# On the Optimal Broadcast Rate of the Two-Sender Unicast Index Coding Problem With Fully-Participated Interactions

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***Abstract*— The problem of two-sender unicast index coding consists of two senders and a set of receivers. Each receiver demands a unique message not demanded by any other receiver and has a subset of messages as its side information. Every demanded message is available with at least one of the senders. The senders avail the knowledge of the side information at all the receivers to reduce the total number of transmissions required to satisfy the demands of all the receivers. The objective is to find the minimum total of number of transmissions per message length (known as the optimal broadcast rate with finite length messages) and its limiting value as the message length tends to infinity (called the optimal broadcast rate). Achievable broadcast rates are provided for a class of the two-sender unicast index coding problem based on a special graph coloring technique called two-sender graph coloring. This result illustrates the utility of graph products in the two-sender unicast index coding problem for the first time in the literature. For another class, achievable broadcast rates are provided based on the optimal broadcast rates of three single-sender sub-problems with finite message length. This employs a code construction for the two-sender unicast index coding problem using optimal codes (including non-linear codes) of the sub-problems. Optimal broadcast rates are provided for a special class of the TUICP for which only an upper bound was known prior to this work. The optimal broadcast rates presented in this work also consider non-linear coding schemes at the two senders.**

***Index Terms*— Index coding, side-information, two-sender unicast index coding, optimal broadcast rate.**

## I. Introduction

THE index coding problem (ICP) with a single sender was introduced in [1]. Each receiver has some messages as its side information and demands some message it does not have. The sender avails the cognizance of the side information available at all the receivers to minimize the number of transmissions required to satisfy the demands of all the receivers. Many practical scenarios require distributed transmissions where the messages are distributed among multiple senders. Content is delivered in cellular networks using large storage capacity nodes called caching helpers [2], where the messages are distributed among the helpers to reduce the total average delay of all the users. Data is stored over multiple storage nodes to account for any failure in one or more storage nodes in distributed storage networks [3], [4]. Hence, the multi-sender ICP is of practical significance.

A special class of the multi-sender ICP where each receiver knows a unique message not known by any other receiver and demands any subset of other messages was studied in [5]. Different lower bounds on the optimal code length were obtained using an iterative algorithm and optimal codes were presented for a special subclass of the multi-sender ICP. The multi-sender unicast ICP where each receiver demands a unique message which is not demanded by other receivers was studied in [6] as a rank-minimization problem. A heuristic algorithm was proposed to obtain sub-optimal linear multi-sender index codes in general. Many variations of the multi-sender ICP were studied and inner and outer bounds on the capacity region were given [7]–[9]. These works assume that there are independent channels with fixed finite capacities from every sender to every receiver. This is in contrast with the previous works where a single noiseless broadcast channel was assumed, with the transmissions from multiple senders being orthogonal in time.

As a basic case of the multi-sender unicast ICP, the two-sender unicast ICP (TUICP) was first studied by Thapa et al. [10]. Single-sender index coding schemes based on graph theory were extended to the TUICP. Thapa et al. [11] studied the TUICP using a new variation of graph coloring called the two-sender graph coloring to account for the non-availability of some messages at each sender. The TUICP described by the *side information digraph* and the message sets available at the two senders, was analyzed using three sub-problems considered as single-sender unicast ICPs and the relation between them. The TUICP was divided into 36 classes based on the relation among the three single-sender sub-problems. Optimal broadcast rate (which is defined as the optimal code length per message bit) with finite length

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TABLE I

SUMMARY OF THE ACHIEVABILITY RESULTS FOR ANY TUICP WITH FULLY-PARTICIPATED INTERACTIONS BETWEEN THE SUB-DIGRAPHS $\mathcal{D}_1^{k,\mathcal{P}}$, $\mathcal{D}_2^{k,\mathcal{P}}$, AND $\mathcal{D}_3^{k,\mathcal{P}}$ OF THE SIDE INFORMATION DIGRAPH $\mathcal{D}^k$. THE LISTING OF ALL THE INTERACTION DIGRAPHS $\mathcal{H}_k$ AND THE RELATED CASES IS GIVEN IN FIG. 3

| CASE | $p_t(\mathcal{D}^k, \mathcal{P})$ |
|---|---|
| I, $\mathcal{H}_{10}$ | $\beta_t(\mathcal{D}_2^{10,\mathcal{P}}) + \lceil \chi(\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}) * \Gamma_t(\mathcal{D}_3^{10,\mathcal{P}})) \rceil / t$ |
| I, $\mathcal{H}_{12}$ | $\beta_t(\mathcal{D}_2^{12,\mathcal{P}}) + \lceil \chi(\Gamma_t(\mathcal{D}_1^{12,\mathcal{P}}) \circ \Gamma_t(\mathcal{D}_3^{12,\mathcal{P}})) \rceil / t$ |
| I, $\mathcal{H}_{14}$ | $\beta_t(\mathcal{D}_2^{14,\mathcal{P}}) + \lceil \chi(\Gamma_t(\mathcal{D}_3^{14,\mathcal{P}}) \circ \Gamma_t(\mathcal{D}_1^{14,\mathcal{P}})) \rceil / t$ |
| II-D, $\mathcal{H}_k$ | $\max\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{k,\mathcal{P}}), \beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}}), \beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}$ |

messages was obtained using the two-sender graph coloring of a related graph for a special class of the TUICP. Upper bounds on the optimal broadcast rate with finite length messages were obtained for other classes in terms of the optimal broadcast rates (with finite length messages) of the three sub-problems based on different code constructions. A sub-class was identified for which these results are optimal. Upper bounds on the optimal broadcast rate (as the message length tends to infinity) were obtained in terms of the optimal broadcast rates of the sub-problems and a sub-class was identified where the results are optimal. Optimal scalar linear codes were presented for some special classes of the TUICP using the notion of joint extensions of two single-sender unicast ICPs [13]. Optimal linear broadcast rates (considering only linear encoding schemes at the senders) with finite length messages, optimal linear broadcast rates as message length tends to infinity, and optimal code constructions were provided for a special class of the TUICP in [14].

We summarize our results related to the optimal broadcast rates with finite length messages in Table I. Our results related to the optimal broadcast rates and those given in [11] are summarized in Table II. Any TUICP is described by a side information digraph $\mathcal{D}$ and a triple of message sets given by $\mathcal{P}$. The vertices of $\mathcal{D}$ represent the demands of the receivers and the outgoing edges represent their side information. The sub-digraph of $\mathcal{D}$ induced by the messages available with only the first sender is denoted by $\mathcal{D}_1$. Similarly, we define $\mathcal{D}_2$. The sub-digraph induced by the common messages at both the senders is denoted by $\mathcal{D}_3$. The TUICP is classified (as in [11]) into different cases based on the relation between $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ in $\mathcal{D}$, which is captured by its interaction digraph denoted by $\mathcal{H}$. We obtain $\mathcal{H}$ with the vertex set $\{1, 2, 3\}$, by replacing $\mathcal{D}_i$ with the vertex $i$, for all $i \in \{1, 2, 3\}$. The edge $(i, j)$ exists (equivalently we say that the interaction $\mathcal{D}_i \rightarrow \mathcal{D}_j$ exists), if there exists an edge from at least one vertex in $\mathcal{D}_i$ to some vertex in $\mathcal{D}_j$, $i, j \in \{1, 2, 3\}$. The interaction $\mathcal{D}_i \rightarrow \mathcal{D}_j$ is said to be fully-participated if there are edges from every vertex of $\mathcal{D}_i$ to every vertex of $\mathcal{D}_j$. There exist 36 possible interaction digraphs listed in Fig. 3. All interaction digraphs without cycles constitute Case I. All interaction digraphs with the edges $(1, 2)$ and $(2, 1)$, and no outgoing edges from vertex 3 constitute Case II-A. All interaction digraphs with the edges $(1, 3)$, $(3, 1)$, $(2, 3)$ and $(3, 2)$ constitute Case II-B. All interaction digraphs with the edges $(1, 3)$ and $(3, 1)$, and possibly having one of the edges $(2, 3)$ and $(3, 2)$ (not both) constitute Case II-C. The remaining interaction digraphs constitute Case II-D. The side information digraphs $\mathcal{D}$ and $\{\mathcal{D}_i\}_{i \in \{1,2,3\}}$ have the superscript $k$, if they are associated with the interaction digraph $\mathcal{H}_k$, for $k \in \{1, 2, \cdots, 36\}$. The notation $\beta_t$ and $p_t$ denote the optimal broadcast rate and an achievable broadcast rate with $t$-bit messages respectively. The notation $\beta$ denotes the asymptotic value of $\beta_t$ as $t$ tends to infinity. The results given in Tables I and II hold for TUICPs with all the existing interactions being fully-participated. The notation and definitions required to understand the results in detail are given in Sections II and III. Note that the complexity of finding the optimal broadcast rate for this class of the TUICP (with fully-participated interactions) is reduced to that of finding the optimal broadcast rate for the single-sender unicast ICP, which is an NP-hard problem in general.

The key results of this paper are summarized as follows.

- We identify some symmetries of the confusion graph (Lemma 6, Section III) and exploit it to color the same using two-sender graph coloring. This yields an achievable broadcast rate with finite length messages for any TUICP with fully-participated interactions and the interaction digraph being $\mathcal{H}_k$, $k \in \{10, 12, 14\}$ (Theorem 1, Section IV). Using a by-product of the above result (Corollary 1, Section IV), we also note that the coloring yields a non-linear index code, if certain conditions on the optimal broadcast rates (with finite length messages) of the sub-problems are satisfied (Note 1, Section IV). As far as the authors' knowledge, this is the first result on deterministic finite length non-linear index codes for the TUICP.
- We then establish the optimal broadcast rates of TUICPs belonging to Case II-C with fully-participated interactions, in terms of those of their sub-problems, for which only upper bounds were given in [11] (Theorem 2, Section V.A). This result exploits a known result on the criticality of side information in single-sender ICP. We also identify that the results also hold for some TUICPs with specific partially-participated interactions belonging to Case II-C (Note 2, Section V.A).
- We then establish an achievable broadcast rate with finite length messages for all TUICPs with fully-participated interactions belonging to Case II-D, in terms of the optimal broadcast rates (with finite length messages) of their sub-problems (Theorem 3, Section V.B). This serves as a tighter upper bound on the optimal broadcast rate with finite length messages, compared to the one established in [11].
- We then establish the optimal broadcast rate of TUICPs with fully-participated interactions belonging to Case II-D, in terms of those of their sub-problems, for which only upper bounds were given in [11] (Theorem 4, Section V.C). We also identify that the results also hold for TUICPs with specific partially-participated interactions with the interaction digraph being $\mathcal{H}_{34}$ (Note 3, Section V.C).
- The results of Theorems 2 and 4 in conjunction with the results established in [11], complete the characterization

TABLE II

OPTIMAL BROADCAST RATES FOR ANY TUICP WITH FULLY-PARTICIPATED INTERACTIONS BETWEEN THE SUB-DIGRAPHS $\mathcal{D}_1^{k,\mathcal{P}}$, $\mathcal{D}_2^{k,\mathcal{P}}$, AND $\mathcal{D}_3^{k,\mathcal{P}}$ OF THE SIDE INFORMATION DIGRAPH $\mathcal{D}^k$. THE LISTING OF ALL THE CASES IS GIVEN IN FIG. 3

| CASE | $\beta(\mathcal{D}^k,\mathcal{P})$ | Comments |
|---|---|---|
| I | $\beta(\mathcal{D}_1^{k,\mathcal{P}})+\beta(\mathcal{D}_2^{k,\mathcal{P}})+\beta(\mathcal{D}_3^{k,\mathcal{P}})$ | Also holds for any partially-participated interactions [11] |
| II-A | $\beta(\mathcal{D}_1^{k,\mathcal{P}})+\beta(\mathcal{D}_2^{k,\mathcal{P}})+\beta(\mathcal{D}_3^{k,\mathcal{P}})$ | Also holds for any partially-participated interactions [11] |
| II-B | $\max\{\beta(\mathcal{D}_3^{k,\mathcal{P}}),\beta(\mathcal{D}_1^{k,\mathcal{P}})+\beta(\mathcal{D}_2^{k,\mathcal{P}})\}$ | Completely solved in [11] |
| II-C | $\beta(\mathcal{D}_2^{k,\mathcal{P}})+\max\{\beta(\mathcal{D}_1^{k,\mathcal{P}}),\beta(\mathcal{D}_3^{k,\mathcal{P}})\}$ | Solved only for all TUICPs with $\beta(\mathcal{D}_1^{k,\mathcal{P}})\geq\beta(\mathcal{D}_3^{k,\mathcal{P}})$ in [11]. |
| II-D | $\max\{\beta(\mathcal{D}_1^{k,\mathcal{P}})+\beta(\mathcal{D}_2^{k,\mathcal{P}}),\beta(\mathcal{D}_1^{k,\mathcal{P}})$ $+\beta(\mathcal{D}_3^{k,\mathcal{P}}),\beta(\mathcal{D}_2^{k,\mathcal{P}})+\beta(\mathcal{D}_3^{k,\mathcal{P}})\}$ | Solved only for all TUICPs with $\min\{\beta(\mathcal{D}_1^{k,\mathcal{P}}),\beta(\mathcal{D}_2^{k,\mathcal{P}})\}\geq$ $\beta(\mathcal{D}_3^{k,\mathcal{P}})$ in [11]. |

of the optimal broadcast rates of all TUICPs with fully-participated interactions (Table II).

The remainder of the paper is organized as follows. Section II introduces the problem setup and establishes the required definitions and notation. Section III recapitulates the notion of the confusion graph and related two-sender graph coloring. Section IV provides achievable broadcast rates with finite-length messages, for three sub-cases of the TUICP with fully-participated interactions belonging to Case I. Section V provides optimal broadcast rates for all the TUICPs with fully-participated interactions belonging to Cases II-C and II-D, for which only upper bounds were known. Section VI concludes the paper.

## II. Problem Formulation and Definitions

In this section, we formulate the two-sender unicast ICP (TUICP) and establish the notation and definitions used in this paper.

For any natural number $n$, let $[n] \triangleq \{1,2,\cdots,n\}$. A given instance of the TUICP consists of two senders collectively having $m$ independent message symbols given by $\mathcal{M}=\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_m\}$, where $\mathbf{x}_i\in\mathbb{F}_2^{t\times 1}$, $\forall i\in[m]$, and $t\geq 1$. The $s$th sender denoted by $\mathcal{S}_s$, $s\in\{1,2\}$, has the message set $\mathcal{M}_s$, where $\mathcal{M}_s\subseteq\mathcal{M}$, and $\mathcal{M}_1\cup\mathcal{M}_2=\mathcal{M}$. The identity of the messages available with each sender is known to the other. The senders transmit through a noiseless broadcast channel. Transmissions from different senders are orthogonal in time. There are $m$ receivers, each receiving all the transmissions from both the senders. The $i$th receiver demands $\mathbf{x}_i$ and has $\mathcal{K}_i\subseteq\mathcal{M}\setminus\{\mathbf{x}_i\}$ as its side information. The single-sender unicast ICP (SUICP) is a special case of TUICP, where either $\mathcal{M}_1=\mathcal{M}$ or $\mathcal{M}_2=\mathcal{M}$. The goal of the TUICP is to design coding schemes at the two senders (also called a two-sender index code) such that the total number of transmissions from the senders is minimized, while all the receivers are able to decode their demands using their side information. An encoding function for the sender $\mathcal{S}_s$ is given by $\mathbb{E}_s:\mathbb{F}_2^{|\mathcal{M}_s|t\times 1}\rightarrow\mathbb{F}_2^{l_s\times 1}$, such that $\mathcal{C}_s=\mathbb{E}_s(\mathcal{M}_s)$, where $l_s$ is the length of the codeword $\mathcal{C}_s$ transmitted by $\mathcal{S}_s$, $s\in\{1,2\}$. The $i$th receiver has a decoding function given by $\mathbb{D}_i:\mathbb{F}_2^{(l_1+l_2+|\mathcal{K}_i|t)\times 1}\rightarrow\mathbb{F}_2^{t\times 1}$, such that $\mathbf{x}_i=\mathbb{D}_i(\mathcal{C}_1,\mathcal{C}_2,\mathcal{K}_i)$, $\forall i\in[m]$, i.e., it can decode $\mathbf{x}_i$ using its side information and the received codewords $\mathcal{C}_1$ and $\mathcal{C}_2$. Without loss of generality, we assume that for the SUICP only $\mathbb{E}_1$ exists and $l_2=0$.

We state the definitions of the broadcast rate of an index code (single-sender or two-sender), the optimal broadcast rate of an ICP (single-sender or two-sender) with $t$-bit messages for any finite $t$, and the optimal broadcast rate of the same, as given in [11]. Note that the definitions take into account both linear and non-linear encoding schemes.

*Definition 1* (Broadcast rate, [11]): The broadcast rate of an index code (for a single-sender problem or a two-sender problem) described by $(\{\mathbb{E}_j\},\{\mathbb{D}_i\})$ is the total number of transmitted bits per received message bit (for finite length messages), given by $p_t\triangleq\frac{(l_1+l_2)}{t}$.

The optimal (minimum) length of any index code for a given ICP and $t$-bit messages is called the optimal code length. Note that for the SUICP, only $\mathbb{E}_1$ exists and hence $p_t=\frac{l_1}{t}$.

*Definition 2* (Optimal broadcast rate with $t$-bit messages for any finite $t$, [11]): The optimal broadcast rate for a given ICP with $t$-bit messages and any finite $t$ is given by $\beta_t\triangleq\min_{\{\mathbb{E}_j\}}p_t$.

*Definition 3* (Optimal broadcast rate, [11]): The optimal broadcast rate of a given ICP (single-sender or two-sender) is given by $\beta\triangleq\inf_t\beta_t=\lim_{t\rightarrow\infty}\beta_t$.

We state some definitions from graph theory that will be used in this paper.

A directed graph (also called digraph) given by $\mathcal{D}=(\mathcal{V}(\mathcal{D}),\mathcal{E}(\mathcal{D}))$, consists of a set of vertices $\mathcal{V}(\mathcal{D})$, and a set of edges $\mathcal{E}(\mathcal{D})$ which is a set of ordered pairs of vertices. A sub-digraph $\mathcal{G}$ of a digraph $\mathcal{D}$ satisfies $\mathcal{V}(\mathcal{G})\subseteq\mathcal{V}(\mathcal{D})$ and $\mathcal{E}(\mathcal{G})\subseteq\mathcal{E}(\mathcal{D})$. The sub-digraph of $\mathcal{D}$ induced by the vertex set $\mathcal{V}(\mathcal{G})$ is the digraph with the vertex set $\mathcal{V}(\mathcal{G})$, and the edge set given by $\mathcal{E}(\mathcal{G})=\{(u,v):u,v\in\mathcal{V}(\mathcal{G}),(u,v)\in\mathcal{E}(\mathcal{D})\}$. A directed path in a digraph $\mathcal{D}$ is a sequence of distinct vertices $\{v_{i_1},\cdots,v_{i_r}\}$, such that $(v_{i_j},v_{i_{j+1}})\in\mathcal{E}(\mathcal{D})$, $\forall j\in[r-1]$. A cycle in a digraph $\mathcal{D}$ is a sequence of distinct vertices $(v_{i_1},\cdots,v_{i_c})$, such that $(v_{i_j},v_{i_{j+1}})\in\mathcal{E}(\mathcal{D})$, $\forall j\in[c-1]$, and $(v_{i_c},v_{i_1})\in\mathcal{E}(\mathcal{D})$. For an undirected graph, the edge set consists of a set of unordered pairs of vertices. Two vertices are said to be adjacent if there exists an edge between them. A proper graph coloring of an undirected graph $\mathcal{D}$ is an onto function $J:\mathcal{V}(\mathcal{D})\rightarrow\mathcal{J}$ where $\mathcal{J}$ is a set of colors such that, if $(u,v)\in\mathcal{E}(\mathcal{D})$, then $J(u)\neq J(v)$. The minimum number of colors required for any proper coloring of $\mathcal{D}$ is its chromatic number and is denoted by $\chi(\mathcal{D})$. Two undirected graphs $\mathcal{G}$ and $\mathcal{H}$ are said to be isomorphic if there exists a bijection between $\mathcal{V}(\mathcal{G})$ and $\mathcal{V}(\mathcal{H})$ given by $f:\mathcal{V}(\mathcal{G})\rightarrow\mathcal{V}(\mathcal{H})$, such that $(u,v)\in\mathcal{E}(\mathcal{G})$ iff $(f(u),f(v))\in\mathcal{E}(\mathcal{H})$.

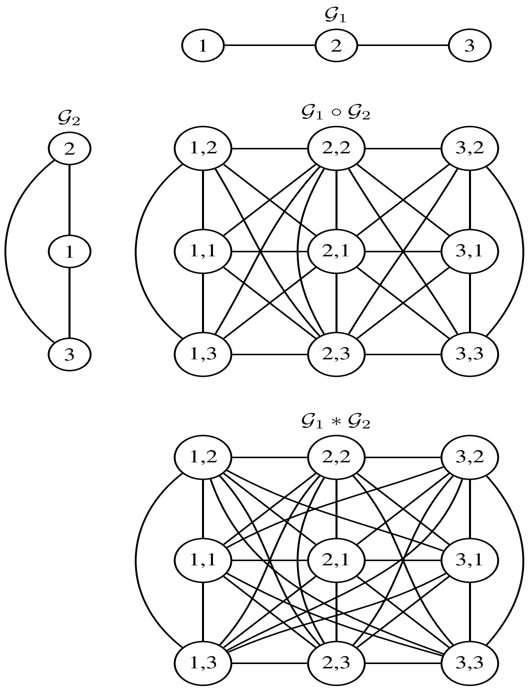


Fig. 1. Lexicographic product and disjunctive product of $\mathcal{G}_1$ and $\mathcal{G}_2$.

The following graph products of any two given undirected graphs are used in this work and illustrated with an example in Fig. 1.

*Definition 4* (Lexicographic product): The lexicographic product $\mathcal{G}$ of two undirected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is denoted by $\mathcal{G}_1 \circ \mathcal{G}_2$, where $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}_1) \times \mathcal{V}(\mathcal{G}_2)$ and $((u_1, u_2), (v_1, v_2)) \in \mathcal{E}(\mathcal{G})$ iff $(u_1, v_1) \in \mathcal{E}(\mathcal{G}_1)$ or $((u_1 = v_1)$ and $(u_2, v_2) \in \mathcal{E}(\mathcal{G}_2))$.

*Definition 5* (Disjunctive product): The disjunctive product $\mathcal{G}$ is denoted by $\mathcal{G}_1 * \mathcal{G}_2$, where $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}_1) \times \mathcal{V}(\mathcal{G}_2)$ and $((u_1, u_2), (v_1, v_2)) \in \mathcal{E}(\mathcal{G})$ iff $(u_1, v_1) \in \mathcal{E}(\mathcal{G}_1)$ or $(u_2, v_2) \in \mathcal{E}(\mathcal{G}_2)$.

For any unicast ICP (single-sender or two-sender), the knowledge of side information and demands of all the receivers is represented by the side information digraph $\mathcal{D} = (\mathcal{V}(\mathcal{D}), \mathcal{E}(\mathcal{D}))$, where the vertex set is given by $\mathcal{V}(\mathcal{D}) = \{v_1, \cdots, v_m\}$. The vertex $v_i$ represents the $i$th receiver which demands the message $\mathbf{x}_i$. Due to the one-to-one relationship between the $i$th receiver and $\mathbf{x}_i$, $v_i$ also represents $\mathbf{x}_i$. Hence, we refer to $v_i$ as the $i$th message, the $i$th receiver and the $i$th vertex interchangeably. The edge set is given by $\mathcal{E}(\mathcal{D}) = \{(v_i, v_j) : \mathbf{x}_j \in \mathcal{K}_i, i, j \in [m]\}$. The sets $\mathcal{P}_1 = \mathcal{M}_1 \setminus \mathcal{M}_2$ and $\mathcal{P}_2 = \mathcal{M}_2 \setminus \mathcal{M}_1$ contain the messages available only at $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. The set of messages available at both the senders is $\mathcal{P}_3 = \mathcal{M}_1 \cap \mathcal{M}_2$. Let $m_i = |\mathcal{P}_i|, i \in \{1, 2, 3\}$. Let $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$. Any TUICP $\mathcal{I}$ can be described in terms of the tuple $(\mathcal{D}, \mathcal{P})$ as $\mathcal{I}(\mathcal{D}, \mathcal{P})$. The optimal broadcast rates $\beta$ and $\beta_t$ of any TUICP $\mathcal{I}(\mathcal{D}, \mathcal{P})$ are denoted by $\beta(\mathcal{D}, \mathcal{P})$ and $\beta_t(\mathcal{D}, \mathcal{P})$ respectively. Similarly, an achievable broadcast rate $p_t$ is denoted by $p_t(\mathcal{D}, \mathcal{P})$. For the single-sender unicast ICP with side information digraph $\mathcal{D}$, the $\beta$ and $\beta_t$ are denoted by $\beta(\mathcal{D})$ and $\beta_t(\mathcal{D})$.

The TUICP has been analyzed using three disjoint sub-digraphs of the side information digraph (equivalently three sub-problems) induced by the three disjoint vertex sets $\mathcal{P}_s$, $s \in \{1, 2, 3\}$ [11]. Let $\mathcal{D}_s$ be the sub-digraph of $\mathcal{D}$ induced by the vertices $\{v_j : \mathbf{x}_j \in \mathcal{P}_s, j \in [m]\}$, where $s \in \{1, 2, 3\}$. If there exists an edge from some vertex in $\mathcal{V}(\mathcal{D}_i)$ to some vertex in $\mathcal{V}(\mathcal{D}_j)$ in the side information digraph $\mathcal{D}$, $i, j \in \{1, 2, 3\}, i \neq j$, then we say that there is an interaction from $\mathcal{D}_i$ to $\mathcal{D}_j$, and denote it by $\mathcal{D}_i \to \mathcal{D}_j$. We say that the interaction $\mathcal{D}_i \to \mathcal{D}_j$ is fully-participated if there are edges from every vertex in $\mathcal{V}(\mathcal{D}_i)$ to every vertex in $\mathcal{V}(\mathcal{D}_j)$. Otherwise, it is said to be a partially-participated interaction. We say that the TUICP has fully-participated interactions if all the existing interactions are fully-participated interactions. For a given TUICP, an associated digraph called the interaction digraph which captures the type of interactions between the sub-digraphs of the side information digraph was introduced in [11] without any name.

*Definition 6* (Interaction digraph, [14]): For a given TUICP $\mathcal{I}(\mathcal{D}, \mathcal{P})$, the digraph $\mathcal{H}$ with $\mathcal{V}(\mathcal{H}) = \{1, 2, 3\}$ and $\mathcal{E}(\mathcal{H}) = \{(i, j) | \mathcal{D}_i \to \mathcal{D}_j, i \neq j, i, j \in \{1, 2, 3\}\}$, is defined as the interaction digraph of the side information digraph $\mathcal{D}$.

Note that a given side information digraph can correspond to different interaction digraphs based on the choice of the message tuple $\mathcal{P}$. The edges $(i, j)$ and $(j, i)$ in any interaction digraph are denoted by a single edge with arrows at both ends, $i, j \in \{1, 2, 3\}$. There are 64 possibilities for the digraph $\mathcal{H}$. As there are a maximum of 6 possible interactions between all pairs of vertices of any interaction digraph, we have $64$ ($= 2^6$) possible interaction digraphs. Out of the 64 digraphs, some can be obtained by swapping the vertices 1 and 2 of other digraphs (correspondingly exchanging the labels of the senders and the sub-digraphs $\mathcal{D}_1$ and $\mathcal{D}_2$). Removing these redundant digraphs, we get 36 unique interaction digraphs as given in Fig. 3, which have been listed and classified in [11]. The number written below each interaction digraph in the figure is used as the subscript to denote the specific interaction digraph. The side information digraph $\mathcal{D}$ describing a given TUICP with the interaction digraph $\mathcal{H}_k$ is denoted by $\mathcal{D}^k$, $k \in [36]$. For any TUICP $\mathcal{I}(\mathcal{D}^k, \mathcal{P})$, the corresponding sub-digraphs $\mathcal{D}_i$, $i \in \{1, 2, 3\}$, are denoted as $\mathcal{D}_i^{k,\mathcal{P}}$. Note that all the possible interaction digraphs are classified into two cases broadly: Case I and Case II. Case I consists of acyclic interaction digraphs (i.e., with no cycles). Case II is further classified into four sub-cases as shown in Fig. 3. The expression for the optimal broadcast rate of any TUICP with fully-participated interactions belonging to any given sub-case of Case II (Cases II-A, II-B, II-C, and II-D), in terms of those of the sub-problems is the same (as listed in Table II). This is the basis for the classification given in Fig. 3. We illustrate the above definitions using an example.

*Example 1:* Consider the TUICP with $m = 5$ messages with the side information digraph and the corresponding interaction digraph given in Fig. 2. Sender $\mathcal{S}_1$ has $\mathcal{M}_1 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5\}$, and $\mathcal{S}_2$ has $\mathcal{M}_2 = \{\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$. Hence, $\mathcal{P}_1 = \{\mathbf{x}_1, \mathbf{x}_2\}$, $\mathcal{P}_2 = \{\mathbf{x}_3, \mathbf{x}_4\}$, and $\mathcal{P}_3 = \{\mathbf{x}_5\}$. The side information of all the receivers are given as follows: $\mathcal{K}_1 = \{\mathbf{x}_2, \mathbf{x}_5\}$, $\mathcal{K}_2 = \{\mathbf{x}_1\}$, $\mathcal{K}_3 = \{\mathbf{x}_4, \mathbf{x}_5\}$, $\mathcal{K}_4 = \{\mathbf{x}_3\}$, $\mathcal{K}_5 = \{\mathbf{x}_1, \mathbf{x}_2\}$.

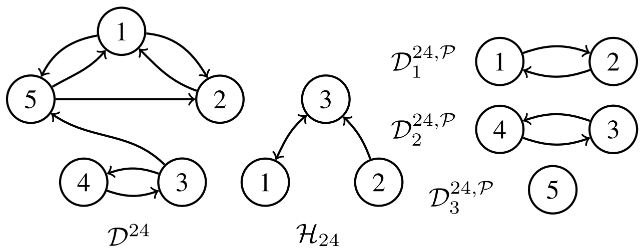


Fig. 2. The interaction digraph and the sub-digraphs of the given side information digraph of the TUICP given in Example 1.

The interaction digraph given in Fig. 2 is $\mathcal{H}_{24}$ as listed in Fig. 3. Hence, the side information digraph is denoted as $\mathcal{D}^{24}$. The vertex-induced sub-digraphs $\mathcal{D}_1^{24,\mathcal{P}}$, $\mathcal{D}_2^{24,\mathcal{P}}$, and $\mathcal{D}_3^{24,\mathcal{P}}$ are also shown in Fig. 2. Note that the interaction $\mathcal{D}_3^{24,\mathcal{P}} \rightarrow \mathcal{D}_1^{24,\mathcal{P}}$ is fully-participated. Others are partially-participated interactions.

The following notations are required for the construction of a two-sender index code from single-sender index codes. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two codewords of length $l_1$ and $l_2$ respectively. $\mathcal{C}_1 \oplus \mathcal{C}_2$ denotes the bit-wise XOR of $\mathcal{C}_1$ and $\mathcal{C}_2$ after zero-padding the shorter codeword at the least significant positions to match the length of the longer codeword. For example, if $\mathcal{C}_1 = 1010$, and $\mathcal{C}_2 = 110$, then $\mathcal{C}_1 \oplus \mathcal{C}_2 = 0110$. The notation $\mathcal{C}[a:b]$ denotes the vector obtained by picking the bits from bit position $a$ to bit position $b$, starting from the most significant position of the codeword $\mathcal{C}$, with $a, b \in [l]$, $l$ being the length of $\mathcal{C}$. For example $\mathcal{C}_1[2:4] = 010$.

## III. Confusion Graphs and the Two-Sender Graph Coloring

In this section, we review confusion graphs and recapitulate some results on the two-sender graph coloring of confusion graphs provided in [11]. We also provide some definitions (Definitions 9 and 10) which are used to describe the symmetries of the confusion graph. Then, we state and prove a lemma which is used to establish the main results in this paper.

Consider the unicast ICP (single-sender or two-sender) described by the side information digraph $\mathcal{D}$ with $m$ messages. Let $\mathbf{x} = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ and $\mathbf{x}' = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be two tuples of realizations of $m$ messages, where $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_2^t, \forall i \in [m]$. The tuples $\mathbf{x}$ and $\mathbf{x}'$ are said to be confusable at the $i$th receiver, if $\mathbf{u}_i \neq \mathbf{v}_i$ and $\mathbf{u}_j = \mathbf{v}_j$ for all $j$ such that $\mathbf{x}_j \in \mathcal{K}_i$. Two tuples are said to be confusable if they are confusable at some receiver. Confusion at a receiver refers to existence of confusable tuples at the receiver. Two tuples of realizations of $m$ messages that are confusable cannot be encoded to the same codeword as one of the receivers cannot decode the demanded message successfully. The confusion graph has been originally introduced in [15] and defined as follows for the unicast ICP.

*Definition 7* (Confusion graph, [11]): The confusion graph of a side information digraph $\mathcal{D}$ with $m$ vertices and $t$-bit messages is an undirected graph, denoted by $\Gamma_t(\mathcal{D}) = (\mathcal{V}(\Gamma_t(\mathcal{D})), \mathcal{E}(\Gamma_t(\mathcal{D})))$, where $\mathcal{V}(\Gamma_t(\mathcal{D})) = \{\mathbf{x} : \mathbf{x} \in \mathbb{F}_2^{mt}\}$ and $\mathcal{E}(\Gamma_t(\mathcal{D})) = \{(\mathbf{x}, \mathbf{x}') : \mathbf{x} \text{ and } \mathbf{x}' \text{ are confusable}\}$.

We use the following notation used in [11], in the context of confusion graphs. Each realization of the bits of concatenated messages belonging to $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, (i.e., each element of $\mathbb{F}_2^{tm_1}$, $\mathbb{F}_2^{tm_2}$ and $\mathbb{F}_2^{tm_3}$ respectively), is represented by unique tuples $\mathbf{b}_{\mathcal{P}_1}^i$, $\mathbf{b}_{\mathcal{P}_2}^j$, and $\mathbf{b}_{\mathcal{P}_3}^k$. Superscripts $i, i' \in [2^{tm_1}]$, $j, j' \in [2^{tm_2}]$, and $k, k' \in [2^{tm_3}]$ are used to represent possible realizations of concatenation of all the messages belonging to $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$ of $tm_1$, $tm_2$, and $tm_3$ bits. Each message tuple $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ can be uniquely written as $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ for some $i, j$, and $k$. Hence, each vertex of the confusion graph is labelled by a unique tuple $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$.

Consider a valid coloring of the confusion graph $\Gamma_t(\mathcal{D})$ with a set of colors $\mathcal{J}$. This results in $|\mathcal{J}|$ sets of vertices, such that all the vertices in a given set are colored with a unique color. Each set of vertices is independent and can be coded into the same codeword, as no pair of vertices in the given set are confusable. Hence, sending a codeword is equivalent to sending the identity of a color. As $\chi(\Gamma_t(\mathcal{D}))$ is the minimum number of colors required, the optimal code length is $\lceil \log_2 \chi(\Gamma_t(\mathcal{D})) \rceil$ bits. The classical graph coloring of the confusion graph will not yield the optimal code length for the TUICP, as there is a constraint on the coloring due to the non-availability of some messages at one of the senders. To account for the encoding done by the two senders, two-sender graph coloring has been introduced in [11].

*Definition 8* (Two-sender graph coloring of $\Gamma_t(\mathcal{D})$, [11]): Let two onto functions $J_1 : \mathbb{F}_2^{tm_1} \times \mathbb{F}_2^{tm_3} \rightarrow \mathcal{J}_1$ and $J_2 : \mathbb{F}_2^{tm_2} \times \mathbb{F}_2^{tm_3} \rightarrow \mathcal{J}_2$ be the coloring functions carried out by senders $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. A proper two-sender graph coloring of $\Gamma_t(\mathcal{D})$ is an onto function $J_0 : \mathbb{F}_2^{tm_1} \times \mathbb{F}_2^{tm_2} \times \mathbb{F}_2^{tm_3} \rightarrow \mathcal{J}_1 \times \mathcal{J}_2$ where $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) = (J_1(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_3}^k), J_2(\mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k))$ such that if $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ and $(\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^{k'})$ are adjacent vertices of $\Gamma_t(\mathcal{D})$, then $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) \neq J_o((\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$.

Note that the two ordered pairs of colors $(c_1, c_2)$ and $(c_1', c_2')$, where $c_i, c_i' \in \mathcal{J}_i$, $i \in \{1, 2\}$ are said to be different iff $c_1 \neq c_1'$ or $c_2 \neq c_2'$. We recapitulate some results on the two-sender graph coloring stated as Lemmas 1 to 4 in [11], which are used in coloring the confusion graph.

*Lemma 1* (Lemma 1, [11]): For any two vertices $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ and $(\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ in $\Gamma_t(\mathcal{D})$ which are confusable, if $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1, c_2)$ and $J_o((\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1', c_2')$ for some $c_1, c_1' \in \mathcal{J}_1$ and $c_2, c_2' \in \mathcal{J}_2$, then we must have $c_1 \neq c_1'$ and $c_2 = c_2'$.

*Lemma 2* (Lemma 2, [11]): For any two vertices $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ and $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^k)$ in $\Gamma_t(\mathcal{D})$ which are confusable, if $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1, c_2)$ and $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1', c_2')$ for some $c_1, c_1' \in \mathcal{J}_1$ and $c_2, c_2' \in \mathcal{J}_2$, then we must have $c_1 = c_1'$ and $c_2 \neq c_2'$.

*Lemma 3* (Lemma 3, [11]): For any two vertices $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ and $(\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^k)$ in $\Gamma_t(\mathcal{D})$ which are confusable due to confusion at some vertices in $\mathcal{D}_1$ and $\mathcal{D}_2$, if $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1, c_2)$ and $J_o((\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1', c_2')$ for some $c_1, c_1' \in \mathcal{J}_1$ and $c_2, c_2' \in \mathcal{J}_2$, then we must have $c_1 \neq c_1'$ and $c_2 \neq c_2'$.

*Lemma 4* (Lemma 4, [11]): For any two vertices $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)$ and $(\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^{k'})$ in $\Gamma_t(\mathcal{D})$ which are confusable, if $J_o((\mathbf{b}_{\mathcal{P}_1}^i, \mathbf{b}_{\mathcal{P}_2}^j, \mathbf{b}_{\mathcal{P}_3}^k)) = (c_1, c_2)$ and

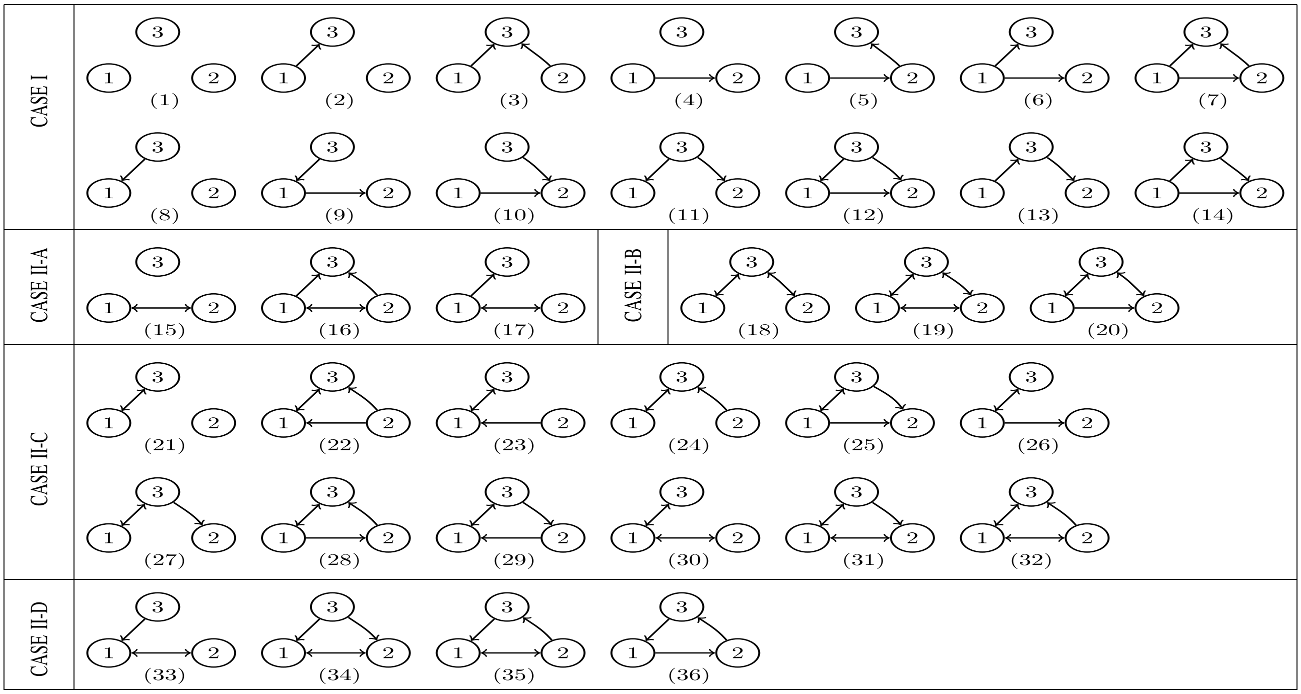


Fig. 3. Enumeration of all unique interactions between the sub-digraphs $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, given by the interaction digraph $\mathcal{H}$.

$J_o((\mathbf{b}^i_{\mathcal{P}_1}, \mathbf{b}^j_{\mathcal{P}_2}, \mathbf{b}^{k'}_{\mathcal{P}_3})) = (c'_1, c'_2)$ for some $c_1, c'_1 \in \mathcal{J}_1$ and $c_2, c'_2 \in \mathcal{J}_2$, then we must have either $c_1 \neq c'_1$, or $c_2 \neq c'_2$.

The optimal broadcast rate for the TUICP with finite length messages is given by Theorem 2 in [11] as follows.

*Lemma 5* (Theorem 2, [11]):

$$\beta_t(\mathcal{D}, \mathcal{P}) = \min_{\mathcal{J}_1, \mathcal{J}_2} \frac{\lceil \log_2 |\mathcal{J}_1| \rceil + \lceil \log_2 |\mathcal{J}_2| \rceil}{t} \tag{1}$$

We illustrate the two-sender graph coloring of the confusion graph using an example.

*Example 2:* Consider the TUICP with $t = 1$ and the side information digraph given in Fig. 4 along with its confusion graph. The message sets at the senders are given by $\mathcal{M}_1 = \{\mathbf{x}_1, \mathbf{x}_3\}$ and $\mathcal{M}_2 = \{\mathbf{x}_2, \mathbf{x}_3\}$. The confusion graph $\Gamma_1(\mathcal{D})$ has $2^m = 8$ vertices representing all possible binary tuples $(\mathbf{b}^i_{\mathcal{P}_1}, \mathbf{b}^j_{\mathcal{P}_2}, \mathbf{b}^k_{\mathcal{P}_3})$, $i, j, k \in \{1, 2\}$, of length three. Edges are drawn between every two confusable tuples. For example, there is an edge between $(0, 1, 0)$ and $(1, 1, 0)$ due to confusion at receiver 1. After the construction of the confusion graph, all the vertices are colored by each sender. In the ordered pair of colors, the first color is associated with $\mathcal{S}_1$ and the second color is associated with $\mathcal{S}_2$. Color RED is denoted as R and BLUE is denoted as B in Fig. 4. Coloring is done based on Lemmas 1 to 4. Hence, if $\mathcal{S}_1$ colors $(0, 1, 0)$ with RED color, it must color $(1, 1, 0)$ with another color, say BLUE. Similarly, we can color other vertices using the two-sender graph coloring. It can be easily verified that only two colors are required at each sender to color the confusion graph. The two-sender coloring shown in Fig. 4 can be easily verified to be a valid two-sender coloring. Hence, $\mathcal{J}_1 = \mathcal{J}_2 = \{$RED, BLUE$\}$. Assuming a map from the colors to binary bits that maps RED to 1 and BLUE to 0, the tuple $(0, 0, 0)$ can be mapped to the codeword 11, the tuple $(0, 0, 1)$ can be mapped to the codeword 00, and so on. Thus the two-sender index code consists of codewords given by $\{00, 01, 10, 11\}$. The first bit of the codeword is sent by $\mathcal{S}_1$, and the second bit is sent by $\mathcal{S}_2$. Thus, $\beta_t(\mathcal{D}, \mathcal{P}) \leq 2$ for any $t \geq 1$. As each sender has a single message which is not present at the other sender, each of them must at least send one bit. Thus, $\beta_t(\mathcal{D}, \mathcal{P}) \geq 2$. Hence, $\beta_t(\mathcal{D}, \mathcal{P}) = 2$.

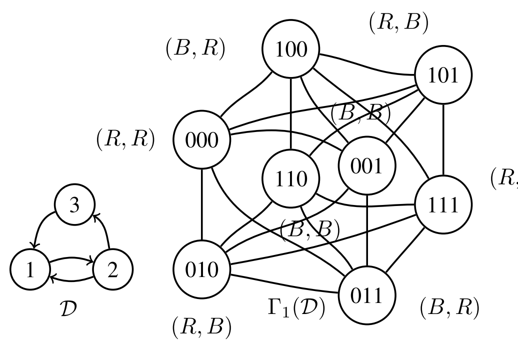


Fig. 4. Side information digraph and two-sender graph coloring of its confusion graph for the TUICP given in Example 2.

To exploit the symmetries of the confusion graph and facilitate the two-sender graph coloring, the vertices of $\Gamma_t(\mathcal{D})$ can be grouped in different ways. We define three ways of grouping the vertices into blocks and the corresponding inter-block edges.

*Definition 9* ($I$-blocks, $J$-blocks, $K$-blocks): Let $\mathcal{B}_{\mathbf{b}^i_{\mathcal{P}_1}} \triangleq \{(\mathbf{b}^i_{\mathcal{P}_1}, \mathbf{b}^j_{\mathcal{P}_2}, \mathbf{b}^k_{\mathcal{P}_3}) : j \in [2^{tm_2}], k \in [2^{tm_3}]\}$ with cardinality $2^{tm_2} \times 2^{tm_3}$ for a given $i \in [2^{tm_1}]$. Similarly, $\mathcal{B}_{\mathbf{b}^j_{\mathcal{P}_2}}$ and $\mathcal{B}_{\mathbf{b}^k_{\mathcal{P}_3}}$ are also defined. The subgraph of $\Gamma_t(\mathcal{D})$ induced by the vertices belonging to $\mathcal{B}_{\mathbf{b}^i_{\mathcal{P}_1}}$ is called the $i$th $I$-block. There are

$2^{tm_1}$ $I$-blocks. The subgraph of $\Gamma_t(\mathcal{D})$ induced by the vertices belonging to $\mathcal{B}_{\mathbf{b}^j_{\mathcal{P}_2}}$ is called the $j$th $J$-block. Similarly, $k$th $K$-block $\mathcal{B}_{\mathbf{b}^k_{\mathcal{P}_3}}$ is also defined.

*Definition 10* (Inter-block edges): An edge between two vertices, each belonging to a different $I$-block of $\Gamma_t(\mathcal{D})$ is called an inter-$I$-block edge. An edge between two vertices, each belonging to a different $J$-block of $\Gamma_t(\mathcal{D})$ is called an inter-$J$-block edge. An edge between two vertices, each belonging to a different $K$-block of $\Gamma_t(\mathcal{D})$ is called an inter-$K$-block edge.

We require the following lemma to exploit the symmetries of the confusion graph in two-sender graph coloring.

*Lemma 6:* All $I$-blocks in a given confusion graph are isomorphic to each other. Similarly, all $J$-blocks are isomorphic to each other, and all $K$-blocks are isomorphic to each other, in a given confusion graph.

*Proof:* We prove the lemma for all $I$-blocks. The proof follows on similar lines for all $J$-blocks and all $K$-blocks. Every vertex in any $i$th $I$-block induced by the vertices in $\mathcal{B}_{\mathbf{b}^i_{\mathcal{P}_1}}$ has the same $\mathbf{b}^i_{\mathcal{P}_1}$ sub-label, $i \in [2^{tm_1}]$. Thus, any edge in any $i$th $I$-block is only due to confusion at the vertices (i.e, receivers) belonging to $\mathcal{V}(\mathcal{D}_2 \cup \mathcal{D}_3)$. Every $I$-block has $2^{tm_2} \times 2^{tm_3}$ vertices. If there is an edge given by $((\mathbf{b}^i_{\mathcal{P}_1}, \mathbf{b}^j_{\mathcal{P}_2}, \mathbf{b}^k_{\mathcal{P}_3}), (\mathbf{b}^i_{\mathcal{P}_1}, \mathbf{b}^{j'}_{\mathcal{P}_2}, \mathbf{b}^{k'}_{\mathcal{P}_3}))$ in $i$th $I$-block induced by $\mathcal{B}_{\mathbf{b}^i_{\mathcal{P}_1}}$, then there is an edge given by $((\mathbf{b}^{i'}_{\mathcal{P}_1}, \mathbf{b}^j_{\mathcal{P}_2}, \mathbf{b}^k_{\mathcal{P}_3}), (\mathbf{b}^{i'}_{\mathcal{P}_1}, \mathbf{b}^{j'}_{\mathcal{P}_2}, \mathbf{b}^{k'}_{\mathcal{P}_3}))$ in $i'$th $I$-block induced by $\mathcal{B}_{\mathbf{b}^{i'}_{\mathcal{P}_1}}$, $i \neq i'$ and vice versa, as the confusion is only due to tuples $(\mathbf{b}^j_{\mathcal{P}_2}, \mathbf{b}^k_{\mathcal{P}_3})$ and $(\mathbf{b}^{j'}_{\mathcal{P}_2}, \mathbf{b}^{k'}_{\mathcal{P}_3})$, at some vertex belonging to $\mathcal{V}(\mathcal{D}_2 \cup \mathcal{D}_3)$. Hence, all $I$-blocks are isomorphic to each other. ■

## IV. An Achievable Broadcast Rate With Finite Length Messages for Some Sub-Cases of CASE I with Fully-Participated Interactions

In this section, we provide an achievable broadcast rate with $t$-bit messages for any TUICP with the side information digraph $\mathcal{D}^k$, $k \in \{10, 12, 14\}$, having fully-participated interactions between its sub-digraphs $\mathcal{D}^{k,\mathcal{P}}_i$, $i \in \{1, 2, 3\}$ using a valid two-sender graph coloring of the confusion graph. No non-trivial achievable broadcast rate was known for these sub-cases. We first review the related results known prior to this paper. The following conjecture was stated in [12].

*Conjecture 1* (Conjecture 1, [12]): For any side information digraph $\mathcal{D}^k$, $k \in \{8, 9, \cdots, 14\}$, having any type of interaction (i.e., either fully-participated or partially-participated) between its sub-digraphs $\mathcal{D}^{k,\mathcal{P}}_i$, $i \in \{1, 2, 3\}$, for any $\mathcal{P}$, and $t$-bit messages for any finite $t$, $\beta_t(\mathcal{D}^k, \mathcal{P}) = \beta_t(\mathcal{D}^{k,\mathcal{P}}_1) + \beta_t(\mathcal{D}^{k,\mathcal{P}}_2) + \beta_t(\mathcal{D}^{k,\mathcal{P}}_3) + \epsilon/t$, for some $\epsilon \in \{-2, -1, 0\}$.

The conjecture was stated considering that a minimum of $\chi(\Gamma_t(\mathcal{D}^{k,\mathcal{P}}_1))\chi(\Gamma_t(\mathcal{D}^{k,\mathcal{P}}_2))\chi(\Gamma_t(\mathcal{D}^{k,\mathcal{P}}_3))$ ordered pairs of colors are required to color the confusion graph $\Gamma_t(\mathcal{D}^k)$, $k \in \{1, 2, \cdots, 7\}$, according to the two-sender graph coloring. The following theorem provides non-trivial achievable broadcast rates with finite length messages based on a two-sender coloring of the confusion graphs $\Gamma_t(\mathcal{D}^k)$, $k \in \{10, 12, 14\}$.

*Theorem 1:* For any TUICP with the side information digraph $\mathcal{D}^k$, $k \in \{10, 12, 14\}$, having fully-participated interactions between its sub-digraphs $\mathcal{D}^{k,\mathcal{P}}_i$, $i \in \{1, 2, 3\}$, for any $\mathcal{P}$, and $t$-bit messages for any finite $t$, the following broadcast rates are achievable.

$$(i)\ p_t(\mathcal{D}^{10}, \mathcal{P}) = \beta_t(\mathcal{D}^{10,\mathcal{P}}_2) + \frac{\lceil \chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_1) * \Gamma_t(\mathcal{D}^{10,\mathcal{P}}_3)) \rceil}{t},$$

$$(ii)\ p_t(\mathcal{D}^{12}, \mathcal{P}) = \beta_t(\mathcal{D}^{12,\mathcal{P}}_2) + \frac{\lceil \chi(\Gamma_t(\mathcal{D}^{12,\mathcal{P}}_1) \circ \Gamma_t(\mathcal{D}^{12,\mathcal{P}}_3)) \rceil}{t},$$

$$(iii)\ p_t(\mathcal{D}^{14}, \mathcal{P}) = \beta_t(\mathcal{D}^{14,\mathcal{P}}_2) + \frac{\lceil \chi(\Gamma_t(\mathcal{D}^{14,\mathcal{P}}_3) \circ \Gamma_t(\mathcal{D}^{14,\mathcal{P}}_1)) \rceil}{t}.$$

*Proof:* See Appendix A. ■

We require the following lemmas to prove Corollary 1, which illustrates the significance of the results of Theorem 1 and the two-sender graph coloring given in its proof.

*Lemma 7* (Lemma A9, [11]): For any real numbers $a$ and $b$, $\lceil a + b \rceil = \lceil a \rceil + \lceil b \rceil + \epsilon$, for some $\epsilon \in \{-1, 0\}$.

*Lemma 8* (Theorem 1, [18]): For any two undirected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, $\chi(\mathcal{G}_1 * \mathcal{G}_2) \leq \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$.

*Lemma 9* (Corollary 3.4.2, [19]): For any two undirected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, $\chi(\mathcal{G}_1 \circ \mathcal{G}_2) \leq \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$.

*Corollary 1:* For any TUICP with the side-information digraph $\mathcal{D}^h$, $h \in \{10, 12, 14\}$, having fully-participated interactions between its sub-digraphs $\mathcal{D}^{h,\mathcal{P}}_i$, $i \in \{1, 2, 3\}$, for any $\mathcal{P}$, and $t$-bit messages for any finite $t$, we have,

$$p_t(\mathcal{D}^h, \mathcal{P}) \leq \beta_t(\mathcal{D}^{h,\mathcal{P}}_1) + \beta_t(\mathcal{D}^{h,\mathcal{P}}_2) + \beta_t(\mathcal{D}^{h,\mathcal{P}}_3) + \epsilon'/t, \quad (2)$$

for some $\epsilon' \in \{-1, 0\}$, where $p_t(\mathcal{D}^h, \mathcal{P})$ is the broadcast rate given in Theorem 1.

*Proof:* We provide the proof for the case with $h = 10$. The proof for other cases follows on similar lines by employing Lemma 9 instead of Lemma 8. For the case with $h = 10$, using Lemma 8 we have

$$\chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_1) * \Gamma_t(\mathcal{D}^{10,\mathcal{P}}_3)) \leq \chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_1))\chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_3)). \quad (3)$$

Taking logarithm on both the sides of (3) and using Lemma 7 and Theorem 1, we have

$$t \times p_t(\mathcal{D}^h, \mathcal{P}) \leq \epsilon' + \lceil \log_2 \chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_1)) \rceil + \lceil \log_2 \chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_2)) \rceil + \lceil \log_2 \chi(\Gamma_t(\mathcal{D}^{10,\mathcal{P}}_3)) \rceil, \quad (4)$$

for some $\epsilon' \in \{-1, 0\}$. Dividing both the sides of (4) by $t$, we have the result of (2). ■

Corollary 1 shows that there is a possibility to achieve a broadcast rate lesser than that stated in Conjecture 1 in [12]. Comparing the optimal broadcast rate stated in Conjecture 1 and the achievable broadcast rate in Corollary 1, the conjecture is disproved if it is possible to find digraphs $\mathcal{D}^{h,\mathcal{P}}_i$, $i \in \{1, 2, 3\}$, $h \in \{10, 12, 14\}$ such that $p_t(\mathcal{D}^h, \mathcal{P})$ given in Theorem 1 is strictly less than $\beta_t(\mathcal{D}^{h,\mathcal{P}}_1) + \beta_t(\mathcal{D}^{h,\mathcal{P}}_2) + \beta_t(\mathcal{D}^{h,\mathcal{P}}_3) - 2/t$.

*Note 1:* Note that the optimal broadcast rates given in Theorem 1 also consider non-linear encoding schemes at the senders. From the optimal linear broadcast rates with finite length messages (which consider only linear encoding schemes at the senders) given in [14], we also note that if $p_t(\mathcal{D}^h, \mathcal{P})$ given in Theorem 1 is strictly less than $\beta_t(\mathcal{D}_1^{h,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{h,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{h,\mathcal{P}})$, $h \in \{10, 12, 14\}$, then the two-sender graph coloring results in a non-linear index code.

*Remark 1:* Let $\mathcal{D}_{ij}^h$, $i \neq j$, $i, j \in \{1, 2, 3\}$, be the side information sub-digraphs of $\mathcal{D}^h$, $h \in \ [36]$, induced by the messages in $\mathcal{P}_i \cup \mathcal{P}_j$. Then the following upper bounds on the optimal broadcast rate of any TUICP with finite length messages can be obtained using two-sender graph coloring by exploiting the symmetries of the confusion graph, as in the proof of Theorem 1.

$$\beta_t(\mathcal{D}^h, \mathcal{P}) \leq \beta_t(\mathcal{D}_1^{h,\mathcal{P}}) + \beta_t(\mathcal{D}_{23}^h). \tag{5}$$

$$\beta_t(\mathcal{D}^h, \mathcal{P}) \leq \beta_t(\mathcal{D}_2^{h,\mathcal{P}}) + \beta_t(\mathcal{D}_{13}^h). \tag{6}$$

$$\beta_t(\mathcal{D}^h, \mathcal{P}) \leq \beta_t(\mathcal{D}_3^{h,\mathcal{P}}) + \beta_t(\mathcal{D}_{12}^h, \mathcal{P}_1 \cup \mathcal{P}_2). \tag{7}$$

To obtain (5), we view $\Gamma_t(\mathcal{D}^h)$ as the union of all $I$-blocks connected by inter-$I$-block edges and color it according to two-sender graph coloring as done in the proof of Theorem 1. All other upper bounds are obtained similarly.

## V. Optimal Broadcast Rates for Cases II-C and II-D with Fully-Participated Interactions

In this section, we provide the optimal broadcast rate for any TUICP belonging to Cases II-C and II-D, with fully-participated interactions, for which only upper bounds were given in [11]. Optimal broadcast rate for any TUICP with $\mathcal{D}^k$ such that $k \in \{21, 22, \cdots, 36\}$, given in [11], depend on the relation between the optimal broadcast rates of the individual single-sender sub-problems described by the three sub-digraphs of the side information digraph. The results given in this section along with those given in [11] provide a complete characterization of the optimal broadcast rate of any TUICP with fully-participated interactions. We also provide an achievable broadcast rate with finite length messages, for all the sub-cases of Case II-D with fully-participated interactions, using a code construction based on optimal codes of the single-sender sub-problems. This provides a tighter upper bound on $\beta_t(\mathcal{D}^k, \mathcal{P})$, $k \in \{33, 34, 35, 36\}$, when compared to that given in [11]. The code constructions used to obtain the optimal broadcast rates in this section are same as those used in [14] with the linear codes of the sub-problems replaced by non-linear codes.

### A. Optimal Broadcast Rate for CASE II-C

We first establish the optimal broadcast rate for any TUICP belonging to Case II-C. We require the following lemma which is a part of Theorem 3 in [17] to derive our results.

*Lemma 10* (Theorem 3, [17]): Consider any single-sender unicast ICP described by a side-information digraph. Removing edges not lying on any directed cycle does not change the optimal broadcast rate.

*Theorem 2* (CASE II-C): For any TUICP with the side information digraph $\mathcal{D}^k$, $k \in \{21, 22, \cdots, 32\}$, having fully-participated interactions between its sub-digraphs $\mathcal{D}_i^{k,\mathcal{P}}$, $i \in \{1, 2, 3\}$, and for any $\mathcal{P}$, we have

$$\beta(\mathcal{D}^k, \mathcal{P}) = \max\{\beta(\mathcal{D}_1^{k,\mathcal{P}}), \beta(\mathcal{D}_3^{k,\mathcal{P}})\} + \beta(\mathcal{D}_2^{k,\mathcal{P}}). \tag{8}$$

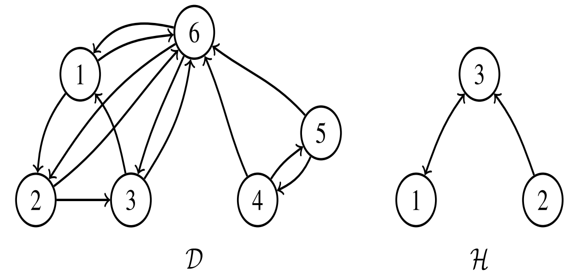


Fig. 5. Example of a two-sender problem belonging to Case II-C.

*Proof:* The result is proved in [11], for the case when $\beta(\mathcal{D}_1^{k,\mathcal{P}}) \geq \beta(\mathcal{D}_3^{k,\mathcal{P}})$. Hence, we prove the result for the case with $\beta(\mathcal{D}_1^{k,\mathcal{P}}) < \beta(\mathcal{D}_3^{k,\mathcal{P}})$ by first providing a lower bound and then providing a matching upper bound. Removing the vertices belonging to $\mathcal{D}_1^{k,\mathcal{P}}$ from $\mathcal{D}^k$, we obtain a digraph $\mathcal{D}_{23}^{k,\mathcal{P}}$ which defines a TUICP. This can be considered as a single-sender unicast ICP as both $\mathcal{P}_2$ and $\mathcal{P}_3$ are with $\mathcal{S}_2$. Hence, we have

$$\beta(\mathcal{D}^k, \mathcal{P}) \geq \beta(\mathcal{D}_{23}^{k,\mathcal{P}}). \tag{9}$$

As there are only unidirectional edges from $\mathcal{V}(\mathcal{D}_2^{k,\mathcal{P}})$ to $\mathcal{V}(\mathcal{D}_3^{k,\mathcal{P}})$ or vice-versa (depending on the particular value of $k$), using Lemma 10, we have

$$\beta(\mathcal{D}_{23}^{k,\mathcal{P}}) = \beta(\mathcal{D}_2^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}}). \tag{10}$$

From (9) and (10), we have $\beta(\mathcal{D}^k, \mathcal{P}) \geq \beta(\mathcal{D}_2^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}})$. Using Theorem 7 in [11], $\beta_t(\mathcal{D}^k, \mathcal{P}) \leq \beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}})$. Dividing both the sides by $t$, and taking the limit as $t \to \infty$ in the previous inequality, we have $\beta(\mathcal{D}^k, \mathcal{P}) \leq \beta(\mathcal{D}_2^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}})$, which is a matching upper bound. ■

*Note 2:* From the proof of Theorem 2, we notice that the optimal broadcast rates for problems belonging to Case II-C given by Theorem 2 remain the same as long as the interactions between $\mathcal{D}_1$ and $\mathcal{D}_3$ are fully-participated. Other interactions need not be fully-participated.

We illustrate Theorem 2 using an example.

*Example 3:* Consider a TUICP with $m = 6$. Let $\mathcal{M}_1 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_6\}$ and $\mathcal{M}_2 = \{\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6\}$. The side information of all the receivers are given as follows: $\mathcal{K}_1 = \{\mathbf{x}_2, \mathbf{x}_6\}, \mathcal{K}_2 = \{\mathbf{x}_3, \mathbf{x}_6\}, \mathcal{K}_3 = \{\mathbf{x}_1, \mathbf{x}_6\}, \mathcal{K}_4 = \{\mathbf{x}_5, \mathbf{x}_6\}, \mathcal{K}_5 = \{\mathbf{x}_4, \mathbf{x}_6\}, \mathcal{K}_6 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. The side information digraph and the interaction digraph are shown in Fig. 5. It is easy to verify that the associated interaction digraph is $\mathcal{H}_{24}$. Note that all the interactions are fully-participated interactions. We also know that $\mathcal{D}_1^{24,\mathcal{P}}$ and $\mathcal{D}_2^{24,\mathcal{P}}$ are cycles on vertex sets $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and $\{\mathbf{x}_4, \mathbf{x}_5\}$ respectively. Hence for any $t \geq 1$, we have $\beta_t(\mathcal{D}_1^{24,\mathcal{P}}) = 2$, $\beta_t(\mathcal{D}_2^{24,\mathcal{P}}) = 1$, and $\beta_t(\mathcal{D}_3^{24,\mathcal{P}}) = 1$. Note that the optimal broadcast rates with $t = 1$ are also equal to the optimal broadcast rates of the respective problems. According to Theorem 2, we have $\beta(\mathcal{D}^{24}, \mathcal{P}) = 1 + \max\{2, 1\} = 3$. We provide the code for $t = 1$. Sender $\mathcal{S}_1$ transmits $\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{x}_6$ and $\mathbf{x}_2 \oplus \mathbf{x}_3$. Sender $\mathcal{S}_2$ transmits $\mathbf{x}_4 \oplus \mathbf{x}_5$. Receiver 1 decodes $\mathbf{x}_1$ using $\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{x}_6$ and its side information $\mathbf{x}_2$ and $\mathbf{x}_6$. Receiver 4 decodes $\mathbf{x}_4$ using $\mathbf{x}_4 \oplus \mathbf{x}_5$ and its side information $\mathbf{x}_5$. Similarly, other receivers decode their demands.

### *B. An Achievable Broadcast Rate with Finite Length Messages for CASE II-D with Fully-Participated Interactions*

The following achievable broadcast rate with $t$-bit messages, for any TUICP belonging to Case II-D with fully-participated interactions was stated in Theorem 8 in [11] as an upper bound on $\beta_t(\mathcal{D}^k, \mathcal{P})$ with $k \in \{33, 34, 35, 36\}$.

$$\beta_t(\mathcal{D}^k, \mathcal{P}) \leq \max\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}), \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\} + \max\{\beta_t(\mathcal{D}_2^{k,\mathcal{P}}), \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}. \quad (11)$$

The following theorem provides an achievable broadcast rate for any TUICP belonging to Case II-D, by providing a code construction which uses optimal codes of the sub-problems described by the three sub-digraphs of the side information digraph. This provides a tighter upper bound compared to the one given in [11] and stated in (11).

*Theorem 3:* For any TUICP with the side information digraph $\mathcal{D}^k$, $k \in \{33, 34, 35, 36\}$, having fully-participated interactions between its sub-digraphs $\mathcal{D}_i^{k,\mathcal{P}}$, $i \in \{1, 2, 3\}$, for any $\mathcal{P}$, and $t$-bit messages for any finite $t$, the following broadcast rate is achievable.

$$p_t(\mathcal{D}^k, \mathcal{P}) = \max\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{k,\mathcal{P}}), \beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}}), \beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}. \quad (12)$$

*Proof:* We provide a code construction for $t$-bit messages for any finite $t$ and show that the constructed code satisfies the demands of all the receivers. For the case with $\beta_t(\mathcal{D}_3^{k,\mathcal{P}}) \leq \min\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}), \beta_t(\mathcal{D}_2^{k,\mathcal{P}})\}$, the broadcast rate $p_t(\mathcal{D}^k, \mathcal{P}) = \beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{k,\mathcal{P}})$, has been shown to be achievable in Theorem 8 of [11]. Without loss of generality, we assume that $\beta_t(\mathcal{D}_2^{k,\mathcal{P}}) \leq \min\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}), \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}$. The case with $\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) \leq \min\{\beta_t(\mathcal{D}_2^{k,\mathcal{P}}), \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}$ can be proved similarly. Let $\mathcal{C}_i$ be a code with the optimal broadcast rate with $t$-bit messages for any finite $t$ given by $\beta_t(\mathcal{D}_i^{k,\mathcal{P}})$ for the SUICP described by $\mathcal{D}_i^{k,\mathcal{P}}$, $i \in \{1, 2, 3\}$. Our code for the original TUICP $\mathcal{I}(\mathcal{D}^k, \mathcal{P})$ is given as follows:

$\mathcal{C}_3[1 + t\beta_t(\mathcal{D}_2^{k,\mathcal{P}}) : t\beta_t(\mathcal{D}_3^{k,\mathcal{P}})]$ sent by any one of $\mathcal{S}_1$ or $\mathcal{S}_2$, $\mathcal{C}_1 \oplus \mathcal{C}_3[1 : t\beta_t(\mathcal{D}_2^{k,\mathcal{P}})]$ by $\mathcal{S}_1, \mathcal{C}_2 \oplus \mathcal{C}_3[1 : t\beta_t(\mathcal{D}_2^{k,\mathcal{P}})]$ by $\mathcal{S}_2$.

The overall length of the two-sender code is given by $t(\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + (\beta_t(\mathcal{D}_3^{k,\mathcal{P}}) - \beta_t(\mathcal{D}_2^{k,\mathcal{P}}))) = t(\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}}))$, with the broadcast rate $\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}})$.

We provide the decoding procedure for receivers in the side information digraphs $\mathcal{D}^k$ with $k \in \{33, 34, 35\}$. The decoding procedure for those in the side information digraph $\mathcal{D}^k$ with $k = 36$ is similar. Receivers belonging to $\mathcal{D}_1^{k,\mathcal{P}}$ and $\mathcal{D}_2^{k,\mathcal{P}}$ recover their demanded messages using $(\mathcal{C}_2 \oplus \mathcal{C}_3[1 : t\beta_t(\mathcal{D}_2^{k,\mathcal{P}})]) \oplus (\mathcal{C}_1 \oplus \mathcal{C}_3[1 : t\beta_t(\mathcal{D}_2^{k,\mathcal{P}})]) = \mathcal{C}_1 \oplus \mathcal{C}_2$, and their side information. Receivers belonging to $\mathcal{D}_3^{k,\mathcal{P}}$ recover their demanded messages using $\mathcal{C}_3[t\beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + 1 : t\beta_t(\mathcal{D}_3^{k,\mathcal{P}})]$ and either $\mathcal{C}_2 \oplus \mathcal{C}_3[1 : t\beta_t(\mathcal{D}_2^{k,\mathcal{P}})]$ or $\mathcal{C}_1 \oplus \mathcal{C}_3[1 : t\beta_t(\mathcal{D}_2^{k,\mathcal{P}})]$, and their side information, depending on the presence of the interaction $\mathcal{D}_3^{k,\mathcal{P}} \rightarrow \mathcal{D}_2^{k,\mathcal{P}}$ or $\mathcal{D}_3^{k,\mathcal{P}} \rightarrow \mathcal{D}_1^{k,\mathcal{P}}$ respectively. ■

We illustrate the theorem using an example.

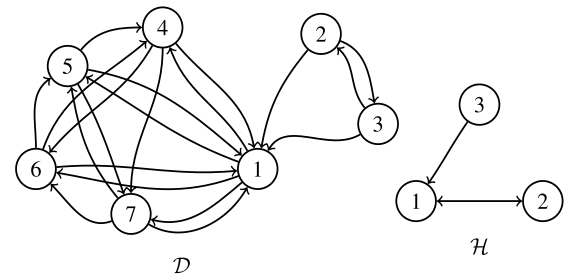

Fig. 6. Example of a two-sender problem belonging to Case II-D.

*Example 4:* Consider a TUICP with $m = 7$. Let $\mathcal{M}_1 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and $\mathcal{M}_2 = \{\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7\}$. The side information of all the receivers are given as follows: $\mathcal{K}_1 = \{\mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7\}$, $\mathcal{K}_2 = \{\mathbf{x}_1, \mathbf{x}_3\}$, $\mathcal{K}_3 = \{\mathbf{x}_1, \mathbf{x}_2\}$, $\mathcal{K}_4 = \{\mathbf{x}_1, \mathbf{x}_6, \mathbf{x}_7\}$, $\mathcal{K}_5 = \{\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_7\}$, $\mathcal{K}_6 = \{\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_5\}$, $\mathcal{K}_7 = \{\mathbf{x}_1, \mathbf{x}_5, \mathbf{x}_6\}$. The side information digraph and the interaction digraph are given in Fig. 6. It is easy to verify that the interaction digraph is $\mathcal{H}_{33}$. We observe that $\mathcal{D}_1$ is a side information digraph with one vertex, $\mathcal{D}_2$ is a problem solved in [16], and $\mathcal{D}_3$ is a clique with two vertices. The optimal broadcast rate of a clique and a side information digraph with a single vertex is 1. For any $t \geq 1$ we have, $\beta_t(\mathcal{D}_1^{33,\mathcal{P}}) = 1$, $\beta_t(\mathcal{D}_2^{33,\mathcal{P}}) = 2$, and $\beta_t(\mathcal{D}_3^{33,\mathcal{P}}) = 1$. Hence, according to Theorem 3, we have $p_t(\mathcal{D}^{33}, \mathcal{P}) = \max\{2+1, 2+1, 1+1\} = 3$ as an achievable broadcast rate. We provide the code for $t = 1$. Sender $\mathcal{S}_1$ transmits $\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{x}_3$ and $\mathcal{S}_2$ transmits $\mathbf{x}_2 \oplus \mathbf{x}_3 \oplus \mathbf{x}_4 \oplus \mathbf{x}_6$ and $\mathbf{x}_5 \oplus \mathbf{x}_7$. Receiver 1 decodes $\mathbf{x}_1$ using $\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{x}_3$, $\mathbf{x}_2 \oplus \mathbf{x}_3 \oplus \mathbf{x}_4 \oplus \mathbf{x}_6$, and its side information $\mathcal{K}_1 = \{\mathbf{x}_2, \mathbf{x}_3\}$. Reciever 4 decodes $\mathbf{x}_4$ using $\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus \mathbf{x}_3$, $\mathbf{x}_2 \oplus \mathbf{x}_3 \oplus \mathbf{x}_4 \oplus \mathbf{x}_6$, and its side information $\mathcal{K}_4 = \{\mathbf{x}_1, \mathbf{x}_6\}$. Similarly, other receivers can decode their demanded messages.

*Remark 2:* Note that the upper bound on $\beta_t(\mathcal{D}^k, \mathcal{P})$, $k \in \{33, 34, 35, 36\}$, stated in (11) can also be written as follows.

$$\beta_t(\mathcal{D}^k, \mathcal{P}) \leq \max\{\ \beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{k,\mathcal{P}}), \beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}}), \beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}}), 2\beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}.$$

Comparing this upper bound with the achievable broadcast rate given in Theorem 3, we see that the achievable broadcast rate given in Theorem 3 is a tighter upper bound.

### *C. Optimal Broadcast Rate for CASE II-D*

We now present our results for Case II-D using the result of Case II-C. We require the following lemma to prove our results.

*Lemma 11:* For any $\mathcal{D}$, $\mathcal{P}$ and finite $t$, if a side-information digraph $\mathcal{D}'$ is obtained by adding more directed edges to $\mathcal{D}$, we have $\beta_t(\mathcal{D}, \mathcal{P}) \geq \beta_t(\mathcal{D}', \mathcal{P})$ and $\beta(\mathcal{D}, \mathcal{P}) \geq \beta(\mathcal{D}', \mathcal{P})$.

*Proof:* Consider an optimal code for the two-sender problem $\mathcal{I}(\mathcal{D}, \mathcal{P})$ with $t$-bit messages with broadcast rate $\beta_t(\mathcal{D}, \mathcal{P})$. Note that this code also solves the two-sender problem $\mathcal{I}(\mathcal{D}', \mathcal{P})$. Hence, $\beta_t(\mathcal{D}, \mathcal{P}) \geq \beta_t(\mathcal{D}', \mathcal{P})$. Taking the limit as $t \rightarrow \infty$, in the definition of the optimal broadcast rate, we have $\beta(\mathcal{D}, \mathcal{P}) \geq \beta(\mathcal{D}', \mathcal{P})$. ■

*Theorem 4* (CASE II-D): For any TUICP with the side information digraph $\mathcal{D}^k$, $k \in \{33, 34, 35, 36\}$, having fully-participated interactions between its sub-digraphs $\mathcal{D}_i^{k,\mathcal{P}}$, $i \in \{1, 2, 3\}$, and for any $\mathcal{P}$, we have

$$\beta(\mathcal{D}^k, \mathcal{P}) = \max\{\beta(\mathcal{D}_1^{k,\mathcal{P}}) + \beta(\mathcal{D}_2^{k,\mathcal{P}}), \beta(\mathcal{D}_1^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}}), \beta(\mathcal{D}_2^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}})\}. \quad (13)$$

*Proof:* We first provide a lower bound using the result of Case II-C. Then, we provide a matching upper bound using the result of Theorem 3. Given any side information digraph $\mathcal{D}^k$ with $k \in \{33, 34, 35, 36\}$, with fully-participated interactions among its sub-digraphs $\mathcal{D}_1^{k,\mathcal{P}}$, $\mathcal{D}_2^{k,\mathcal{P}}$, and $\mathcal{D}_3^{k,\mathcal{P}}$, we can get $(i)$ one of the side information digraphs $\mathcal{D}^{k'}, k' \in \{31, 32\}$ and $(ii)$ one of the side information digraphs $\mathcal{D}^{k''}$ with the interaction digraph $\mathcal{H}$ obtained by swapping the labels of vertices 1 and 2 in $\mathcal{H}_{k'}$, $k' \in \{31, 32\}$, with the same sub-digraphs $\mathcal{D}_1^{k,\mathcal{P}}$, $\mathcal{D}_2^{k,\mathcal{P}}$, and $\mathcal{D}_3^{k,\mathcal{P}}$ having fully-participated interactions, by adding appropriate edges between the sub-digraphs of $\mathcal{D}^k$. From Lemma 11, we have,

$$\beta(\mathcal{D}^k, \mathcal{P}) \geq \beta(\mathcal{D}^{k'}, \mathcal{P}), \quad (14)$$

$$\beta(\mathcal{D}^k, \mathcal{P}) \geq \beta(\mathcal{D}^{k''}, \mathcal{P}). \quad (15)$$

Combining the result of Theorem 2 using (14) and (15), we get,

$$\beta(\mathcal{D}^k, \mathcal{P}) \geq \max\{\beta(\mathcal{D}_1^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}}), \beta(\mathcal{D}_2^{k,\mathcal{P}}) + \beta(\mathcal{D}_3^{k,\mathcal{P}}), \beta(\mathcal{D}_1^{k,\mathcal{P}}) + \beta(\mathcal{D}_2^{k,\mathcal{P}})\}. \quad (16)$$

Using the result of Theorem 3, we have

$$\beta_t(\mathcal{D}^k, \mathcal{P}) \leq \max\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_2^{k,\mathcal{P}}), \beta_t(\mathcal{D}_1^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}}), \beta_t(\mathcal{D}_2^{k,\mathcal{P}}) + \beta_t(\mathcal{D}_3^{k,\mathcal{P}})\}. \quad (17)$$

Taking the limit as $t \to \infty$ in the definition of $\beta(\mathcal{D}^k, \mathcal{P})$, we obtain the matching upper bound. ∎

*Note 3:* From the proof of Theorem 4, we notice that the optimal broadcast rates for problems belonging to Case II-D with the interaction digraph $\mathcal{H}_{34}$ remain the same as long as one of the two interactions given by $\mathcal{D}_3 \to \mathcal{D}_1$ and $\mathcal{D}_3 \to \mathcal{D}_2$ is fully-participated.

We illustrate the theorem using an example.

*Example 5:* Consider a TUICP with $m = 8$. Let $\mathcal{M}_1 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_8\}$ and $\mathcal{M}_2 = \{\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_8, \mathbf{x}_7\}$. The side information of all the receivers are given as follows: $\mathcal{K}_1 = \{\mathbf{x}_3, \mathbf{x}_7\}$, $\mathcal{K}_2 = \{\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_7\}$, $\mathcal{K}_3 = \{\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_7\}$, $\mathcal{K}_4 = \{\mathbf{x}_2, \mathbf{x}_7\}$, $\mathcal{K}_5 = \{\mathbf{x}_4, \mathbf{x}_6, \mathbf{x}_7\}$, $\mathcal{K}_6 = \{\mathbf{x}_8, \mathbf{x}_7\}$, $\mathcal{K}_7 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, $\mathcal{K}_8 = \{\mathbf{x}_5, \mathbf{x}_7\}$. The side information digraph and the interaction digraph are given in Fig. 7. Note that the interaction digraph of the side information digraph $\mathcal{D}$ is $\mathcal{H}_{34}$. We observe that $\mathcal{D}_1^{34,\mathcal{P}}$ is a problem that is solved using the results in [16] with optimal broadcast rate 2, and that $\mathcal{D}_2^{34,\mathcal{P}}$ is a vertex and $\mathcal{D}_3^{34,\mathcal{P}}$ is a cycle. For any $t \geq 1$ we have, $\beta_t(\mathcal{D}_1^{34,\mathcal{P}}) = 2$, $\beta_t(\mathcal{D}_2^{34,\mathcal{P}}) = 1$, and $\beta_t(\mathcal{D}_3^{34,\mathcal{P}}) = 2$. The optimal broadcast rates with $t = 1$ are also equal to optimal broadcast rates of the respective problems. Hence, according to Theorem 4, we have $\beta(\mathcal{D}^{34}, \mathcal{P}) = \max\{2+2, 2+1, 2+1\} = 4$. We provide the code for $t = 1$. Sender $\mathcal{S}_1$ transmits $\mathbf{x}_1 \oplus \mathbf{x}_3 \oplus \mathbf{x}_5 \oplus \mathbf{x}_6$, $\mathbf{x}_2 \oplus \mathbf{x}_4$, and $\mathbf{x}_6 \oplus \mathbf{x}_8$, and $\mathcal{S}_2$ transmits $\mathbf{x}_5 \oplus \mathbf{x}_6 \oplus \mathbf{x}_7$.

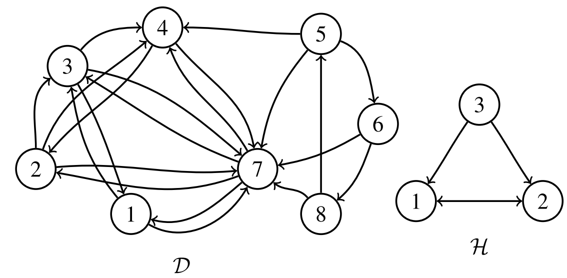


Fig. 7. Example of a two-sender problem belonging to Case II-D.

Note that the results related to optimal broadcast rates given in this section are summarized in Table II in Section I, and given in terms of those of the sub-problems. The optimal broadcast rates of SUICPs are known only for some special cases [5], [16]. Hence, the complexity of solving the TUICP with fully-participated interactions is reduced to that of solving the SUICP.

*Remark 3:* For Case II-D, [11] provided upper bound for the optimal broadcast rate with $t$-bit messages when $\beta_t(\mathcal{D}_3^{k,\mathcal{P}}) > \min\{\beta_t(\mathcal{D}_1^{k,\mathcal{P}}), \beta_t(\mathcal{D}_2^{k,\mathcal{P}})\}$ and optimal broadcast rate when $\beta(\mathcal{D}_3^{k,\mathcal{P}}) > \min\{\beta(\mathcal{D}_1^{k,\mathcal{P}}), \beta(\mathcal{D}_2^{k,\mathcal{P}})\}$. However, we have shown that the given upper bounds in [11] are loose, and Theorem 4 provides the optimal broadcast rates for Case II-D.

## VI. Conclusion and Future Work

This paper establishes the optimal broadcast rates for all the cases of the TUICP with fully-participated interactions, for which only upper bounds were known, in terms of the corresponding results of the three single-sender sub-problems. Achievable broadcast rate with finite length messages is given for some cases of the TUICP with fully-participated interactions, using two-sender graph coloring of the confusion graph. No results on non-trivial achievable broadcast rates were known for these cases. Finding non-trivial achievable broadcast rate with $t$-bit messages for any finite $t$, for the sub-cases of Case I without known optimal broadcast rates with finite length messages is an interesting problem. Finding the optimal broadcast rate of the TUICP with partially-participated interactions is also an open problem. Further, extension of these results to index coding problems with any number of senders is open.

## Appendix A
## Proof of Theorem 1

We first prove statement $(i)$ of Theorem 1. The proofs of statements $(ii)$ and $(iii)$ follow on similar lines and only the required changes are mentioned. We first identify the edges in the confusion graph $\Gamma_t(\mathcal{D}^h)$ due to confusions at the vertices (receivers) belonging to each of the sub-digraphs $\mathcal{D}_1^{h,\mathcal{P}}$, $\mathcal{D}_2^{h,\mathcal{P}}$, and $\mathcal{D}_3^{h,\mathcal{P}}$, $h \in \{10, 12, 14\}$. To avail the symmetries of the confusion graph in coloring it according to two-sender graph coloring, we view $\Gamma_t(\mathcal{D}^h)$ as the union of all the $J$-blocks connected by inter-$J$-block edges. The number of ordered pairs of colors required to color the confusion graph is used to calculate an achievable broadcast rate with $t$-bit messages. Throughout the proof we assume that the superscripts $i, i' \in [2^{tm_1}]$, $j, j' \in [2^{tm_2}]$, $k, k' \in [2^{tm_3}]$.

*Proof of* $(i)$ *in Theorem 1.* We first list all the edges of $\Gamma_t(\mathcal{D}^{10})$.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_1^{10,\mathcal{P}})$</u>: If $\mathbf{b}_{\mathcal{P}_1}^{i}$ and $\mathbf{b}_{\mathcal{P}_1}^{i'}$ are confusable at some vertex in $\mathcal{V}(\mathcal{D}_1^{10,\mathcal{P}})$, then the corresponding edges in $\Gamma_t(\mathcal{D}^{10})$ due to the confusion at the same vertex in $\mathcal{V}(\mathcal{D}^{10})$ are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$, as the vertex has all the messages represented by $\mathcal{V}(\mathcal{D}_2^{10,\mathcal{P}})$ as side information, and has no side information in $\mathcal{V}(\mathcal{D}_3^{10,\mathcal{P}})$. Hence, confusion at any vertex in $\mathcal{V}(\mathcal{D}_1^{10,\mathcal{P}})$ does not contribute to inter-$J$-block edges.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_2^{10,\mathcal{P}})$</u>: If $\mathbf{b}_{\mathcal{P}_2}^{j}$ and $\mathbf{b}_{\mathcal{P}_2}^{j'}$ are confusable at some vertex in $\mathcal{V}(\mathcal{D}_2^{10,\mathcal{P}})$, then the corresponding edges in $\Gamma_t(\mathcal{D}^{10})$ are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j'}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$. Hence, confusion at any vertex in $\mathcal{V}(\mathcal{D}_2^{10,\mathcal{P}})$ results in inter-$J$-block edges.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_3^{10,\mathcal{P}})$</u>: If $\mathbf{b}_{\mathcal{P}_3}^{k}$ and $\mathbf{b}_{\mathcal{P}_3}^{k'}$ are confusable at some vertex in $\mathcal{V}(\mathcal{D}_3^{10,\mathcal{P}})$, then the corresponding edges in $\Gamma_t(\mathcal{D}^{10})$ are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$. Hence, confusion at any vertex in $\mathcal{V}(\mathcal{D}_3^{10,\mathcal{P}})$ does not result in inter-$J$-block edges.

<u>Coloring the confusion graph $\Gamma_t(\mathcal{D}^{10})$</u>: From Lemma 6, we know that all the $J$-blocks are isomorphic to each other. From the listing of all the edges of the confusion graph, we know that the inter-$J$-block edges are only due to the confusions at the receivers belonging to $\mathcal{V}(\mathcal{D}_2^{10,\mathcal{P}})$. Hence, in order to color the confusion graph according to the two-sender graph coloring, we find an optimal classical graph coloring of any $J$-block and associate the resulting colors with sender $\mathcal{S}_1$. This can be done, as the edges within any $J$-block are only due to the confusions at the vertices belonging to $\mathcal{V}(\mathcal{D}_1^{10,\mathcal{P}} \cup \mathcal{D}_3^{10,\mathcal{P}})$, and $\mathcal{S}_1$ alone has all the messages in $\mathcal{P}_1 \cup \mathcal{P}_3$. The same set of colors can be used by $\mathcal{S}_1$ to color every $J$-block identically. This resolves all the confusions at all the receivers in $\mathcal{V}(\mathcal{D}_1^{10,\mathcal{P}} \cup \mathcal{D}_3^{10,\mathcal{P}})$.

Note that there is an edge given by $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$, belonging to any $j$th $J$-block iff either the edge $(\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_1}^{i'}) \in \mathcal{E}(\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}))$ or the edge $(\mathbf{b}_{\mathcal{P}_3}^{k}, \mathbf{b}_{\mathcal{P}_3}^{k'}) \in \mathcal{E}(\Gamma_t(\mathcal{D}_3^{10,\mathcal{P}}))$. From the definition of the disjunctive graph product, we observe that each $J$-block is isomorphic to $\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}) * \Gamma_t(\mathcal{D}_3^{10,\mathcal{P}})$. Hence, $\mathcal{S}_1$ requires a minimum of $\chi(\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}) * \Gamma_t(\mathcal{D}_3^{10,\mathcal{P}}))$ colors to color any $J$-block. The confusions associated with inter-$J$-block edges can be resolved by $\mathcal{S}_2$ alone, as all such confusions are associated with vertices in $\mathcal{V}(\mathcal{D}_2^{10,\mathcal{P}})$ and only $\mathcal{S}_2$ has all the messages in $\mathcal{P}_2$. Observe that there are inter-$J$-block edges between any $j$th and any $j'$th $J$-blocks iff $(\mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_2}^{j'})$ is an edge of $\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}})$. We know that a minimum of $\chi(\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}}))$ colors are required to color $\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}})$. By assigning the color given to $\mathbf{b}_{\mathcal{P}_2}^{j}$ in $\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}})$ to the $j$th $J$-block (to all the vertices in the $j$th $J$-block) for all $j \in [2^{tm_2}]$, we observe that all the confusions associated with all the inter-$J$-block edges are resolved. Hence, a minimum of $\chi(\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}}))$ colors are sufficient for $\mathcal{S}_2$ to color the confusion graph.

This is a valid two-sender graph coloring of $\Gamma_t(\mathcal{D}^{10})$ requiring a total of $\big(\chi(\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}) * \Gamma_t(\mathcal{D}_3^{10,\mathcal{P}}))\big) \times \chi(\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}}))$ ordered pairs of colors. The total length of the two-sender index code is thus given by the sum of the lengths of codewords transmitted by the senders as $t \times p_t(\mathcal{D}^{10}, \mathcal{P}) = \lceil \log_2(\chi(\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}) * \Gamma_t(\mathcal{D}_3^{10,\mathcal{P}})))\rceil + \lceil \log_2(\chi(\Gamma_t(\mathcal{D}_2^{10,\mathcal{P}})))\rceil$. Hence, the associated broadcast rate is given by $p_t(\mathcal{D}^{10}, \mathcal{P}) = \lceil \log_2(\chi(\Gamma_t(\mathcal{D}_1^{10,\mathcal{P}}) * \Gamma_t(\mathcal{D}_3^{10,\mathcal{P}})))\rceil / t + \beta_t(\mathcal{D}_2^{10,\mathcal{P}})$. ■

*Proof of* $(ii)$ *in Theorem 1.* We first list all the edges of $\Gamma_t(\mathcal{D}^{12})$ as follows. Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_1^{12,\mathcal{P}})$: If $\mathbf{b}_{\mathcal{P}_1}^{i}$ and $\mathbf{b}_{\mathcal{P}_1}^{i'}$ are confusable at some vertex in $\mathcal{V}(\mathcal{D}_1^{12,\mathcal{P}})$, then the edges are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_2^{12,\mathcal{P}})$</u>: Same as in proof of $(i)$ in Theorem 1.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_3^{12,\mathcal{P}})$</u>: The edges are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$, where $\mathbf{b}_{\mathcal{P}_3}^{k}$ and $\mathbf{b}_{\mathcal{P}_3}^{k'}$ are confusable at some receiver in $\mathcal{V}(\mathcal{D}_3^{12,\mathcal{P}})$.

<u>Coloring the confusion graph $\Gamma_t(\mathcal{D}^{12})$</u>: Observe that there is an edge given by $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$, belonging to any $j$th $J$-block iff either the edge $(\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_1}^{i'}) \in \mathcal{E}(\Gamma_t(\mathcal{D}_1^{12,\mathcal{P}}))$, or $(\mathbf{b}_{\mathcal{P}_3}^{k}, \mathbf{b}_{\mathcal{P}_3}^{k'}) \in \mathcal{E}(\Gamma_t(\mathcal{D}_3^{12,\mathcal{P}}))$ and $\mathbf{b}_{\mathcal{P}_1}^{i} = \mathbf{b}_{\mathcal{P}_1}^{i'}$. From the definition of the lexicographic graph product, each $J$-block is isomorphic to $\Gamma_t(\mathcal{D}_1^{12,\mathcal{P}}) \circ \Gamma_t(\mathcal{D}_3^{12,\mathcal{P}})$. As seen in the proof of statement $(i)$ of this theorem, we have the associated broadcast rate given by $p_t(\mathcal{D}^{12}, \mathcal{P}) = \lceil \chi(\Gamma_t(\mathcal{D}_1^{12,\mathcal{P}}) \circ \Gamma_t(\mathcal{D}_3^{12,\mathcal{P}}))\rceil / t + \beta_t(\mathcal{D}_2^{12,\mathcal{P}})$. ■

*Proof of* $(iii)$ *in Theorem 1.* We list all the edges of $\Gamma_t(\mathcal{D}^{14})$ as follows.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_1^{14,\mathcal{P}})$</u>: If $\mathbf{b}_{\mathcal{P}_1}^{i}$ and $\mathbf{b}_{\mathcal{P}_1}^{i'}$ are confusable at some vertex in $\mathcal{V}(\mathcal{D}_1^{14,\mathcal{P}})$, then the edges are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}))$.

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_2^{14,\mathcal{P}})$: Same as in proof of $(i)$ in Theorem 1.</u>

<u>Edges due to confusions at the vertices in $\mathcal{V}(\mathcal{D}_3^{14,\mathcal{P}})$</u>: The edges are of the form $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$, where $\mathbf{b}_{\mathcal{P}_3}^{k}$ and $\mathbf{b}_{\mathcal{P}_3}^{k'}$ are confusable at some receiver in $\mathcal{V}(\mathcal{D}_3^{14,\mathcal{P}})$.

<u>Coloring the confusion graph $\Gamma_t(\mathcal{D}^{14})$</u>: Note that there is an edge given by $((\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k}), (\mathbf{b}_{\mathcal{P}_1}^{i'}, \mathbf{b}_{\mathcal{P}_2}^{j}, \mathbf{b}_{\mathcal{P}_3}^{k'}))$, belonging to any $j$th $J$-block iff either the edge $(\mathbf{b}_{\mathcal{P}_1}^{i}, \mathbf{b}_{\mathcal{P}_1}^{i'}) \in \mathcal{E}(\Gamma_t(\mathcal{D}_1^{14,\mathcal{P}}))$ and $\mathbf{b}_{\mathcal{P}_3}^{k} = \mathbf{b}_{\mathcal{P}_3}^{k'}$, or the edge $(\mathbf{b}_{\mathcal{P}_3}^{k}, \mathbf{b}_{\mathcal{P}_3}^{k'}) \in \mathcal{E}(\Gamma_t(\mathcal{D}_3^{14,\mathcal{P}}))$. Observe that each $J$-block is isomorphic to $\Gamma_t(\mathcal{D}_3^{14,\mathcal{P}}) \circ \Gamma_t(\mathcal{D}_1^{14,\mathcal{P}})$. As seen in the proof of statement $(i)$ of this theorem, we have the associated broadcast rate given by $p_t(\mathcal{D}^{14}, \mathcal{P}) = \lceil \chi(\Gamma_t(\mathcal{D}_3^{14,\mathcal{P}}) \circ \Gamma_t(\mathcal{D}_1^{14,\mathcal{P}}))\rceil / t + \beta_t(\mathcal{D}_2^{14,\mathcal{P}})$. ■

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