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An analogue of Ingham's theorem on the Heisenberg group

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Abstract

We prove an exact analogue of Ingham's uncertainty principle for the group Fourier transform on the Heisenberg group. This is accomplished by explicitly constructing compactly supported functions on the Heisenberg group whose operator valued Fourier transforms have suitable Ingham type decay and proving an analogue of Chernoff's theorem for the family of special Hermite operators.

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1 Introduction

Roughly speaking, the uncertainty principle for the Fourier transform on $\mathbb{R}^n$ says that a function f and its Fourier transform $\widehat{f}$ cannot both have rapid decay. Several manifestations of this principle are known: Heisenberg–Pauli–Weyl inequality, Paley–Wiener theorem and Hardy's uncertainty principle are some of the most well known.

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But there are lesser known results such as theorems of Ingham and Levinson. The best decay a non trivial function can have is vanishing identically outside a compact set and for such functions it is well known that their Fourier transforms extend to $\mathbb{C}^n$ as entire functions and hence cannot vanish on any open set. For any such function of compact support, its Fourier transform cannot have any exponential decay for a similar reason: if $|\widehat{f}(\xi)| \leq Ce^{-a|\xi|}$ for some $a > 0$, then it follows that f extends to a tube domain in $\mathbb{C}^n$ as a holomorphic function and hence it cannot have compact support. So it is natural to ask the question: what is the best possible decay, on the Fourier transform side, that is allowed of a function of compact support? An interesting answer to this question was provided by Ingham [14] in 1934 who proved the following theorem.

Theorem 1.1 (Ingham) *Let $\Theta(y)$ be a nonnegative even function on $\mathbb{R}$ such that $\Theta(y)$ decreases to zero when $y \to \infty$. There exists a nonzero continuous function f on $\mathbb{R}$, equal to zero outside an interval $(-a, a)$ whose Fourier transform $\widehat{f}$ satisfies the estimate $|\widehat{f}(y)| \leq Ce^{-|y|\Theta(y)}$ if and only if $\int_1^\infty \Theta(t)t^{-1}dt < \infty$.*

This theorem of Ingham and its close relatives Paley -Wiener ([25–27]) and Levinson ([19]) theorems have received considerable attention in recent years. In [2], Bhowmik et al proved analogues of the above theorem for $\mathbb{R}^n$, the n-dimensional torus $\mathbb{T}^n$ and step two nilpotent Lie groups. See also the recent work of Bowmik–Pusti–Ray [3] for a version of Ingham's theorem for the Fourier transform on Riemannian symmetric spaces of non-compact type. As we are interested in Ingham's theorem on the Heisenberg group, let us recall the result proved in [2]. Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group. For an integrable function f on $\mathbb{H}^n$, let $\widehat{f}(\lambda)$ be the operator valued Fourier transform of f indexed by non-zero reals λ. Measuring the decay of the Fourier transform in terms of the Hilbert-Schmidt operator norm $\|\widehat{f}(\lambda)\|_{HS}$ Bhowmik et al. have proved the following result.

Theorem 1.2 (Bhowmik-Ray-Sen) *Let $\Theta(\lambda)$ be a nonnegative even function on $\mathbb{R}$ such that $\Theta(\lambda)$ decreases to zero when $\lambda \to \infty$. There exists a nonzero, compactly supported continuous function f on $\mathbb{H}^n$, whose Fourier transform satisfies the estimate $\|\widehat{f}(\lambda)\|_{HS} \leq C|\lambda|^{n/2}e^{-|\lambda|\Theta(\lambda)}$ if the integral $\int_1^\infty \Theta(t)t^{-1}dt < \infty$. On the other hand, if the above estimate is valid for a function f and the integral $\int_1^\infty \Theta(t)t^{-1}dt$ diverges, then the vanishing of f on any set of the form $\{z \in \mathbb{C}^n : |z| < \delta\} \times \mathbb{R}$ forces f to be identically zero.*

As the Fourier transform on the Heisenberg group is operator valued, it is natural to measure the decay of $\widehat{f}(\lambda)$ by comparing it with the Hermite semigroup $e^{-aH(\lambda)}$ generated by $H(\lambda) = -\Delta_{\mathbb{R}^n} + \lambda^2|x|^2$. In this connection, let us recall the following two versions of Hardy's uncertainty principle. Let $p_a(z, t)$ stand for the heat kernel associated to the sublaplacian $\mathcal{L}$ on the Heisenberg group whose Fourier transform turns out to be the Hermite semigroup $e^{-aH(\lambda)}$. The version in which one measures the decay of $\widehat{f}(\lambda)$ in terms of its Hilbert-Schmidt operator norm reads as follows. If

$$|f(z,t)| \leq Ce^{-a(|z|^2+t^2)}, \quad \|\widehat{f}(\lambda)\|_{HS} \leq Ce^{-b\lambda^2} \tag{1.1}$$

then $f = 0$ whenever $ab > 1/4$. This is essentially a theorem in the t-variable and can be easily deduced from Hardy's theorem on $\mathbb{R}$, see Theorem 2.9.1 in [37]. Compare this with the following version [37, Theorem 2.9.2]. If

$$|f(z,t)| \leq C p_a(z,t), \;\; \widehat{f}(\lambda)^* \widehat{f}(\lambda) \leq C e^{-2bH(\lambda)} \tag{1.2}$$

then $f = 0$ whenever $a < b$. Here and throughout the rest of this paper, we use the following standard operator theoretic notation: Given two self-adjoint operators A and B on a Hilbert space H, we write $A \leq B$ whenever $B - A \geq 0$ or equivalently, $(Bu, u) \geq (Au, u)$ for all $u \in H$. Coming back to the discussion, this latter version is the exact analogue of Hardy's theorem for the Heisenberg group, which we can view not merely as an uncertainty principle but also as a characterization of the heat kernel. Hardy's theorem in the context of semi-simple Lie groups and non-compact Riemannian symmetric spaces are also to be viewed in this perspective.

We remark that the Hermite semigroup has been used to measure the decay of the Fourier transform in connection with the heat kernel transform [17], Pfannschmidt's theorem [39] and the extension problem for the sublaplacian [29] on the Heisenberg group. In connection with the study of Poisson integrals, it has been noted in [38] that when the Fourier transform of f satisfies an estimate of the form $\widehat{f}(\lambda)^* \widehat{f}(\lambda) \leq Ce^{-a\sqrt{H(\lambda)}}$, then the function extends to a tube domain in the complexification of $\mathbb{H}^n$ as a holomorphic function and hence the vanishing of f on an open set forces it to vanish identically. It is therefore natural to ask if the same conclusion can be arrived at by replacing the constant a in the above estimate by an operator $\Theta(\sqrt{H(\lambda)})$ for a function Θ decreasing to zero at infinity. Our investigations have led us to the following exact analogue of Ingham's theorem for the Fourier transform on $\mathbb{H}^n$.

Theorem 1.3 *Let $\Theta(\lambda)$ be a nonnegative function on $[0, \infty)$ which decreases to zero as $\lambda \to \infty$. Then there exists a nonzero compactly supported continuous function f on $\mathbb{H}^n$ whose Fourier transform $\widehat{f}$ satisfies the estimate*

$$\widehat{f}(\lambda)^* \widehat{f}(\lambda) \leq C e^{-2\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}, \;\; \lambda \neq 0, \tag{1.3}$$

if and only if Θ satisfies the condition $\int_1^\infty \Theta(t) t^{-1} dt < \infty$.

Under the assumption that $\int_1^\infty \Theta(t) t^{-1} dt = \infty$, the above theorem demonstrates that any compactly supported function whose Fourier transform satisfies (1.3) vanishes identically. This can be viewed as an uncertainty principle in the sense mentioned in the first paragraph. Recently this aspect of Ingham's theorem has been proved in the context of higher dimensional Euclidean spaces and Riemannian symmetric spaces with a much weaker hypothesis on the function. As observed in [11], for the Heisenberg group case, the hypothesis can be weakened considerably if we slightly strengthen the condition (1.3). More precisely, the second and the last author proved the following theorem in this context.

Theorem 1.4 [11] *Let $\Theta(\lambda)$ be a nonnegative function on $[0, \infty)$ such that it decreases to zero as $\lambda \to \infty$, and satisfies the conditions $\int_1^\infty \Theta(t) t^{-1} dt = \infty$. Let f be an* *integrable function on $\mathbb{H}^n$ whose Fourier transform satisfies the estimate*

$$\hat{f}(\lambda)^* \hat{f}(\lambda) \leq C\, e^{-2|\lambda|\, \Theta(|\lambda|)} e^{-2\sqrt{H(\lambda)}\Theta(\sqrt{H(\lambda)})}. \tag{1.4}$$

Then f cannot vanish on any nonempty open set unless it is identically zero.

Comparing the decay condition (1.3) and (1.4), it is not difficult to see that the Theorem 1.3 is a significant improvement of the Theorem 1.4 in terms of the Ingham type decay condition. However, we believe that the necessary part of the Theorem 1.3 is true under the weaker hypothesis on the function as in the Theorem 1.4. In what follows, we shed more light on the difficulties in this regard.

The sufficiency part of Theorem 1.3 is proved in Section 4.1 by explicitly constructing compactly supported functions whose Fourier transforms satisfy the stated decay condition. Though at present we are not able to prove the necessary part of the theorem under the assumption that f vanishes on an open set, a slightly different version can be proved. Recall that the Fourier transform $\widehat{f}$ is defined by integrating f against the Schrödinger representations π_λ:

$$\widehat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)dz\, dt.$$

Since $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z,0)$, it follows that $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$, where $f^\lambda(z)$ is the inverse Fourier transform of $f(z,t)$ in the central variable and

$$W_\lambda(f^\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z,0)dz$$

is the Weyl transform of f^λ. With these notations we prove the following improvement on the necessary part of Theorem 1.3.

Theorem 1.5 *Let $\Theta(\lambda)$ be a nonnegative function on $[0,\infty)$ such that it decreases to zero when $\lambda \to \infty$, and satisfies the condition $\int_1^\infty \Theta(t)t^{-1}dt = \infty$. Let f be an integrable function on $\mathbb{H}^n$ whose Fourier transform $\widehat{f}$ satisfies the estimate*

$$\widehat{f}(\lambda)^* \widehat{f}(\lambda) \leq C e^{-2\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}},\ \lambda \neq 0. \tag{1.5}$$

If for every $\lambda \neq 0$, there exists an open set $U_\lambda \subset \mathbb{C}^n$ on which f^λ vanishes, then $f = 0$.

Remark 1.1 Note that when f is compactly supported the function f^λ is also compactly supported and hence vanishes on an open set. The same is true if we assume that f is supported on a cylindrical set $\{z \in \mathbb{C}^n : |z| < a\} \times \mathbb{R}$. As $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$, the above can be considered as a result for the Weyl transform of functions on $\mathbb{C}^n$.

Theorem 1.1 was proved in [14] by Ingham by making use of Denjoy–Carleman theorem on quasi-analytic functions. In [2], the authors have used Radon transform and a several variable extension of Denjoy–Carleman theorem due to Bochner and

Taylor [5] in order to prove the n-dimensional version of Theorem 1.1. An L^2 variant of the result of Bochner–Taylor which was proved by Chernoff in [8] has turned out to be very useful in establishing Ingham type theorems.

Theorem 1.6 [8, Chernoff] *Let f be a smooth function on $\mathbb{R}^n$. Assume that $\Delta^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and that $\sum_{m=1}^{\infty} \|\Delta_{\mathbb{R}^n}^m f\|_2^{-\frac{1}{2m}} = \infty$. If f and all its partial derivatives vanish at* 0*, then f is identically zero.*

As the Laplacian is translation invariant, 0 can be replaced by any other point in the above theorem. As a matter of fact, this theorem shows how partial differential operators generate the class of quasi-analytic functions. Recently, Bhowmik–Pusti–Ray [3] have established an analogue of Chernoff's theorem for the Laplace-Beltrami operators on non-compact Riemannian symmetric spaces and use the same in proving a version of Ingham's theorem for the Helgason Fourier transform.

In the context of the Heisenberg group, we prove Theorem 1.5, and hence Theorem 1.3, by using the following analogue of Chernoff's theorem for the family of special Hermite operators L_λ. These operators on $\mathbb{C}^n$ are defined via the relation $\mathcal{L}(f(z)e^{i\lambda t}) = e^{i\lambda t} L_\lambda f(z)$ where $\mathcal{L}$ is the sublaplacian on $\mathbb{H}^n$. It turns out that when $\lambda = 0$, the special Hermite operator L_λ reduces to the Laplacian $\Delta_{\mathbb{C}^n}$ on $\mathbb{C}^n$. We refer the reader to Sect. 2.3 for more details.

Theorem 1.7 *For any fixed $\lambda \in \mathbb{R}$, let $f \in C^\infty(\mathbb{C}^n)$ be such that $L_\lambda^m f \in L^2(\mathbb{C}^n)$ for all $m \geq 0$ and that $\sum_{m=1}^{\infty} \|L_\lambda^m f\|_2^{-\frac{1}{2m}} = \infty$. If f and all its partial derivatives vanish at some $w \in \mathbb{C}^n$, then f is identically zero.*

When $\lambda = 0$, the above is just Chernoff's theorem for the Laplacian on $\mathbb{C}^n$. For $\lambda = 1$, a weaker version of the theorem, namely under the assumption that f vanishes on an open set, has been proved in [10, Theorem 4.1]. The weaker version is in fact good enough to prove Theorems 1.5 and 1.3. However, in this paper, we prove the above improvement which is the exact analogue of Theorem 1.6 for the special Hermite operators and the second main result of this article.

Remark 1.2 The interest in Ingham type theorems for the Fourier transforms in various settings was revived by Bhowmik and his collaborators in a series of papers [2–4]. These works mainly dealt with the Fourier transform on Riemannian symmetric spaces. The second and the last authors treated the case of Fourier transform on Heisenberg groups and eigenfunction expansions in [10] and [11]. In order to help the reader to get a better understanding of the status of the investigations in this interesting area of research, we would like to conclude this introduction with the following remarks.

(i)*Ingham's theorem:* One of the two parts of Ingham's theorem is the construction of a compactly supported function with prescribed decay on the Fourier transform side. This construction is not only required to prove the sharpness of the theorem, but also plays a major role in the proof of the direct part of the theorem. In the context of higher dimensional Euclidean spaces and Riemannian symmetric spaces, this construction easily follows from the original construction of Ingham on $\mathbb{R}$. On the other hand, the construction on the Heisenberg group is much more involved and difficult as the Fourier transform is operator valued.

Let us mention in passing that this new construction has already been used without proof in the work of [11]. More precisely, the authors in [11] convolved the function constructed in this paper with a suitable compactly supported function in the central variable in order to construct a function satisfying (1.4). See [11, Sect. 4] for more details.

Furthermore, in the construction of functions in [10] where the second and the last author proved (among other things) analogues of Ingham's theorem for the spectral projections associated with Hermite and special Hermite operators, the example constructed in this paper played a very important role. Exploiting the connection between the Weyl transform and the Fourier transform on Heisenberg group coupled with a periodization technique, the example for the special Hermite case has been derived from the function constructed in this paper. Also, the function for the Hermite case has been deduced from the special Hermite case. For a detailed account of this, we refer the reader to the article [10, Sect. 6].

We would also like to point out that, in proving Theorem 1.4, the authors used a weaker version of Chernoff's theorem for the full Laplacian on the Heisenberg group. This explains why they have demanded the slightly stronger Ingham-type decay (1.4) in their work. As opposed to this, in this paper, we use a version of Chernoff's theorem for the family of special Hermite operators $\{L_\lambda\}_{\lambda \neq 0}$. As we shall see later, this allows us to dispense with the extra decay corresponding to the central variable, getting a significantly improved analogue of Ingham's theorem on the Heisenberg group.

(ii)*Chernoff's theorem:* Following the earlier works on Ingham's theorem we also use Chernoff's theorem as an important tool in our proof. Apart from this, another idea used is the reduction technique which allows us consider only radial functions. In the earlier work [10, Theorem 4.1] the authors used the twisted spherical means to effect this reduction and then used a Chernoff type theorem for the radial part of the special Hermite operator. This technique demanded the restriction of the vanishing condition, resulting in a Chernoff-type theorem for L_1 with a strong vanishing condition as mentioned above. As opposed to this, in this paper, we use bi-graded spherical harmonics along with Hecke–Bochner type identity for the special Hermite projections to reduce the matter to the radial case. This allows us to replace the stronger vanishing condition with a weaker one, retaining the quasi-analytic nature of the theorem. Unlike the previous one, this technique has also been proven to be beneficial in the contexts of rank one Reimannian symmetric spaces. See [12] for further details in this regard.

In proving Theorem 1.7 in [11] the authors have used Chernoff's theorem for the full Laplacian on the Heisenberg which required the slightly stronger hypothesis on the Fourier transform side. In this paper the proof of Ingham's theorem is based on Chernoff's theorem for the family of special Hermite operators L_λ. Thus, we only need to assume the decay condition of $\widehat{f}(\lambda)$ for each λ fixed. Moreover, when f is compactly supported, the same is true for of f^λ for each λ as functions on $\mathbb{C}^n$. These two ideas allowed us to prove a stronger version of Ingham's theorem in this paper. We would also like to remark that the proof of Chernoff's theorem uses a theorem of de Jeu (see Theorem 2.3 in [15]) related to the moment problem. Thus the similarity between proofs of Chernoff's theorem for operators in different settings is not a coincidence. However, details differs in terms of the degree of difficulties involved.

Let us mention that proving the exact analogue of Chernoff's theorem still remains as an interesting open problem. Our attempts to follow the original ideas of Chernoff in the case of Heisenberg group met with serious difficulties even for the full Laplacian on $\mathbb{H}^n$. Thus, Chernoff's theorem for the full Laplacian and sublaplacian on $\mathbb{H}^n$ remains as an interesting open problem worthy of further investigation.

Here is a brief outline of the organization of the paper. After recalling the required preliminaries regarding harmonic analysis on Heisenberg group in Sect. 2, we prove an analogue of Chernoff's theorem for the special Hermite operators (Theorem 1.7) in Sect. 3. In Sect. 4, we prove the Ingham's theorems on the Heisenberg group, namely Theorems 1.3, and 1.5.

2 Preliminaries on Heisenberg groups

In this section, we collect the results which are necessary for the study of uncertainty principles for the Fourier transform on the Heisenberg group. We refer the reader to the two classical books Folland [9] and Taylor [34] for the preliminaries of harmonic analysis on the Heisenberg group. However, we will be closely following the notations of the books of Thangavelu [36] and [37].

2.1 Heisenberg group and Fourier transform

Let $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ denote the $(2n+1)$-dimensional Heisenberg group equipped with the group law

$$(z, t).(w, s) := \left(z + w, t + s + \frac{1}{2}\Im(z.\bar{w})\right), \ \forall (z, t), (w, s) \in \mathbb{H}^n.$$

This is a step two nilpotent Lie group where the Lebesgue measure $dzdt$ on $\mathbb{C}^n \times \mathbb{R}$ serves as the Haar measure. The representation theory of $\mathbb{H}^n$ is well-studied in the literature. In order to define Fourier transform, we use the Schrödinger representations as described below.

For each non-zero real number λ, we have an infinite dimensional representation π_λ realised on the Hilbert space $L^2(\mathbb{R}^n)$. These are explicitly given by

$$\pi_\lambda(z, t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$. These representations are known to be unitary and irreducible. Moreover, by a theorem of Stone and Von-Neumann (see e.g., [9]), these account, upto unitary equivalence, for all the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ which act as $e^{i\lambda t}I, \ \lambda \neq 0$, on the center. Also, there is another class of one dimensional irreducible representations that corresponds to the case $\lambda = 0$. As they do not contribute to the Plancherel measure we will not describe them here.

The Fourier transform of a function $f \in L^1(\mathbb{H}^n)$ is the operator valued function obtained by integrating f against π_λ:

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

Note that $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. Now, by definition of π_λ and $\hat{f}(\lambda)$, it is easy to see that

$$\widehat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z,0)dz,$$

where f^λ stands for the inverse Fourier transform of f in the central variable:

$$f^\lambda(z) := \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)dt.$$

This motivates the following definition. Given a function g on $\mathbb{C}^n$, we consider the following operator defined by

$$W_\lambda(g) := \int_{\mathbb{C}^n} g(z)\pi_\lambda(z,0)dz.$$

With these notations, we note that $\hat{f}(\lambda) = W_\lambda(f^\lambda)$. These transforms are called the Weyl transforms and for $\lambda = 1$, they are simply denoted by $W(g)$ instead of $W_1(g)$. We have the following Plancherel formula for the Weyl transforms (See [37, 2.2.9, Page no-49])

$$\|W_\lambda(g)\|_{HS}^2|\lambda|^n = (2\pi)^n\|g\|_2^2,\ g \in L^2(\mathbb{C}^n). \tag{2.1}$$

This, in view of the relation between the group Fourier transform and the Weyl transform, proves that when $f \in L^1 \cap L^2(\mathbb{H}^n)$, its Fourier transform is actually a Hilbert-Schmidt operator and one has

$$\int_{\mathbb{H}^n} |f(z,t)|^2 dzdt = (2\pi)^{-(n+1)} \int_{-\infty}^{\infty} \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda,$$

where $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm. The above allows us to extend the Fourier transform as a unitary operator between $L^2(\mathbb{H}^n)$ and the Hilbert space of Hilbert-Schmidt operator valued functions on $\mathbb{R}$ which are square integrable with respect to the Plancherel measure $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$. We polarize the above identity to obtain

$$\int_{\mathbb{H}^n} f(z,t)\overline{g(z,t)}dzdt = \int_{-\infty}^{\infty} tr(\widehat{f}(\lambda)\widehat{g}(\lambda)^*)\, d\mu(\lambda).$$

Also for suitable functions f on $\mathbb{H}^n$ we have the inversion formula

$$f(z,t)=\int_{-\infty}^{\infty} tr(\pi_\lambda(z,t)^*\widehat{f}(\lambda))d\mu(\lambda).$$

Moreover, the Fourier transform behaves well with the convolution of two functions defined by

$$f*g(x):=\int_{\mathbb{H}^n} f(xy^{-1})g(y)dy.$$

In fact, for any $f,g\in L^1(\mathbb{H}^n)$, it follows from the definition that

$$\widehat{f*g}(\lambda)=\hat{f}(\lambda)\hat{g}(\lambda).$$

We end this subsection by recording an important property of the group Fourier transform. Let δ_r stand for the non-isotropic dilation on $\mathbb{H}^n$ defined by $\delta_r(z,t)=(rz,r^2t)$ for $(z,t)\in\mathbb{H}^n$. Given a function f on $\mathbb{H}^n$, we denote the dilation of f by $\delta_r f$ defined by $\delta_r f(z,t):=f(\delta_r(z,t))$. The group Fourier transforms of $\delta_r f$ and f are connected via the relation

$$\widehat{\delta_r f}(\lambda)=r^{-(2n+2)}d_r\circ\widehat{f}(r^{-2}\lambda)\circ d_r^{-1} \tag{2.2}$$

where d_r is the standard dilation on $\mathbb{R}^n$ given by $d_r\varphi(x)=\varphi(rx)$. This can be obtained by an easy calculation. Indeed, first observe that

$$\pi_\lambda(rz,0)=d_r^{-1}\circ\pi_{\lambda r^2}(z,0)\circ d_r \tag{2.3}$$

which can be easily checked using the definition of π_λ. Now a simple change of variable yields

$$\widehat{\delta_r f}(\lambda)=\int_{\mathbb{H}^n} f(rz,r^2t)\pi_\lambda(z,t)dzdt=r^{-2n-2}\int_{\mathbb{C}^n} f^{\lambda/r^2}(z)\pi_\lambda(r^{-1}z,0)dz.$$

But in view of the above observation (2.3), we see that the right hand side of the above equals to

$$r^{-2n-2}d_r\circ\left(\int_{\mathbb{C}^n} f^{\lambda/r^2}(z)\pi_{\lambda/r^2}(z,0)dz\right)\circ d_r^{-1}$$

from which follows (2.2) immediately.

In the following subsection, we describe the role of special functions in the harmonic analysis on $\mathbb{H}^n$ and show that the group Fourier transform of a suitable class of functions take a nice form.

2.2 Special functions and fourier transform

For each $\lambda \neq 0$, we consider the following family of scaled Hermite functions indexed by $\alpha \in \mathbb{N}^n$:

$$\Phi_\alpha^\lambda(x) := |\lambda|^{\frac{n}{4}} \Phi_\alpha(\sqrt{|\lambda|}x), \ x \in \mathbb{R}^n,$$

where Φ_α denote the $n-$dimensional Hermite functions (see [35]). It is well-known that these scaled functions Φ_α^λ are eigenfunctions of the scaled Hermite operator $H(\lambda) := -\Delta_{\mathbb{R}^n} + \lambda^2|x|^2$ with eigenvalue $(2|\alpha| + n)|\lambda|$ and $\{\Phi_\alpha^\lambda : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. As a consequence,

$$\|\widehat{f}(\lambda)\|_{HS}^2 = \sum_{\alpha \in \mathbb{N}^n} \|\widehat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2.$$

In view of this, the Plancheral formula takes the following very useful form

$$\int_{\mathbb{H}^n} |f(z,t)|^2 dz dt = \int_{-\infty}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \|\widehat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \, d\mu(\lambda).$$

Given $\sigma \in U(n)$, we define $R_\sigma f(z,t) = f(\sigma.z,t)$. We say that a function f on $\mathbb{H}^n$ is radial if it is invariant under the action of $U(n)$ i.e., $R_\sigma f = f$ for all $\sigma \in U(n)$. The Fourier transforms of such radial integrable functions are functions of the Hermite operator $H(\lambda)$. In fact, if $H(\lambda) = \sum_{k=0}^\infty (2k+n)|\lambda|P_k(\lambda)$ is the spectral decomposition of this operator, then for a radial intrgrable function f we have

$$\widehat{f}(\lambda) = \sum_{k=0}^\infty R_k(\lambda, f)P_k(\lambda).$$

Here, $P_k(\lambda)$ stands for the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the k^{th} eigenspace spanned by scaled Hermite functions Φ_α^λ with $|\alpha| = k$. The coefficients $R_k(\lambda, f)$ are given by

$$R_k(\lambda, f) = \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} f^\lambda(z)\varphi_{k,\lambda}^{n-1}(z)\, dz. \tag{2.4}$$

In the above formula, $\varphi_{k,\lambda}^{n-1}$ are the Laguerre functions of type $(n-1)$:

$$\varphi_{k,\lambda}^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right) e^{-\frac{1}{4}|\lambda||z|^2},$$

where L_k^{n-1} denotes the Laguerre polynomial of type $(n-1)$. For more about Laguerre functions, we refer the reader to Sect. 2.4. Furthermore, given two radial integrable functions f and g on $\mathbb{H}^n$, in view of the formula $\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda)$, from basic spectral theory it follows that

$$\widehat{f * g}(\lambda) = \sum_{k=0}^{\infty} R_k(\lambda, f) R_k(\lambda, g) P_k(\lambda)$$

which yields

$$R_k(\lambda, f * g) = R_k(\lambda, f) R_k(\lambda, g), \ \forall k \geq 0. \tag{2.5}$$

2.3 The sublaplacian and special Hermite operators

We let $\mathfrak{h}_n$ stand for the Heisenberg Lie algebra consisting of left invariant vector fields on $\mathbb{H}^n$. A basis for $\mathfrak{h}_n$ is provided by the $2n + 1$ vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \ \ Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \ \ j = 1, 2, ..., n, \text{ and } T = \frac{\partial}{\partial t}.$$

These correspond to certain one parameter subgroups of $\mathbb{H}^n$. The sublaplacian on $\mathbb{H}^n$ is defined by

$$\mathcal{L} := -\sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

which can be explicitly calculated as

$$\mathcal{L} = -\Delta_{\mathbb{C}^n} - \frac{1}{4}|z|^2 \frac{\partial^2}{\partial t^2} + N \frac{\partial}{\partial t},$$

where $\Delta_{\mathbb{C}^n}$ stands for the Laplacian on $\mathbb{C}^n$ and N is the rotation operator defined by

$$N = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

This is a sub-elliptic operator and homogeneous of degree 2 with respect to the non-isotropic dilation given by $\delta_r(z, t) = (rz, r^2 t)$. The sublaplacian is also invariant under rotation i.e., $R_\sigma \circ \mathcal{L} = \mathcal{L} \circ R_\sigma, \ \sigma \in U(n)$. For each $\lambda \neq 0$, special Hermite operator L_λ is defined via the relation

$$\mathcal{L}(e^{i\lambda t} f(z)) = e^{i\lambda t} L_\lambda f(z).$$

Furthermore, it is not hard to see that $(\mathcal{L} f)^\lambda(z) = L_\lambda f^\lambda(z)$. It turns out that L_λ is explicitly given by

$$L_\lambda = -\Delta_{\mathbb{C}^n} + \frac{1}{4}\lambda^2 |z|^2 + i\lambda N.$$

This family of special Hermite operators has a useful translation invariance property coming from the sublaplacian.

Recall that the sublaplacian $\mathcal{L}$ is invariant under the left translations defined by $\tau_y f(x) := f(y^{-1}x),\ x, y \in \mathbb{H}^n$. In other words, $\tau_y(\mathcal{L}f) = \mathcal{L}(\tau_y f)$. Now, with $x = (w, 0) \in \mathbb{H}^n$, taking inverse Fourier transform in the central variable gives us

$$(\tau_x(\mathcal{L}f))^\lambda(z) = L_\lambda(\tau_x f)^\lambda(z)$$

which, after simplification leads to

$$e^{\frac{i\lambda}{2}\Im(w.\bar{z})}L_\lambda f^\lambda(z-w) = L_\lambda(e^{\frac{i\lambda}{2}\Im(w.\bar{z})}f^\lambda(z-w)).$$

This observation in turn implies that the special Hermite operator L_λ is invariant under the λ-twisted translation $T_w^\lambda,\ w \in \mathbb{C}^n$, defined by

$$T_w^\lambda g(z) := e^{\frac{i\lambda}{2}\Im(w.\bar{z})}g(z-w). \tag{2.6}$$

In other words,

$$T_w^\lambda(L_\lambda g) = L_\lambda(T_w^\lambda g),\ w \in \mathbb{C}^n. \tag{2.7}$$

It is also known that these L_λ's are elliptic operators on $\mathbb{C}^n$ with an explicit spectral decomposition. The spectrum consists of the real numbers of the form $(2k+n)|\lambda|, k \geq 0$, and the eigenspaces associated to each of these eigenvalues are infinite dimensional.

In the following, we describe the spectral decomposition for the case when $\lambda = 1$. For the sake of simplicity, we write L instead of L_1. In this regard, we also need to introduce twisted convolution $f \times g$ defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\Im(z\cdot\bar{w})}dw.$$

It is known that ([37, page no. 58]) the special Hermite expansion of a function $f \in L^2(\mathbb{C}^n)$ and Parseval's identity reads as

$$f(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k^{n-1}(z),\quad \|f\|_2^2 = (2\pi)^{-n}\sum_{k=0}^{\infty}\|f \times \varphi_k^{n-1}\|_2^2 \tag{2.8}$$

and each $f \times \varphi_k^{n-1}$ is an eigenfunction of the operator L with eigenvalue $(2k+n)$. Now if $g(z) = g_0(|z|)$ is a radial function on $\mathbb{C}^n$, then $L_\lambda g$ takes the form $L_\lambda g = L_{\lambda,n-1}g_0$ where $L_{\lambda,n-1}$ is the scaled Laguerre operator of type $(n-1)$ given by

$$L_{\lambda,n-1} := -\frac{d^2}{dr^2} - \frac{2n-1}{r}\frac{d}{dr} + \frac{1}{4}\lambda^2 r^2.$$

In what follows, when $\lambda = 1$, we simply denote the radial part of the special Hermite operator $L_{1,n-1}$ by L_{n-1}. Also, in order to prove Chernoff's theorem for the special

Hermite operator, we need to use Laguerre operators of more general type which are obtained by replacing $(n-1)$ by any $\delta \geq -\frac{1}{2}$ and eigenfunction expansion associated with them. In the following subsection, we develop notations and record required results related to Laguerre expansions in this connection.

2.4 Laguerre expansions

To start with, we first recall the definition of Laguerre polynomials. For any $\delta \geq -\frac{1}{2}$, the Laguerre polynomials of type δ are defined by

$$e^{-t}t^{\delta}L_k^{\delta}(t) = \frac{1}{k!}\frac{d^k}{dt^k}(e^{-t}t^{k+\delta})$$

for $t > 0$, and $k \geq 0$. The explicit form of $L_k^{\delta}(t)$ which is a polynomial of degree k, is given by

$$L_k^{\delta}(t) = \sum_{j=0}^{k} \frac{\Gamma(k+\delta+1)}{\Gamma(j+\delta+1)\Gamma(k-j+1)} \frac{(-t)^j}{j!}.$$

We now introduce the normalised Laguerre functions $\mathcal{L}_k^{\delta}$ defined as follows.

$$\mathcal{L}_k^{\delta}(t) = \left(\frac{\Gamma(k+1)}{\Gamma(k+1+\delta)}\right)^{\frac{1}{2}} e^{-\frac{t}{2}}t^{\frac{\delta}{2}}L_k^{\delta}(t), \;\; t > 0.$$

Then it is well-known that for any fixed $\delta \geq -\frac{1}{2}$, $\left\{\mathcal{L}_k^{\delta}\right\}_{k=0}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R}^+, dt)$. Now, fix $\delta \geq -\frac{1}{2}$ and consider the following Laguerre functions of type δ defined by

$$\varphi_k^{\delta}(r) := L_k^{\delta}\left(\frac{1}{2}r^2\right)e^{-\frac{1}{4}r^2}, \;\; r > 0.$$

For any $\lambda \neq 0$, the λ-scaled Laguerre function is defined by the relation $\varphi_{k,\lambda}^{\delta}(r) := \varphi_k^{\delta}(\sqrt{|\lambda|}r)$. However, we also use the the following normalised Laguerre functions of type δ :

$$\psi_k^{\delta}(r) := \frac{\Gamma(k+1)\Gamma(\delta)}{\Gamma(k+\delta+1)}L_k^{\delta}\left(\frac{1}{2}r^2\right)e^{-\frac{1}{4}r^2}, \;\; r > 0$$

so that $\psi_k^{\delta}(0) = 1$. It turns out that these are eigenfunctions of the following Laguerre operator of type δ given by

$$L_{\delta} := -\frac{d^2}{dr^2} - \frac{2\delta+1}{r}\frac{d}{dr} + \frac{1}{4}r^2$$

with eigenvalue $(2k+\delta+1)$ i.e., $L_\delta \psi_k^\delta = (2k+\delta+1)\psi_k^\delta$. This can be checked using the relations [35, 1.1.48, 1.1.49] satisfied by the Laguerre polynomials. We will see later that for $\delta = n-1$, L_δ corresponds to the radial part of the special Hermite operator. Now, using the orthogonality property of the functions $\mathcal{L}_k^\delta$ (mentioned above), it is not difficult to see that $\{\psi_k^\delta : k \geq 0\}$ forms an orthogonal basis for $L^2(\mathbb{R}^+, r^{2\delta+1}dr)$. In view of this, for $f \in L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ we have

$$f(r) = \sum_{k=0}^{\infty} c_k^\delta \mathcal{R}_k^\delta(f)\psi_k^\delta(r), \quad \|f\|_2^2 = \sum_{k=0}^{\infty} c_k^\delta |\mathcal{R}_k^\delta(f)|^2, \tag{2.9}$$

where $(c_k^\delta)^{-1} := \int_0^\infty |\psi_k^\delta(r)|^2 r^{2\alpha+1} dr$, and $\mathcal{R}_k^\delta(f)$ denotes the Laguerre coefficients of f given by

$$\mathcal{R}_k^\delta(f) = \int_0^\infty f(r)\psi_k^\delta(r) r^{2\delta+1} dr, \; k \geq 0. \tag{2.10}$$

We have the following Chernoff type theorem for L_δ:

Theorem 2.1 *Let $\delta \geq -\frac{1}{2}$ and $f \in L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ be such that $L_\delta^m f \in L^2(\mathbb{R}^+, r^{2\delta+1}dr)$ for all $m \geq 0$, and satisfies the Carleman condition $\sum_{m=1}^\infty \|L_\delta^m f\|_2^{-1/(2m)} = \infty$. If $L_\delta^m f(0) = 0$ for all $m \geq 0$, then f is identically zero.*

For a proof of this result, we refer the reader to Theorem 2.4 and the Remark 2.5 after that in [10].

We end this subsection by recalling the following asymptotic properties of Laguerre functions which are needed in estimating the Fourier transforms of radial functions. We state them here for the general case though we need them only for the Laguerre functions $\varphi_{k,\lambda}^{n-1}$ of type $(n-1)$. Asymptotic properties of $\mathcal{L}_k^\delta(r)$ are well-known in the literature, see [35, Lemma 1.5.3]. The estimates in [35, Lemma 1.5.3] are sharp, see [20, Sect. 2] and [21, Sect. 7]. For our convenience, we restate the result in terms of $\varphi_{k,\lambda}^{n-1}(r)$.

Lemma 2.2 *Let $\nu(k) = 2(2k+n)$ and $C_{k,n} = \left(\frac{k!(n-1)!}{(k+n-1)!}\right)^{\frac{1}{2}}$. For $\lambda \neq 0$, we have the estimates*

$$C_{k,n}\, |\varphi_{k,\lambda}^{n-1}(r)|$$
$$\leq C(r\sqrt{|\lambda|})^{-(n-1)} \begin{cases} (\frac{1}{2}\nu(k)r^2|\lambda|)^{(n-1)/2}, & 0 \leq r \leq \frac{\sqrt{2}}{\sqrt{\nu(k)|\lambda|}} \\ (\frac{1}{2}\nu(k)r^2|\lambda|)^{-\frac{1}{4}}, & \frac{\sqrt{2}}{\sqrt{\nu(k)|\lambda|}} \leq r \leq \frac{\sqrt{\nu(k)}}{\sqrt{|\lambda|}} \\ \nu(k)^{-\frac{1}{4}}(\nu(k)^{\frac{1}{3}} + |\nu(k) - \frac{1}{2}|\lambda|r^2|)^{-\frac{1}{4}}, & \frac{\sqrt{\nu(k)}}{\sqrt{|\lambda|}} \leq r \leq \frac{\sqrt{3\nu(k)}}{\sqrt{|\lambda|}} \\ e^{-\frac{1}{2}\gamma r^2|\lambda|}, & r \geq \frac{\sqrt{3\nu(k)}}{\sqrt{|\lambda|}}, \end{cases}$$

where $\gamma > 0$ is a fixed constant and C is independent of k and λ.

3 An analogue of Chernoff's theorem for the special Hermite operator

Our next aim is to prove Theorem 1.7. For the sake of simplicity, we assume that $\lambda = 1$ and prove the Theorem 1.7 for L. In proving the weaker version of Chernoff's theorem for L, in [10], the authors used twisted spherical means and a Chernoff type theorem for its radial part which is a Laguerre operator of type $(n-1)$. However, in this case, we have to consider Laguerre operators of general type δ as well as the eigenfunction expansion that goes with them, which has already been described at end of the previous section. Furthermore, we will use Hecke-Bochner type identity for special Hermite projections, which requires some preparations. To begin with, closely following the notations of [37, Sect. 5, Chapter 2] we describe bi-graded spherical harmonics on $\mathbb{C}^n$.

Bi-graded spherical harmonics: Let p and q be two non-negative integers. Suppose $\mathcal{P}_{p,q}$ denotes the set of all polynomials in z and $\bar{z}$ of the form

$$P(z) = \sum_{|\alpha|=p,\ |\beta|=q} c_{\alpha,\beta} z^{\alpha} \bar{z}^{\beta}$$

which clearly has the following homogeneity property: $P(\lambda z) = \lambda^p \bar{\lambda}^q P(z),\ \lambda \in \mathbb{C}$. Now, in terms of the vector fields $\frac{\partial}{\partial z_j},\ \frac{\partial}{\partial \bar{z}_j},\ j = 1, 2, .., n$, the Laplacian on $\mathbb{C}^n$ has the form $\Delta_{\mathbb{C}^n} = 4\sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$. In view of this, it can be checked that $\Delta_{\mathbb{C}^n} : \mathcal{P}_{p,q} \to \mathcal{P}_{p-1,q-1}$. We denote the kernel of $\Delta_{\mathbb{C}^n}$ by $\mathcal{H}_{p,q}$. More precisely,

$$\mathcal{H}_{p,q} := \{P \in \mathcal{P}_{p,q} : \Delta_{\mathbb{C}^n} P = 0\},$$

which is called the set of all bi-graded solid harmonics of degree (p, q). We define

$$\mathcal{S}_{p,q} := \{P|_{\mathbb{S}^{2n-1}} : P \in \mathcal{H}_{p,q}\}.$$

The elements of $\mathcal{S}_{p,q}$ are called the bi-graded spherical harmonics of degree (p, q). This turns out be a Hilbert space under the usual inner-product of $L^2(\mathbb{S}^{2n-1})$. Let $d(p, q)$ denote the dimension of this Hilbert space. Now, it is well-known that we can choose an orthonormal basis $\mathcal{B}_{p,q} := \{S^j_{p,q} : 1 \le j \le d(p, q)\}$ for $\mathcal{S}_{p,q}$, for each pair of non-negative integers (p, q) such that $\mathcal{B} := \cup_{p,q\ge 0}\mathcal{B}_{p,q}$ forms an orthonormal basis for $L^2(\mathbb{S}^{2n-1})$. For our purpose, we require the following Hecke-Bochner identity in the context of special Hermite projections.

Theorem 3.1 *Suppose $f \in L^1(\mathbb{C}^n)$ has the form $f = Pg$ where g is radial and $P \in \mathcal{H}_{p,q}$ for some $p, q \ge 0$. Then $f \times \varphi_k^{n-1} = 0$ unless $k \ge p$, in which case*

$$f \times \varphi_k^{n-1}(z) = (2\pi)^{-n} g \times \varphi_{k-p}^{n+p+q-1}(z) P(z),$$

where the twisted convolution on the right hand side is on $\mathbb{C}^{n+p+q}$.

For a proof of this result, we refer the reader to [37, Theorem 2.6.1]. Hecke-Bochner identity for the Weyl transform was first proved by Geller in [13] from which the above theorem can be deduced. In [37] a different proof has been given. Both proofs are long and involved and we refer the reader to these references for details. We are now in a position to prove the Theorem 1.7.

Proof of Theorem 1.7 Let f be as in the statement. The main idea is to reduce the matters to radial case by expanding f in terms of bi-graded spherical harmonics and then use Chernoff's theorem for Laguerre operator of suitable type. The proof will be completed in the following steps.

Step 1:(Reduction of vanishing condition) Suppose f and all its partial derivatives vanish at a point $0 \neq w \in \mathbb{C}^n$. Consider the function g defined by $g = T^1_{-w} f$, which is nothing but the twisted translation of f by $-w$ (See (2.6)). In the following, we will be using standard multi-index notations. Using the product rule of partial derivatives, an easy calculation shows that $\partial^\alpha g(z)$ is equal to

$$\begin{aligned}\partial^\alpha(e^{-\frac{i}{2}\Im(w.\bar{z})} f(z+w)) &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta(e^{-\frac{i}{2}\Im(w.\bar{z})}) \partial^{\alpha-\beta}(f(z+w)) \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (e^{-\frac{i}{2}\Im(w.\bar{z})}) P_\beta(w,\bar{w}) \partial^{\alpha-\beta}(f(z+w)),\end{aligned}$$

where $P_\beta(w, \bar{w})$ is some polynomial in w and $\bar{w}$ whose explicit form is not required for our purpose. Note that for any multi-index α, we have from the equation above

$$\partial^\alpha g(0) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} P_\beta(w,\bar{w}) \partial^{\alpha-\beta} f(w) = 0$$

by the the assumption that $\partial^\alpha f(w) = 0$ for all α. Furthermore, using the twisted translation invariance of L (See (2.7)), it is not hard to see that $\|L^m g\|_2 = \|L^m f\|_2$, whence $\|L^m g\|_2$ also satisfy the Carleman condition. Therefore, if f and all its partial derivatives vanish at any point, we can simply work with a suitable twisted translate of f. So, there is no loss of generality in assuming that f and all its partial derivatives vanish at 0.

Step 2: (Spherical harmonic coefficients of $L^m f$) The spherical harmonic expansion of f reads as

$$f(z) = \sum_{p,q=0}^{\infty} \sum_{j=1}^{d(p,q)} \langle f(r.), S^j_{p,q} \rangle_{L^2(\mathbb{S}^{2n-1})} S^j_{p,q}(\omega), \ z = r\omega.$$

Writing $f^j_{p,q}(r) = r^{-p-q} \langle f(r.), S^j_{p,q} \rangle_{L^2(\mathbb{S}^{2n-1})}$, and $P^j_{p,q}(z) = |z|^{p+q} S^j_{p,q}(\omega)$, we observe from the above that

$$L^m f(z) = \sum_{p,q=0}^{\infty} \sum_{j=1}^{d(p,q)} L^m(f^j_{p,q} P^j_{p,q})(z) = \sum_{p,q=0}^{\infty} \sum_{j=1}^{d(p,q)} L^m F^j_{p,q}(z),$$

where we have written $F^j_{p,q}(z) := f^j_{p,q}(|z|)P^j_{p,q}(z)$. Let us calculate the special Hermite projections of $F^j_{p,q}$. In view of the Theorem 3.1, we see that for $k \geq p$,

$$F^j_{p,q} \times \varphi^{n-1}_k(z) = (2\pi)^{-n} P^j_{p,q}(z)\big(f^j_{p,q} \times \varphi^{n+p+q-1}_{k-p}(z)\big)$$
$$= (2\pi)^{-n} P^j_{p,q}(z)\, \mathcal{R}^{\delta(p,q)}_{k-p}(f^j_{p,q})\, \varphi^{\delta(p,q)}_{k-p}(z)$$

where $\delta(p,q) := n+p+q-1$. In the last equality, we have used the fact that $f^j_{p,q}$ can be thought of as a radial function on $\mathbb{C}^{n+p+q}$, and the notation introduced in (2.10). Therefore, we obtain from the special Hermite expansion (see (2.8)) of $F^j_{p,q}$ that

$$\begin{aligned}
&L^m F^j_{p,q}(z)\\
&= (2\pi)^{-n} \sum_{k=0}^{\infty}(2k+n)^m F^j_{p,q} \times \varphi^{n-1}_k(z)\\
&= (2\pi)^{-2n} P^j_{p,q}(z) \sum_{k=p}^{\infty} (2k+n)^m \mathcal{R}^{\delta(p,q)}_{k-p}(f^j_{p,q})\, \varphi^{\delta(p,q)}_{k-p}(z)\\
&= (2\pi)^{-2n}\ P^j_{p,q}(z) \sum_{k=0}^{\infty}(2k+2p+n)^m \mathcal{R}^{\delta(p,q)}_{k}(f^j_{p,q})\, \varphi^{\delta(p,q)}_{k}(z)\\
&= (2\pi)^{-2n}\ S^j_{p,q}(\omega)\, r^{p+q} \sum_{k=0}^{\infty}(2k+2p+n)^m \mathcal{R}^{\delta(p,q)}_{k}(f^j_{p,q})\, \varphi^{\delta(p,q)}_{k}(r), \quad z = r\omega.
\end{aligned} \tag{3.1}$$

Thus, for a fixed m the spherical harmonic coefficients of $L^m f(r\cdot)$ are given by

$$\begin{aligned}
G^j_{p,q}(r) &:= \langle L^m f(r.), S^j_{p,q}\rangle_{L^2(\mathbb{S}^{2n-1})}\\
&= (2\pi)^{-2n}\ r^{p+q} \sum_{k=0}^{\infty}(2k+2p+n)^m \mathcal{R}^{\delta(p,q)}_{k}(f^j_{p,q})\, \varphi^{\delta(p,q)}_{k}(r)
\end{aligned} \tag{3.2}$$

for any $r > 0$. Now, by using the orthogonality of the Laguerre functions, we get from (3.2) that

$$\begin{aligned}
&\int_0^{\infty} G^j_{p,q}(r)\varphi^{\delta(p,q)}_k(r) r^{2n+p+q-1} dr\\
&\quad= (2\pi)^{-2n}(2k+2p+n)^m \mathcal{R}^{\delta(p,q)}_{k}(f^j_{p,q})\, \|\varphi^{\delta(p,q)}_k\|_2^2\\
&\quad= c_n\, (2k+2p+n)^m \frac{k!(n+p+q-1)!}{(k+n+p+q-1)!}\mathcal{R}^{\delta(p,q)}_{k}(f^j_{p,q}).
\end{aligned} \tag{3.3}$$

Step 3:(Carleman condition) We consider the function $f^j_{p,q}$ for fixed j, p and q. With $\delta(p,q)$ as above, in view of the Plancherel formula (2.9), we see that for any $m \geq 1$,

$$\|L^m_{\delta(p,q)} f^j_{p,q}\|_2^2 = \sum_{k=0}^{\infty}(2k+\delta(p,q)+1)^{2m}\, c_k^{\delta(p,q)}\, |\mathcal{R}_k^{\delta(p,q)}(f^j_{p,q})|^2$$
$$= \sum_{k=0}^{\infty} C(k,m,p,q,n)\left|\int_0^\infty G^j_{p,q}(r)\,\varphi_k^{\delta(p,q)}(r)\, r^{2n+p+q-1}dr\right|^2, \tag{3.4}$$

where we have used (3.3). Here, $C(k,m,p,q,n)$ is given by

$$C(k,m,p,q,n) := \left(\frac{2k+\delta(p,q)+1}{2k+2p+1}\right)^{2m} c_k^{\delta(p,q)} \left(\frac{(k+n+p+q-1)!}{k!(n+p+q-1)!}\right)^2.$$

Now, using the value of $\delta(p,q)$, we see that

$$\frac{2k+\delta(p,q)+1}{2k+2p+1} \le 1+\frac{q}{p} := a_{p,q}.$$

Using this, we have from (3.4) that

$$\|L^m_{\delta(p,q)} f^j_{p,q}\|_2^2 \le a_{p,q}^{2m}\sum_{k=0}^{\infty} c_k^{\delta(p,q)}\left|\int_0^\infty r^{-p-q} G^j_{p,q}(r)\,\psi_k^{\delta(p,q)}(r)\, r^{2n+2p+2q-1}dr\right|^2 \tag{3.5}$$

where we have used the notation that $\psi_k^{\delta(p,q)}(r) = \frac{(k+n+p+q-1)!}{k!(n+p+q-1)!}\varphi_k^{\delta(p,q)}(r)$ (see Sect. 2.4). This allows us to observe that the expression inside the modulus sign on the right hand side is the Laguerre coefficient $\mathcal{R}_k^{\delta(p,q)}(r^{-p-q}G^j_{p,q})$. Therefore, by the Plancherel formula (2.9), we obtain

$$\|L^m_{\delta(p,q)} f^j_{p,q}\|_2^2 \le a_{p,q}^{2m}\int_0^\infty |r^{-p-q}G^j_{p,q}(r)|^2 r^{2n+2p+2q-1}dr$$
$$= a_{p,q}^{2m}\int_0^\infty |G^j_{p,q}(r)|^2\, r^{2n-1}dr.$$

Recalling the definition of $G^j_{p,q}(r)$, we observe that

$$|G^j_{p,q}(r)| = |(L^m f(r.), S^j_{p,q})_{L^2(\mathbb{S}^{2n-1})}| \le \|L^m f(r.)\|_{L^2(\mathbb{S}^{2n-1})}.$$

Using this in the equation above and integrating in polar coordinates, we obtain

$$\|L^m_{\delta(p,q)} f^j_{p,q}\|_2^2 \le a_{p,q}^{2m}\|L^m f\|_2^2. \tag{3.6}$$

Thus, the Carleman condition on $L^m f$ implies the Carleman condition

$$\sum_{m=1}^{\infty} \|L^m_{\delta(p,q)} f^j_{p,q}\|_2^{-1/(2m)} = \infty \tag{3.7}$$

for any spherical harmonic coefficient $f^j_{p,q}$.

Step 4:(Vanishing condition) We have assumed that f and all its partial derivatives vanish at the origin. However, for our purpose, it is more convenient to work with the following equivalent vanishing condition written in terms of polar coordinates:

$$\left(\frac{d}{dr}\right)^m f(r\omega)|_{r=0} = 0, \quad \text{for all } \omega \in \mathbb{S}^{2n-1}, \ m \geq 0. \tag{3.8}$$

Indeed, it can be checked that

$$\left(\frac{d}{dr}\right)^k f(r\omega) = \sum_{|\alpha|=k} \partial^\alpha f(r\omega)\, \omega^\alpha.$$

Hence, $(\frac{d}{dr})^k f(r\omega)|_{r=0} = 0$, for all k if and only if $\partial^\alpha f(0) = 0$, for all α. We recall that $f^j_{p,q}$ is explicitly given by

$$f^j_{p,q}(r) = r^{-p-q} \int_{\mathbb{S}^{2n-1}} f(r\omega) S^j_{p,q}(\omega) d\sigma(\omega).$$

In view of the vanishing condition (3.8), a calculation using repeated application of L'Hospital rule, we verify that all the derivatives of $f^j_{p,q}$ at 0 are zero. Thus, $L^m_{\delta(p,q)} f^j_{p,q}(0) = 0$, for all $m \geq 0$. Hence, by Chernoff's theorem for $L_{\delta(p,q)}$ (See Theorem 2.1), we have $f^j_{p,q} = 0$, for all j, p, q. Therefore, we conclude that $f = 0$, thereby completing the proof. □

4 Ingham's theorem on the Heisenberg group

In this section we prove Theorems 1.3, and 1.5 using Chernoff's theorem for the special Hermite operator. We first show the existence of a compactly supported function f on $\mathbb{H}^n$ whose Fourier transform has a prescribed decay as stated in Theorem 1.3. This proves the sufficiency part of the condition on the function Θ appearing in the hypothesis. We then use this part of the theorem to prove the necessity of the condition on Θ. We begin with some preparations.

4.1 Construction of *F*

The Koranyi norm of $x = (z, t) \in \mathbb{H}^n$, is defined by $|x| = |(z, t)| = (|z|^4 + t^2)^{\frac{1}{4}}$. In what follows, we work with the following left invariant metric defined by $d(x, y) :=$

$|x^{-1}y|,\ x, y \in \mathbb{H}^n$. Given $a \in \mathbb{H}^n$ and $r > 0$, the open ball of radius r with centre at a is defined by

$$B(a, r) := \{x \in \mathbb{H}^n : |a^{-1}x| < r\}.$$

With this definition, we note that if $f, g : \mathbb{H}^n \to \mathbb{C}$ are such that supp$(f) \subset B(0, r_1)$ and supp$(g) \subset B(0, r_2)$, then we have

$$\text{supp}(f * g) \subset B(0, r_1).B(0, r_2) \subset B(0, r_1 + r_2),$$

where $f * g(x) = \int_{\mathbb{H}^n} f(xy^{-1})g(y)dy$ is the convolution of f with g.

Suppose $\{\rho_j\}_j$ and $\{\tau_j\}_j$ are two sequences of positive real numbers such that both the series $\sum_{j=1}^{\infty} \rho_j$ and $\sum_{j=1}^{\infty} \tau_j$ are convergent. We let $B_{\mathbb{C}^n}(0, r)$ stand for the ball of radius r centered at 0 in $\mathbb{C}^n$ and let χ_S denote the characteristic function of a set S. For each $j \in \mathbb{N}$, we define functions f_j on $\mathbb{C}^n$ and g_j on $\mathbb{R}$ by

$$\begin{aligned} f_j(z) &:= \rho_j^{-2n} \chi_{B_{\mathbb{C}^n}(0, a\rho_j)}(z), \quad z \in \mathbb{C}^n; \\ g_j(t) &:= \tau_j^{-2} \chi_{[-\tau_j^2/2, \tau_j^2/2]}(t),\ t \in \mathbb{R}, \end{aligned} \tag{4.1}$$

where the positive constant a is chosen so that $\|f_j\|_{L^1(\mathbb{C}^n)} = 1$. We now consider the functions $F_j : \mathbb{H}^n \to \mathbb{C}$ defined by

$$F_j(z, t) := f_j(z)g_j(t),\ (z, t) \in \mathbb{H}^n.$$

In the following lemma, we record some useful, but easily proven properties of these functions.

Lemma 4.1 *Let F_j be as above and define $G_N = F_1 * F_2 * ..... * F_N$. Then we have*

(1) $\|F_j\|_{L^\infty(\mathbb{H}^n)} \le \rho_j^{-2n} \tau_j^{-2}, \quad \|F_j\|_{L^1(\mathbb{H}^n)} = 1,$
(2) *supp*$(F_j) \subset B_{\mathbb{C}^n}(0, a\rho_j) \times [-\tau_j^2/2, \tau_j^2/2] \subset B(0, a\rho_j + c\tau_j)$*, where* $4c^4 = 1$.
(3) For any $N \in \mathbb{N}$*, supp*$(G_N) \subset B(0, a \sum_{j=1}^N \rho_j + c \sum_{j=1}^N \tau_j),\ \|G_N\|_1 = 1.$
(4) Given $x \in \mathbb{H}^n$*, and* $N \in \mathbb{N}$*,* $F_2 * F_3 ..... * F_N(x) \le \rho_2^{-2n} \tau_2^{-2}$.

We also recall a result about Hausdörff measure which will be used in the proof of the next theorem. Let $\mathcal{H}^n(A)$ denote the n-dimensional Hausdorff measure of $A \subset \mathbb{R}^n$. Hausdörff measure coincides with the Lebesgue measure for Lebesgue measurable sets. For sets in $\mathbb{R}^n$ with sufficiently nice boundaries, the $(n-1)$-dimensional Hausdorff measure is same as the surface measure. For more about this, we refer the reader to [33, Chapter 7]. Let $A \Delta B$ stand for the symmetric difference between any two sets A and B. See [31] for a proof of the following theorem.

Theorem 4.2 *Let $A \subset \mathbb{R}^n$ be a bounded set. Then for any $\xi \in \mathbb{R}^n$,*

$$\mathcal{H}^n(A \Delta (A + \xi)) \le |\xi| \mathcal{H}^{n-1}(\partial A),$$

where $A + \xi$ is the translation of A by ξ and ∂A is the boundary of A.

Theorem 4.3 *The sequence defined by* $G_k = F_1 * F_2 * ..... * F_k$ *converges in* $L^2(\mathbb{H}^n)$, *as well as in* $L^1(\mathbb{H}^n)$, *to a compactly supported non-trivial function* F.

Proof In order show that $\{G_k\}$ is Cauchy in $L^2(\mathbb{H}^n)$, we first estimate $\|G_{k+1} - G_k\|_{L^\infty(\mathbb{H}^n)}$. As all the functions F_j have unit L^1 norm, we have for any $x \in \mathbb{H}^n$

$$\begin{aligned} G_{k+1}(x) - G_k(x) &= \int_{\mathbb{H}^n} G_k(xy^{-1}) F_{k+1}(y)dy - G_k(x) \int_{\mathbb{H}^n} F_{k+1}(y)dy \\ &= \int_{\mathbb{H}^n} \Big(G_k(xy^{-1}) - G_k(x) \Big) F_{k+1}(y)dy. \end{aligned}$$

Since F_j's are even, we can change y into y^{-1} in the above and estimate the same as

$$|G_{k+1}(x) - G_k(x)| \leq \int_{\mathbb{H}^n} |G_k(xy) - G_k(x)| \, F_{k+1}(y)dy. \tag{4.2}$$

By defining $H_{k-1} = F_2 * F_3 ...... * F_k$, we note that $G_k = F_1 * H_{k-1}$. Thus,

$$G_k(xy) - G_k(y) = \int_{\mathbb{H}^n} \Big(F_1(xyu^{-1}) - F_1(xu^{-1}) \Big) H_{k-1}(u)du.$$

Using the estimate (4) in Lemma 4.1, we get that

$$|G_k(xy) - G_k(x)| \leq \rho_2^{-2n} \tau_2^{-2} \int_{\mathbb{H}^n} \Big| F_1(xyu^{-1}) - F_1(xu^{-1}) \Big| \, du. \tag{4.3}$$

The change of variables $u \to ux$ transforms the integral in the right hand side of the inequality above into

$$\int_{\mathbb{H}^n} \Big| F_1(xyu^{-1}) - F_1(xu^{-1}) \Big| \, du = \int_{\mathbb{H}^n} \Big| F_1(xyx^{-1}u^{-1}) - F_1(u^{-1}) \Big| \, du.$$

Since the group $\mathbb{H}^n$ is unimodular, another change of variables $u \to u^{-1}$ yields

$$\int_{\mathbb{H}^n} \Big| F_1(xyx^{-1}u^{-1}) - F_1(u^{-1}) \Big| \, du = \int_{\mathbb{H}^n} \Big| F_1(xyx^{-1}u) - F_1(u) \Big| \, du.$$

Let $x = (z, t) = (z, 0)(0, t)$, $y = (w, s) = (w, 0)(0, s)$. As $(0, t)$ and $(0, s)$ belong to the center of $\mathbb{H}^n$, an easy calculation shows that $xyx^{-1} = (w, 0)(0, s + \Im(z \cdot \bar{w}))$. With $u = (\zeta, \tau)$ we have

$$xyx^{-1}u = (w + \zeta, 0)(0, \tau + s + \Im(z \cdot \bar{w}) - (1/2)\Im(\zeta \cdot \bar{w})).$$

Since $F_1(z, t) = f_1(z)g_1(t)$, we see that the integrand $F_1(xyx^{-1}u) - F_1(u)$ in the above integral takes the form

$$f_1(w + \zeta)g_1(\tau + s + \Im(z \cdot \bar{w}) - (1/2)\Im(\zeta \cdot \bar{w})) - f_1(\zeta)g_1(\tau).$$

By setting $b = b(s, z, w, \zeta) = s + \Im(z \cdot \bar{w}) - (1/2)\Im(\zeta \cdot \bar{w})$, we can rewrite the above as

$$\big(f_1(w + \zeta) - f_1(\zeta)\big)g_1(\tau + b) + f_1(\zeta)\big(g_1(\tau + b) - g_1(\tau)\big). \tag{4.4}$$

In order to estimate the contribution of the second term in (4.4) to the integral under consideration, we first estimate the integral in τ-variable as follows:

$$\int_{-\infty}^{\infty} |g_1(\tau + b) - g_1(\tau)|d\tau = \tau_1^{-2}|(-b + K_\tau)\Delta K_\tau|,$$

where $K_\tau = [-\frac{1}{2}\tau_1^2, \frac{1}{2}\tau^2]$ is the support of g_1. For ζ in the support of f_1, we have $|\zeta| \le a\rho_1$, and hence

$$|(-b + K_\tau)\Delta K_\tau| \le 2|b(s, z, w, \zeta)| \le (2|s| + |z||w| + a\rho_1|w|).$$

Thus, we have proved the following estimate

$$\int_{\mathbb{H}^n} f_1(\zeta)|g_1(\tau + b) - g_1(\tau)|d\zeta d\tau \le C\big(2|s| + (a\rho_1 + |z|)|w|\big). \tag{4.5}$$

As the integral of g_1 is one, the contribution of the first term in (4.4) is given by

$$\int_{\mathbb{C}^n} |f_1(w + \zeta) - f_1(\zeta)|d\zeta = \rho_1^{-2n}\mathcal{H}^{2n}\left((-w + B_{\mathbb{C}^n}(0, a\rho_1))\Delta B_{\mathbb{C}^n}(0, a\rho_1)\right).$$

By appealing to Theorem 4.2 in estimating the above, we obtain

$$\int_{\mathbb{H}^n} |f_1(w + \zeta) - f_1(\zeta)|\, g(\tau + b)\, d\zeta d\tau \le C|w|. \tag{4.6}$$

Using the estimates (4.5) and (4.6) in (4.3) we obtain

$$|G_k(xy) - G_k(x)| \le C\rho_2^{-2n}\tau_2^{-2}\big(|s| + (c_1 + c_2|z|)|w|\big).$$

This estimate, when used in (4.2), in turn gives us

$$|G_{k+1}(z, t) - G_k(z, t)| \le C\int_{\mathbb{H}^n} \big(|s| + (c_1 + c_2|z|)|w|\big)F_{k+1}(w, s)\, dw\, ds \tag{4.7}$$

where the constants c_1, c_2 and C depend only on n. Recalling that on the support of $F_{k+1}(w, s) = f_{k+1}(w)g_{k+1}(s)$, $|w| \le \rho_{k+1}$ and $|s| \le \tau_{k+1}^2$, the above yields the estimate

$$|G_{k+1}(z, t) - G_k(z, t)| \le C\big(\tau_{k+1}^2 + (c_1 + c_2|z|)\rho_{k+1}\big). \tag{4.8}$$

It is easily seen that the support of $G_{k+1} - G_k$ is contained in $B(0, a\rho + c\tau)$ where $\rho = \sum_{j=1}^{\infty}\rho_j$ and $\tau = \sum_{\tau_j}$. Consequently, from the above we conclude that

$$\|G_{k+1} - G_k\|_2 \le \|G_{k+1} - G_k\|_\infty\big(|B(0, a\rho + c\tau)|\big)^{1/2} \le C\big(\tau_{k+1}^2 + c_3\rho_{k+1}\big).$$

From the above, it is clear that $\{G_k\}$ is Cauchy in $L^2(\mathbb{H}^n)$, and hence converges to a function $F \in L^2(\mathbb{H}^n)$ whose support is contained in $B(0, a\rho + c\tau)$. The same argument shows that $\{G_k\}$ converges to F in L^1. As $\|G_k\|_1 = 1$ for any k, it follows that $\|F||_1 = 1$ and hence F is nontrivial. □

4.2 Estimating the Fourier transform of *F*

Suppose now that Θ is an even, decreasing function on $\mathbb{R}$ for which $\int_1^\infty \Theta(t)t^{-1}dt < \infty$. We want to choose two sequences of positive real numbers $\{\rho_j\}$ and $\{\tau_j\}$ in terms of Θ so that the series $\sum_{j=1}^\infty \rho_j$ and $\sum_{j=1}^\infty \tau_j$ both converge. We can then construct a function F as in Theorem 4.3 which will be compactly supported. Having done the construction we now want to compute the Fourier transform of the constructed function F and compare it with $e^{-\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}$. This can be achieved by a judicious choice of the sequences $\{\rho_j\}$ and $\{\tau_j\}$. As Θ is given to be decreasing, it follows that $\sum_{j=1}^\infty \frac{\Theta(j)}{j} < \infty$. It is then possible to choose a decreasing sequence $\{\rho_j\}$ such that $\rho_j \geq c_n^2 e^2 \frac{\Theta(j)}{j}$ (for a constant c_n to be chosen later) and $\sum_{j=1}^\infty \rho_j < \infty$. Similarly, we choose another decreasing sequence $\{\tau_j\}$ such that $\sum_{j=1}^\infty \tau_j < \infty$.

In the proof of the following lemma we require good estimates for the Laguerre coefficients of the function $f_j(z) = \rho_j^{-2n}\chi_{B_{\mathbb{C}^n}(0,a\rho_j)}(z)$ where a chosen so that $\|f_j\|_1 = 1$. These coefficients are defined by

$$R_k^{n-1}(\lambda, f_j) = \frac{k!(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n} f_j(z)\varphi_{k,\lambda}^{n-1}(z)dz. \tag{4.9}$$

Lemma 4.4 *There exists a constant* $c_n > 0$ *such that*

$$|R_k^{n-1}(\lambda, f_j)| \leq c_n\big(\rho_j\sqrt{(2k+n)|\lambda|}\big)^{-n+1/2}.$$

Proof By abuse of notation we write $\varphi_{k,\lambda}^{n-1}(r)$ in place of $\varphi_{k,\lambda}^{n-1}(z)$ when $|z| = r$. As f_j is defined as the dilation of a radial function, the Laguerre coefficients are given by the integral

$$R_k^{n-1}(\lambda, f_j) = \frac{2\pi^n}{\Gamma(n)}\frac{k!(n-1)!}{(k+n-1)!}\int_0^a \varphi_{k,\lambda}^{n-1}(\rho_j r)r^{2n-1}dr. \tag{4.10}$$

When $a \leq (\rho_j\sqrt{(2k+n)|\lambda|})^{-1}$, we use the bound $\frac{k!(n-1)!}{(k+n-1)!}|\varphi_{k,\lambda}^{n-1}(r)| \leq 1$ (See [35]) to estimate

$$\frac{2\pi^n}{\Gamma(n)}\frac{k!(n-1)!}{(k+n-1)!}\int_0^a \varphi_{k,\lambda}^{n-1}(\rho_j r)r^{2n-1}dr \leq \frac{\pi^n a^{n+1/2}}{\Gamma(n+1)}\big(\rho_j\sqrt{(2k+n)|\lambda|}\big)^{-n+1/2}.$$

When $a > (\rho_j\sqrt{(2k+n)|\lambda|})^{-1}$, we split the integral into two parts, one of which gives the same estimate as above. To estimate the integral taken over $(\rho_j\sqrt{(2k+n)|\lambda|})^{-1} <$

$r < a$, we use the bound stated in Lemma 2.2 which leads to the estimate

$$\frac{2\pi^n}{\Gamma(n)} \frac{k!(n-1)!}{(k+n-1)!} \int_{(\rho_j\sqrt{(2k+n)|\lambda|})^{-1}}^{a} \varphi_{k,\lambda}^{n-1}(\rho_j r) r^{2n-1} dr$$
$$\leq C_n\big(\rho_j\sqrt{(2k+n)|\lambda|}\big)^{-n+1/2} \int_0^a r^{n-1/2} dr = C_n' a^{n+1/2}\big(\rho_j\sqrt{(2k+n)|\lambda|}\big)^{-n+1/2}.$$

Combining the two estimates we get the lemma. □

Theorem 4.5 *Let* $\Theta : \mathbb{R} \to [0, \infty)$ *be an even, decreasing function with* $\lim_{\lambda\to\infty} \Theta(\lambda) = 0$ *for which* $\int_1^\infty \frac{\Theta(\lambda)}{\lambda} d\lambda < \infty$*. Let* ρ_j *and* τ_j *be chosen as above. Then the Fourier transform of the function* F *constructed in Theorem 4.3 satisfies the estimate*

$$\widehat{F}(\lambda)^* \widehat{F}(\lambda) \leq e^{-2\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}, \ \lambda \neq 0.$$

Proof Observe that F is radial since each F_j is radial and hence the Fourier transform $\widehat{F}(\lambda)$ is a function of the Hermite opertaor $H(\lambda)$. More precisely,

$$\widehat{F}(\lambda) = \sum_{k=0}^{\infty} R_k^{n-1}(\lambda, F) P_k(\lambda) \tag{4.11}$$

where the Laguerre coefficients are explicitly given by (see (2.4.7) in [37]. There is a typo- the factor $|\lambda|^{n/2}$ should not be there)

$$R_k^{n-1}(\lambda, F) = \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} F^\lambda(z) \varphi_{k,\lambda}^{n-1}(z) dz.$$

In the above, $F^\lambda(z)$ stands for the inverse Fourier transform of $F(z, t)$ in the t variable. Expanding any $\varphi \in L^2(\mathbb{R}^n)$ in terms of Φ_α^λ it is easy to see that the conclusion $\widehat{F}(\lambda)^* \widehat{F}(\lambda) \leq e^{-2\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}$ follows once we show that

$$(R_k^{n-1}(\lambda, F))^2 \leq C e^{-2\Theta(\sqrt{(2k+n)|\lambda|})\sqrt{(2k+n)|\lambda|}}$$

for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}^*$. Now note that, by definitions of f_j and g_j and the choice of a, (see (4.1)) we have

$$|\widehat{g}_j(\lambda)| = \left| \frac{\sin(\frac{1}{2}\tau_j^2\lambda)}{\frac{1}{2}\tau_j^2\lambda} \right| \leq 1, \quad |R_k^{n-1}(\lambda, f_j)| \leq 1.$$

The bound on $R_k^{n-1}(\lambda, f_j)$ follows from the fact that $|\varphi_{k,\lambda}^{n-1}(z)| \leq \frac{(k+n-1)!}{k!(n-1)!}$. This shows that for any j we also have $|R_k^{n-1}(\lambda, F_j)| \leq 1$.

In estimating $R_k^{n-1}(\lambda, F)$ we make use of the following facts about the map $f \to R_k^{n-1}(\lambda, f)$ where $f \in L^1(\mathbb{H}^n)$ is radial. From the definition of $R_k^{n-1}(\lambda, f)$ and the uniform estimate for $\varphi_{k,\lambda}^{n-1}(z)$ it follows that

$$|R_k^{n-1}(\lambda, f)| \leq \int_{\mathbb{C}^n} |f^\lambda(z)| dz \leq \|f\|_1.$$

Since F is constructed as the L^1 limit of the N-fold convolution $G_N = F_1 * F_2 ...... * F_N$ we observe that $R_k^{n-1}(\lambda, F)$ is the limit of $R_k^{n-1}(\lambda, G_N)$ as N tends to infinity. Moreover, in view of the fact that $R_k^{n-1}(\lambda, f * g) = R_k^{n-1}(\lambda, f) R_k^{n-1}(\lambda, g)$ (See (2.5)) and the bounds $|R_k^{n-1}(\lambda, F_j)| \leq 1$ we see that

$$(R_k^{n-1}(\lambda, G_{N+1}))^2 = (\Pi_{j=1}^{N+1} R_k^{n-1}(\lambda, F_j))^2 \leq R_k^{n-1}(\lambda, G_N))^2$$

and hence $(R_k^{n-1}(\lambda, G_N))^2$ decreases to $(R_k^{n-1}(\lambda, F)^2$. Thus, for any N

$$(R_k^{n-1}(\lambda, F))^2 \leq (R_k^{n-1}(\lambda, G_N))^2 = (\Pi_{j=1}^{N} R_k^{n-1}(\lambda, F_j))^2.$$

Therefore, it is enough to show that for a given k and λ one can choose $N = N(k, \lambda)$ in such a way that

$$(\Pi_{j=1}^{N} R_k^{n-1}(\lambda, F_j))^2 \leq C e^{-2\Theta(\sqrt{(2k+n)|\lambda|})\sqrt{(2k+n)|\lambda|}}. \tag{4.12}$$

where C is independent of N. From the definition of G_N it follows that

$$\widehat{G_N}(\lambda) = \Pi_{j=1}^{N} \widehat{F_j}(\lambda) = \Pi_{j=1}^{N} \left(\sum_{k=0}^{\infty} R_k^{n-1}(\lambda, F_j) P_k(\lambda) \right)$$

and hence $R_k^{n-1}(\lambda, G_N) = \Pi_{j=1}^{N} R_k^{n-1}(\lambda, F_j)$. As $F_j(z, t) = f_j(z) g_j(t)$, we have

$$R_k^{n-1}(\lambda, G_N) = \big(\Pi_{j=1}^{N} \widehat{g_j}(\lambda)\big)\big(\Pi_{j=1}^{N} R_k^{n-1}(\lambda, f_j)\big).$$

As the first factor is bounded by one, it is enough to consider the product $\Pi_{j=1}^{N} R_k^{n-1}(\lambda, f_j)$.

We now choose ρ_j satisfying $\rho_j \geq c_n^2 e^{2\frac{\Theta(j)}{j}}$, where c_n is the same constant appearing in Lemma 4.4. We then take $N = \lfloor \Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}} \rfloor$, and consider

$$\Pi_{j=1}^{N} R_k^{n-1}(\lambda, f_j) \leq \Pi_{j=1}^{N} c_n (\rho_j \sqrt{(2k+n)|\lambda|})^{-n+1/2}$$

where we have used the estimates proved in Lemma 4.4. As $\{\rho_j\}$ is decreasing

$$\Pi_{j=1}^{N} c_n (\rho_j \sqrt{(2k+n)|\lambda|})^{-n+1/2} \leq c_n^N \big(\rho_N \sqrt{(2k+n)|\lambda|}\big)^{-(n-1/2)N}. \tag{4.13}$$

By the choice of ρ_j, it follows that

$$\rho_N^2(2k+n)|\lambda| \geq c_n^4 e^4 \frac{\Theta(N)^2}{N^2}(2k+n)|\lambda|.$$

As Θ is decreasing and $N \leq \sqrt{(2k+n)|\lambda|}$, we have $\Theta(N) \geq \Theta(\sqrt{(2k+n)|\lambda|})$ and so

$$\Theta(N)^2(2k+n)|\lambda| \geq \Theta\big(\sqrt{(2k+n)|\lambda|}\big)^2(2k+n)|\lambda| \geq N^2$$

which proves that $\rho_N^2(2k+n)|\lambda| \geq c_n^4 e^4$. Using this in (4.13) we obtain

$$\Pi_{j=1}^N c_n\big(\rho_j\sqrt{(2k+n)|\lambda|}\big)^{-n+1/2} \leq (c_n^2e^2)^{-(n-1)N}e^{-N}.$$

Finally, as $N+1 \geq \Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}}$, we obtain the estimate (4.12). □

4.3 Ingham's theorem

We can now complete the proofs of Theorems 1.3, and 1.5. Since half of the theorem has been already proved, as already mentioned in Sect. 1, we only need to prove the Theorem 1.5.

Proof of Theorem 1.5 Fix $\lambda \neq 0$. By the hypothesis, f^λ vanishes on an open set U_λ in $\mathbb{C}^n$. First we assume that $\Theta(\lambda) \geq c\,|\lambda|^{-\frac{1}{2}},\ |\lambda| \geq 1$. In view of Plancherel formula (2.1) for the Weyl transform, we have

$$(2\pi)^n\|L_\lambda^m f^\lambda\|_2^2 = |\lambda|^n\|W_\lambda(L_\lambda^m f^\lambda)\|_{HS}^2 = |\lambda|^n\|\widehat{f}(\lambda)H(\lambda)^m\|_{HS}^2.$$

Using the formula for Hilbert-Schmidt norm of an operator we have

$$(2\pi)^n\|L_\lambda^m f^\lambda\|_2^2 = |\lambda|^n\sum_\alpha((2|\alpha|+n)|\lambda|)^{2m}\|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2.$$

Now, the given condition on the Fourier transform leads to the estimate

$$\begin{aligned}(2\pi)^n\|L_\lambda^m f^\lambda\|_2^2 &\leq C|\lambda|^n\sum_\alpha((2|\alpha|+n)|\lambda|)^{2m}e^{-2\Theta(((2|\alpha|+n)|\lambda|)^{\frac{1}{2}})((2|\alpha|+n)|\lambda|)^{\frac{1}{2}}}\\ &\leq C|\lambda|\sum_{k=0}^\infty((2k+n)|\lambda|)^{2m+n-1}e^{-2\Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}}}.\end{aligned} \tag{4.14}$$

We write the last sum as $I_1 + I_2$, where

$$I_1 := \sum_{k\geq 0,\,(2k+n)|\lambda|\leq m^8} ((2k+n)|\lambda|)^{2m+n-1} e^{-2\Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}}},$$

and

$$I_2 := \sum_{k\geq 0,\,(2k+n)|\lambda|> m^8} ((2k+n)|\lambda|)^{2m+n-1} e^{-2\Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}}}.$$

Now, we estimate each sum separately. Notice that when $(2k+n)|\lambda| \leq m^8$, we have $\Theta(((2k+n)|\lambda|)^{\frac{1}{2}}) \geq \Theta(m^4)$, as Θ is decreasing. This shows that

$$I_1 \leq \sum_{k\geq 0,\,(2k+n)|\lambda|\leq m^8} ((2k+n)|\lambda|)^{2m+n-1} e^{-2\Theta(m^4)((2k+n)|\lambda|)^{\frac{1}{2}}}$$

which can be dominated by

$$\sum_{k\geq 0,\,(2k+n)|\lambda|\leq m^8} ((2k+n)|\lambda|)^{n} \int_{(2k+n)|\lambda|}^{(\sqrt{(2k+n)|\lambda|}+1)^2} x^{2m-1} e^{-2\Theta(m^4)(\sqrt{x}-1)} dx$$
$$\leq e^{2\Theta(m^4)} m^{8n} \int_0^\infty x^{2m-1} e^{-2\Theta(m^4)\sqrt{x}} dx.$$

The change of variable $y = 2\Theta(m^4)\sqrt{x}$ transform the last expression into

$$e^{2\Theta(m^4)} \frac{m^{8n}}{(2\Theta(m^4))^{4m}} \int_0^\infty y^{4m-1} e^{-y} dy = e^{2\Theta(m^4)} \frac{m^{8n}}{(2\Theta(m^4))^{4m}} \Gamma(4m).$$

This along with the fact that $\Theta(m^4) \leq \Theta(1)$ shows that

$$I_1 \leq C \frac{m^{8n}}{(2\Theta(m^4))^{4m}} \Gamma(4m).$$

Using Stirling's formula (see Ahlfors [1]) $\Gamma(x) = \sqrt{2\pi}\, x^{x-1/2}\, e^{-x} e^{\theta(x)/12x}$, $0 < \theta(x) < 1$, which is valid for $x > 0$, for large m, we observe that

$$I_1 \leq C \left(\frac{2m}{\Theta(m^4)}\right)^{4m}. \tag{4.15}$$

Now, to estimate I_2, we make use of the initial assumption that $\Theta(t) \geq c\,t^{-1/2}$ for $t \geq 1$. Following the same procedure as above, we observe that I_2 is dominated by

$$\sum_{k\geq 0,(2k+n)|\lambda|>m^8} \int_{(2k+n)|\lambda|}^{((2k+n)|\lambda|)^2} x^{2m+n-1} e^{-2c\sqrt{x}} dx$$
$$\leq e^{-c\,m^4} \sum_{k\geq 0,(2k+n)|\lambda|>m^8} \int_{(2k+n)|\lambda|}^{((2k+n)|\lambda|)^2} x^{2m+n-1} e^{-c\sqrt{x}} dx$$
$$= e^{-c\,m^4} \int_0^\infty x^{2m+n-1} e^{-2c\sqrt{x}} dx.$$

Again the change of variables $y = c\sqrt{x}$ transforms the above integral into

$$2c^{-(4m+2n-2)} \int_0^\infty y^{4m+2n-1} e^{-y} dy = 2c^{-(4m+2n-2)} \Gamma(4m + 2n).$$

Hence, we obtain

$$I_2 \leq 2c^{-(4m+2n-2)} \Gamma(4m + 2n) e^{-cm^4}.$$

Now, for large m, using the fact that $\Gamma(4m+2n) \leq \Gamma(5m)$, and Stirling's formula we have

$$I_2 \leq C(c^{-4}(5m)^5 e^{-cm^3})^m.$$

But the right hand side of above goes to zero as $m \to \infty$. Hence, in view of (4.15), for large m, we conclude that

$$I_1 + I_2 \leq C \left(\frac{2m}{\Theta(m^4)}\right)^{4m} \tag{4.16}$$

which from (4.14) yields for large m

$$(2\pi)^n \|L_\lambda^m f^\lambda\|_2^2 \leq C|\lambda| \left(\frac{2m}{\Theta(m^4)}\right)^{4m}.$$

The hypothesis on Θ, namely $\int_1^\infty \frac{\Theta(t)}{t} dt = \infty$, implies that $\int_1^\infty \frac{\Theta(y^4)}{y} dy = \infty$. Hence, by integral test we get $\sum_{m=1}^\infty \frac{\Theta(m^4)}{m} = \infty$. Therefore, it follows that

$$\sum_{m=1}^\infty \|L_\lambda^m f^\lambda\|_2^{-\frac{1}{2m}} = \infty.$$

Since f^λ vanishes on an open set, by the Theorem 1.7 (analogue of Chernoff's theorem for L_λ) we conclude that $f^\lambda = 0$ which is true for all $\lambda \neq 0$. Hence $f = 0$.

Now, we consider the general case. In other words, we now proceed to remove the assumption that $\Theta(t) \geq ct^{-1/2}$ for $t \geq 1$. Notice that the function $\Psi(y) = (1+|y|)^{-1/2}$ satisfies $\int_1^\infty \frac{\Psi(y)}{y} dy < \infty$. By Theorem 4.3 we can construct a compactly supported radial function $F \in L^2(\mathbb{H}^n)$ such that

$$\hat{F}(\lambda)^* \hat{F}(\lambda) \leq e^{-2\Psi(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}, \ \lambda \neq 0.$$

We can further arrange that $\text{supp}(F) \subset B_{\mathbb{C}^n}(0,\delta) \times (-a,a)$ for some $\delta, a > 0$. We now consider the function $h = f * F$. Notice that

$$h^\lambda(z) = (f * F)^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z-w) F^\lambda(w) e^{\frac{i\lambda}{2}\Im(z.\bar{w})} dw.$$

As f^λ is assumed to vanish on U_λ, the function h^λ vanishes on a smaller open set $U_{\lambda,\delta} \subset U_\lambda$. We now claim that

$$\widehat{h}(\lambda)^* \widehat{h}(\lambda) \leq e^{-2\Phi(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}, \tag{4.17}$$

where $\Phi(y) = \Theta(y) + \Psi(y)$. As $\widehat{h}(\lambda) = \widehat{f}(\lambda)\widehat{F}(\lambda)$, for any $\varphi \in L^2(\mathbb{R}^n)$ we have

$$\langle \widehat{h}(\lambda)^* \widehat{h}(\lambda)\varphi, \varphi\rangle = \langle \widehat{f}(\lambda)^* \widehat{f}(\lambda)\widehat{F}(\lambda)\varphi, \widehat{F}(\lambda)\varphi\rangle.$$

The hypothesis on f gives us the estimate

$$\langle \widehat{f}(\lambda)^* \widehat{f}(\lambda)\widehat{F}(\lambda)\varphi, \widehat{F}(\lambda)\varphi\rangle \leq C\langle e^{-2\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}} \widehat{F}(\lambda)\varphi, \widehat{F}(\lambda)\varphi\rangle.$$

As F is radial, $\widehat{F}(\lambda)$ commutes with any function of $H(\lambda)$ and hence the right hand side can be estimated using the decay of $\widehat{F}(\lambda)$:

$$\langle \widehat{F}(\lambda)^* \widehat{F}(\lambda) e^{-\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}\varphi, e^{-\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}\varphi\rangle \leq C\langle e^{-2(\Theta+\Psi)(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}\varphi, \varphi\rangle.$$

This proves our claim (4.17) on $\widehat{h}(\lambda)$ with $\Phi = \Theta + \Psi$. As $\Phi(y) \geq |y|^{-1/2}$, by the already proved part of the theorem we conclude that $h = 0$. In order to conclude that $f = 0$ we proceed as follows.

Given F as above, let us consider $\delta_r F(z,t) = F(rz, r^2 t)$. It has been shown (see (2.2)) that

$$\widehat{\delta_r F}(\lambda) = r^{-(2n+2)} d_r \circ \widehat{F}(r^{-2}\lambda) \circ d_r^{-1}$$

where recall that d_r is the standard dilation on $\mathbb{R}^n$ given by $d_r\varphi(x) = \varphi(rx)$. The property of the function F, namely $\hat{F}(\lambda)^*\hat{F}(\lambda) \leq e^{-2\Psi(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}$ gives us

$$\widehat{\delta_r F}(\lambda)^*\widehat{\delta_r F}(\lambda) \leq Cr^{-2(2n+2)} d_r \circ e^{-2\Psi\left(\sqrt{H(\lambda/r^2)}\right)\sqrt{H(\lambda/r^2)}} \circ d_r^{-1}.$$

Testing against Φ_α^λ we can simplify the right hand side which gives us

$$\widehat{\delta_r F}(\lambda)^*\widehat{\delta_r F}(\lambda) \leq Cr^{-2(2n+2)} e^{-2\Psi_r(\sqrt{H(\lambda)})\sqrt{H(\lambda)}},$$

where $\Psi_r(y) = \frac{1}{r}\Psi(y/r)$. If we let $F_\varepsilon(x) = \varepsilon^{-(2n+2)}\delta_{\varepsilon^{-1}}F(x)$, then it follows that F_ε is an approximate identity. Moreover, F_ε is compactly supported and satisfies the same hypothesis as F with $\Psi(y)$ replaced by $\varepsilon\Psi(\varepsilon y)$ which has the same integrability and decay conditions. Hence, working with F_ε we can conclude that $f * F_\varepsilon = 0$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ and noting that $f * F_\varepsilon$ converges to f in $L^1(\mathbb{H}^n)$, we conclude that $f = 0$. This completes the proof. □

Remark 4.1 It would be interesting to see whether the conclusion of the Theorem 1.5 still holds true under the assumption that the function vanishes on a non-empty open subset of $\mathbb{H}^n$. A moment's thought staring at the above proof reveals that this can be achieved if we use an analogue of the Theorem 1.6 for the sublaplacian instead of special Hermite operators. But it turns out that proving an analogue of Theorem 1.6 is a very interesting and difficult open problem. We hope to revisit this in the near future.

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References

1. Ahlfors, Lars V.: Complex Analysis: An introduction to the theory of analytic functions of one complex variable. 3rd Edition, McGraw-Hill, Inc
2. Bhowmik, M., Ray, S.K., Sen, S.: Around theorems of Ingham-type regarding decay of Fourier transform on $\mathbb{R}^n$, $\mathbb{T}^n$ and two step nilpotent Lie Groups. Bull. Sci. Math **155**, 33–73 (2019)
3. Bhowmik, M., Pusti, S., Ray, S.K.: Theorems of Ingham and Chernoff on Riemannian symmetric spaces of noncompact type. Journal of Functional Analysis, **279**(11), (2020)
4. Bhowmik, M., Sen, S.: Uncertainty principles of Ingham and Paley-wiener on semisimple Lie groups. Israel J. Math. **225**, 193–221 (2018)
5. Bochner, S., Taylor, A.E.: Some theorems on quasi-analyticity for functions of several variables. Amer. J. Math. **61**(2), 303–329 (1939)
6. Bochner, S.: Quasi-analytic functions. Laplace operator, positive kernels Ann. of Math. (2) **51**, 68–91 (1950). (**MR0032708 (11,334g)**)
7. Chernoff, P.R.: Some remarks on quasi-analytic vectors. Trans. Amer. Math. Soc. **167**, 105–113 (1972)
8. Chernoff, P.R.: Quasi-analytic vectors and quasi-analytic functions. Bull. Amer. Math. Soc. **81**, 637–646 (1975)

9. Folland, G.B.: Harmonic Analysis in Phase Space, Ann. Math. Stud., vol. 122. Princeton University Press, Princeton, N.J (1989)
10. Ganguly, P., Thangavelu, S.: Theorems of Chernoff and Ingham for certain eigenfunction expansions. Adv. Math., **386**, (2021). https://doi.org/10.1016/j.aim.2021.107815
11. Ganguly, P., Thangavelu, S.: Analogues of theorems of Chernoff and Ingham on the Heisenberg group. Accepted for publication in J. Anal. Math. (2021). arXiv:2106.02704
12. Ganguly, P., Manna, R., Thangavelu, S.: On a theorem of Chernoff on rank one Riemannian symmetric spaces. J. Funct. Anal. **282**(5), 109351 (2022)
13. Geller, D.: Spherical harmonics, the Weyl transform and the Fourier transform on the heisenberg group. Canad. J. Math. **36**(4), 625–684 (1984)
14. Ingham, A.E.: A Note on Fourier Transforms. J. Lond. Math. Soc. **S1–9**(1), 29–32 (1934). (**MR1574706**)
15. de M, Jeu: Determinate multidimensional measures, the extended Carleman theorem and quasi-analytic weights. Ann. Probab. **31**(3), 1205–1227 (2003). (**MR1988469 (2004f:44006)**)
16. Krötz, B., Olafasson, G., Stanton, R.J.: The image of heat kernel transform on Riemannian symmetric spaces of the noncompact type. Int.Math.Res.Not., (22), 1307-1329 (2005)
17. Krötz, B., Thangavelu, S., Xu, Y.: Heat kernel transform for nilmanifolds associated to the Heisenberg group. Rev. Mat. Iberoam. **24**(1), 243–266 (2008)
18. Lakshmi Lavanya, R., Thangavelu, S.: Revisiting the Fourier transform on the Heisenberg group. Publ. Mat. **58**, 47–63 (2014)
19. Levinson, N.: On a Class of Non-Vanishing Functions. Proc. Lond. Math. Soc. **41**(1), 393–407 (1936). (**MR1576177**)
20. Markett, C.: Mean Cesàro summability of Laguerre expansions and norm estimates with shifted parameter. Anal. Math. **8**(1), 19–37 (1982)
21. Muckenhoupt, B.: Mean convergence of Hermite and Laguerre series. II Trans. Am. Math. Soc. **147**, 433–460 (1970)
22. Masson, D., McClary, W.: Classes of C^∞ vectors and essential self-adjointness. J. Funct. Anal. **10**, 19–32 (1972)
23. Nussbaum, A.E.: Quasi-analytic vectors. Ark. Mat. **6**, 179–191 (1965). (**MR 33 No.3105**)
24. Nussbaum, A.E.: A note on quasi-analytic vectors. Studia Math. **33**, 305–309 (1969)
25. Paley, R.E.A.C., Wiener, N.: Notes on the theory and application of Fourier transforms. I. Trans. Am. Math. Soc. **35**(2), 348–355 (1933)
26. Paley, R.E.A.C., Wiener, N.: Notes on the theory and application of Fourier transforms. II. Trans. Am. Math. Soc. **35**(2), 348–355 (1933)
27. Paley, R.E.A.C., Wiener, N.: Fourier transforms in the complex domain (Reprint of the 1934 original) American Mathematical Society Colloquium Publications, 19. American Mathematical Society, Providence, RI (1987)
28. Reed, M., Simon, B.: Methods of modern mathematical physics I: Functional Analysis, Academic Press, INC. (London) LTD
29. Roncal, L., Thangavelu, S.: An extension problem and trace Hardy inequality for the sublaplacian on the H-type gropus. Int.Math.Res.Not. (14), 4238-4294 (2020)
30. Rudin, W.: Functional Analysis. 2nd edition, McGraw-Hill, Inc
31. Schymura, D.: An upper bound on the volume of the symmetric difference of a body and a congruent copy. Adv. Geom. **14**(2), 287–298 (2014)
32. Simon, B.: The theory of semi-analytic vectors: A new proof of a theorem of Masson and McClary. Indiana Univ. Math. J. **20**(1970/71), 1145-1151. MR 44 No.7357
33. Stein, E.M., Shakarchi, R.: Real Analysis: Measure Thery, Integration and Hilbert spaces, Princeton Lectures in Analysis III, Princeton University Press
34. Taylor, M.E.: Non-commutative Harmonic Analysis. Amer. Math. Soc, Providence, RI (1986)
35. Thangavelu, S.: Lectures on Hermite and Laguerre expansions, Mathematical Notes 42. Princeton University Press, Princeton, NJ (1993)
36. Thangavelu, S.: Harmonic Analysis on the Heisenberg group, Progress in Mathematics 159. Birkhäuser, Boston, MA (1998)
37. Thangavelu, S.: An introduction to the uncertainty principle. Hardy's theorem on Lie groups. With a foreword by Gerald B. Folland, Progress in Mathematics 217. Birkhäuser, Boston, MA (2004)
38. Thangavelu, S.: Gutzmer's formula and Poisson integrals on the Heisenberg group. Pacific J. Math. **231**(1), 217–237 (2007)

39. Thangavelu, S.: An analogue of Pfannschmidt's theorem for the Heisenberg group, The Journal of. Analysis **26**, 235–244 (2018)