support the arguments against the CAPM (see, for example, Blume and Friend 1973; Fama and MacBeth 1973; Stambaugh 1982; Fama and French 1992, 2004; Dempsey 2013). Further, our results also find lacunae in the empirical models like FF3F and FF5F in estimating average returns. One of the reasons for the difference in findings between our work and previous studies may be due to different sampling strategies. Previous studies often consider much longer samples and over these longer periods, the gradual long-term increase in the market returns is found to be reflected similarly and linearly in the large portfolios. However, this fails to encapsulate the dynamics of market behaviour or to reflect the practical situation where investors may hold smaller portfolios over smaller holding

[10]The values of the $actual_1$ average returns and the CAPM betas are available upon request.
[11]The plots of RF_{1a} are very like those of RF_{1b}, but we plot the latter, as this version is simpler and more accurate, especially for long time-span data. However, all results are available upon request.

Table 6. Correlations and Sum of Squared Errors (SSE) between Actual$_1$ Average Returns (calculated as ratios of average prices) and Estimated Average Returns across P1 to P-full.

	CAPM			FF3F			FF5F			RF_{1a}			RF_{1b}		
Sample Portfolios	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Correlation	t-stats	SSE
S1	−81.74%	−2.46	1.01E-04	−81.68%	−2.45	1.27E-04	−55.51%	−1.16	9.22E-05	99.58%	18.77	1.25E-06	99.57%	18.72	1.22E-06
S2	−79.49%	−2.27	9.59E-05	−81.55%	−2.44	1.19E-04	−60.03%	−1.30	9.53E-05	99.58%	18.95	1.21E-06	99.58%	18.94	1.19E-06
S3	−79.36%	−2.26	1.20E-04	−68.78%	−1.64	1.15E-04	77.48%	2.12	5.40E-05	99.36%	15.25	1.20E-06	99.36%	15.24	1.23E-06
S4	−77.19%	−2.10	1.08E-04	−60.82%	−1.33	1.05E-04	83.23%	2.60	5.14E-05	99.36%	15.27	1.22E-06	99.36%	15.26	1.24E-06
S5	46.82%	0.92	2.31E-07	44.88%	0.87	2.24E-07	72.15%	1.80	1.61E-07	98.91%	11.65	1.45E-08	98.92%	11.67	1.38E-08
S6	52.44%	1.07	9.67E-07	50.13%	1.00	7.77E-07	72.49%	1.82	4.96E-07	98.90%	11.60	9.18E-09	98.90%	11.57	8.93E-09
S7	33.23%	0.61	2.19E-07	30.46%	0.55	1.78E-07	49.76%	0.99	7.41E-08	93.80%	4.69	1.08E-08	93.73%	4.66	7.84E-09
S8	−75.70%	−2.01	5.35E-07	−65.33%	−1.49	3.37E-07	3.23%	0.06	9.99E-08	91.13%	3.83	1.89E-08	91.11%	3.83	1.86E-08
S9	−46.40%	−0.91	4.16E-05	−43.23%	−0.83	4.84E-05	58.83%	1.26	2.60E-05	99.50%	17.27	5.26E-07	99.50%	17.26	5.19E-07
S10	7.28%	0.13	2.95E-07	−0.01%	−0.00	2.92E-07	37.54%	0.70	1.70E-07	96.00%	5.94	1.70E-08	95.95%	5.90	1.32E-08
S11	−31.71%	−0.58	3.36E-07	−5.48%	−0.09	2.01E-07	63.98%	1.44	7.88E-08	94.81%	5.16	1.49E-08	94.79%	5.16	1.45E-08
S12	−76.81%	−2.08	8.86E-05	−81.77%	−2.46	9.39E-05	−17.56%	−0.31	8.37E-05	99.69%	21.82	2.50E-07	99.69%	21.82	2.49E-07
S13	34.61%	0.74	5.29E-05	54.86%	1.14	2.91E-05	86.78%	3.03	2.97E-05	99.33%	14.88	2.36E-07	99.33%	17.20	3.16E-07
S14	23.42%	0.48	5.46E-05	39.60%	0.75	3.11E-05	65.04%	1.48	3.50E-05	99.00%	12.15	2.90E-07	99.01%	14.14	4.36E-07
S15	−22.79%	−0.47	6.01E-05	19.88%	0.35	3.52E-05	89.06%	3.39	3.94E-05	99.76%	24.94	1.41E-07	99.77%	29.43	2.94E-07
S16	−15.68%	−0.32	5.78E-05	27.98%	0.50	3.45E-05	93.81%	4.69	2.90E-05	99.61%	19.52	2.88E-07	99.61%	22.65	3.77E-07
S17	−36.48%	−0.78	5.33E-05	0.40%	0.01	3.95E-05	65.85%	1.52	3.54E-05	99.59%	19.18	2.99E-07	99.61%	22.66	3.76E-07
S18	35.51%	0.76	4.71E-05	55.35%	1.15	2.18E-05	74.98%	1.96	2.61E-05	99.03%	12.34	2.60E-07	99.05%	14.41	3.45E-07
S19	27.37%	0.49	7.37E-07	28.12%	0.51	8.28E-07				99.32%	14.77	9.89E-09	99.33%	14.88	1.13E-08
S20	65.35%	1.50	2.18E-07	75.89%	2.02	2.73E-07				96.31%	6.20	2.86E-08	96.29%	6.18	2.98E-08
S21	39.22%	0.74	9.64E-05	17.13%	0.30	1.12E-04				99.90%	39.54	7.50E-07	99.90%	39.54	7.46E-07

Table 7. Correlations and Sum of Squared Errors (SSE) between Actual$_2$ Average Returns (calculated as averages of time series of continuous returns) and Estimated Average Returns across P1 to P-full.

	CAPM			FF3F			FF5F			RF_{1a}			RF_{1b}		
Sample Portfolios	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Correlation	t-stats	SSE
S1	−81.86%	−2.47	8.66E-05	−82.50%	−2.53	1.13E-04	−57.90%	−1.23	7.92E-05	99.22%	13.78	1.62E-06	99.22%	13.76	1.66E-06
S2	−79.65%	−2.28	8.08E-05	−82.52%	−2.53	1.03E-04	−62.44%	−1.38	8.33E-05	99.23%	13.92	1.23E-06	99.23%	13.91	1.24E-06
S3	−79.85%	−2.30	1.02E-04	−69.26%	−1.66	9.71E-05	77.72%	2.14	4.09E-05	99.23%	13.86	2.29E-06	99.23%	13.86	2.19E-06
S4	−77.69%	−2.14	8.70E-05	−61.39%	−1.35	8.34E-05	83.47%	2.62	3.85E-05	99.23%	13.87	2.22E-06	99.23%	13.87	2.16E-06
S5	43.02%	0.83	2.48E-07	41.67%	0.79	2.37E-07	75.61%	2.00	1.64E-07	99.40%	15.78	7.33E-09	99.41%	15.81	7.05E-09
S6	48.78%	0.97	1.05E-06	47.15%	0.93	8.45E-07	75.42%	1.99	5.33E-07	99.40%	15.71	8.72E-09	99.39%	15.66	7.70E-09
S7	34.54%	0.64	1.85E-07	32.29%	0.59	2.33E-07	48.46%	0.96	1.07E-07	93.96%	4.76	2.35E-08	93.90%	4.73	1.21E-08
S8	−73.59%	−1.88	5.12E-07	−62.69%	−1.39	3.39E-07	7.16%	0.12	1.00E-07	90.74%	3.74	2.07E-08	90.72%	3.73	1.91E-08
S9	−53.91%	−1.11	3.38E-05	−49.62%	−0.99	4.08E-05	56.43%	1.18	2.12E-05	99.03%	12.36	1.27E-06	99.03%	12.37	1.29E-06
S10	8.35%	0.15	2.59E-07	1.32%	0.02	3.43E-07	35.92%	0.67	2.00E-07	96.02%	5.96	2.89E-08	95.97%	5.91	1.68E-08
S11	−28.20%	−0.51	3.21E-07	−1.78%	−0.03	2.11E-07	65.80%	1.51	8.48E-08	95.42%	5.52	1.76E-08	95.40%	5.51	1.58E-08
S12	−61.01%	−1.33	6.68E-05	−70.00%	−1.70	7.19E-05	−10.29%	−0.18	6.30E-05	98.12%	8.81	2.66E-06	98.12%	8.81	2.67E-06
S13	43.03%	0.95	3.55E-05	62.87%	1.40	1.72E-05	89.39%	3.45	1.78E-05	98.73%	10.77	2.97E-06	98.74%	12.46	3.29E-06
S14	17.83%	0.36	4.07E-05	31.04%	0.57	2.33E-05	55.67%	1.16	2.55E-05	97.49%	7.58	2.44E-06	97.50%	8.77	2.94E-06
S15	−27.06%	−0.56	4.45E-05	16.07%	0.28	2.57E-05	88.51%	3.29	2.62E-05	99.00%	12.18	2.31E-06	99.07%	14.59	2.96E-06
S16	−12.61%	−0.25	4.63E-05	30.93%	0.56	2.67E-05	93.58%	4.60	2.01E-05	99.66%	20.98	1.83E-06	99.67%	24.40	2.11E-06
S17	−34.59%	−0.74	3.97E-05	3.23%	0.06	3.01E-05	68.76%	1.64	2.38E-05	99.62%	19.69	2.03E-06	99.63%	23.34	2.43E-06
S18	43.75%	0.97	3.68E-05	63.49%	1.42	1.48E-05	80.35%	2.34	1.80E-05	96.63%	6.50	1.81E-06	96.68%	7.57	2.07E-06
S19	25.07%	0.45	7.72E-07	25.78%	0.46	8.72E-07				99.04%	12.41	1.33E-08	99.05%	12.49	1.39E-08
S20	64.65%	1.47	1.97E-07	76.18%	2.04	2.48E-07				96.50%	6.37	2.27E-08	96.48%	6.36	2.37E-08
S21	42.11%	0.80	8.41E-05	40.19%	0.76	3.52E-05				96.97%	6.88	4.66E-05	96.97%	6.88	4.62E-05

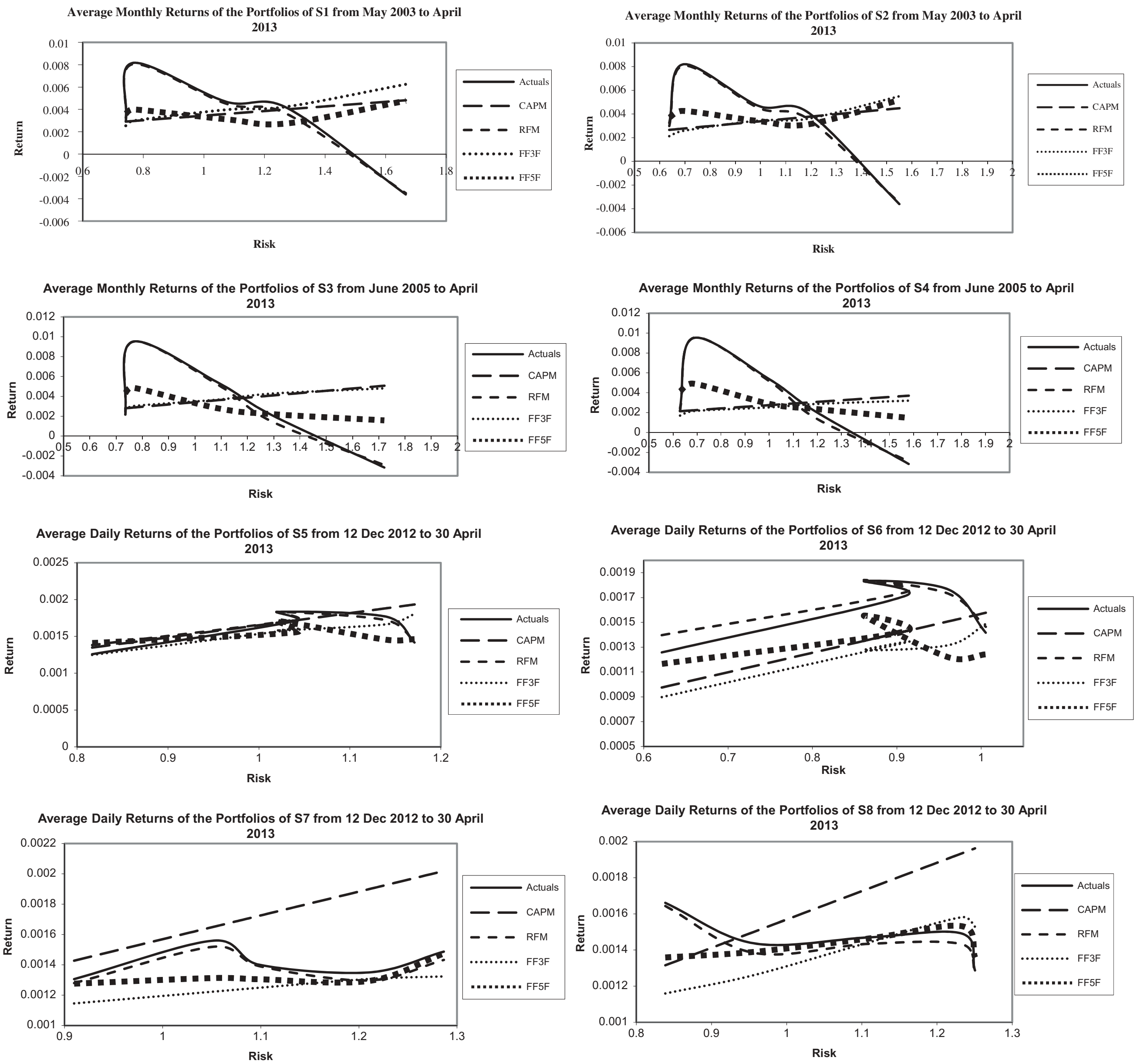

Figure 1. Charts of the actual_1 average returns and the estimated average returns across increasing risk according to CAPM, FF3F, FF5F and RF_{1b} Equation (17b).

periods. It is in this context that our findings are particularly pertinent.

Another key implication of the RF_1 models arises from the nature of our results, which show that although the sorting factors do not affect the accuracy of the RF_1 models they might influence the level of average returns. We find that although risk, as measured by either return variance or idiosyncratic volatility, does not influence average returns in any specific way, it is reasonable to believe that other market factors like size, investment, reversals, momentum, operating profits, etc. might influence the magnitude of average returns. However, since the CAPM, FF3F and FF5F models do not provide accurate estimates of average returns, the RF_1 model can be used instead for estimating average returns once the stocks have been sorted for the relevant financial factors.

As already mentioned, we have studied not only asset returns but also asset volumes. We have reported the correlations and the Sum of Squared

Errors (SSE) between the actual and the estimated average change in volumes in Table 8. It can be seen that the correlations for RF_{1c} are all positive and quite high with all the values being above 90% except for sample S21, where the correlation is 85.5% for the Indian market. The t-statistics of all these correlations are also very high indicating that these correlations are significant and hence the estimates of the average change in volumes by the RF_{1c} model are reasonably accurate. The correlations of the RF_{1c} are higher than those of the LW2F for ten samples and slightly lower for four samples. However, it can be seen that the correlations of the LW2F are not consistent and takes on both negative and positive values, indicating that sometimes the LW2F estimates can be misleading. Looking at the SSE values, it can be seen that all the SSE values of the RF_{1c} model are lower than those for the

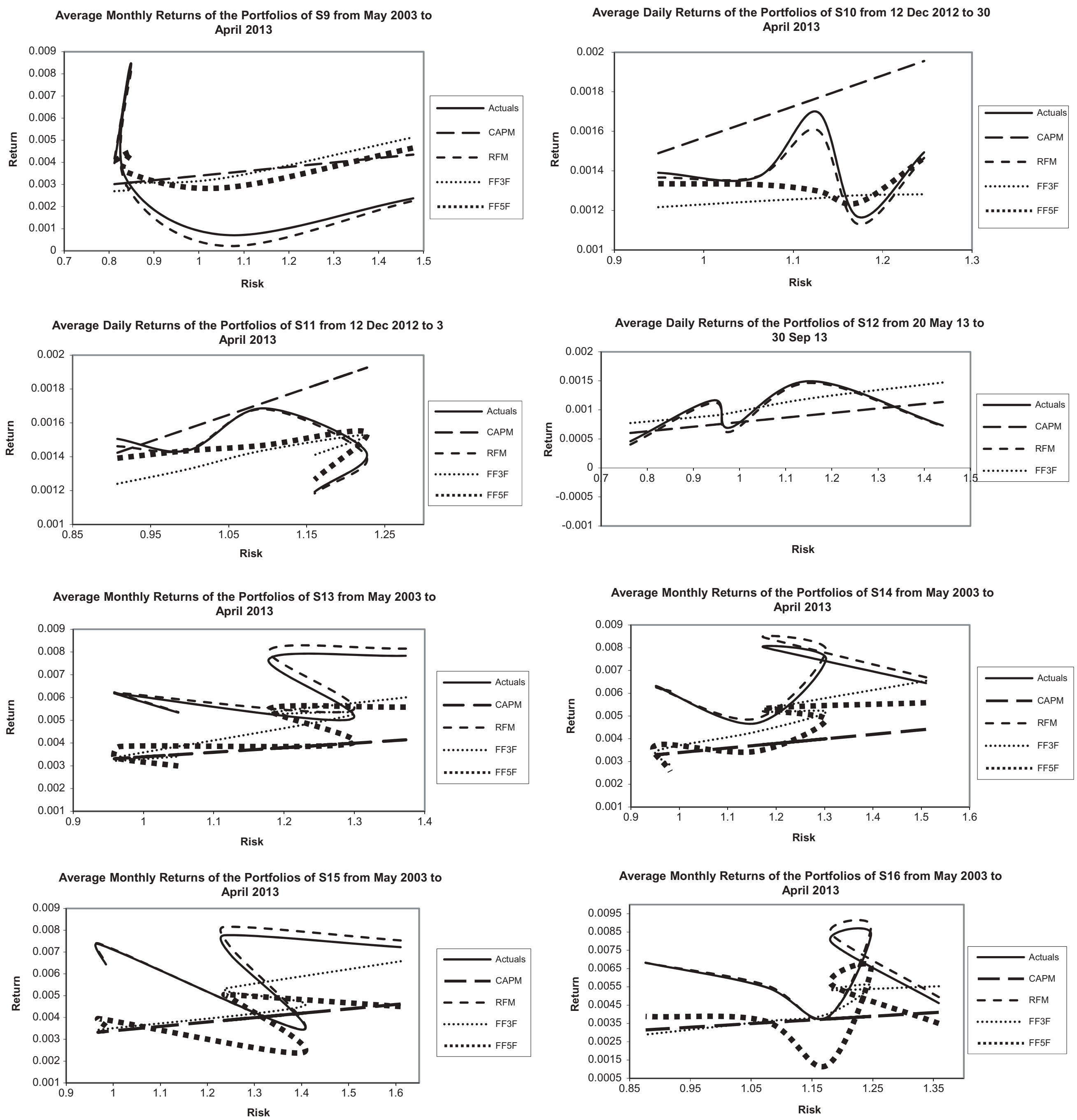

Figure 1. (Continued).

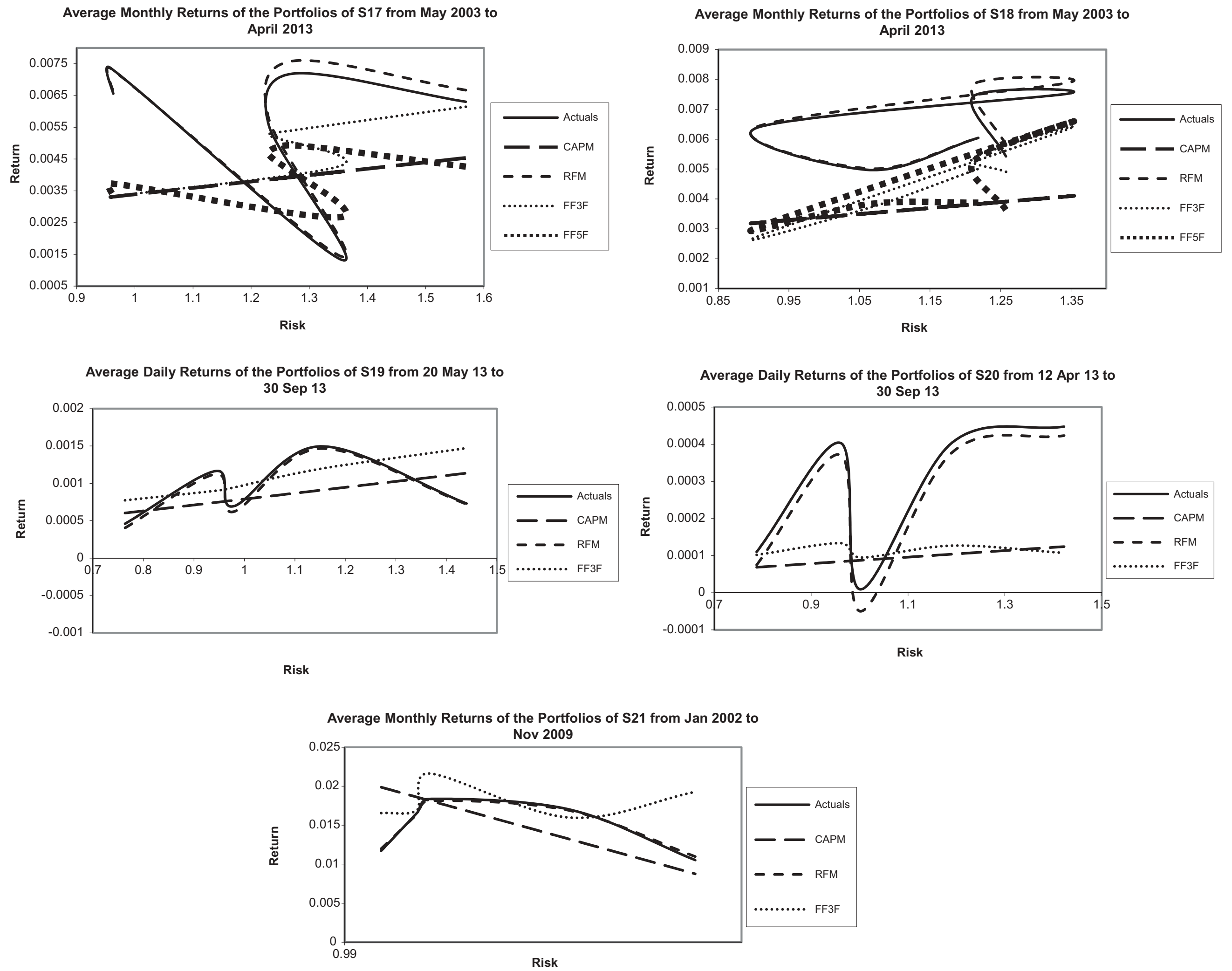

Figure 1. (Continued).

Table 8. Correlations and Sum of Squared Errors (SSE) between Actual Average Change in Volumes (calculated as ratios of average volumes) and Estimated Average Change in Volumes across P1 to P-full.

	LW2F			RF_{1c}			
Names	Correlation	t-stats	SSE	Correlation	t-stats	SSE	Percentage improvement in SSE between RF_{1c} and LW2F
S1	75.43%	1.99	5.25E-02	99.12%	12.94	7.12E-04	98.64%
S2	75.43%	1.99	5.25E-02	99.13%	13.05	6.79E-04	98.71%
S3	−28.30%	−0.51	2.78E-02	99.50%	17.27	4.74E-04	98.29%
S4	−28.30%	−0.51	2.78E-02	99.45%	16.52	3.35E-04	98.79%
S5	99.26%	14.12	7.87E-03	98.87%	11.42	2.39E-04	96.96%
S6	99.24%	14.01	7.87E-03	98.48%	9.82	5.15E-04	93.45%
S7	97.66%	7.86	4.49E-02	94.49%	5.00	6.72E-03	85.03%
S8	96.39%	6.27	1.00E-01	98.29%	9.24	1.64E-02	83.57%
S9	47.75%	0.94	7.24E-03	90.88%	3.77	1.65E-03	77.18%
S10	97.98%	8.48	3.60E-02	99.99%	120.27	1.19E-04	99.67%
S11	97.71%	7.96	3.74E-02	94.78%	5.15	5.25E-03	85.95%
S19	−76.73%	−2.07	2.19E-02	99.79%	26.54	7.49E-05	99.66%
S20	−76.68%	−2.07	1.84E-02	91.06%	3.82	2.90E-03	84.23%
S21	28.70%	0.52	3.90E-01	85.49%	2.85	1.33E-01	65.91%

LW2F model (being improvements in the range of 65.9% to 99.7%) indicating that the RF_{1c} estimates are more accurate.

We now consider the continuous returns, which behave approximately linearly across time and hence have been modeled directly

Table 9. Correlations between the Actual and the Estimated Continuous Returns.

Sample	Models	Correlations of Estimated Returns with Actual Returns					
		P1	P2	P3	P4	P5	P-full or P6
S1	CAPM	83.77%	86.53%	92.09%	89.16%	89.04%	98.49%
	FF3F	85.39%	86.58%	92.44%	89.50%	91.93%	98.73%
	FF5F	86.26%	87.21%	92.72%	90.07%	92.04%	98.76%
	RF_{2a}	86.04%	87.21%	92.78%	89.65%	91.82%	98.76%
	RF_{2b}	86.76%	87.59%	93.03%	90.25%	91.95%	98.80%
S2	CAPM	77.60%	84.72%	92.12%	89.99%	89.33%	97.67%
	FF3F	80.42%	85.14%	92.44%	90.00%	91.61%	97.90%
	FF5F	82.29%	86.91%	92.51%	90.06%	91.57%	98.06%
	RF_{2a}	81.53%	86.23%	92.62%	90.17%	91.62%	98.00%
	RF_{2b}	82.85%	87.41%	92.68%	90.23%	91.64%	98.13%
S3	CAPM	84.86%	88.35%	93.32%	90.88%	89.32%	98.83%
	FF3F	86.51%	88.36%	93.63%	90.90%	93.36%	99.11%
	FF5F	87.67%	89.04%	93.88%	91.34%	94.25%	99.13%
	RF_{2a}	87.84%	89.08%	93.88%	91.06%	93.62%	99.13%
	RF_{2b}	88.70%	89.60%	94.09%	91.55%	94.38%	99.16%
S4	CAPM	79.08%	86.66%	93.41%	91.28%	89.16%	97.68%
	FF3F	82.05%	86.86%	93.76%	91.52%	92.74%	98.19%
	FF5F	84.23%	88.49%	93.81%	91.56%	93.32%	98.30%
	RF_{2a}	83.67%	87.62%	93.79%	91.73%	93.24%	98.28%
	RF_{2b}	85.11%	88.95%	93.87%	91.79%	93.63%	98.38%
S5	CAPM	77.83%	85.85%	88.97%	87.02%	76.86%	97.84%
	FF3F	80.28%	86.42%	89.07%	87.46%	80.74%	98.09%
	FF5F	81.23%	86.46%	89.89%	88.85%	82.68%	98.25%
	RF_{2a}	83.19%	86.79%	89.41%	88.40%	81.45%	98.20%
	RF_{2b}	84.28%	86.82%	90.38%	89.79%	83.58%	98.37%
S6	CAPM	69.39%	89.32%	88.98%	86.65%	77.87%	97.16%
	FF3F	73.02%	89.62%	89.68%	86.68%	80.59%	97.38%
	FF5F	75.33%	89.93%	91.73%	87.04%	81.46%	97.44%
	RF_{2a}	75.89%	89.81%	89.80%	87.24%	81.08%	97.33%
	RF_{2b}	78.48%	90.13%	92.17%	87.64%	82.09%	97.41%
S7	CAPM	95.98%	93.78%	92.97%	90.77%	86.32%	93.81%
	FF3F	97.02%	96.83%	96.67%	96.28%	94.16%	97.99%
	FF5F	97.46%	96.94%	97.01%	96.50%	94.50%	98.21%
	RF_{2a}	97.09%	96.86%	96.67%	96.57%	94.99%	98.10%
	RF_{2b}	97.52%	97.00%	97.01%	96.80%	95.33%	98.32%
S8	CAPM	95.33%	97.10%	96.74%	96.72%	91.55%	97.92%
	FF3F	95.36%	97.39%	97.35%	97.85%	95.67%	98.83%
	FF5F	96.40%	97.79%	97.38%	97.87%	95.93%	98.85%
	RF_{2a}	95.43%	97.39%	97.37%	97.92%	96.36%	98.87%
	RF_{2b}	96.56%	97.80%	97.40%	97.95%	96.65%	98.89%
S9	CAPM	87.24%	90.70%	88.32%	87.14%	86.47%	97.41%
	FF3F	87.66%	91.73%	88.61%	87.23%	87.90%	97.60%
	FF5F	88.41%	92.75%	88.88%	87.77%	87.98%	97.77%
	RF_{2a}	88.29%	91.99%	88.85%	87.77%	88.73%	97.71%
	RF_{2b}	89.19%	92.78%	89.12%	88.14%	88.73%	97.86%
S10	CAPM	96.83%	94.34%	91.85%	90.03%	86.53%	93.85%
	FF3F	97.69%	97.15%	96.44%	95.38%	94.20%	98.00%
	FF5F	98.09%	97.37%	96.57%	95.55%	94.69%	98.21%
	RF_{2a}	97.70%	97.14%	96.42%	96.11%	94.85%	98.11%
	RF_{2b}	98.10%	97.37%	96.57%	96.26%	95.35%	98.33%
S11	CAPM	96.86%	97.25%	97.32%	96.58%	91.25%	97.92%
	FF3F	96.89%	97.48%	97.82%	98.19%	94.86%	98.83%
	FF5F	97.44%	97.73%	97.83%	98.20%	95.15%	98.85%
	RF_{2a}	96.91%	97.48%	97.83%	98.22%	95.63%	98.87%
	RF_{2b}	97.49%	97.74%	97.85%	98.22%	95.94%	98.89%
S12	CAPM	93.52%	89.80%	93.39%	78.37%	94.26%	99.48%
	FF3F	94.56%	89.85%	96.75%	79.13%	96.66%	99.72%
	FF5F	95.22%	90.44%	96.80%	80.27%	96.68%	99.73%
	RF_{2a}	94.85%	89.78%	96.77%	80.14%	96.76%	99.72%
	RF_{2b}	95.40%	90.31%	96.83%	81.08%	96.81%	99.73%
S13	CAPM	91.65%	92.34%	92.67%	97.46%	99.02%	97.76%
	FF3F	99.13%	99.41%	99.02%	97.53%	99.02%	98.02%
	FF5F	99.54%	99.43%	99.56%	99.00%	99.08%	98.89%
	RF_{2a}	99.18%	99.46%	99.11%	97.88%	99.01%	98.12%
	RF_{2b}	99.58%	99.47%	99.57%	99.04%	99.06%	98.90%
S14	CAPM	90.97%	91.86%	92.69%	93.15%	98.63%	93.27%
	FF3F	98.62%	98.71%	97.77%	94.19%	98.94%	94.92%
	FF5F	98.87%	98.72%	98.21%	95.29%	98.94%	95.68%
	RF_{2a}	98.63%	98.79%	97.93%	94.62%	98.99%	95.05%
	RF_{2b}	98.88%	98.80%	98.35%	95.80%	98.99%	95.71%

(Continued)

Table 9. (Continued).

Sample	Models	Correlations of Estimated Returns with Actual Returns					
		P1	P2	P3	P4	P5	P-full or P6
S15	CAPM	88.13%	92.93%	89.97%	88.96%	98.03%	91.44%
	FF3F	93.52%	98.96%	96.86%	89.39%	98.12%	92.81%
	FF5F	94.05%	99.01%	96.88%	90.34%	98.15%	92.81%
	RF_{2a}	94.07%	98.97%	96.87%	90.47%	98.13%	92.90%
	RF_{2b}	94.43%	99.02%	96.90%	91.04%	98.16%	92.93%
S16	CAPM	92.14%	92.53%	91.54%	96.54%	99.17%	98.65%
	FF3F	99.00%	99.42%	99.02%	96.90%	99.24%	99.07%
	FF5F	99.70%	99.43%	99.27%	98.79%	99.24%	99.30%
	RF_{2a}	99.01%	99.40%	99.04%	97.11%	99.24%	99.00%
	RF_{2b}	99.70%	99.42%	99.32%	98.84%	99.25%	99.27%
S17	CAPM	91.78%	93.27%	91.13%	93.20%	98.27%	92.96%
	FF3F	97.44%	99.19%	97.22%	93.42%	98.41%	93.00%
	FF5F	97.86%	99.30%	97.29%	93.89%	98.40%	93.03%
	RF_{2a}	97.67%	99.19%	97.37%	93.69%	98.36%	93.17%
	RF_{2b}	97.98%	99.31%	97.44%	94.01%	98.37%	93.26%
S18	CAPM	91.33%	92.34%	91.02%	97.39%	98.22%	94.00%
	FF3F	99.23%	99.60%	99.56%	99.19%	98.55%	98.61%
	FF5F	99.54%	99.61%	99.59%	99.18%	98.55%	98.99%
	RF_{2a}	99.26%	99.58%	99.57%	99.14%	98.55%	98.64%
	RF_{2b}	99.56%	99.59%	99.60%	99.14%	98.55%	99.02%
S19	CAPM	90.73%	94.09%	92.13%	92.20%	86.15%	96.90%
	FF3F	91.66%	94.16%	92.49%	93.09%	87.47%	97.60%
	RF_{2a}	91.89%	94.56%	92.77%	93.16%	88.04%	97.78%
S20	CAPM	90.79%	93.27%	91.12%	88.93%	81.71%	95.93%
	FF3F	91.34%	93.31%	92.62%	92.50%	84.14%	97.36%
	RF_{2a}	91.37%	93.37%	92.80%	92.59%	84.30%	97.42%
S21	CAPM	87.10%	91.21%	93.49%	93.08%	91.94%	98.82%
	FF3F	87.10%	91.91%	93.91%	93.18%	93.05%	98.93%
	RF_{2a}	87.37%	92.13%	94.12%	93.52%	93.30%	99.00%

through linear regression relationships as shown in Equations (19a) and (19b). Table 9 reports that the correlations between the actual continuous returns and the RF_{2a} estimates are marginally, but consistently, higher than those of the CAPM and the FF3F estimates. Similarly, the correlations between the actual continuous returns and the RF_{2b} estimates are marginally, but consistently, higher than those of the FF5F estimates. Thus, the results in Table 9 show that the RF_{2b} estimates are the most accurate for continuous returns.

Similar conclusions are provided by the average Sum of Squared Errors (SSE) between the actual and the estimated continuous returns as reported in Table 10 which show that the RF_{2b} model outperforms the CAPM, FF3F and the FF5F models in estimating continuous returns. It can be seen that the RF_{2b} model is better than the FF5F model within a range of 3.88% to 8.86% while it outperforms the CAPM estimates by 13.64% to 89.42%. The two bottom rows of Table 10 show paired *t*-tests for no improvement in the average SSE for the continuous returns estimated by different models. The p-values of the paired *t*-tests show that the null hypothesis of no improvement for RF_{2b} model's average estimates of continuous returns over the CAPM and the FF5F average estimates of continuous returns can be rejected safely at *p*-values of 8.4E-05% and 3.0E-09%, respectively, indicating that the RF_{2b} model estimates for continuous returns are the most accurate.

V. Practical implications

One of the main contributions of this paper is that it has developed separate models for estimating average and continuous returns. The findings of this study indicate that the behaviour of actual average returns across increasing levels of risk (considering both returns variance and idiosyncratic volatility) is non-linear. Further, for stocks sorted on financial factors like industry, size, investment, profitability, momentum, reversals, etc. risk, as measured by the market-beta, does not increase uniformly across the sub-portfolios. This can be seen clearly in Figure 1, from the plots of the $actual_1$ average returns. It can also be seen that unlike the CAPM, FF3F and FF5F estimates,

Table 10. Average SSEs between the Actual and the Estimated Continuous Returns across Risk-Time Format.

Portfolios	CAPM	FF3F	FF5F	RF_{2a}	RF_{2b}	$(SSE_{CAPM} - SSE_{FF5F})/ SSE_{CAPM}$	$(SSE_{CAPM} - SSE_{RF2b})/ SSE_{CAPM}$	$(SSE_{FF5F} - SSE_{RF2b})/SSE_{FF5F}$
S1	5.66E-02	5.00E-02	4.77E-02	4.74E-02	4.58E-02	15.76%	19.03%	3.88%
S2	5.87E-02	5.30E-02	5.11E-02	5.01E-02	4.88E-02	12.94%	16.73%	4.36%
S3	4.88E-02	3.99E-02	3.61E-02	3.70E-02	3.43E-02	26.19%	29.68%	4.73%
S4	5.18E-02	4.30E-02	3.97E-02	3.98E-02	3.78E-02	23.38%	27.02%	4.75%
S5	1.23E-03	1.12E-03	1.05E-03	1.06E-03	9.78E-04	15.04%	20.61%	6.56%
S6	1.26E-03	1.17E-03	1.10E-03	1.12E-03	1.04E-03	13.25%	17.68%	5.10%
S7	1.01E-03	4.59E-04	4.24E-04	4.24E-04	3.89E-04	57.94%	61.46%	8.35%
S8	4.69E-04	3.13E-04	2.86E-04	2.88E-04	2.62E-04	39.04%	44.12%	8.32%
S9	6.08E-02	5.71E-02	5.49E-02	5.41E-02	5.25E-02	9.63%	13.64%	4.43%
S10	9.98E-04	4.62E-04	4.28E-04	4.24E-04	3.90E-04	57.18%	60.95%	8.81%
S11	4.24E-04	2.85E-04	2.67E-04	2.65E-04	2.47E-04	37.00%	41.73%	7.51%
S12	3.71E-02	3.01E-02	2.86E-02	2.85E-02	2.72E-02	22.93%	26.73%	4.94%
S13	3.55E-02	7.58E-03	4.46E-03	6.79E-03	4.06E-03	87.43%	88.54%	8.86%
S14	4.93E-02	1.82E-02	1.55E-02	1.71E-02	1.43E-02	68.51%	71.01%	7.93%
S15	7.46E-02	4.51E-02	4.23E-02	4.17E-02	3.99E-02	43.22%	46.52%	5.81%
S16	3.61E-02	8.24E-03	4.68E-03	7.67E-03	4.28E-03	87.05%	88.16%	8.61%
S17	5.23E-02	2.59E-02	2.41E-02	2.43E-02	2.29E-02	53.97%	56.12%	4.69%
S18	4.36E-02	5.74E-03	4.89E-03	5.52E-03	4.61E-03	88.80%	89.42%	5.62%
S19	1.46E-03	1.34E-03		1.29E-03				
S20	2.37E-03	1.99E-03		1.97E-03				
S21	1.18E-01	9.80E-02		9.45E-02				
	Paired t-test					H_0: $(SSE_{CAPM} - SSE_{FF5F})/ SSE_{CAPM} \le 0$	H_0: $(SSE_{CAPM} - SSE_{RF2b})/ SSE_{CAPM} \le 0$	H_0: $(SSE_{FF5F} - SSE_{RF2b})/ SSE_{FF5F} \le 0$
	t-statistic					6.35	7.13	14.39
	p-value					3.6E-06	8.4E-07	3.0E-11

the RF_{1b} average returns consistently follow the $actual_1$ average returns. This is because of the non-linear nature and the multiplicity of factors used in estimating asset prices by the RF_1 models. The CAPM and the FF3F models are linear and thus are unable to capture the curvilinear behaviour of the actual average returns. The FF5F model generates better estimates of average returns than the CAPM and the FF3F models but is still less accurate than the RF_1 models.

Tables 6 and 7 report that the CAPM, the FF3F and the FF5F models sometimes generate estimates that have negative correlations with both $actual_1$ and $actual_2$ average returns. This means that these models could sometimes lead to misleading inferences. On the other hand, the RF_1 models generate estimates that consistently have correlations of above 90% with the actual average returns. This clearly shows that the RF_1 models are more accurate tools that can be used to evaluate investments over multiple time-periods and can help in making choices that are more risk-return efficient. Another key implication from our work is that sorting factors, such as size, value and momentum might be used to build portfolios that can generate higher average returns, but the analysis of such portfolios and the generation of their estimated returns should be conducted through the RF approach, which provides greater accuracy. This combination of approaches allows us to accurately identify and select portfolios that have higher mean-variance efficiency for average asset returns. However, it would be interesting to further examine the ability of the financial factors used in factor models like the FF5F, to sort stocks into an order that gives increasing average returns, in the new light of the RF theory. In addition, the RF_{2b} model for continuous returns can help investors in making better single time-period assessments of their investments through more accurate monitoring of the changing asset returns on a contemporaneous basis.

The RF theory also provides a reasonably accurate model to estimate average change in asset volume $V_{i,t}$, as can be seen from Table 8, where the RF_{1c} estimates are found to be more accurate than the LW2F estimates. The average change in asset volume $V_{i,t}$, together with average return $R_{i,t}$ indicate the average change in market value $\Delta MV_{i,t}$ of an asset. Since change in asset volume $V_{i,t}$ shows the change in liquidity of an asset thereby indicating the ease of realizing the return for that asset, it is an important factor. The $\Delta MV_{i,t}$ of an asset being an arithmetic product of return and change in volume, flows from both $R_{i,t}$ and $V_{i,t}$. Hence, investors who want to maximize wealth instead of just profits should choose assets that give maximum mean $\Delta MV_{i,t}$ for minimum variances in $R_{i,t}$ and $V_{i,t}$ for a given time period.

VI. Summary and conclusions

A significant part of the existing literature emphasizes that the estimates of the market premium as obtained from the CAPM do not match the actual values. While much of the subsequent research has sought to remedy this through additional factors, no model has yet been able to provide a sufficiently accurate description of the data nor the theoretical reasons underlying the above discrepancy. In this paper, we argue that this discrepancy arises from the assumption that asset returns can be directly averaged within a portfolio or across time. Returning to the basics involved in estimating asset prices, we present an alternative model for estimating average returns based upon the concept of Rational Functions (i.e., an asset return is a ratio of price polynomials). In addition, we distinguish between the modeling requirements of average returns and continuous returns. For estimating continuous returns, we identify three factors – change in index volume, time and the preceding asset return that can be used in combination with an existing factor model (e.g. FF5F) to improve its performance. We also provide a model to estimate average change in volume which together with average return provides estimate for average change in market value of an asset. Further, we empirically test the Rational Function (RF) model for both average and continuous returns and compare them with the CAPM, FF3F and FF5F models using twenty-one samples based on both monthly and daily data from three markets - the USA, Australia and India.

Our empirical results show that the RF models provide the most accurate descriptions of actual returns compared to all the established asset pricing models. This is because the RF models are able to capture the non-linear dynamics within the behaviour of average returns and introduce additional relevant factors for estimating continuous returns. The charts of the average returns indicate that the risk-return-efficient investments should be carefully selected from such charts as average returns plot non-linear across risk and sometimes the lower risk assets offer higher returns. Stocks can be sorted on various relevant financial factors like size, profitability, etc. and then the average returns for the portfolios should be estimated by the RF model. The RF estimates of the average change in volumes are also more accurate than the estimates of an extant asset volume model. These results prove the empirical authenticity of the RF theory and affirm that price and volume are two complementary forces of the market. As the market value of an asset flows through both of these variables, maximizing the former requires maximizing both the latter variables.

Our contribution is not limited to introducing and testing the RF models but also to differentiating between the behaviours of the average and continuous returns and modeling them separately. However, we recognize that further empirical testing is required to substantiate the RF theory. Hence, it would be interesting to consider this approach with more comprehensive empirical datasets from different markets. However, even now, the RF theory and the preliminary empirical evidence provided here indicate that the RF models are of both academic interest as well as of practical value to the investment community.

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