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The first negative eigenvalue of Yoshida lifts

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Abstract

We prove that for any given $\epsilon > 0$, the first negative eigenvalue of the Yoshida lift F of a pair of elliptic cusp forms f, g having square-free levels (where g has weight 2 and satisfies $(\log Q_g)^2 \ll \log Q_f$), occurs before $c_\epsilon \cdot Q_F^{1/2-2\theta+\epsilon}$; where Q_F, Q_f, Q_g are the analytic conductors of F, f, g respectively, $\theta < 1/4$, and c_ϵ is a constant depending only on ϵ.

Keywords: Sign changes, Eigenvalues, Yoshida lifts

Mathematics Subject Classification: 11F46, 11F66

1 Introduction

Eigenvalues of Hecke eigenforms are of considerable interest to number theorists, in particular their distribution (e.g., with respect to the Sato–Tate measure), magnitude (Ramanujan–Petersson conjecture), and more recently study of their signs have been the focus of intensive research. In this paper we are interested about the signs of eigenvalues of the so-called Yoshida lifts, whose definition would be recalled below. Let us first briefly discuss the setting of the problem and the results existing in the literature. The best known result for the first sign change of an elliptic newform was given by Matomäki [11] : if Q_f is the analytic conductor (see Sect. 2 for the definition) of an elliptic newform f of level N and weight $k \geq 2$ (so $Q_f \asymp k^2 N$, i.e., has the same order of magnitude as $k^2 N$), then the first negative eigenvalue a_n occurs for some $n \ll Q_f^{3/8}$. See also [3] for related results.

There are far fewer results available in the context of Siegel modular forms. For a Siegel Hecke eigenform F on $\mathrm{Sp}_2(\mathbf{Z})$, which is not a Maaß lift (so that $k \geq 20$), it is known [8] that its eigenvalues change signs infinitely many often. Related results are available in [4, 14]. Concerning the first negative eigenvalue problem, the best known result [9] due to Kohnen and Sengupta says that the first negative eigenvalue $\lambda_F(n)$ (with F as above) occurs for

$$n \ll Q_F \log^{20} Q_F, \tag{1.1}$$

the implied constant being absolute. Here Q_F denotes the analytic conductor of F, defined in Sect. 2. This result was generalised to the case of higher levels (which were held fixed throughout the paper, but both Maaß lifts and non-lifts were considered) by Brown [2], who got the same bound as above. Note that in both of these cases one has $Q_F \asymp k^2$. Improving these results seem to be a rather difficult problem. One of the main reasons behind this is that the Hecke relations between the eigenvalues of a Siegel-Hecke eigenform are more complicated than those for an elliptic newform. In this paper we restrict our attention to the case of the Yoshida lift of two elliptic cusp forms and show that one can improve the above results considerably (cf. (1.1), (1.2)).

The setting of this paper is as follows (see e.g. [15] for a more detailed discussion). Let $S_\kappa(\mathscr{L})$ denote the space of cusp forms of even weight $\kappa \geq 2$ and level $\mathscr{L}$. Let $f \in S_k(N_1)$ and $g \in S_2(N_2)$ be normalised newforms with $N_1, N_2 \geq 1$ squarefree and $M := \gcd(N_1, N_2) > 1$. Assume that the Atkin-Lehner eigenvalues of f and g coincide for all p dividing M. To this data, one can associate a Siegel modular form $F = F_{f,g} \in S_{k/2+1}(\Gamma_0^2(N))$, where $N = \mathrm{lcm}(N_1, N_2)$, which is called the Yoshida lift attached to f, g (see [15,17]). Let the Hecke eigenvalues of F be $\lambda_F(n)$. Further, let θ denote any saving over the exponent of convexity (so that $\theta < 1/4$, see [13]) bound for the normalised L-functions $L(f,s), L(g,s)$ on the critical line. We prove the following theorem.

Theorem 1.1 *Let $\epsilon > 0$ be given and the notation and setting be as in the above paragraph. Suppose $(\log Q_g)^2 \ll \log Q_f$. Then there exists $n \in \mathbf{N}$ with*

$$n \ll_\epsilon Q_F^{1/2-2\theta+\epsilon} \text{ such that } \lambda_F(n) < 0. \tag{1.2}$$

This result suggests that for a generic Siegel Hecke eigenform (not necessarily a lift) of degree 2, the bound $Q_F^{1/2+\epsilon}$ could be plausible (since the Yoshida lift satisfies the Ramanujan–Petersson conjecture, which seems crucial in these problems), this is got by taking $\theta = 0$ in the above theorem. Perhaps the stronger exponent $1/2 - \delta$ could be true.

Conjecture 1 For an arbitrary Siegel Hecke eigenform F of degree 2 and weight $k \geq 20$, for any given $\epsilon > 0$, the first negative eigenvalue $\lambda_F(n)$ occurs at $n \ll_\epsilon Q_F^{1/2+\epsilon}$, with the implied constant being absolute and depending only on ϵ.

Our proof of Theorem 1.1 uses the factorisation of the spinor L-function of F as a product of two GL_2 L-functions, subconvexity estimates of the GL_2 L-functions in question, and the Hecke relations for the eigenvalues of the elliptic newforms f, g. The use of the subconvexity bound is not crucial for us, Theorem 1.1 with $\theta = 0$ is already an improvement of Kohnen and Senupta's bound in [9]. Like all other results on this topic, we consider upper and lower bounds for a suitable weighted sum of the eigenvalues of F:

$$S(F,x) := \sum_{n \leq x,\, (n,N)=1} \lambda_F(n) \log\left(\frac{x}{n}\right) \tag{1.3}$$

in terms of Q_F and x. Using standard analytic techniques, we obtain an upper bound $Q_F^{1/4-\theta+\epsilon} \cdot x^{1/2}$ with an implied constant depending only on ϵ. The main point is to get a suitable lower bound by exploiting the non-negativity of $\lambda_F(n)$, say up to x, and exploiting the Hecke relations. The two bounds combined would give the desired upper bound $Q_F^{1/2-2\theta+\epsilon}$ in Theorem 1.1. Let us mention here that our method of exploiting the Hecke relations between eigenvalues is rather different from those existing in the literature for any other 'Linnik-type' problem on determining the first sign change in the sequence of Hecke eigenvalues of an eigenform. See the paragraph below, and for more details, Sect. 3.2.

For all y such that $\log y \gg (\log \mathscr{L})^2$ and any given elliptic newform $h \in S_2(\mathscr{L})$ (2 can be replaced by any fixed weight ≥ 2), we prove a non-trivial upper bound of $\sum_{p \leq y, p \nmid \mathscr{L}} |\lambda_h(p)|$

(one trivial bound is $2\pi(y)$ with $\pi(\cdot)$ being the prime counting function, and we show that one can reduce the constant here to $11/10$); up to the best of our knowledge, this result has not been written down explicitly in the literature. This follows from the holomorphy of the symmetric power L-functions (cf. [7]) and can be used to provide point-wise upper bounds for $\lambda_h(p)$ on sets of primes with positive natural density (in an effective and explicit manner, in particular the set of primes may depend mildly on $\mathscr{L}$, see Corollary 3.5). In particular we avoid using the Sato–Tate theorem not only because such an advanced machinery is not required (and so perhaps this method may generalise to other situations), but more so because we need explicit and controllable dependence on the parameter $\mathscr{L}$.

2 Notation and preliminaries

2.1 General notation

Let $\mathscr{A}$ be a subset of $\mathbf{N}$ and $a_n \in \mathbf{C}$, we define

$$\sum_{n\in\mathscr{A}}{}^{N} a_n := \sum_{n\in\mathscr{A},\,(n,N)=1} a_n.$$

Let $u(x), v(x)$ be two real functions defined on a subset $\mathscr{B}$ of $\mathbf{R}$. Whenever we write $u(x) \ll v(x)$ or $u(x) = O(v(x))$ or $u(x) \ll_\epsilon v(x)$, it will always mean that,

$$|u(x)| \le M \cdot h(x), \quad \text{for all } x \in \mathscr{B} \text{ and for some } M > 0,$$

where M is an absolute constant unless specified otherwise, and in the last case M may depend on ϵ. The notation $u(x) = o(v(x))$ means that

$$\lim_{x\to\infty} \frac{u(x)}{v(x)} = 0.$$

2.2 Spinor L-function

Whenever $H(s)$ is a Dirichlet series with an Euler product, having Euler factors $H_p(s)$ at primes p, we will write for $\Re s \gg 1$, $\mathscr{L} \ge 1$ and primes p

$$H_{\mathscr{L}}(s) := \prod_{p\nmid\mathscr{L}} H_p(s) \quad \text{and} \quad H^{\mathscr{L}}(s) := \prod_{p\mid\mathscr{L}} H_p(s);$$

so that $H(s) = H_{\mathscr{L}}(s) \cdot H^{\mathscr{L}}(s)$. Let the notation be as in the introduction. We can attach to the Yoshida lift $F \in S_\kappa(\Gamma_0^2(N))$ the spinor L-function $Z(F, s)$ (in the sense of Langlands) which is given by a certain Euler-product defined in terms of the Satake parameters of F. In this paper, we would always work with Euler-products away from N.

Useful information about the Euler factors of $Z(F, s)$ away from the level is given, for instance, in (see [1]). Let us consider the Euler factor $Z_{F,p}(s)$ of $Z(F, s)$ at a prime $p \nmid N$. For $\Re s > 1$ we have that

$$Z_N(F, s) := \prod_{p\nmid N} Z_{F,p}(s), \quad \text{where} \quad Z_{F,p}(s) := \prod_{1\le i\le 4} (1 - \beta_{i,p} p^{-s})^{-1}.$$

Here $\beta_{1,p} := \alpha_{0,p}$, $\beta_{2,p} := \alpha_{0,p}\alpha_{1,p}$, $\beta_{3,p} := \alpha_{0,p}\alpha_{2,p}$, $\beta_{4,p} := \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}$, and the complex numbers $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ are the Satake parameters of F at p (see [2]).

The main information that we require about a Yoshida-lift is the following. Firstly, for $(n, N) = 1$, $Z_N(F, s)$ is related to the eigenvalues $\lambda_F(n)$ of the Hecke operator $T(n)$ acting on $S_\kappa(\Gamma_0^2(N))$ by the following relation (see [2, Proposition 4.4])

$$\sum_{n\in\mathbf{N}}{}^{N} \frac{\lambda_F(n)}{n^s} = \frac{Z_N(F, s)}{\zeta_N(1 + 2s)}, \tag{2.1}$$

where $\zeta_N(s) := \prod_{p\nmid N}(1-p^{-s})^{-1}$ for $\Re s > 1$.

Secondly, letting $\lambda_f(n)$, $\lambda_g(n)$ denote the Hecke eigenvalues of f and g and $L(f,s)$, $L(g,s)$ denote the normalised L-functions of f and g, so that their functional equations relate s with $1-s$:

$$L(f,s) := \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}, \quad L(g,s) := \sum_{n=1}^{\infty} \lambda_g(n)n^{-s};$$

we have the relation (see [1, Corollary 6.1], also [15, Proposition 3.1])

$$Z_N(F,s) = L_N(f,s) \cdot L_N(g,s) = \frac{L(f,s)}{L^N(f,s)} \cdot \frac{L(g,s)}{L^N(g,s)}, \tag{2.2}$$

where $L^N(f,s)$ and $L^N(g,s)$ are given by

$$L^N(f,s) := \prod_{p|N}(1-\lambda_f(p)p^{-s})^{-1} \quad \text{and} \quad L^N(g,s) := \prod_{p|N}(1-\lambda_g(p)p^{-s})^{-1}.$$

2.3 Analytic conductor

Let $L(h,s) = \sum_{n\geq 1}\lambda_h(n)n^{-s}$ be a normalised 'L-function' of a modular form h in the sense of [5, Chap. 5]. Further assume that $L(h,s)$ has an Euler product of degree d with the p-Euler polynomial given by $\prod_{1\leq i\leq d}(1-\alpha_i(p)X)$ for some complex numbers $\alpha_i(p)$ for all p. The completed L-function

$$\Lambda(h,s) = q(h)^{\frac{s}{2}}\pi^{\frac{-ds}{2}}\prod_{j=1}^{d}\Gamma\left(\frac{s+k_j}{2}\right)L(h,s)$$

satisfies a functional equation relating s with $1-s$, has meromorphic continuation to $\mathbf{C}$, where $q(h) \geq 1$ is the arithmetic conductor and $k_j \in \mathbf{C}$. When $p \nmid q(h)$, one has $\alpha_i(p) < p$. Then we define the analytic conductor (see [5] for a more detailed discussion) Q_h of $L(h,s)$ (or h for brevity) as

$$Q_h := q(h)\prod_{j=1}^{d}(|k_j|+3).$$

For $f \in S_k(N_1)$ and $g \in S_2(N_2)$ it is well-known that $Q_f \asymp k^2N_1$ and $Q_g \asymp N_2$. Letting Q_F denote the analytic conductor of F, from (2.2) we have $Q_F = Q_f \cdot Q_g$. So

$$Q_F \asymp k^2N_1N_2. \tag{2.3}$$

For $x \geq 1$, we put

$$S(F,x) := \sum_{n\leq x}{}^{N} \lambda_F(n)\log\left(\frac{x}{n}\right). \tag{2.4}$$

3 Upper and lower bounds for $S(F,x)$

Let $\lambda_F(n) \geq 0$ for all $n \leq x$. We will estimate x by comparing the upper and lower bounds of $S(F,x)$. We will first work with f, g and finally transfer everything to F using (2.2).

3.1 Upper bound

Let Q_h be the analytic conductor of an elliptic Hecke newform h. From the subconvexity bound for GL_2 L-functions (see [13, Theorem 1.1]) we have for any $t \in \mathbf{R}$,

$$\left|L\left(h,\frac{1}{2}+it\right)\right| \ll Q_h^{1/4-\theta}\left|\frac{1}{2}+it\right|^{1/2-2\theta}, \tag{3.1}$$

for some $1/4 > \theta > 0$. From Perron's formula, (2.1) and (2.2) we can write

$$\sum_{n\leq x}^{N} \lambda_F(n) \log\left(\frac{x}{n}\right) = \frac{1}{2\pi i}\int_{(2)} \frac{1}{\zeta_N(1+2s)} L_{N_1}(f,s) L_{N_2}(g,s)\frac{x^s}{s^2}ds. \tag{3.2}$$

For $p|N_1$, we have $|\lambda_f(p)| \leq 1$ (see Theorem 3, [10]). Also recall that N_1 is squarefree. Let us now note the following estimate that will be used in the next paragraph.

$$\begin{aligned} \left|L^N\left(f,\frac{1}{2}+it\right)\right|^{-1} &= \prod_{p|N}\left|1-\lambda_f(p)p^{-\frac{1}{2}-it}\right| \\ &\leq \prod_{p|N}\left(1+\frac{|\lambda_f(p)|}{p^{\frac{1}{2}}}\right) \leq \sum_{d|N}\frac{1}{d^{\frac{1}{2}}} = O_\epsilon(N^\epsilon). \end{aligned} \tag{3.3}$$

Similarly we get, $|L^N(g, \frac{1}{2}+it)|^{-1} = O_\epsilon(N^\epsilon)$.

From (2.2), (3.1), (3.3) and the fact that $\zeta_N(2+2it)^{-1}$ is absolutely bounded, it is clear that the integral in (3.2) is absolutely convergent on the critical line $\Re s = \frac{1}{2}$. Shifting the line of integration to $\Re(s) = \frac{1}{2}$ gives (the bound in (3.1) implies that the horizontal integrals do not contribute)

$$S(F,x) = \frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)} \frac{1}{\zeta_N(1+2s)} L_N(f,s) L_N(g,s)\frac{x^s}{s^2}ds.$$

Now again using (2.2), (2.3), (3.1), (3.3) we estimate in a standard way that

$$S(F,x) \ll_\epsilon Q_F^{\frac{1}{4}-\theta+\epsilon} x^{\frac{1}{2}}. \tag{3.4}$$

Remark 3.1 One has (see Theorem I.5.5, [16]), $\sum_{d|N_1}\frac{1}{d^{\frac{1}{2}}} \leq e^{2+o(1)\frac{\log^{\frac{1}{2}} N_1}{\log_2 N_1}}$. So we would achieve a slightly better bound using this inequality. But for simplicity we are using the bound N_1^ϵ here.

3.2 Lower bound

From (2.1) and (2.2), comparing the Euler factors we have that for $p \nmid N$

$$\begin{aligned} &\lambda_F(p) = \lambda_f(p) + \lambda_g(p) \quad \text{and} \\ &\lambda_F(p^2) = \lambda_f(p^2) + \lambda_g(p^2) + \lambda_f(p)\lambda_g(p) - \frac{1}{p}. \end{aligned} \tag{3.5}$$

Hence, for $p \nmid N$, using the Hecke relation

$$\lambda_f(p^2) = \lambda_f(p)^2 - 1 \quad \text{and} \quad \lambda_g(p^2) = \lambda_g(p)^2 - 1,$$

we get from (3.5)

$$\lambda_F(p)^2 - \lambda_F(p^2) = 2 + \frac{1}{p} + \lambda_f(p)\lambda_g(p). \tag{3.6}$$

We look for lower bounds for $\lambda_F(p)$ from (3.6), exploiting the nonnegativity of $\lambda_F(p^2)$ for $p \leq x^{1/2}$. This leads us to look for a sizeable set of primes on which both $\lambda_f(p), \lambda_g(p)$ are small. To this end, we shall first prove the following lemma which will lead us to the required lower bound. See [12, Lemma 3.1 (iv)] for a result related to this lemma (where a lower bound version has been done).

Lemma 3.2 *Let $\pi(y,\mathscr{L}) := \#\{p : p \le y, p \nmid \mathscr{L}\}$. Let $h \in S_\kappa(\mathscr{L})$ be a newform and $\lambda_h(n)$ denote its normalised Fourier coefficients. Then for any $y \ge 2$, we have*

$$\sum_{p\le y}^{\mathscr{L}} |\lambda_h(p)| \le \sum_{p\le y}^{\mathscr{L}} \left(\frac{11}{10} - \frac{57}{1000}\lambda_h(p^4) + \frac{399}{1000}\lambda_h(p^2)\right).$$

Proof We know that for $p \nmid \mathscr{L}$, the Ramanujan bound for $|\lambda_h(p)|$ is 2 and

$$\lambda_h(p^2) = \lambda_h(p)^2 - 1 \quad \textit{and} \quad \lambda_h(p^4) = \lambda_h(p)^4 - 3\lambda_h(p)^2 + 1.$$

Before proceeding further, let us first discuss the idea of the proof. If we can find $\delta, \alpha, \beta \in \mathbf{R}$, with $\delta > 0$ as small as possible, such that

$$t \le \delta + \alpha(t^4 - 3t^2 + 1) + \beta(t^2 - 1) \tag{3.7}$$

for all $0 \le t \le 2$, then we would have that

$$\sum_{p\le y}^{\mathscr{L}} |\lambda_h(p)| \le \sum_{p\le y}^{\mathscr{L}} (\delta + \alpha\lambda_h(p^4) + \beta\lambda_h(p^2)). \tag{3.8}$$

We put $\beta = \alpha\Upsilon$ and rewrite the polynomial in the right hand side of (3.7) as

$$q(t) := \delta + \alpha(t^4 + (-3+\Upsilon)t^2 + 1 - \Upsilon). \tag{3.9}$$

Let us also define $r(t) := q(t) - t$. We want to find α, Υ, such that $r(t) > 0$ for all $t \in [0, 2]$. Now note that if the derivative of $r(t)$ has no root in $(0, 2)$ and if $r(0), r(2) > 0$, then $r(t) > 0$ for all $t \in [0, 2]$. We have

$$r'(t) = 4\alpha t^3 + 2\alpha t(-3+\Upsilon) - 1.$$

For given α, Υ, when t is very close to zero, $r'(t)$ is negative valued. Hence $r'(t)$ has to be negative valued for all $t \in (0, 2)$. To ensure this, we want to see, whether there exist α, Υ, such that $r'(t)$ has a maximum in $t > 0$ (note, as a degree 3 polynomial it can have at most one maximum) and at the point of maximum, $r'(t)$ is negative. We observe that *if α is negative and* $\Upsilon < 3$, then $r'(t)$ has a maximum at $t = (\frac{3-\Upsilon}{6})^{1/2}$. To ensure that $r'(t)$ is negative for all $t \in (0, 2)$ we also want $r'(t)|_{(\frac{3-\Upsilon}{6})^{\frac{1}{2}}} < 0$. This yields to the condition

$$-8\alpha < \frac{6^{\frac{3}{2}}}{(3-\Upsilon)^{\frac{3}{2}}}. \tag{3.10}$$

The conditions $q(0) > 0$ and $q(2) > 2$ are equivalent to

$$\delta + (1-\Upsilon)\alpha > 0, \quad \delta + (5+3\Upsilon)\alpha > 2. \tag{3.11}$$

Using techniques from non-linear programming we get many solutions to this set of simultaneous inequalities (3.10), (3.11) together with the condition $\alpha < 0$ and $\Upsilon < 3$. We take the solution

$$\delta = \frac{11}{10}, \quad \alpha = -\frac{57}{1000}, \quad \Upsilon = -7.$$

Hence we get

$$\frac{11}{10} - \frac{57}{1000}\lambda_h(p^4) + \frac{399}{1000}\lambda_h(p^2) \ge |\lambda_h(p)|$$

for all $p \nmid \mathscr{L}$. Hence the lemma follows. □

Remark 3.3 $\frac{11}{10}$ is not the optimal choice for δ. Since the optimal value for δ improves the lower bound of $S(F, x)$ only up to a constant, we keep $\delta = \frac{11}{10}$.

Recall that $Q_g \asymp N_2$. Let $L(sym^2 g, s)$ and $L(sym^4 g, s)$ be the symmetric square and symmetric fourth power L-functions associated with g. It is well known that (see e.g., [5, Chaps. 5.1, 5.12])

(i) $Q_{sym^2 g} \asymp N_2^2$ and $Q_{sym^4 g} \asymp N_2^4$ and
(ii) $\lambda_g(p^2), \lambda_g(p^4)$ are p-th coefficients of the Dirichlet series which represent $L(sym^2 g, s)$ and $L(sym^4 g, s)$ respectively.

Proposition 3.4 *There exists an absolute constant $c_1 > 0$ such that if $c_1 \log y \geq (\log Q_g)^2$, then one has*

$$\sum_{p \leq y}^{N_2} |\lambda_g(p)| \leq \left(\frac{11}{10} + O\left(\frac{1}{\log y} \right) \right) \pi(y, N_2).$$

Proof From the holomorphy and non-vanishing at $s = 1$ of the symmetric square and the symmetric fourth power L-functions (see [7], [5, Chap. 5.12]) and the prime number theorem of L-functions (see [5], pp 110–111), we have for some absolute constant $c_0 > 0$ that

$$\sum_{p \leq y}^{N_2} \lambda_g(p^2) \log p = O(N_2\, y \, \exp(-c_0 \sqrt{\log y})) \quad \text{and}$$
$$\sum_{p \leq y}^{N_2} \lambda_g(p^4) \log p = O(N_2^2\, y \, \exp(-c_0 \sqrt{\log y})).$$

So, when $N_2 \ll \exp(\frac{c_0}{4} \sqrt{\log y})$ using Abel's summation formula we obtain

$$\sum_{p \leq y}^{N_2} \lambda_g(p^2) = O\left(\frac{y}{\log^2 y} \right) \quad \text{and} \quad \sum_{p \leq y}^{N_2} \lambda_g(p^4) = O\left(\frac{y}{\log^2 y} \right).$$

The proposition now follows immediately from Lemma 3.2. □

For any $\gamma > 0$, let us define $V(y, \gamma) := \{p : p \leq y, p \nmid N_2, |\lambda_g(p)| \leq \gamma\}$.

Corollary 3.5 *Let the assumptions be as in Proposition 3.4. Then at least one of these two following inequalities holds true:*

$$\frac{\# V(y, \frac{19}{20})}{\pi(y, N_2)} \geq \frac{1}{100} \quad \text{or} \quad \frac{\# V(y, \frac{13}{10})}{\pi(y, N_2)} \geq \frac{51}{100}.$$

Proof We appeal to Proposition 3.4 and choose y large enough so that the $O(1/\log y)$ term is less than $1/1000$. Suppose none of the the inequalities mentioned above holds. From the negation of the first inequality we get

$$\frac{\#\{p : |\lambda_g(p)| > \frac{19}{20}, p \leq y, p \nmid N_2\}}{\pi(y, N_2)} > \frac{99}{100}. \tag{3.12}$$

From the negation of second inequality we get

$$\frac{\#\{p : |\lambda_g(p)| > \frac{13}{10}, p \leq y, p \nmid N_2\}}{\pi(y, N_2)} > \frac{49}{100}. \tag{3.13}$$

Hence, combining (3.12) and (3.13) we get

$$\sum_{p \leq y}^{N_2} |\lambda_g(p)| > \left(\frac{49}{100} \cdot \frac{13}{10} + \frac{50}{100} \cdot \frac{19}{20} \right) \pi(y, N_2) = \frac{1112}{1000} \pi(y, N_2).$$

This is a contradiction with Proposition 3.4 and hence the corollary follows. □

We note that

$$S(F,x) = \sum_{n\le x}^{N} \lambda_F(n)\log\left(\frac{x}{n}\right) \gg \sum_{n\le \frac{x}{2}}^{N} \lambda_F(n).$$

Thus it is enough to find a lower bound for $\sum_{n\le x,(n,N)=1}\lambda_F(n)$. Recall that $N = \mathrm{lcm}(N_1, N_2)$.

Proposition 3.6 *For all x such that $\log x \gg (\log Q_g)^2$. Then we have, under the assumption that $\lambda_F(n) \ge 0$ for all $n \le x$, that*

$$\sum_{n\le x}^{N} \lambda_F(n) \gg \frac{x}{\log^2 x}. \tag{3.14}$$

Proof We appeal to Corollary 3.5 with $y = x^{\frac{1}{2}}$ (Here we assume that $\frac{c_1}{2}\log x \ge (\log Q_g)^2$, with c_1 being the same absolute constant as in Proposition 3.4. This allows us to choose $y = x^{\frac{1}{2}}$). As the number of distinct prime factors of N_2 is at most $\log N_2/\log 2$, we get

$$\pi\left(x^{\frac{1}{2}}, N_2\right) \ge \pi\left(x^{\frac{1}{2}}\right) - \frac{\log N_2}{\log 2} \gg \frac{x^{\frac{1}{2}}}{\log x}.$$

Here the implied constant is absolute. To see this, let c_1 be as above, and $x > 1$ be such that $x^{\frac{1}{2}}/\log x \ge (\log x)^{\frac{1}{2}}$.The absoluteness follows from the assumption $\frac{c_1}{2}\log x \ge (\log Q_g)^2$.

If g satisfies the first inequality in Corollary 3.5, then one has $\# V(x^{\frac{1}{2}}, \frac{19}{20}) \gg \frac{x^{\frac{1}{2}}}{\log x}$. So in this case, for $p \in V(x^{\frac{1}{2}}, \frac{19}{20})$, from (3.6) we have $\lambda_F(p) > 10^{-1/2}$. That gives us

$$\sum_{n\le x}^{N} \lambda_F(n) \ge \sum_{\substack{p_1\ne p_2\\ p_1,p_2\in V\left(x^{\frac{1}{2}},\frac{19}{20}\right)}} \lambda_F(p_1p_2) \gg \left(\sum_{p\in V\left(x^{\frac{1}{2}},\frac{19}{20}\right)} 1\right)^2 - \sum_{p\in V\left(x^{\frac{1}{2}},\frac{19}{20}\right)} 1 \gg \frac{x}{\log^2 x}.$$

If g satisfies the second inequality in Corollary 3.5, then one has $\# V(x^{\frac{1}{2}}, \frac{13}{10}) \gg x^{\frac{1}{2}}/\log x$. We split $V(x^{\frac{1}{2}}, \frac{13}{10})$ into two disjoint sets:

(I) those p for which $|\lambda_f(p)| \ge 14/10$, in which case from (3.5) one gets $\lambda_F(p) \ge 1/10$;
(II) those p for which $|\lambda_f(p)| < 14/10$, in which case from (3.6) one gets $\lambda_F(p) > 3\sqrt{2}/10$.

Thus combining all the cases above we have,

$$\sum_{n\le x}^{N} \lambda_F(n) \ge \sum_{\substack{p_1\ne p_2\\ p_1,p_2\in V\left(x^{1/2},\frac{13}{10}\right)}} \lambda_F(p_1p_2) \gg \left(\sum_{p\in V\left(x^{\frac{1}{2}},\frac{13}{10}\right)} 1\right)^2 - \sum_{p\in V\left(x^{\frac{1}{2}},\frac{13}{10}\right)} 1 \gg \frac{x}{\log^2 x}.$$

□

3.3 Proof of Theorem 1.1

Suppose to the contrary that the first sign change in the sequence $\lambda_F(n)$ occurs after x and that

$$x \gg_\epsilon Q_F^{1/2-2\theta+\epsilon} \tag{3.15}$$

for a given $\epsilon > 0$. We then look at the upper bound (3.4) for $S(F, x)$ (replacing ϵ by $\epsilon/8$ there) and the lower bound from (3.14) (note that from our hypothesis in Theorem 1.1, namely that $(\log Q_g)^2 \leq c \log Q_f$ for some absolute constant c, it follows easily that $\frac{c_2}{2} \log x \geq (\log Q_g)^2$ for some absolute constant c_2 large enough, so that we can apply Proposition 3.6). This leads to the inequality

$$\frac{x}{\log^4 x} \ll_\epsilon Q_F^{\frac{1}{2}-2\theta+\epsilon/4}.$$

As $\epsilon > 0$ is arbitrary, applying Lemma 4 of [3] we conclude

$$x \ll_\epsilon Q_F^{\frac{1}{2}-2\theta+\epsilon/4} (\log Q_F)^4 \ll Q_F^{\frac{1}{2}-2\theta+\epsilon/2}. \tag{3.16}$$

We arrive at a contradiction with (3.15). This completes the proof of Theorem 1.1.

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