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Nonzero-Sum Risk-Sensitive Continuous-Time Stochastic Games with Ergodic Costs

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Abstract
We study nonzero-sum stochastic games for continuous time Markov decision processes on a denumerable state space with risk-sensitive ergodic cost criterion. Transition rates and cost rates are allowed to be unbounded. Under a Lyapunov type stability assumption, we show that the corresponding system of coupled HJB equations admits a solution which leads to the existence of a Nash equilibrium in stationary strategies. We establish this using an approach involving principal eigenvalues associated with the HJB equations. Furthermore, exploiting appropriate stochastic representation of principal eigenfunctions, we completely characterize Nash equilibria in the space of stationary Markov strategies.

Keywords Nonzero-sum game · Risk-sensitive ergodic cost criterion · Stationary strategies · Coupled HJB equations · Fan's fixed point theorem · Nash equilibrium

1 Introduction

We consider a nonzero-sum stochastic game on the infinite time horizon for continuous time Markov decision processes (CTMDPs) on a denumerable state space. The

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performance evaluation criterion is exponential of integral cost which addresses the decision makers (i.e., players) attitude towards risk. In other words we address the problem of nonzero-sum risk sensitive stochastic games involving continuous time Markov decision processes. In the literature of stochastic games involving continuous/discrete time Markov decision processes, one usually studies the integral of the cost [11, 16–18, 37] which is the so called risk-neutral situation. In [11] the author has studied N-person nonzero-sum stochastic games for discrete time Markov chain and in [16–18, 37] the authors have studied zero-sum/nonzero-sum game problems for continuous time Markov chain. In the exponential of integral cost, the evaluation criterion is multiplicative as opposed to the additive nature of evaluation criterion in the integral of cost case. This difference makes the risk sensitive case significantly different from its risk neutral counterpart. The study of risk sensitive criterion was first introduced in [26,p. 125] also see [38,Part-II] and the references therein. This criterion is studied extensively in the context of MDP both in discrete and continuous times; see, for instance [7–9, 12, 21, 24, 33, 39], and the references therein. The corresponding results for stochastic (dynamic) games are limited. Notable exceptions are [4, 5, 14]. In discrete time and discrete state space the risk-sensitive zero-sum stochastic games with bounded cost and transition rates have been studied by Basu and Ghosh [4] and nonzero-sum games in [5]. For CTMDPs, zero-sum stochastic games with risk-sensitive costs for bounded cost and bounded transition rates have been studied in [14]. One can see [15, 36], and the references therein for finite horizon risk-sensitive stochastic games for CTMDPs, where [15] deals with zero-sum game and [36] deals with nonzero-sum game. Recently risk sensitive continuous time Markov decision processes for ergodic cost criterion have been studied in [6, 13, 20, 29, 30]. In the above five papers the authors have studied risk-sensitive stochastic optimal control problem, where controller is trying to control the state dynamics by choosing appropriate controls. When there is more than one controller the stochastic control problems become stochastic game problems. In this paper we have extended the results of the above five papers from one controller case to multi-controller case where the controllers are non-cooperative. More specifically, we study ergodic nonzero sum risk-sensitive stochastic (non-cooperative) games. Using principal eigenvalue approach, under a Lyapunov type stability assumption, we have shown that the corresponding system of coupled HJB equations admits a solution which in turn leads to the existence of Nash equilibrium in stationary strategies. Also, exploiting the stochastic representation of principal eigenfunction we completely characterize all possible Nash equilibria in the space of stationary Markov strategies.

The main motivation for studying this kind of games arises from their applications to many interesting problems, such as controlled birth-and-death systems, telecommunication and queueing systems in which the transition and cost rates may be both unbounded.

Our main contribution in this paper is the following. We establish the existence and characterization of Nash equilibria under a blanket Lyapunov type stability assumption. To be more specific, we study ergodic nonzero sum risk-sensitive stochastic games for CTMDPs having the following features: (a) the transition and the cost rates may be unbounded (b) state space is countable (c) at any state of the system the space of admissible actions is compact (d) the strategies are (state) feedback. To our knowledge, these results are new in the literature of ergodic non-zero sum risk-sensitive games for CTMDPs.

The rest of this paper is organized as follows: Sect. 2 deals with the problem description and preliminaries. The ergodic cost criterion is analyzed in Sect. 3. Under a Lyapunov type stability assumption(s), we first establish the existence of a solution to the corresponding coupled Hamilton-Jacobi-Bellman (HJB) equations. This in turn leads to the existence of a Nash equilibrium in stationary strategies (see Theorem 3.2). In Sect. 4, we present an illustrative example.

2 The Game Model

For the sake of notational simplicity we treat two player game. The N-player game for $N \geq 3$, is analogous. The continuous-time two-person nonzero-sum stochastic game model which consists of the following elements

$$\{S, U_1, U_2, (U_1(i) \subset U_1, U_2(i) \subset U_2, i \in S), \bar{\pi}_{ij}(u_1, u_2), \bar{c}_1(i, u_1, u_2), \bar{c}_2(i, u_1, u_2)\}, \tag{2.1}$$

where each component is described below:

- S, called the state space, is assumed to be the set of all positive integers endowed with the discrete topology, i.e. $S =: \{1, 2, \cdots\}$.
- U_1 and U_2 are the action sets for players 1 and 2, respectively. The action spaces U_1 and U_2 are assumed to be Borel spaces with the Borel σ-algebras $\mathcal{B}(U_1)$ and $\mathcal{B}(U_2)$, respectively.
- For each $i \in S$, $U_1(i) \in \mathcal{B}(U_1)$ and $U_2(i) \in \mathcal{B}(U_2)$ denote the sets of admissible actions for players 1 and 2 in state i, respectively. Let $K := \{(i, u_1, u_2) | i \in S, u_1 \in U_1(i), u_2 \in U_2(i)\}$, which is a Borel subset of $S \times U_1 \times U_2$.
  Throughout this paper, we assume that
  (A1)(a) For each $i \in S$, the admissible action spaces $U_k(i)$, $k = 1, 2$, are nonempty and compact subsets of U_k.
- The transition rates $\bar{\pi}_{ij}(u_1, u_2)$, $(u_1, u_2) \in U_1(i) \times U_2(i)$, $i, j \in S$, satisfy the condition $\bar{\pi}_{ij}(u_1, u_2) \geq 0$ for all $i \neq j$, $(u_1, u_2) \in U_1(i) \times U_2(i)$. Also, we assume that:
  (A1)(b) The transition rates $\bar{\pi}_{ij}(u_1, u_2)$ are conservative, i.e.,

  $$\sum_{j \in S} \bar{\pi}_{ij}(u_1, u_2) = 0 \text{ for } i \in S \text{ and } (u_1, u_2) \in U_1(i) \times U_2(i)$$

  and satisfy the following stability condition

  $$\bar{\pi}_i := \sup_{(u_1, u_2) \in U_1(i) \times U_2(i)} [-\bar{\pi}_{ii}(u_1, u_2)] < \infty\,.$$

- Finally, the measurable function $\bar{c}_k : K \to \mathbb{R}_+$ denotes the cost rate function for player k, $k = 1, 2$.

The game is played as follows. The players observe continuously the current state of the system. When the system is in state $i \in S$ at time $t \geq 0$, the players independently choose actions $u_1(t) \in U_1(i)$ and $u_2(t) \in U_2(i)$ according to some strategies, respectively. As a consequence of this, the following happens:

- player 1 (resp. 2) pays an immediate cost at rate $\bar{c}_1(i, u_1(t), u_2(t))$ (resp. $\bar{c}_2(i, u_1(t), u_2(t)))$;
- the system stays in state i for a random time, with rate of leaving i given by $-\bar{\pi}_{ii}(u_1(t), u_2(t))$, and then jumps to a new state $j \neq i$ with the probability determined by $\dfrac{\bar{\pi}_{ij}(u_1(t), u_2(t))}{-\bar{\pi}_{ii}(u_1(t), u_2(t))}$ (see Proposition B. 8 in [19,p. 205] for details).

The whole process then repeats from the new state j. Cost accumulates throughout the course of the game. The planning horizon is infinite, and each player wants to minimize his infinite-horizon risk-sensitive cost with respect to some performance criterion $\rho_k^{\xi_1,\xi_2}$, $k = 1, 2$, which in our present case is defined by (2.3), below. To formalize what is described above, below we describe the construction of continuous time Markov decision processes (CTMDPs) under admissible feedback strategies. We consider a continuous time Markov decision processes (CTMDPs) $\{Y(t)\}_{t\geq 0}$ with state space S and controlled rate matrix $\Pi_{u_1,u_2} = (\bar{\pi}_{ij}(u_1, u_2))$. To construct the underlying CTMDPs $Y(t)$ (as in [22, 27, 35]) we introduce some notations: let $S_\Delta := S \cup \{\Delta\}$ (for some isolated point $\Delta \notin S$), $\Omega_0 := (S \times (0, \infty))^\infty$, $\Omega_m := (S \times (0, \infty))^m \times S \times (\{\infty\} \times \{\Delta\})^\infty$ for $m \geq 1$ and $\Omega := \cup_{m=0}^{\infty} \Omega_m = (S \times \mathbb{R}_+)^\infty \cup \{(i_0^{'}, \theta_1, i_1^{'}, \cdots, \theta_m, i_m^{'}, \infty, \Delta, \infty, \Delta, \cdots) |\ m \geq 0, i_l^{'} \in S, \theta_l \in \mathbb{R}_+\ \forall\ 0 \leq l \leq m\}$, where $\mathbb{R}_+ = (0, \infty)$.

Let $\mathscr{F}$ be the Borel σ-algebra on Ω. Then we obtain the measurable space $(\Omega, \mathscr{F})$. For some $m \geq 1$, and sample $\omega := (i_0^{'}, \theta_1, i_1^{'}, \cdots, \theta_m, i_m^{'}, \cdots) \in \Omega$, let $\hat{h}_m(\omega) := (i_0^{'}, \theta_1, i_1^{'}, \cdots, \theta_m, i_m^{'})$ denote the m-component internal history, and define

$$T_0(\omega) := 0,\ T_m(\omega) := T_{m-1}(\omega) + \theta_m,\ T_\infty(\omega) := \lim_{m\to\infty} T_m(\omega).$$

Using $\{T_m\}$, we define the state process $\{Y(t)\}_{t\geq 0}$ as

$$Y(t)(w) := \sum_{m\geq 0} I_{\{T_m \leq t < T_{m+1}\}} i_m^{'} + I_{\{t \geq T_\infty\}} \Delta,\ \text{ for } t \geq 0 \text{ (with } T_0 := 0). \tag{2.2}$$

Here, I_E denotes the indicator function of a set E, and we use the convention that $0 + z =: z$ and $0z =: 0$ for all $z \in S_\Delta$. Obviously, $Y(t)$ is right-continuous on $[0, \infty)$. From (2.2), we see that $T_m(\omega)$ $(m \geq 1)$ denotes the m-th jump moment of $\{Y(t)\}_{t\geq 0}$ and $i_{m-1}^{'}$ is the state of the process on $[T_{m-1}(\omega), T_m(\omega))$, $\theta_m(\omega) = T_m(\omega) - T_{m-1}(\omega)$ plays the role of sojourn time at state $i_{m-1}^{'}$, and the sample path $\{Y(t)(\omega)\}_{t\geq 0}$ has at most denumerable states $i_m^{'} (m = 0, 1, \cdots)$. The process after T_∞ is regarded to be absorbed in the state Δ. Thus, let $\bar{\pi}_{\Delta j}(u_1^\Delta, u_2^\Delta) := 0$, for all $j \in S$, where $u_1^\Delta \notin U_1$, $u_2^\Delta \notin U_2$ are isolated points; $U_1^\Delta := U_1 \cup \{u_1^\Delta\}$, $U_2^\Delta := U_2 \cup \{u_2^\Delta\}$, $U_1(\Delta) := \{u_1^\Delta\}$, $U_2(\Delta) := \{u_2^\Delta\}$. Also, assume that $\bar{c}_k(\Delta, u_1, u_2) := 0$ ($\bar{c}_k$ is the running cost function for kth player) for all $(u_1, u_2) \in U_1^\Delta \times U_2^\Delta$. Moreover, let

$\mathscr{F}_t := \sigma(\{T_m \le s, Y(T_m) \in S\} : 0 \le s \le t, m \ge 0)$ for all $t \ge 0$, $\mathscr{F}_{s-} =: \bigvee_{t<s} \mathscr{F}_t$, and $\tilde{\mathscr{F}} := \sigma(A \times \{0\}, B \times (s, \infty) : A \in \mathscr{F}_0, B \in \mathscr{F}_{s-})$ which denotes the σ-algebra of predictable sets on $\Omega \times [0, \infty)$ related to $\{\mathscr{F}_t\}_{t\ge 0}$.
To complete the specification of a risk-sensitive stochastic game problem, we need, of course, to introduce an optimality criterion. This requires to define the class of strategies as below.

Definition 2.1 An admissible (feedback) strategy for player 1, denoted by $\xi_1 = \{\xi_1(t)\}_{t\ge 0}$, is a transition probability $\xi_1(du_1|\omega, t)$ from $(\Omega \times [0, \infty), \tilde{\mathscr{F}})$ onto $(U_1^\Delta, \mathcal{B}(U_1^\Delta))$, such that $\xi_1(U_1(Y(t-)(\omega))|\omega, t) = 1$. An admissible strategy ξ_1, is usually determined by a sequence $\{\xi_1^m, m \ge 0\}$ of stochastic kernels on U_1 such that

$$
\begin{aligned}
\xi_1(t)(\omega) =&\xi_1(du_1|\omega, t)\\
=&I_{\{t=0\}}(t)\xi_1^0(du_1|i_0^{'}, 0)\\
&+\sum_{m\ge 0} I_{\{T_m<t\le T_{m+1}\}}\xi_1^m(du_1|i_0^{'}, \theta_1, i_1^{'}, \dots, \theta_m, i_m^{'}, t-T_m)\\
&+I_{\{t\ge T_\infty\}}\delta_{u_1^\Delta}(du_1),
\end{aligned}
$$

where $\xi_1^0(du_1|i_0^{'}, 0)$ is a stochastic kernel on U_1 given S which satisfies $\xi_1^0(U_1(i_0^{'})|i_0^{'}, 0) = 1$, $\xi_1^m (m \ge 1)$ are stochastic kernels on U_1 given $(S \times (0, \infty))^{m+1}$ which satisfy $\xi_1^m(U_1(i_m^{'})|i_0^{'}, \theta_1, i_1^{'}, \cdots, \theta_m, i_m^{'}, t - T_m) = 1$, and $\delta_{u_1^\Delta}(du_1)$ denotes the Dirac measure at the point u_1^Δ. The set of all admissible strategies for player 1 is denoted by $\mathcal{U}_1^{Ad}$. For more details see [23,Definition 2.1, Remark 2.2], [36, 39].

An admissible strategy $\xi_1 \in \mathcal{U}_1^{Ad}$, is called a Markov strategy for player 1 if $\xi_1(t)(\omega) = \xi_1(t, Y(t-)(\omega))$, i.e., $\xi_1(du_1|\omega, t) = \xi_1(du_1|Y(t-)(\omega), t)$ for every $\omega \in \Omega$ and $t \ge 0$, where $Y(t-)(\omega) := \lim_{s\uparrow t} Y(s)(\omega)$. We denote by $\mathcal{U}_1^M$ the family of all Markov strategies for player 1. If the Markov strategy ξ_1 for player 1 does not have any explicit time dependency then it is called a stationary Markov strategy. The set of such strategies for player 1 is denoted by $\mathcal{U}_1^{SM}$. The sets of all admissible strategies $\mathcal{U}_2^{Ad}$, all Markov strategies $\mathcal{U}_2^M$ and all stationary strategies $\mathcal{U}_2^{SM}$ for player 2 are defined similarly.

Some comments are in order.

Remark 2.1 In the definition of strategies we do not include the entire history of the game, i.e., past and present states, past sojourn times and past actions taken by the players. If players use general strategies (i.e., history dependent non-anticipative strategies) there may not be a probability measure over plays; see Proposition 1 in [32]. See also [31]. Thus it is imperative for us to confine our attention to specific classes of strategies. In this paper we restrict our attention to feedback strategies only, i.e., at any point of time each player has access to past and present states and past sojourn times. Though the past and present states and past sojourn times implicitly contain the past actions of the players, explicit inclusions thereof in the strategies run into unassailable technical issues as explained clearly in [32].

To avoid possible explosion of the state process $\{Y(t)\}_{t\geq 0}$, we make the following Lyapunov stability assumption imposed on the transition rates, which had been widely used in CTMDPs; see, for instance, [21–24] and the references therein.

Assumption 2.1 There exists a non-constant function $\tilde{W} : S \to [1, \infty)$ such that

(i) $\sum_{j\in S} \tilde{W}(j)\bar{\pi}_{ij}(u_1, u_2) \leq C_1\tilde{W}(i) + C_2$ for all $(u_1, u_2) \in U_1(i) \times U_2(i)$ and $i \in S$ with some constants $C_1 \neq 0$, $C_2 \geq 0$;
(ii) $\bar{\pi}_i \leq C_3\tilde{W}(i)$ for all $i \in S$ with some positive constant C_3.

For the rest of this article Assumption 2.1 is in force. Note that if $\sup_{i\in S} \bar{\pi}_i < \infty$ then Assumption 2.1 holds. In this case we can choose $\tilde{W}$ to be a suitable constant. Also note that under Assumption 2.1, for any initial state $i \in S$ and any pair of strategies $(\xi_1, \xi_2) \in \mathcal{U}_1^{Ad} \times \mathcal{U}_2^{Ad}$, Theorem 4.27 in [28] yields the existence of a unique probability measure denoted by $P_i^{\xi_1,\xi_2}$ on $(\Omega, \mathscr{F})$. Let $E_i^{\xi_1,\xi_2}$ be the expectation operator with respect to $P_i^{\xi_1,\xi_2}$. Also, from [19,pp. 13–15], we know that $\{Y(t)\}_{t\geq 0}$ is a Markov process under any $(\xi_1, \xi_2) \in \mathcal{U}_1^M \times \mathcal{U}_2^M$ (in fact, strong Markov).

For any compact metric space A, let $\mathcal{P}(A)$ denote the space of probability measures on A with the topology of weak convergence. Let $V_k = \mathcal{P}(U_k)$ and $V_k(i) = \mathcal{P}(U_k(i))$ for $i \in S$ and $k = 1, 2$. Since $U_k(i)$ is a compact set for each $i \in S$, we have $V_k(i)$ is a compact metric space for $k = 1, 2$. For each $i, j \in S$, $k = 1, 2$, $v_1 \in V_1(i)$ and $v_2 \in V_2(i)$, the associated transition and cost rates are defined, respectively, as follows:

$$\pi_{ij}(v_1, v_2) := \int_{U_1(i)} \int_{U_2(i)} \bar{\pi}_{ij}(u_1, u_2)v_1(du_1)v_2(du_2),$$
$$c_k(i, v_1, v_2) := \int_{U_1(i)} \int_{U_2(i)} \bar{c}_k(i, u_1, u_2)v_1(du_1)v_2(du_2).$$

Note that for $k = 1, 2$, $\xi_k \in \mathcal{U}_k^{SM}$ can be identified with a map $\xi_k : S \to V_k$ such that for each $j \in S$, $\xi_k(j) \in V_k(j)$ for each $j \in S$. Thus, we have $\mathcal{U}_1^{SM} = \Pi_{i\in S}\mathcal{P}(U_1(i))$ and $\mathcal{U}_2^{SM} = \Pi_{i\in S}\mathcal{P}(U_2(i))$ i.e., the sets $\mathcal{U}_1^{SM}$ and $\mathcal{U}_2^{SM}$ are endowed with the product topology. Therefore by Tychonoff theorem, the sets $\mathcal{U}_1^{SM}$ and $\mathcal{U}_2^{SM}$ are compact metric spaces.
For $j = 1, 2$, let $\mathcal{M}(U_j(i)), i = 1, 2, \ldots,$ be the space of finite signed measures on $U_j(i)$ endowed with the topology of weak convergence. Then $\mathcal{M}(U_j(i))$ is a locally convex topological vector space which is metrizable [34]. Thus for $j = 1, 2$, $\Pi_{i\in S}\mathcal{M}(U_j(i))$ is a locally convex topological vector space which is metrizable as well. Moreover for $j = 1, 2$, $\mathcal{U}_j^{SM}$ is a compact, convex subset of $\Pi_{i\in S}\mathcal{M}(U_j(i))$. For more details along these lines we refer to [11].

We list the commonly used notations below.

- Given any real-valued function $\mathcal{V} \geq 1$ on S, we define a Banach space $(L^\infty_{\mathcal{V}}, \|\cdot\|^\infty_{\mathcal{V}})$ of $\mathcal{V}$-weighted functions by

$$L^\infty_{\mathcal{V}} = \left\{u : S \to \mathbb{R} \mid \|u\|^\infty_{\mathcal{V}} := \sup_{i\in S} \frac{|u(i)|}{\mathcal{V}(i)} < \infty\right\}.$$

- $L^{1,\infty}_{\mathcal{V}}$ denotes the subset of $L^{\infty}_{\mathcal{V}}$ consists of function u such that $\|u\|^{\infty}_{\mathcal{V}} \leq 1$.

For a pair of admissible strategies $(\xi_1, \xi_2) \in \mathcal{U}_1^{Ad} \times \mathcal{U}_2^{Ad}$, the risk-sensitive ergodic cost for player k, $k = 1, 2$, is given by

$$\rho_k^{\xi_1,\xi_2}(i) := \limsup_{T\to\infty} \frac{1}{T} \ln E_i^{\xi_1,\xi_2}\left[e^{\int_0^T c_k(Y(t),\xi_1(t),\xi_2(t))dt}\right], \tag{2.3}$$

where $Y(t)$ is the CTMDP corresponding to $(\xi_1, \xi_2) \in \mathcal{U}_1^{Ad} \times \mathcal{U}_2^{Ad}$ and $E_i^{\xi_1,\xi_2}$ denotes the expectation with respect to the law of the process $Y(t)$ with initial condition $Y(0) = i$.

Definition 2.2 A function $f : S \to \mathbb{R}$ is said to be norm-like if for every $k \in \mathbb{R}$, the set $\{i : f(i) \leq k\}$ is either empty or finite.

Definition 2.3 A time-homogeneous Markov process Y with rate matrix $Q = [\bar{\pi}_{ij}]$ is irreducible if for any $i, j \in S$, $i \neq j$, there exist distinct states $i_1, i_2 \cdots, i_k \in S$ satisfying $\bar{\pi}_{ii_1} \cdots \bar{\pi}_{i_k j} > 0$ (see, [19,p. 107]).

Since we are allowing our transition and cost rates to be unbounded, to guarantee the finiteness of $\rho_k^{\xi_1,\xi_2}$ for $k = 1, 2$, we make the following Lyapunov stability Assumption.

Assumption 2.2 We assume that the CTMDP $\{Y(t)\}_{t\geq 0}$ is irreducible under every pair of stationary Markov strategies $(\xi_1, \xi_2) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$. Furthermore, suppose there exist a constant $C_4 > 0$ and a function $W : S \to [1, \infty)$ such that one of the following hold.

(a) **When the running cost is bounded:** For some positive constant $\gamma > \max\{\|\bar{c}_1\|_\infty, \|\bar{c}_2\|_\infty\}$ and a finite set $\mathscr{K}$ it holds that

$$\sup_{(u_1,u_2)\in U_1(i)\times U_2(i)} \sum_{j\in S} W(j)\bar{\pi}_{ij}(u_1, u_2) \leq C_4 I_{\mathscr{K}}(i) - \gamma W(i) \ \forall i \in S,$$

where $\|\bar{c}_k\|_\infty := \sup_{(i,u_1,u_2)\in K} \bar{c}_k(i, u_1, u_2)$ for $k = 1, 2$.

(b) **When the running cost is unbounded:** For some norm-like function $\ell : S \to \mathbb{R}_+$ and a finite set $\mathscr{K}$ it holds that

$$\sup_{(u_1,u_2)\in U_1(i)\times U_2(i)} \sum_{j\in S} W(j)\bar{\pi}_{ij}(u_1, u_2) \leq C_4 I_{\mathscr{K}}(i) - \ell(i)W(i) \ \forall i \in S.$$

Also, the functions $\ell(\cdot) - \max_{(u_1,u_2)\in U_1(\cdot)\times U_2(\cdot)} \bar{c}_k(\cdot, u_1, u_2)$, $k = 1, 2$, are norm-like.

This type of Foster-Lyapunov condition on the dynamics is quite common in the literature to study the continuous-time risk-sensitive ergodic control problems, for example, see [2, 3] for controlled diffusion case and [6, 20] for Markov chain case.

Definition 2.4 A pair of strategies $(\xi_1^*, \xi_2^*) \in \mathcal{U}_1^{Ad} \times \mathcal{U}_2^{Ad}$ is called a Nash equilibrium if

$$\rho_1^{\xi_1^*,\xi_2^*}(i) \leq \rho_1^{\xi_1,\xi_2^*}(i) \text{ for all } \xi_1 \in \mathcal{U}_1^{Ad} \text{ and } i \in S$$

and

$$\rho_2^{\xi_1^*,\xi_2^*}(i) \leq \rho_2^{\xi_1^*,\xi_2}(i) \text{ for all } \xi_2 \in \mathcal{U}_2^{Ad} \text{ and } i \in S.$$

We wish to establish the existence of a Nash equilibrium in stationary strategies. To ensure the existence of a Nash equilibrium, we assume the following:

Assumption 2.3 (i) For any fixed $i, j \in S$, k=1,2, $\bar{\pi}_{ij}(u_1, u_2)$ and $\bar{c}_k(i, u_1, u_2)$ are continuous in $(u_1, u_2) \in U_1(i) \times U_2(i)$.

(ii) $\sum_{j \in S} W(j)\bar{\pi}_{ij}(u_1, u_2)$ is continuous in $(u_1, u_2) \in U_1(i) \times U_2(i)$ for any given $i \in S$, where W is as Assumption 2.2.

(iii) There exists $i_0 \in S$ such that $\bar{\pi}_{i_0 j}(u_1, u_2) > 0$ for all $j \neq i_0$ and for all $(u_1, u_2) \in U_1(j) \times U_2(j)$.

It is also possible to consider other type of condition instead Assumption 2.3(iii). We refer to Remark 3.1 for further discussion.

We now proceed to establish the existence of a Nash equilibrium in stationary strategies. To this end we first outline a standard procedure for establishing the existence of a Nash equilibrium. Suppose player 2 announces that he/she is going to employ a strategy $\xi_2 \in \mathcal{U}_2^{SM}$. In such a scenario, player 1 attempts to minimize

$$\rho_1^{\xi_1,\xi_2}(i) = \limsup_{T\to\infty} \frac{1}{T} \ln E_i^{\xi_1,\xi_2}\Big[e^{\int_0^T c_1(Y(t),\xi_1(t),\xi_2(Y(t-)))dt}\Big],$$

over $\xi_1 \in \mathcal{U}_1^{Ad}$. Thus for player 1 it is a continuous time Markov decision problem (CTMDP) with risk sensitive ergodic cost. This problem has been studied in [6, 13, 29, 30]. In particular under certain assumptions, it is shown in [6, 29, 30], that the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \rho_1 \hat{\psi}_1(i) = \inf_{v_1 \in V_1(i)} \Big[\sum_{j\in S} \pi_{ij}(v_1, \xi_2(i))\hat{\psi}_1(j) + c_1(i, v_1, \xi_2(i))\hat{\psi}_1(i)\Big] \\ \hat{\psi}_1(\hat{i}_0) = 1, \end{cases}$$

has a suitable solution $(\rho_1, \hat{\psi}_1)$, where ρ_1 is a scalar and $\hat{\psi}_1 : S \to \mathbb{R}$ has suitable growth rate; and $\hat{i}_0$ is some fixed state in S. Furthermore it is shown in [6, 29, 30] that

$$\rho_1 = \inf_{\xi_1 \in \mathcal{U}_1^{Ad}} \limsup_{T\to\infty} \frac{1}{T} \ln E_i^{\xi_1,\xi_2}\Big[e^{\int_0^T c_1(Y(t),\xi_1(t),\xi_2(Y(t-)))dt}\Big],$$

and if $\xi_1^* \in \mathcal{U}_1^{SM}$ is such that for all $i \in S$

$$\inf_{v_1 \in V_1(i)} \Big[\sum_{j \in S} \pi_{ij}(v_1, \xi_2(i)) \hat{\psi}_1(j) + c_1(i, v_1, \xi_2(i)) \hat{\psi}_1(i) \Big] = \sum_{j \in S} \pi_{ij}(\xi_1^*(i), \xi_2(i)) \hat{\psi}_1(j) + c_1(i, \xi_1^*(i), \xi_2(i)) \hat{\psi}_1(i),$$

then $\xi_1^* \in \mathcal{U}_1^{SM}$ is an optimal control for player 1, i.e., for any $i \in S$

$$\rho_1 = \limsup_{T \to \infty} \frac{1}{T} \ln E_i^{\xi_1^*, \xi_2} \Big[e^{\int_0^T c_1(Y(t), \xi_1^*(Y(t-)), \xi_2(Y(t-)))dt} \Big].$$

In [30], the ergodic case is treated via the limit of the corresponding finite horizon risk-sensitive continuous-time MDP. For the latter, the HJB equation is an infinite system of coupled ODEs. Then as the length of the horizon tends to ∞, the above equation is derived using limiting horizon asymptotics. In [29], the existence of ergodic optimal control is established by using vanishing discount approach. In [6], the ergodic case is studied directly by an approach involving the principal eigenpair associated with the above equation. In this paper we follow the approach of [6].

In view of the foregoing it follows that given that player 2 is using the strategy $\xi_2 \in \mathcal{U}_2^{SM}$, $\xi_1^* \in \mathcal{U}_1^{SM}$ is an optimal response for player 1. Clearly ξ_1^* depends on ξ_2 and moreover there may be several optimal responses for player 1 in $\mathcal{U}_1^{SM}$. Analogous results holds for player 2 if player 1 announces that he is going to use a strategy $\xi_1 \in \mathcal{U}_1^{SM}$. Hence given a pair of strategies $(\xi_1, \xi_2) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, we can find a set of pairs of optimal responses $\{(\xi_1^*, \xi_2^*) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}\}$ via the appropriate pair of HJB equations described above. This defines a set-valued map. Clearly any fixed point of this set-valued map is a Nash equilibrium.

The above discussion leads to the following procedure for finding a pair of Nash equilibrium strategies. Suppose that there exist a pair of stationary strategies $(\xi_1^*, \xi_2^*) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, a pair of scalars (ρ_1^*, ρ_2^*) and a pair of functions $(\hat{\psi}_1^*, \hat{\psi}_2^*)$ with appropriate growth conditions, satisfying the following coupled HJB equations:

$$\begin{cases} \rho_1^* \, \hat{\psi}_1^*(i) = \inf_{v_1 \in V_1(i)} \Big[\sum_{j \in S} \pi_{ij}(v_1, \xi_2^*(i)) \hat{\psi}_1^*(j) + c_1(i, v_1, \xi_2^*(i)) \hat{\psi}_1^*(i) \Big] \\ \qquad = \sum_{j \in S} \pi_{ij}(\xi_1^*(i), \xi_2^*(i)) \hat{\psi}_1^*(j) + c_1(i, \xi_1^*(i), \xi_2^*(i)) \hat{\psi}_1^*(i) \\ \hat{\psi}_1^*(\hat{i}_0) = 1, \\ \rho_2^* \, \hat{\psi}_2^*(i) = \inf_{v_2 \in V_2(i)} \Big[\sum_{j \in S} \pi_{ij}(\xi_1^*(i), v_2) \hat{\psi}_2^*(j) + c_2(i, \xi_1^*(i), v_2) \hat{\psi}_2^*(i) \Big] \\ \qquad = \sum_{j \in S} \pi_{ij}(\xi_1^*(i), \xi_2^*(i)) \hat{\psi}_2^*(j) + c_2(i, \xi_1^*(i), \xi_2^*(i)) \hat{\psi}_2^*(i) \\ \hat{\psi}_2^*(\hat{i}_0) = 1, \end{cases}$$

where as before $\hat{i}_0 \in S$ is a fixed state. Then it can be shown that (ξ_1^*, ξ_2^*) is a pair of Nash equilibrium and (ρ_1^*, ρ_2^*) is the pair of corresponding Nash values. Thus the main result of our paper is to establish that the above coupled HJB equations has suitable solutions.

Remark 2.2 Note that similar stochastic optimal control problems have been studied in [13, 30] for bounded cost and bounded transition rates. But in our game model transition and cost rates are allowed to be unbounded. Analogous MDP problems are treated in [6].

3 Coupled HJB Equations and Existence of Nash Equilibrium

By the definition of weak convergence of probability measures, one can easily get the following result, which will be crucial for the existence of Nash equilibrium; for details we refer to [16,Lemma 7.2].

Lemma 3.1 *Under Assumptions 2.1, 2.2, and 2.3, the functions*

$$c_k(i, v_1, v_2),\ k = 1, 2 \ \textit{and} \ \sum_{j \in S} \pi_{ij}(v_1, v_2)\phi(j)$$

are continuous on $V_1(i) \times V_2(i)$ *for each fixed* $\phi \in L_W^\infty$ *and* $i \in S$.

For any finite set $\mathcal{D} \subset S$, we define

$$\mathcal{B}_{\mathcal{D}} = \{f : S \to \mathbb{R} \mid f \ \text{ is a Borel measurable function and } \ f(i) = 0 \ \ \forall \ i \in \mathcal{D}^c\}.$$

Also, $\mathcal{B}_{\mathcal{D}}^+ \subset \mathcal{B}_{\mathcal{D}}$ denotes the cone of all nonnegative functions vanishing outside $\mathcal{D}$.

Let $\mathcal{D}_n \subset S$ be an increasing sequence of finite sets such that $\cup_n \mathcal{D}_n = S$ such that $i_0 \in \mathcal{D}_n$ for each $n \geq 1$, where $i_0 \in S$ is a fixed state as in Assumption 2.3. Also, we denote $\succeq$ as the partial ordering in $\mathcal{B}_{\mathcal{D}_n}$ with respect to the closed cone $\mathcal{B}_{\mathcal{D}_n}^+$, i.e., for $f, g \in \mathcal{B}_{\mathcal{D}_n}$, $f \succeq g$ if and only if $f - g \in \mathcal{B}_{\mathcal{D}_n}^+$. Now using Krein-Rutman theorem we prove the existence of an eigenpair to a Dirichlet problem in $\mathcal{D}_n$ for each $n \in \mathbb{N}$. In the next lemma we show the existence of eigenpairs to certain equations in $\mathcal{D}_n$ for each $n \in \mathbb{N}$.

Lemma 3.2 *Suppose that Assumptions 2.1, 2.2, and 2.3 are satisfied. Then for each* $n \in \mathbb{N}$*, the following hold.*

(1) For $\hat{\xi}_2 \in \mathcal{U}_2^{SM}$*, there exists an eigenpair* $(\rho_{1,n}, \psi_{1,n}) \in \mathbb{R} \times \mathcal{B}_{\mathcal{D}_n}^+$*, satisfying*

$$\begin{cases} \rho_{1,n}\psi_{1,n}(i) = \inf_{v_1 \in V_1(i)} \\ \qquad \left[\sum_{j \in S} \psi_{1,n}(j)\pi_{ij}(v_1, \hat{\xi}_2(i)) + c_1(i, v_1, \hat{\xi}_2(i))\psi_{1,n}(i)\right] \textit{for } i \in \mathcal{D}_n, \\ \psi_{1,n}(i_0) = 1. \end{cases} \tag{3.1}$$

Moreover, we have

$$
\begin{aligned}
0 \le \liminf_{n\to\infty} \rho_{1,n} &\le \limsup_{n\to\infty} \rho_{1,n} \\
&\le \inf_{\xi_1 \in \mathcal{U}_1^{Ad}} \limsup_{T\to\infty} \frac{1}{T} \ln E_{i_0}^{\xi_1,\hat{\xi}_2}\Big[e^{\int_0^T c_1(Y(t),\xi_1(t),\hat{\xi}_2(Y(t-)))dt}\Big],
\end{aligned}
\tag{3.2}
$$

and $\{\rho_{1,n}\}$ *is a bounded sequence.*

(2) Similarly, for $\hat{\xi}_1 \in \mathcal{U}_1^{SM}$*, there exists an eigenpair* $(\rho_{2,n}, \psi_{2,n}) \in \mathbb{R} \times \mathcal{B}^+_{\mathcal{D}_n}$*, satisfying*

$$
\begin{cases}
\rho_{2,n}\psi_{2,n}(i) = \inf\limits_{v_2 \in V_2(i)} \\
\quad \Big[\sum\limits_{j\in S} \psi_{2,n}(j)\pi_{ij}(\hat{\xi}_1(i), v_2) + c_2(i, \hat{\xi}_1(i), v_2)\psi_{2,n}(i)\Big] \text{for } i \in \mathcal{D}_n, \\
\psi_{2,n}(i_0) = 1.
\end{cases}
\tag{3.3}
$$

Moreover, we have

$$
\begin{aligned}
0 \le \liminf_{n\to\infty} \rho_{2,n} &\le \limsup_{n\to\infty} \rho_{2,n} \\
&\le \inf_{\xi_2 \in \mathcal{U}_2^{Ad}} \limsup_{T\to\infty} \frac{1}{T} \ln E_{i_0}^{\hat{\xi}_1,\xi_2}\Big[e^{\int_0^T c_2(Y(t),\hat{\xi}_1(Y(t-)),\xi_2(t))dt}\Big],
\end{aligned}
\tag{3.4}
$$

and $\{\rho_{2,n}\}$ *is a bounded sequence.*

Proof We prove part (1); part (2) follows by analogous arguments. Fix $\hat{\xi}_2 \in \mathcal{U}_2^{SM}$. Let $\delta > 0$. Set $\tilde{c}_1(i, v_1, v_2) = c_1(i, v_1, v_2) - k_n - \delta$, where $k_n = \sup\{c_1(i, v_1, v_2) \mid i \in \mathcal{D}_n, v_1 \in V_1(i), v_2 \in V_2(i)\}$. From [6,Proposition 3.1], it is easy to see that for each $g \in \mathcal{B}_{\mathcal{D}_n}$ the following equation

$$
-g(i) = \inf_{v_1 \in V_1(i)} \Big[\sum_{j\in S} \phi_1(j)\pi_{ij}(v_1, \hat{\xi}_2(i)) + \tilde{c}_1(i, v_1, \hat{\xi}_2(i))\phi_1(i)\Big] \text{ for } i \in \mathcal{D}_n,
$$

admits a unique solution $\phi_1 \in \mathcal{B}_{\mathcal{D}_n}$ and ϕ_1 is given by

$$
\phi_1(i) = \inf_{\xi_1 \in \mathcal{U}_1^{Ad}} E_i^{\xi_1,\hat{\xi}_2}\Big[\int_0^{\tau(\mathcal{D}_n)} e^{\int_0^t \tilde{c}_1(Y(s),\xi_1(s),\hat{\xi}_2(Y(s-)))ds} g(Y(t))dt\Big], \; i \in S,
$$

where $\tau(\mathcal{D}_n) := \inf\{t > 0 : Y(t) \notin \mathcal{D}_n\}$. Therefore, the operator $T : \mathcal{B}_{\mathcal{D}_n} \to \mathcal{B}_{\mathcal{D}_n}$ given by

$$T(g)(i) := \phi_1(i) = \inf_{\xi_1 \in \mathcal{U}_1^{Ad}} E_i^{\xi_1, \hat{\xi}_2}\left[\int_0^{\tau(\mathcal{D}_n)} e^{\int_0^t \tilde{c}_1(Y(s), \xi_1(s), \hat{\xi}_2(Y(s-)))ds} g(Y(t))dt\right],$$
$$i \in \mathcal{D}_n,\ g \in \mathcal{B}_{\mathcal{D}_n}$$

with $T(g)(i) = 0$ for $i \in \mathcal{D}_n^c$ is well defined. Then by similar arguments as in [6,Lemma 3.1], the map T is order-preserving, 1-homogeneous, completely continuous and for some nonzero function $g \in \mathcal{B}^+_{\mathcal{D}_n}$, there exists $M > 0$, such that $MT(g) \succeq g$, i.e., it satisfies all conditions of Krein-Rutman theorem. Hence by a version of nonlinear Krein-Rutman theorem [1,Sect. 3.1], there exist nontrivial $\psi_{1,n} \in \mathcal{B}^+_{\mathcal{D}_n}$ and $\lambda_{\mathcal{D}_n} > 0$, satisfying $T\psi_{1,n} = \lambda_{\mathcal{D}_n}\psi_{1,n}$. Let $\tilde{\rho}_{1,n} = -[\lambda_{\mathcal{D}_n}]^{-1}$. Then we have that the pair $(\tilde{\rho}_{1,n}, \psi_{1,n})$ satisfies

$$\inf_{v_1 \in V_1(i)}\left[\sum_{j \in S} \pi_{ij}(v_1, \hat{\xi}_2(i))\psi_{1,n}(j) + \tilde{c}_1(i, v_1, \hat{\xi}_2(i))\psi_{1,n}(i)\right] = \tilde{\rho}_{1,n}\psi_{1,n}(i),\ \ \forall i \in \mathcal{D}_n\,.$$

Now, let $\rho_{1,n} = \tilde{\rho}_{1,n} + k_n + \delta$, then it is easy to see that the pair $(\rho_{1,n}, \psi_{1,n})$ satisfies the following equation

$$\rho_{1,n}\psi_{1,n}(i) = \inf_{v_1 \in V_1(i)}$$
$$\left[\sum_{j \in S} \psi_{1,n}(j)\pi_{ij}(v_1, \hat{\xi}_2(i)) + c_1(i, v_1, \hat{\xi}_2(i))\psi_{1,n}(i)\right] \text{for } i \in \mathcal{D}_n,\ \psi_{1,n} \gneq 0.$$

From Assumption 2.3(iii) and using the above equation we have $\psi_{1,n}(i_0) > 0$. Thus by normalizing $\psi_{1,n}$ we obtain $\psi_{1,n}(i_0) = 1$. Therefore, it follows that the pair $(\rho_{1,n}, \psi_{1,n})$ satisfies the required HJB equation (3.1).

Now, following [6,Lemma 3.3] one can show that $\rho_{1,n}$ satisfies (3.2) and $\{\rho_{1,n}\}$ is a bounded sequence. □

Next by taking limit $n \to \infty$ we show that the limiting equations admit eigenpairs in appropriate spaces. In particular, we have the following theorem.

Theorem 3.1 *Suppose that Assumptions 2.1, 2.2, and 2.3 are satisfied. Then the following hold.*

(1) *For $\hat{\xi}_2 \in \mathcal{U}_2^{SM}$, there exists a unique minimal eigenpair $(\rho_1, \psi_1) \in \mathbb{R}_+ \times L_W^{1,\infty}$, $\psi_1 > 0$, satisfying*

$$\begin{cases} \rho_1\psi_1(i) = \inf_{v_1 \in V_1(i)}\left[\sum_{j \in S}\psi_1(j)\pi_{ij}(v_1, \hat{\xi}_2(i)) + c_1(i, v_1, \hat{\xi}_2(i))\psi_1(i)\right] \textit{for } i \in S, \\ \psi_1(i_0) = 1. \end{cases} \tag{3.5}$$

Moreover, we have

$$\rho_1 = \inf_{\xi_1\in\mathcal{U}_1^{Ad}} \limsup_{T\to\infty} \frac{1}{T} \ln E_i^{\xi_1,\hat{\xi}_2}\Big[e^{\int_0^T c_1(Y(t),\xi_1(t),\hat{\xi}_2(Y(t-)))dt}\Big]$$
$$(:= \rho_1^{\hat{\xi}_2} = \inf_{\xi_1\in\mathcal{U}_1^{Ad}} \rho_1^{\xi_1,\hat{\xi}_2}), \tag{3.6}$$

and there exists a finite set $\mathscr{B}_1 \supset \mathscr{K}$, such that

$$\psi_1(i) = \inf_{\xi_1\in\mathcal{U}_1^{SM}} E_i^{\xi_1,\hat{\xi}_2}\Big[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}(c_1(Y(t),\xi_1(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_1)dt}\psi_1(Y(\hat{\tau}(\mathscr{B}_1)))\Big]$$
$$(:= \psi_1^{\hat{\xi}_2}(i))\ \forall i \in \mathscr{B}_1^c, \tag{3.7}$$

where $\hat{\tau}(\mathscr{B}_1) = \tau(\mathscr{B}_1^c) = \inf\{t : Y(t) \in \mathscr{B}_1\} =: \tilde{\tau}_1$.

(2) *Similarly, for $\hat{\xi}_1 \in \mathcal{U}_1^{SM}$, there exists a unique minimal eigenpair $(\rho_2, \psi_2) \in \mathbb{R}_+ \times L_W^{1,\infty}$, $\psi_2 > 0$ satisfying*

$$\begin{cases} \rho_2\psi_2(i) = \inf_{v_2\in V_2(i)}\Big[\sum_{j\in S}\psi_2(j)\pi_{ij}(\hat{\xi}_1(i), v_2) + c_2(i,\hat{\xi}_1(i), v_2)\psi_2(i)\Big] \text{ for } i \in S, \\ \psi_2(i_0) = 1. \end{cases} \tag{3.8}$$

Moreover, we have

$$\rho_2 = \inf_{\xi_2\in\mathcal{U}_2^{Ad}} \limsup_{T\to\infty} \frac{1}{T} \ln E_i^{\hat{\xi}_1,\xi_2}\Big[e^{\int_0^T c_2(Y(t),\hat{\xi}_1(Y(t-)),\xi_2(t))dt}\Big]$$
$$(:= \rho_2^{\hat{\xi}_1} = \inf_{\xi_2\in\mathcal{U}_2^{Ad}} \rho_2^{\hat{\xi}_1,\xi_2}), \tag{3.9}$$

and there exists a finite set $\mathscr{B}_2 \supset \mathscr{K}$, such that

$$\psi_2(i) = \inf_{\xi_2\in\mathcal{U}_2^{SM}} E_i^{\hat{\xi}_1,\xi_2}\Big[e^{\int_0^{\hat{\tau}(\mathscr{B}_2)}(c_2(Y(t),\hat{\xi}_1(Y(t-)),\xi_2(Y(t-)))-\rho_2)dt}\psi_2(Y(\hat{\tau}(\mathscr{B}_2)))\Big]$$
$$(:= \psi_2^{\hat{\xi}_1}(i))\ \forall i \in \mathscr{B}_2^c, \tag{3.10}$$

where $\hat{\tau}(\mathscr{B}_2) = \tau(\mathscr{B}_2^c) = \inf\{t : Y(t) \in \mathscr{B}_2\} =: \tilde{\tau}_2$.

Proof Since $c_1 \geq 0$, using Assumption 2.2, we deduce that there exists a finite set $\mathscr{B}_1$ containing $\mathscr{K}$ such that

- under Assumption 2.2 (a), since $\gamma > \|\bar{c}_1\|_\infty$, we have

$$\sup_{(u_1,u_2)\in U_1(i)\times U_2(i)} \bar{c}_1(i,u_1,u_2) - \rho_{1,n} < \gamma \quad \forall\ i \in \mathscr{B}_1^c \quad \text{and all } n \text{ large enough}.$$

- under Assumption 2.2 (b), since the function $\ell(\cdot) - \max_{(u_1,u_2)\in U_1(\cdot)\times U_2(\cdot)} \bar{c}_1(\cdot, u_1, u_2)$ is norm-like, we have

$$\sup_{(u_1,u_2)\in U_1(i)\times U_2(i)} \bar{c}_1(i, u_1, u_2) - \rho_{1,n} < \ell(i) \quad \forall\ i \in \mathscr{B}_1^c \quad \text{and all } n \text{ large enough}\,.$$

Let $\xi_1 \in \mathcal{U}_1^{SM}$. Then applying Itô-Dynkin formula, from Assumption 2.2, we prove the following estimates:

- Under Assumption 2.2(a):

$$E_i^{\xi_1,\hat{\xi}_2}\left[e^{\hat{\tau}(\mathscr{B}_1)\gamma} W(Y(\hat{\tau}(\mathscr{B}_1)))\right] \leq W(i) \ \ \forall i \in \mathscr{B}_1^c\,. \tag{3.11}$$

- Under Assumption 2.2(b):

$$E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} \ell(Y(t))dt} W(Y(\hat{\tau}(\mathscr{B}_1)))\right] \leq W(i) \ \ \forall i \in \mathscr{B}_1^c\,. \tag{3.12}$$

It is easy to see that the proof of (3.11) is analogous to that the proof of (3.12) when we replace ℓ with γ. So, we prove only (3.12). Suppose Assumption 2.2 (b) holds. Let n be large enough so that $\mathscr{B}_1 \subset \mathcal{D}_n$. Applying Dynkin's formula [19,Appendix C.3], for $i \in \mathscr{B}_1^c \cap \mathcal{D}_n$ and $T > 0$, we have

$$\begin{aligned}
&E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)\wedge T\wedge\tau(\mathcal{D}_n)} \ell(Y(s))ds} W(\hat{\tau}(\mathscr{B}_1) \wedge T \wedge \tau(\mathcal{D}_n))\right] - W(i) \\
&= E_i^{\xi_1,\hat{\xi}_2}\bigg[\int_0^{\hat{\tau}(\mathscr{B}_1)\wedge T\wedge\tau(\mathcal{D}_n)} e^{\int_0^t \ell(Y(s))ds}[\ell(Y(t))W(Y(t)) \\
&\quad + \sum_{j\in S} \pi_{Y(t)j}(\xi_1(Y(t-)), \hat{\xi}_2(Y(t-)))W(j)]dt\bigg] \\
&\leq E_i^{\xi_1,\hat{\xi}_2}\left[\int_0^{\hat{\tau}(\mathscr{B}_1)\wedge T\wedge\tau(\mathcal{D}_n)} e^{\int_0^t \ell(Y(s))ds} C_4 I_{\mathscr{K}}(Y(t))dt\right] = 0,
\end{aligned}$$

where $\tau(\mathcal{D}_n) = \inf\{t > 0 : Y(t) \notin \mathcal{D}_n\}$ (as defined in Lemma 3.2). Now by Fatou's lemma, taking first $n \to \infty$ and then $T \to \infty$, we get (3.12). Now we scale $\psi_{1,n}$ in such a way that it touches W from below. Define

$$\hat{\theta}_n = \sup\{k > 0 : (W - k\psi_{1,n}) > 0 \text{ in } S\}.$$

Then we see that $\hat{\theta}_n$ is finite as $\psi_{1,n}$ vanishes in $\mathcal{D}_n^c$ and $\psi_{1,n} \gneq 0$. Also, it is easy to see that $\hat{\theta}_n\psi_{1,n} \leq W$. We claim that if we replace $\psi_{1,n}$ by $\hat{\theta}_n\psi_{1,n}$, then $\psi_{1,n}$ touches W inside $\mathscr{B}_1$. If not, then for some state $\hat{i} \in \mathscr{B}_1^c$, $(W - \psi_{1,n})(\hat{i}) = 0$ and $W - \psi_{1,n} > 0$

in $\mathscr{B}_1 \cup \mathcal{D}_n^c$. Then by Dynkin formula, we get (under Assumption 2.2 (b))

$$\psi_{1,n}(\hat{i}) \leq E_{\hat{i}}^{\xi_1,\hat{\xi}_2}\Bigg[e^{\int_0^{T\wedge\hat{\tau}(\mathscr{B}_1)}(c(Y(s),\xi_1(Y(s-)),\hat{\xi}_2(Y(s-)))-\rho_{1,n})ds}\psi_{1,n}(Y(T\wedge\hat{\tau}(\mathscr{B}_1)))$$
$$I_{\{T\wedge\hat{\tau}(\mathscr{B}_1)<\tau(\mathcal{D}_n)\}}\Bigg]$$
$$\leq E_{\hat{i}}^{\xi_1,\hat{\xi}_2}\Bigg[e^{\int_0^{T\wedge\hat{\tau}(\mathscr{B}_1)}\ell(Y(s))ds}\psi_{1,n}(Y(T\wedge\hat{\tau}(\mathscr{B}_1)))I_{\{T\wedge\hat{\tau}(\mathscr{B}_1)<\tau(\mathcal{D}_n)\}}\Bigg].$$

Since $\psi_{1,n} \leq W$, in view of (3.12), by the dominated convergence theorem, taking $T \to \infty$, we get

$$\psi_{1,n}(\hat{i}) \leq E_{\hat{i}}^{\xi_1,\hat{\xi}_2}\Bigg[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}\ell(Y(s))ds}\psi_{1,n}(Y(\hat{\tau}(\mathscr{B}_1)))\Bigg].$$

Using this and (3.12), we have

$$0=(W-\psi_{1,n})(\hat{i}) \geq E_{\hat{i}}^{\xi_1,\hat{\xi}_2}\Bigg[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}\ell(Y(s))ds}(W-\psi_{1,n})(Y(\hat{\tau}(\mathscr{B}_1)))\Bigg] > 0.$$

Hence we arrive at a contradiction. Thus $\psi_{1,n}$ touches W inside $\mathscr{B}_1$. Similar conclusion holds under Assumption 2.2 (a). Therefore, $\psi_{1,n} \leq W$ and at some point $\hat{i}^* \in \mathscr{B}_1$, $\psi_{1,n}(\hat{i}^*) = W(\hat{i}^*)$.

Since $\psi_{1,n} \leq W$ for all n large enough, by diagonalization argument we deduce that along a suitable subsequence $\psi_{1,n}(i) \to \psi_1(i)$ for all $i \in S$, for some $\psi_1 \in L_W^{1,\infty}$. Also, from Lemma 3.2, we have $\{\rho_{1,n}\}$ is a bounded sequence. Thus along a further subsequence we have $\rho_{1,n} \to \rho_1$ as $n \to \infty$. Let $\tilde{\xi}_1^n \in \mathcal{U}_1^{SM}$ be a minimizing selector of (3.1), i.e., we have

$$\rho_{1,n}\psi_{1,n}(i) = \Bigg[\sum_{j\in S}\psi_{1,n}(j)\pi_{ij}(\tilde{\xi}_1^n(i),\hat{\xi}_2(i)) + c_1(i,\tilde{\xi}_1^n(i),\hat{\xi}_2(i))\psi_{1,n}(i)\Bigg] \text{ for } i \in \mathcal{D}_n\,. \tag{3.13}$$

Since $\mathcal{U}_1^{SM}$ is compact along further subsequence $\tilde{\xi}_1^n \to \tilde{\xi}_1$ in $\mathcal{U}_1^{SM}$. Therefore, by generalized Fatou's lemma [25,Lemma 8.3.7], letting $n \to \infty$, from (3.13) it follows that

$$\rho_1\psi_1(i) \geq \inf_{v_1\in V_1(i)}\Bigg[\sum_{j\in S}\psi_1(j)\pi_{ij}(v_1,\hat{\xi}_2(i)) + c_1(i,v_1,\hat{\xi}_2(i))\psi_1(i)\Bigg] \text{ for } i \in S\,. \tag{3.14}$$

Also, from (3.1), for any $v_1 \in V_1(i)$, we have

$$\rho_{1,n}\psi_{1,n}(i) \leq \Bigg[\sum_{j\in S}\psi_{1,n}(j)\pi_{ij}(v_1,\hat{\xi}_2(i)) + c_1(i,v_1,\hat{\xi}_2(i))\psi_{1,n}(i)\Bigg] \text{ for } i \in \mathcal{D}_n\,.$$

Since $\psi_{1,n} \leq W$, by the dominated convergence theorem, letting $n \to \infty$ we deduce

$$\rho_1 \psi_1(i) \leq \Big[\sum_{j \in S} \psi_1(j) \pi_{ij}(v_1, \hat{\xi}_2(i)) + c_1(i, v_1, \hat{\xi}_2(i)) \psi_1(i) \Big]. \tag{3.15}$$

Therefore, combining (3.14) and (3.15), it follows that the pair $(\rho_1, \psi_1) \in \mathbb{R}_+ \times L^{1,\infty}_W$, $\psi_1 \geq 0$ satisfies

$$\rho_1 \psi_1(i) = \inf_{v_1 \in V_1(i)} \Big[\sum_{j \in S} \psi_1(j) \pi_{ij}(v_1, \hat{\xi}_2(i)) + c_1(i, v_1, \hat{\xi}_2(i)) \psi_1(i) \Big] \text{ for } i \in S.$$

Since at some point in $\mathscr{B}_1$ we have $(W - \psi_{1,n}) = 0$, for all large n, we have $(W - \psi_1)(\hat{i}^*) = 0$ for some $\hat{i}^* \in \mathscr{B}_1$. Since $W \geq 1$, it is clear that ψ_1 is nontrivial. Now we claim that $\psi_1 > 0$. If not, we must have $\psi_1(\tilde{i}) = 0$ for some $\tilde{i} \in S$. Then, for any minimizing selector $\tilde{\xi}_1^* \in \mathcal{U}_1^{SM}$ of (3.5), it follows that

$$\rho_1 \psi_1(\tilde{i}) = \Big[\sum_{j \in S} \psi_1(j) \pi_{\tilde{i}j}(\tilde{\xi}_1^*(\tilde{i}), \hat{\xi}_2(\tilde{i})) + c(\tilde{i}, \tilde{\xi}_1^*(\tilde{i}), \hat{\xi}_2(\tilde{i})) \psi_1(\tilde{i}) \Big].$$

This implies

$$\sum_{j \neq \tilde{i}} \psi_1(j) \pi_{\tilde{i}j}(\tilde{\xi}_1^*(\tilde{i}), \hat{\xi}_2(\tilde{i})) = 0.$$

Since the Markov chain $Y(t)$ is irreducible under $(\tilde{\xi}_1^*, \hat{\xi}_2) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, from the above equation, it follows that $\psi_1 \equiv 0$. So, we arrive at a contradiction. This proves that (ρ_1, ψ_1) is an eigenpair to (3.5).

By truncating the running cost c_1, one can show that ρ_1 satisfies (3.6) (see, [6,Lemma 3.5]). Next we prove the stochastic representation (3.7).

Applying Itô-Dynkin formula for any minimizing selector ξ_1^* of (3.5) and any $T > 0$, we have

$$\psi_1(i) = E_i^{\xi_1^*, \hat{\xi}_2} \Big[e^{\int_0^{\hat{\tau}(\mathscr{B}_1) \wedge T} (c_1(Y(t), \xi_1^*(Y(t-)), \hat{\xi}_2(Y(t-))) - \rho_1) dt} \psi_1(Y(\hat{\tau}(\mathscr{B}_1) \wedge T)) \Big] \; \forall i \in \mathscr{B}_1^c.$$

Then applying Fatou's lemma, by taking $T \to \infty$, we get

$$\begin{aligned} \psi_1(i) &\geq E_i^{\xi_1^*, \hat{\xi}_2} \Big[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \xi_1^*(Y(t-)), \hat{\xi}_2(Y(t-))) - \rho_1) dt} \psi_1(Y(\hat{\tau}(\mathscr{B}_1))) \Big] \\ &\geq \inf_{\xi_1 \in \mathcal{U}_1^{SM}} E_i^{\xi_1, \hat{\xi}_2} \Big[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \xi_1(Y(t-)), \hat{\xi}_2(Y(t-))) - \rho_1) dt} \psi_1(Y(\hat{\tau}(\mathscr{B}_1))) \Big] \; \forall i \in \mathscr{B}_1^c. \end{aligned} \tag{3.16}$$

Again, by applying Itô-Dynkin formula, from (3.1) for any $\xi_1 \in \mathcal{U}_1^{SM}$, $T > 0$ and $i \in \mathcal{D}_n \cap \mathscr{B}_1^c$ it follows that

$$\begin{aligned}
\psi_{1,n}(i) &\leq E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)\wedge\tau(\mathcal{D}_n)\wedge T}(c_1(Y(t),\xi_1(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_{1,n})dt}\psi_{1,n}(Y(\hat{\tau}(\mathscr{B}_1)\wedge\tau(\mathcal{D}_n)\wedge T))\right] \\
&\leq E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}(c_1(Y(t),\xi_1(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_{1,n})dt}\psi_{1,n}(Y(\hat{\tau}(\mathscr{B}_1)))I_{\{\hat{\tau}(\mathscr{B}_1)\leq\tau(\mathcal{D}_n)\wedge T\}}\right] \\
&\quad + E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{T}(c_1(Y(t),\xi_1(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_{1,n})dt}\psi_{1,n}(Y(T))I_{\{T\leq\hat{\tau}(\mathscr{B}_1)\wedge\tau(\mathcal{D}_n)\}}\right]. \qquad (3.17)
\end{aligned}$$

Under Assumption 2.2 (a), the estimate (3.11) and the fact that $\psi_{1,n} \leq W$ (from the construction of $\hat{\theta}_n$, it is clear that if we replace $\psi_{1,n}$ by $\hat{\theta}_n\psi_{1,n}$, we get this inequality), we have

$$\begin{aligned}
&E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{T}(c_1(Y(t),\xi_1(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_{1,n})dt}\psi_{1,n}(Y(T))I_{\{T\leq\hat{\tau}(\mathscr{B}_1)\wedge\tau(\mathcal{D}_n)\}}\right] \\
&\leq e^{(\|c_1\|_\infty-\rho_{1,n}-\gamma)T}E_i^{\xi_1,\hat{\xi}_2}\left[e^{T\gamma}W(Y(T))I_{\{T\leq\hat{\tau}(\mathscr{B}_1)\wedge\tau(\mathcal{D}_n)\}}\right] \\
&\leq e^{(\|c_1\|_\infty-\rho_{1,n}-\gamma)T}W(i)\,.
\end{aligned}$$

Thus, letting $T \to \infty$ from (3.17) we get

$$\psi_{1,n}(i) \leq E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}(c_1(Y(t),\xi_1((Y(t-))),\hat{\xi}_2(Y(t-)))-\rho_{1,n})dt}\psi_{1,n}(Y(\hat{\tau}(\mathscr{B}_1)))I_{\{\hat{\tau}(\mathscr{B}_1)\leq\tau(\mathcal{D}_n)\}}\right].$$

Again, since $\psi_{1,n} \leq W$, using (3.11) and applying the dominated convergence theorem it follows that

$$\psi_1(i) \leq E_i^{\xi_1,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}(c_1(Y(t),\xi_1(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_1)dt}\psi_1(Y(\hat{\tau}(\mathscr{B}_1)))\right]\ \forall i \in \mathscr{B}_1^c\,. \qquad (3.18)$$

Since $\xi_1 \in \mathcal{U}_1^{SM}$ is arbitrary, combining (3.16) and (3.18), we obtain (3.7). Also, it it clear from the proof that for any minimizing selector ξ_1^* of (3.5) we have

$$\psi_1(i) = E_i^{\xi_1^*,\hat{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}(c_1(Y(t),\xi_1^*(Y(t-)),\hat{\xi}_2(Y(t-)))-\rho_1)dt}\psi_1(Y(\hat{\tau}(\mathscr{B}_1)))\right]\ \forall i \in \mathscr{B}_1^c\,. \qquad (3.19)$$

Using (3.12) it is easy to check that the same conclusion holds under Assumption 2.2(b).

Now exploiting the stochastic representation (3.7), we show that $(\rho_1, \psi_1) \in \mathbb{R}_+ \times L_W^{1,\infty}$ is the minimal eigenpair. Suppose $(\hat{\rho}_1, \hat{\psi}_1) \in \mathbb{R}_+ \times L_W^{1,\infty}$, $\hat{\psi}_1 > 0$ is an eigenpair satisfying

$$\begin{cases} \hat{\rho}_1 \hat{\psi}_1(i) = \inf_{v_1 \in V_1(i)} \left[\sum_{j \in S} \hat{\psi}_1(j) \pi_{ij}(v_1, \hat{\xi}_2(i)) + c_1(i, v_1, \hat{\xi}_2(i)) \hat{\psi}_1(i) \right] \text{ for } i \in S, \\ \hat{\psi}_1(i_0) = 1. \end{cases} \tag{3.20}$$

We want to show that $\rho_1 \leq \hat{\rho}_1$. If not suppose that $\rho_1 > \hat{\rho}_1$. Then, for any minimizing selector $\hat{\xi}_1^*$ of (3.20), applying Itô-Dynkin formula and Fatou's lemma, we obtain

$$\hat{\psi}_1(i) \geq E_i^{\hat{\xi}_1^*, \hat{\xi}_2} \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \hat{\xi}_1^*(Y(t-)), \hat{\xi}_2(Y(t-))) - \hat{\rho}_1) dt} \hat{\psi}_1(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c. \tag{3.21}$$

On the other hand from (3.7), we have

$$\psi_1(i) \leq E_i^{\hat{\xi}_1^*, \hat{\xi}_2} \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \hat{\xi}_1^*(Y(t-)), \hat{\xi}_2(Y(t-))) - \hat{\rho}_1) dt} \psi_1(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c. \tag{3.22}$$

Let $\hat{\kappa} := \min_{\mathscr{B}_1} \frac{\hat{\psi}_1}{\psi_1}$. Hence, from (3.21) and (3.22) it follows that $(\hat{\psi}_1 - \hat{\kappa} \psi_1) \geq 0$ in S and $(\hat{\psi}_1 - \hat{\kappa} \psi_1)(\tilde{i}_0) = 0$ for some $\tilde{i}_0 \in \mathscr{B}_1$. Now, combining (3.5) and (3.20) we deduce that

$$\left[\sum_{j \neq \tilde{i}_0} (\hat{\psi}_1 - \hat{\kappa} \psi_1)(j) \pi_{\tilde{i}_0 j}(\hat{\xi}_1^*(\tilde{i}_0), \hat{\xi}_2(\tilde{i}_0)) \right] \equiv 0. \tag{3.23}$$

Since $Y(t)$ is irreducible under $(\hat{\xi}_1^*, \hat{\xi}_2)$, in view of (3.23) it is clear that $(\hat{\psi}_1 - \hat{\kappa} \psi_1) \equiv 0$. Again, since $\hat{\psi}_1(i_0) = \psi_1(i_0) = 1$, we get $\hat{\psi}_1 \equiv \psi_1$. But, this is a contradiction to the fact that $\rho_1 > \hat{\rho}_1$. Thus we deduce that $(\rho_1, \psi_1) \in \mathbb{R}_+ \times L_W^{1,\infty}$ is the minimal eigenpair. Following the above argument one can show that any eigenfunction satisfying (3.7) is unique up to a scalar multiplication. Also, by the similar argument, one can show that there exists a minimal eigenpair $(\rho_2, \psi_2) \in \mathbb{R}_+ \times L_W^{1,\infty}$ satisfying (3.8), (3.9) and (3.10). This completes the proof. □

Remark 3.1 We can replace Assumption 2.3 (iii) by other similar assumption. For example, if the killed process communicates with every state from i_0 before leaving the domain $\mathcal{D}_n$, for large n, then our method applies. More precisely, for every $\mathcal{D}_n$, $(\xi_1, \xi_2) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$ and for every $j \in \mathcal{D}_n \backslash \{i_0\}$, if there exists distinct $i_1, i_2, \cdots, i_m \in \mathcal{D}_n \backslash \{i_0\}$ satisfying

$$\pi_{i_0 i_1}(\xi_1(i_0), \xi_2(i_0)) \pi_{i_1 i_2}(\xi_1(i_1), \xi_2(i_1)) \cdots \pi_{i_m j}(\xi_1(i_m), \xi_2(i_m)) > 0,$$

then we get $\psi_{1,n}(i_0) > 0$ in $\mathcal{D}_n$ (see Lemma 3.2). Also, the conclusion of Theorem 3.1 holds.

To proceed further we establish some technical results needed later.

Lemma 3.3 *Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then the maps* $\hat{\xi}_1 \to \psi_2^{\hat{\xi}_1}$ *from* $\mathcal{U}_1^{SM} \to L_W^\infty$, $\hat{\xi}_1 \to \rho_2^{\hat{\xi}_1}$ *from* $\mathcal{U}_1^{SM} \to \mathbb{R}_+$, $\hat{\xi}_2 \to \psi_1^{\hat{\xi}_2}$ *from* $\mathcal{U}_2^{SM} \to L_W^\infty$, *and* $\hat{\xi}_2 \to \rho_1^{\hat{\xi}_2}$ *from* $\mathcal{U}_2^{SM} \to \mathbb{R}_+$ *are continuous.*

Proof Let $\{\xi_{2,n}\}$ be a sequence in $\mathcal{U}_2^{SM}$ such that $\xi_{2,n} \to \tilde{\xi}_2$ in $\mathcal{U}_2^{SM}$, i.e., for each $i \in S$, $\xi_{2,n}(i) \to \tilde{\xi}_2(i)$ in $V_2(i)$. Now by Theorem 3.1, there exists $(\rho_1^{\xi_{2,n}}, \psi_1^{\xi_{2,n}}) \in \mathbb{R}_+ \times L_W^{1,\infty}$, $\psi_1^{\xi_{2,n}} > 0$ satisfying

$$\rho_1^{\xi_{2,n}} \psi_1^{\xi_{2,n}}(i) = \inf_{v_1 \in V_1(i)} \left[\sum_{j \in S} \psi_1^{\xi_{2,n}}(j) \pi_{ij}(v_1, \xi_{2,n}(i)) + c_1(i, v_1, \xi_{2,n}(i)) \psi_1^{\xi_{2,n}}(i) \right], \tag{3.24}$$

with $\psi_1^{\xi_{2,n}}(i_0) = 1$. Now, since $\psi_1^{\xi_{2,n}} \in L_W^{1,\infty}$, by a standard diagonalization argument, there exists a function $\psi_1^* \in L_W^{1,\infty}$ such that $\psi_1^{\xi_{2,n}}(i) \to \psi_1^*(i)$ as $n \to \infty$ for all $i \in S$. Also, $\{\rho_1^{\xi_{2,n}}\}$ is a bounded sequence. Hence, along a suitable subsequence (without loss of generality denoting by the same notation) $\rho_1^{\xi_{2,n}} \to \rho_1^*$. Now from (3.24), for any $v_1 \in V_1(i)$ we deduce that

$$\rho_1^{\xi_{2,n}} \psi_1^{\xi_{2,n}}(i) \le \left[\sum_{j \in S} \psi_1^{\xi_{2,n}}(j) \pi_{ij}(v_1, \xi_{2,n}(i)) + c_1(i, v_1, \xi_{2,n}(i)) \psi_1^{\xi_{2,n}}(i) \right].$$

This implies that

$$\begin{aligned} &\rho_1^{\xi_{2,n}} \psi_1^{\xi_{2,n}}(i) - \psi_1^{\xi_{2,n}}(i) \pi_{ii}(v_1, \xi_{2,n}(i)) \\ &\qquad \le \left[\sum_{j \ne i} \psi_1^{\xi_{2,n}}(j) \pi_{ij}(v_1, \xi_{2,n}(i)) + c_1(i, v_1, \xi_{2,n}(i)) \psi_1^{\xi_{2,n}}(i) \right]. \end{aligned} \tag{3.25}$$

Note that

$$\sum_{j \ne i} \psi_1^{\xi_{2,n}}(j) \pi_{ij}(v_1, \xi_{2,n}(i)) \le \sum_{j \ne i} W(j) \pi_{ij}(v_1, \xi_{2,n}(i)). \tag{3.26}$$

Thus, using Lemma 3.1, generalized Fatou's lemma in [25,Lemma 8.3.7] and taking $n \to \infty$ in (3.25), we get

$$\rho_1^* \psi_1^*(i) \le \left[\sum_{j \in S} \psi_1^*(j) \pi_{ij}(v_1, \tilde{\xi}_2(i)) + c_1(i, v_1, \tilde{\xi}_2(i)) \psi_1^*(i) \right].$$

Hence,

$$\rho_1^* \psi_1^*(i) \le \inf_{v_1 \in V_1(i)} \left[\sum_{j \in S} \psi_1^*(j) \pi_{ij}(v_1, \tilde{\xi}_2(i)) + c_1(i, v_1, \tilde{\xi}_2(i)) \psi_1^*(i) \right]. \tag{3.27}$$

Let $\xi_{1,n}^* \in \mathcal{U}_1^{SM}$ be a minimizing selector of (3.24), i.e.,

$$\rho_1^{\xi_{2,n}} \psi_1^{\xi_{2,n}}(i) = \left[\sum_{j \in S} \psi_1^{\xi_{2,n}}(j) \pi_{ij}(\xi_{1,n}^*(i), \xi_{2,n}(i)) + c_1(i, \xi_{1,n}^*(i), \xi_{2,n}(i)) \psi_1^{\xi_{2,n}}(i) \right]. \tag{3.28}$$

Since $\mathcal{U}_1^{SM}$ is compact under the product topology, there exists $\xi_1^* \in \mathcal{U}_1^{SM}$ such that along a subsequence (without loss of generality denoting by the same notation) $\xi_{1,n}^* \to \xi_1^*$.

Now, using Lemma 3.1, the dominated convergence theorem and passing $n \to \infty$ in (3.28), we obtain

$$\rho_1^* \psi_1^*(i) = \left[\sum_{j \in S} \psi_1^*(j) \pi_{ij}(\xi_1^*(i), \tilde{\xi}_2(i)) + c_1(i, \xi_1^*(i), \tilde{\xi}_2(i)) \psi_1^*(i) \right].$$

Therefore

$$\rho_1^* \psi_1^*(i) \ge \inf_{v_1 \in V_1(i)} \left[\sum_{j \in S} \psi_1^*(j) \pi_{ij}(v_1, \tilde{\xi}_2(i)) + c_1(i, v_1, \tilde{\xi}_2(i)) \psi_1^*(i) \right]. \tag{3.29}$$

Hence, from (3.27), and (3.29), it follows that

$$\rho_1^* \psi_1^*(i) = \inf_{v_1 \in V_1(i)} \left[\sum_{j \in S} \psi_1^*(j) \pi_{ij}(v_1, \tilde{\xi}_2(i)) + c_1(i, v_1, \tilde{\xi}_2(i)) \psi_1^*(i) \right]. \tag{3.30}$$

Since $\rho_1^{\tilde{\xi}_2}$ is the minimal eigenvalue corresponding to $\tilde{\xi}_2$ of (3.30), we have $\rho_1^* \ge \rho_1^{\tilde{\xi}_2}$. Suppose $\rho_1^* > \rho_1^{\tilde{\xi}_2}$. Now, from Theorem 3.1, for any minimizing $\hat{\xi}_1 \in \mathcal{U}_1^{SM}$ of (3.5), there exists a finite set $\mathscr{B}_1 \supset \mathscr{K}$, such that

$$\psi_1(i) = E_i^{\hat{\xi}_1, \tilde{\xi}_2} \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \hat{\xi}_1(Y(t-)), \tilde{\xi}_2(Y(t-))) - \rho_1^{\tilde{\xi}_2}) dt} \psi_1(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c, \tag{3.31}$$

where $\hat{\tau}(\mathscr{B}_1)$ is a stopping time define as in Theorem 3.1. Since $\rho_1^* > \rho_1^{\tilde{\xi}_2}$, by similar arguments as in [6,Lemma 3.4] we deduce that

$$\psi_1^*(i) \le E_i^{\hat{\xi}_1, \tilde{\xi}_2} \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \hat{\xi}_1(Y(t-)), \tilde{\xi}_2(Y(t-))) - \rho_1^{\tilde{\xi}_2}) dt} \psi_1^*(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c. \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$(\psi_1-\psi_1^*)(i)\geq E_i^{\hat{\xi}_1,\tilde{\xi}_2}\left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)}(c_1(Y(t),\hat{\xi}_1(Y(t-)),\tilde{\xi}_2(Y(t-)))-\rho_1^{\tilde{\xi}_2})dt}(\psi_1-\psi_1^*)(Y(\hat{\tau}(\mathscr{B}_1)))\right]\ \forall i\in\mathscr{B}_1^c. \tag{3.33}$$

Now choosing an appropriate constant θ (e.g., $\theta=\max_{\mathscr{B}_1}\frac{\psi_1}{\psi_1^*}$), we have $(\psi_1-\theta\psi_1^*)\geq 0$ in $\mathscr{B}_1$ and for some $\hat{i}_0\in\mathscr{B}_1$, $(\psi_1-\theta\psi_1^*)(\hat{i}_0)=0$. Thus, in view of (3.33), we get $(\psi_1-\theta\psi_1^*)\geq 0$ in S. Now combining (3.5) and (3.30), we get

$$\rho_1^{\tilde{\xi}_2}(\psi_1-\theta\psi_1^*)(\hat{i}_0)\geq\left[\sum_{j\in S}(\psi_1-\theta\psi_1^*)(j)\pi_{\hat{i}_0j}(\hat{\xi}_1(\hat{i}_0),\tilde{\xi}_2(\hat{i}_0))+c_1(\hat{i}_0,\hat{\xi}_1(\hat{i}_0),\tilde{\xi}_2(\hat{i}_0))(\psi_1-\theta\psi_1^*)(\hat{i}_0)\right].$$

This implies that

$$\sum_{j\neq\hat{i}_0}(\psi_1-\theta\psi_1^*)(j)\pi_{\hat{i}_0j}(\hat{\xi}_1(\hat{i}_0),\tilde{\xi}_2(\hat{i}_0))=0\,. \tag{3.34}$$

Since, $\{Y(t)\}_{t\geq 0}$ is irreducible under $(\hat{\xi}_1,\tilde{\xi}_2)\in\mathcal{U}_1^{SM}\times\mathcal{U}_2^{SM}$, from (3.34) it follows that $\psi_1\equiv\theta\psi_1^*$. But, this is a contradiction to the fact that $\rho_1^*>\rho_1^{\tilde{\xi}_2}$. Hence, we deduce that $\rho_1^*=\rho_1^{\tilde{\xi}_2}$. This proves the continuity of the map $\hat{\xi}_2\to\rho_1^{\hat{\xi}_2}$. Since $\psi_1^{\hat{\xi}_{2,n}}(i_0)=1$ for all $n\geq 1$, we have $\psi_1^*(i_0)=1$. Hence by Theorem 3.1, we have ψ_1^* is the unique solution of (3.5). Thus $\psi_1^*=\psi_1^{\tilde{\xi}_2}$. This proves the continuity of the map $\hat{\xi}_2\to\psi_1^{\hat{\xi}_2}$. Continuity of other maps follows by the similar argument. □

Fix $\hat{\xi}_2\in\mathcal{U}_2^{SM}$. For each $i\in S$, $v_1\in V_1(i)$, set

$$\tilde{F}_1(i,v_1,\hat{\xi}_2(i))=\left[\sum_{j\in S}\psi_1^{\hat{\xi}_2}(j)\pi_{ij}(v_1,\hat{\xi}_2(i))+c_1(i,v_1,\hat{\xi}_2(i))\psi_1^{\hat{\xi}_2}(i)\right],$$

where $\psi_1^{\hat{\xi}_2}$ is the solution of (3.5) corresponding to the strategy $\hat{\xi}_2\in\mathcal{U}_2^{SM}$. Let

$$\tilde{H}(\hat{\xi}_2)=\left\{\hat{\xi}_1^*\in\mathcal{U}_1^{SM}:\tilde{F}_1(i,\hat{\xi}_1^*(i),\hat{\xi}_2(i))=\inf_{v_1\in V_1(i)}\tilde{F}_1(i,v_1,\hat{\xi}_2(i))\ \forall\, i\in S\right\}.$$

By Lemma 3.1, we know that the functions $c_1(i,v_1,\hat{\xi}_2(i))\psi_1^{\hat{\xi}_2}(i)$ and $\sum_{j\in S}\psi_1^{\hat{\xi}_2}(j)\pi_{ij}(v_1,\hat{\xi}_2(i))$ are continuous on $V_1(i)\times V_2(i)$ for each $i\in S$. Also since $V_1(i)$ is compact for each $i\in S$, it is easy to see that $\tilde{H}(\hat{\xi}_2)$ is a non empty subset of $\mathcal{U}_1^{SM}$. From the definition of $\tilde{H}(\hat{\xi}_2)$ and the topology of $\mathcal{U}_1^{SM}$, it is clear that $\tilde{H}(\hat{\xi}_2)$ is convex and closed. Since $\mathcal{U}_1^{SM}$ is a compact metric space under the product topology, it follows that $\tilde{H}(\hat{\xi}_2)$ is also compact. Similarly, for $i \in S$, $\hat{\xi}_1 \in \mathcal{U}_1^{SM}$, $v_2 \in V_2(i)$, we set

$$\tilde{F}_2(i, \hat{\xi}_1(i), v_2) = \left[\sum_{j\in S} \psi_2^{\hat{\xi}_1}(j)\pi_{ij}(\hat{\xi}_1(i), v_2) + c_2(i, \hat{\xi}_1(i), v_2)\psi_2^{\hat{\xi}_1}(i)\right],\ i \in S,$$

where $\psi_2^{\hat{\xi}_1}$ is the solution of (3.8) corresponding to the strategy $\hat{\xi}_1 \in \mathcal{U}_1^{SM}$. Let

$$\tilde{H}(\hat{\xi}_1) = \left\{\hat{\xi}_2^* \in \mathcal{U}_2^{SM} : \tilde{F}_2(i, \hat{\xi}_1(i), \hat{\xi}_2^*(i)) = \inf_{v_2\in V_2(i)} \tilde{F}_2(i, \hat{\xi}_1(i), v_2)\ \forall\, i \in S\right\}.$$

Then by analogous arguments, $\tilde{H}(\hat{\xi}_1)$ is a nonempty, convex and compact subset of $\mathcal{U}_2^{SM}$. Next set

$$\tilde{H}(\hat{\xi}_1, \hat{\xi}_2) = \tilde{H}(\hat{\xi}_2) \times \tilde{H}(\hat{\xi}_1).$$

From the above argument it is clear that $\tilde{H}(\hat{\xi}_1, \hat{\xi}_2)$ is a nonempty, convex, and compact subset of $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$. Therefore we may define a map from $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM} \to 2^{\mathcal{U}_1^{SM}\times\mathcal{U}_2^{SM}}$.

3.1 Existence of Nash Equilibria

Next lemma proves the upper semicontinuity of certain set valued map. This result will be useful in establishing the existence of a Nash equilibrium in the space of stationary Markov strategies.

Lemma 3.4 *Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then the map* $(\hat{\xi}_1, \hat{\xi}_2) \to \tilde{H}(\hat{\xi}_1, \hat{\xi}_2)$ *from* $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM} \to 2^{\mathcal{U}_1^{SM}\times\mathcal{U}_2^{SM}}$ *is upper semicontinuous.*

Proof Let $\{(\xi_1^m, \xi_2^m)\} \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$ and $(\xi_1^m, \xi_2^m) \to (\hat{\xi}_1, \hat{\xi}_2)$ in $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, i.e., for each $i \in S$, $(\xi_1^m(i), \xi_2^m(i)) \to (\hat{\xi}_1(i), \hat{\xi}_2(i))$ in $V_1(i) \times V_2(i)$. Let $\bar{\xi}_1^m \in \tilde{H}(\xi_2^m)$. Then $\{\bar{\xi}_1^m\} \subset \mathcal{U}_1^{SM}$. Since $\mathcal{U}_1^{SM}$ is compact, it has a convergent subsequence (denoted by the same sequence by an abuse of notation), such that

$$\bar{\xi}_1^m \to \bar{\xi}_1 \text{ in } \mathcal{U}_1^{SM}.$$

Then $(\bar{\xi}_1^m, \xi_2^m) \to (\bar{\xi}_1, \hat{\xi}_2)$ in $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$. Note that

$$\sum_{j\neq i} \pi_{ij}(\bar{\xi}_1^m(i), \xi_2^m(i))\psi_1^{\xi_2^m}(j) \le \sum_{j\neq i} \pi_{ij}(\bar{\xi}_1^m(i), \xi_2^m(i))W(j).$$

Recall that by Lemma 3.3 the maps $\hat{\xi}_1 \to \psi_2^{\hat{\xi}_1}$, $\hat{\xi}_2 \to \psi_1^{\hat{\xi}_2}$, $\hat{\xi}_1 \to \rho_2^{\hat{\xi}_1}$, $\hat{\xi}_2 \to \rho_1^{\hat{\xi}_2}$ are continuous. Thus by generalized Fatou's lemma [25,Lemma 8.3.7], Assumption 2.3

and the (product) topology of $\mathcal{U}_k^{SM}$, $k = 1, 2$, it follows that for each $i \in S$,

$$\sum_{j \in S} \pi_{ij}(\overline{\xi}_1^m(i), \xi_2^m(i))\psi_1^{\xi_2^m}(j) + c_1(i, \overline{\xi}_1^m(i), \xi_2^m(i))\psi_1^{\xi_2^m}(i)$$

converges to

$$\sum_{j \in S} \pi_{ij}(\overline{\xi}_1(i), \hat{\xi}_2(i))\psi_1^{\hat{\xi}_2}(j) + c_1(i, \overline{\xi}_1(i), \hat{\xi}_2(i))\psi_1^{\hat{\xi}_2}(i).$$

Hence

$$\lim_{m \to \infty} \tilde{F}_1(i, \overline{\xi}_1^m(i), \xi_2^m(i)) = \tilde{F}_1(i, \overline{\xi}_1(i), \hat{\xi}_2(i)). \tag{3.35}$$

Now fix $\tilde{\xi}_1 \in \mathcal{U}_1^{SM}$ and consider the sequence $\{(\tilde{\xi}_1, \xi_2^m)\}$. Using analogous arguments as above, we conclude that

$$\lim_{m \to \infty} \tilde{F}_1(i, \tilde{\xi}_1(i), \xi_2^m(i)) = \tilde{F}_1(i, \tilde{\xi}_1(i), \hat{\xi}_2(i)). \tag{3.36}$$

Since $\overline{\xi}_1^m \in \tilde{H}(\xi_2^m)$, for any m we have

$$\tilde{F}_1(i, \tilde{\xi}_1(i), \xi_2^m(i)) \geq \tilde{F}_1(i, \overline{\xi}_1^m(i), \xi_2^m(i)).$$

Thus, in view of (3.35) and (3.36), taking $m \to \infty$ in the above equation, for any $\tilde{\xi}_1 \in \mathcal{U}_1^{SM}$ we get

$$\tilde{F}_1(i, \tilde{\xi}_1(i), \hat{\xi}_2(i)) \geq \tilde{F}_1(i, \overline{\xi}_1(i), \hat{\xi}_2(i)).$$

Therefore, $\overline{\xi}_1 \in \tilde{H}(\hat{\xi}_2)$. Suppose $\overline{\xi}_2^m \in \tilde{H}(\xi_1^m)$ and along a subsequence $\overline{\xi}_2^m \to \overline{\xi}_2$ in $\mathcal{U}_2^{SM}$. Then, by similar arguments as above one can show that $\overline{\xi}_2 \in \tilde{H}(\hat{\xi}_1)$. This proves that the map $(\hat{\xi}_1, \hat{\xi}_2) \to \tilde{H}(\hat{\xi}_1, \hat{\xi}_2)$ is upper semicontinuous. □

Theorem 3.2 *Suppose that Assumptions 2.1, 2.2, and 2.3 are satisfied. Then there exists a Nash equilibrium in the space of stationary Markov strategies* $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$.

Proof Since $\tilde{H}(\hat{\xi}_1^*, \hat{\xi}_2^*)$ is non-empty, compact and convex, using Lemma 3.4 and Fan's fixed point theorem [10], it follows that there exists a fixed point $(\hat{\xi}_1^*, \hat{\xi}_2^*) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, for the map $(\hat{\xi}_1, \hat{\xi}_2) \to \tilde{H}(\hat{\xi}_1, \hat{\xi}_2)$ from $\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM} \to 2^{\mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}}$, i.e.,

$$(\hat{\xi}_1^*, \hat{\xi}_2^*) \in \tilde{H}(\hat{\xi}_1^*, \hat{\xi}_2^*).$$

This implies that $(\rho_1^{\hat{\xi}_2^*}, \psi_1^{\hat{\xi}_2^*}), (\rho_2^{\hat{\xi}_1^*}, \psi_2^{\hat{\xi}_1^*})$ satisfy the following coupled HJB equations:

$$\begin{cases} \rho_1^{\hat{\xi}_2^*} \psi_1^{\hat{\xi}_2^*}(i) & = \inf_{v_1 \in V_1(i)} \Big[\sum_{j \in S} \pi_{ij}(v_1, \hat{\xi}_2^*(i)) \psi_1^{\hat{\xi}_2^*}(j) + c_1(i, v_1, \hat{\xi}_2^*(i)) \psi_1^{\hat{\xi}_2^*}(i) \Big] \\ & = \Big[\sum_{j \in S} \pi_{ij}(\hat{\xi}_1^*(i), \hat{\xi}_2^*(i)) \psi_1^{\hat{\xi}_2^*}(j) + c_1(i, \hat{\xi}_1^*(i), \hat{\xi}_2^*(i)) \psi_1^{\hat{\xi}_2^*}(i) \Big], \\ \psi_1^{\hat{\xi}_2^*}(i_0) = 1 \end{cases} \tag{3.37}$$

and

$$\begin{cases} \rho_2^{\hat{\xi}_1^*} \psi_2^{\hat{\xi}_1^*}(i) & = \inf_{v_2 \in V_2(i)} \Big[\sum_{j \in S} \pi_{ij}(\hat{\xi}_1^*(i), v_2) \psi_2^{\hat{\xi}_1^*}(j) + c_2(i, \hat{\xi}_1^*(i), v_2) \psi_2^{\hat{\xi}_1^*}(i) \Big] \\ & = \Big[\sum_{j \in S} \pi_{ij}(\hat{\xi}_1^*(i), \hat{\xi}_2^*(i)) \psi_2^{\hat{\xi}_1^*}(j) + c_2(i, \hat{\xi}_1^*(i), \hat{\xi}_2^*(i)) \psi_2^{\hat{\xi}_1^*}(i) \Big], \\ \psi_2^{\hat{\xi}_2^*}(i_0) = 1. \end{cases} \tag{3.38}$$

Now by Theorem 3.1, from (3.37), it follows that

$$\rho_1^{\hat{\xi}_2^*} = \inf_{\xi_1 \in \mathcal{U}_1^{Ad}} \rho_1^{\xi_1, \hat{\xi}_2^*} = \rho_1^{\hat{\xi}_1^*, \hat{\xi}_2^*}. \tag{3.39}$$

Similarly, from (3.38), we have

$$\rho_2^{\hat{\xi}_1^*} = \inf_{\xi_2 \in \mathcal{U}_2^{Ad}} \rho_2^{\hat{\xi}_1^*, \xi_2} = \rho_2^{\hat{\xi}_1^*, \hat{\xi}_2^*}. \tag{3.40}$$

Thus, from equations (3.39) and (3.40), we get

$$\rho_1^{\xi_1, \hat{\xi}_2^*} \geq \rho_1^{\hat{\xi}_1^*, \hat{\xi}_2^*}, \ \forall \, \xi_1 \in \mathcal{U}_1^{Ad},$$
$$\rho_2^{\hat{\xi}_1^*, \xi_2} \geq \rho_2^{\hat{\xi}_1^*, \hat{\xi}_2^*}, \ \forall \, \xi_2 \in \mathcal{U}_2^{Ad}.$$

Hence $(\hat{\xi}_1^*, \hat{\xi}_2^*) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$ is a Nash equilibrium. This completes the proof. □

The above theorem establishes a Nash equilibrium belonging to the space of stationary Markov strategies. Note that the equilibrium thus obtained is a Nash equilibrium among all admissible strategies. However the equilibrium need not be unique. In case the set-valued map (of optimal responses) admits a unique fixed point then the Nash-equilibrium will be unique. For uniqueness stringent conditions may be required on the transition rates and cost functions. Next we prove a converse of Theorem 3.2.

Theorem 3.3 *Suppose Assumptions 2.1, 2.2, and 2.3 hold. If $(\underline{\xi}_1^*, \underline{\xi}_2^*) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$ is a Nash equilibrium, i.e.,*

$$\rho_1^{\xi_1, \underline{\xi}_2^*} \geq \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}, \ \forall \, \xi_1 \in \mathcal{U}_1^{Ad},$$
$$\rho_2^{\underline{\xi}_1^*, \xi_2} \geq \rho_2^{\underline{\xi}_1^*, \underline{\xi}_2^*}, \ \forall \, \xi_2 \in \mathcal{U}_2^{Ad}.$$

Then $\underline{\xi}_1^* \in \mathcal{U}_1^{SM}$ is a minimizing selector of (3.5) (corresponding to fixed strategy $\underline{\xi}_2^* \in \mathcal{U}_2^{SM}$ of player 2) and $\underline{\xi}_2^* \in \mathcal{U}_2^{SM}$ is a minimizing selector of (3.8) (corresponding to fixed strategy $\underline{\xi}_1^* \in \mathcal{U}_1^{SM}$ of player 1).

Proof Applying analogous arguments as in [6,Lemma 3.4 and Remark 3.1], one can prove that for the given pair $(\underline{\xi}_1^*, \underline{\xi}_2^*) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, there exists a eigenpair $(\rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}, \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}) \in \mathbb{R} \times L_W^\infty$, $\psi_1^{\underline{\xi}_1^*} > 0$ and $\rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*} \geq 0$ satisfying

$$\begin{cases} \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*} \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i) = \sum_{j \in S} \pi_{ij}(\underline{\xi}_1^*(i), \underline{\xi}_2^*(i)) \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(j) + c_1(i, \underline{\xi}_1^*(i), \underline{\xi}_2^*(i)) \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i), \\ \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i_0) = 1. \end{cases} \tag{3.41}$$

Also, for given $\underline{\xi}_2^* \in \mathcal{U}_2^{SM}$, there exists a minimal eigenpair $(\rho_1^{\underline{\xi}_2^*}, \psi_1^{\underline{\xi}_2^*}) \in \mathbb{R}_+ \times L_W^\infty$, $\psi_1^{\underline{\xi}_2^*} > 0$, satisfying

$$\begin{cases} \rho_1^{\underline{\xi}_2^*} \psi_1^{\underline{\xi}_2^*}(i) = \inf_{v_1 \in V_1(i)} \Big[\sum_{j \in S} \pi_{ij}(v_1, \underline{\xi}_2^*(i)) \psi_1^{\underline{\xi}_2^*}(j) + c_1(i, v_1, \underline{\xi}_2^*(i)) \psi_1^{\underline{\xi}_2^*}(i) \Big], \\ \psi_1^{\underline{\xi}_2^*}(i_0) = 1. \end{cases} \tag{3.42}$$

Since $\rho_1^{\underline{\xi}_2^*}$ is a minimal eigenvalue of (3.42), corresponding to $\underline{\xi}_2^*$, we have

$$\rho_1^{\underline{\xi}_2^*} = \inf_{\xi_1 \in \mathcal{U}_1^{Ad}} \rho_1^{\xi_1, \underline{\xi}_2^*}. \tag{3.43}$$

Also, we have

$$\rho_1^{\xi_1, \underline{\xi}_2^*} \geq \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}, \ \forall \, \xi_1 \in \mathcal{U}_1^{Ad}.$$

Hence,

$$\inf_{\xi_1 \in \mathcal{U}_1^{Ad}} \rho_1^{\xi_1, \underline{\xi}_2^*} \geq \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}. \tag{3.44}$$

So, by (3.43) and (3.44), we obtain

$$\rho_1^{\underline{\xi}_2^*} \geq \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}.$$

Also, from (3.43), we have

$$\rho_1^{\underline{\xi}_2^*} \leq \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}.$$

Hence, we deduce that

$$\rho_1^{\underline{\xi}_2^*} = \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}. \tag{3.45}$$

Now, applying Ito-Dynkin formula, from (3.41), it follows that

$$\psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i) = E_i^{\underline{\xi}_1^*, \underline{\xi}_2^*} \left[e^{\int_0^{T \wedge \hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \underline{\xi}_1^*(Y(t-)), \underline{\xi}_2^*(Y(t-))) - \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}) dt} \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(Y(T \wedge \hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c,$$

where $\mathscr{B}_1$ is as in Theorem 3.1. Now, by Fatou's Lemma, taking $T \to \infty$ in the above equation, we get

$$\psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i) \geq E_i^{\underline{\xi}_1^*, \underline{\xi}_2^*} \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \underline{\xi}_1^*(Y(t-)), \underline{\xi}_2^*(Y(t-))) - \rho_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}) dt} \psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c. \tag{3.46}$$

Again, using (3.42), from Theorem 3.1, it follows that

$$\psi_1^{\underline{\xi}_2^*}(i) \leq E_i^{\underline{\xi}_1^*, \underline{\xi}_2^*} \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \underline{\xi}_1^*(Y(t-)), \underline{\xi}_2^*(Y(t-))) - \rho_1^{\underline{\xi}_2^*}) dt} \psi_1^{\underline{\xi}_2^*}(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c. \tag{3.47}$$

So, by (3.46) and (3.47), we obtain

$$\begin{aligned} &\psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i) - \psi_1^{\underline{\xi}_2^*}(i) \\ &\geq E_i^{\underline{\xi}_1^*, \underline{\xi}_2^*} \\ &\quad \left[e^{\int_0^{\hat{\tau}(\mathscr{B}_1)} (c_1(Y(t), \underline{\xi}_1^*(Y(t-)), \underline{\xi}_2^*(Y(t-))) - \rho_1^{\underline{\xi}_2^*}) dt} (\psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*} - \psi_1^{\underline{\xi}_2^*})(Y(\hat{\tau}(\mathscr{B}_1))) \right] \forall i \in \mathscr{B}_1^c. \end{aligned} \tag{3.48}$$

Now arguing as in the proof of Lemma 3.3, we obtain $\psi_1^{\underline{\xi}_1^*, \underline{\xi}_2^*}(i) \equiv \psi_1^{\underline{\xi}_2^*}$. Thus, from (3.41) and (3.42) it follows that $\underline{\xi}_1^*$ is a minimizing selector of (3.5) (for fixed strategy

$\underline{\xi}_2^* \in \mathcal{U}_2^{SM}$ of player 2). Following similar arguments one can show that $\underline{\xi}_2^*$ is a minimizing selector of (3.8) (for fixed strategy $\underline{\xi}_1^* \in \mathcal{U}_1^{SM}$ of player 1). This completes the proof. □

4 Example

In this section, we present an illustrative where transition rates are unbounded and cost rates are nonnegative and unbounded.

Example 4.1 Consider a shop which deals with only one type of product for buying and selling. Suppose there are two workers, say, player 1 and player 2 for buying and selling the products, respectively. The number of stocks in the shop is a finite subset of the set of natural numbers $\mathbb{N}$ at each time $t \geq 0$. There are 'natural' buying and selling rates, say $\tilde{\mu}$ and λ, respectively, and buying parameters h_1 controlled by player 1 and selling parameters h_2 controlled by player 2. When the state of the system is $i \in S := \{1, 2, \cdots\}$ (i.e., number of items in the shop), player 1 takes an action u_1 from a given set $U_1(i)$, which may increase ($h_1(i, u_1) \geq 0$) or decrease ($h_1(i, u_1) \leq 0$) the buying rate. These actions produce a payoff denoted by $r_1(i, u_1)$ per unit time. Similarly, if the state is $i \in S$, player 2 takes an action u_2 from a set $U_2(i)$ to decrease ($h_2(i, b) \leq 0$) or to increase ($h_2(i, b) \geq 0$) the selling rate. These actions result in a payoff denoted by $r_2(i, u_2)$ per unit time. We assume that when the stock of items in the shop becomes 1, the first player may buy any number of stocks of that item as much as he/she likes depending upon the availability of cash. In addition, we assume that player k, ($k = 1, 2$) 'gets' a reward $r_k(i) := p_k i$ or incurs a cost $r_k(i) := p_k i$ for each unit of time during which the system remains in the state $i \in S$, where $p_k > 0$ is a fixed reward fee, and $p_k < 0$, a fixed cost fee, per stock, from the owner.
We next formulate this model as a continuous-time Markov game. The corresponding transition rate $\bar{\pi}_{ij}(u_1, u_2)$ and payoff rate $\bar{c}_k(i, u_1, u_2)$ for player k, ($k = 1, 2$) are given as follows: for $(1, u_1, u_2) \in K$ (K as in the game model (2.1)).

$$\bar{\pi}_{1j}(u_1, u_2) > 0 \;\forall j \geq 2, \text{ such that } \sum_{j \in S} \bar{\pi}_{1j}(u_1, u_2) = 0, \text{ and } \bar{\pi}_{1j}(u_1, u_2) \leq e^{-2\theta j} \;\forall\, j \geq 2, \tag{4.1}$$

where $\theta > 0$ is a constant.
Also, for $(i, u_1, u_2) \in K$ with $i \geq 2$,

$$\bar{\pi}_{ij}(u_1, u_2) = \begin{cases} \lambda i + h_2(i, u_2), \text{ if } j = i - 1 \\ -\tilde{\mu} i - \lambda i - h_1(i, u_1) - h_2(i, u_2), \text{ if } j = i \\ \tilde{\mu} i + h_1(i, u_1), \text{ if } j = i + 1 \\ 0, \text{ otherwise.} \end{cases}$$

$$\bar{c}_1(i, u_1, u_2) := i p_1 - r_1(i, u_1), \;\; \bar{c}_2(i, u_1, u_2) = i p_2 - r_2(i, u_2) \;\text{ for }\; (i, u_1, u_2) \in K. \tag{4.2}$$

We now investigate conditions under which there exists a Nash equilibrium. To this end we make following assumptions:

(I) For each $i \in S$, $U_1(i) = U_2(i) = [-L, L]$, $L > 0$ is a constant.
(II) Let $\lambda e^{-\theta} > \tilde{\mu} > 0$, $\tilde{\mu} i + h_1(i, u_1) \geq 0$ and $\lambda i + h_2(i, u_2) \geq 0$ for all $(i, u_1, u_2) \in K$ with $i \geq 2$.
(III) The functions $h_1(i, u_1), h_2(i, u_2), r_1(i, u_1), r_2(i, u_2)$, and $\bar{\pi}_{11}(u_1, u_2)$ are continuous in (u_1, u_2) for each fixed $i \in S$. Suppose there exists a finite set $\mathscr{K}$ such that $h_k(i, u_k) = \frac{u_k}{e^{\theta i}} I_{\mathscr{K}}(i)$ and $1 \in \mathscr{K}$. Also assume that $\inf_{(u_1,u_2) \in U_1(\cdot) \times U_2(\cdot)} r_k(\cdot, u_k)$ is norm like function for $k = 1, 2$.
(IV) Suppose $i p_k - r_k(i, u_k) \geq 0 \; \forall i \in S$, $(u_1, u_2) \in U_1(i) \times U_2(i)$ and $(1 - e^{-\theta})\lambda + (1 - e^{\theta})\tilde{\mu} > p_k$ for $k = 1, 2$.

Proposition 4.1 *Under conditions* (I)–(IV)*, the above controlled system satisfies the Assumptions* 2.1, 2.2*, and* 2.3*. Hence by Theorem* 3.2*, there exists a Nash equilibrium.*

Proof Take a Lyapunov function as $\mathcal{V}(i) := e^{\theta i}$ for $i \in S$ for some $\theta > 0$ as described earlier. Then, we have $\mathcal{V}(i) \geq 1$ for all $i \in S$. Now for each $i \geq 2$, and $(u_1, u_2) \in U_1(i) \times U_2(i)$, we have

$$\begin{aligned}
\sum_{j \in S} \bar{\pi}_{ij}(u_1, u_2)\mathcal{V}(j) &= \bar{\pi}_{i(i-1)}(u_1, u_2)\mathcal{V}(i-1) + \mathcal{V}(i)\bar{\pi}_{ii}(u_1, u_2) + \mathcal{V}(i+1)\bar{\pi}_{i(i+1)}(u_1, u_2) \\
&= e^{\theta i}\Big[(\lambda i + h_2(i, u_2))e^{-\theta} - (i\tilde{\mu} + \lambda i + h_1(i, u_1) + h_2(i, u_2)) + (\tilde{\mu} i + h_1(i, u_1))e^{\theta}\Big] \\
&= e^{\theta i} i\left[\tilde{\mu}(e^{\theta} - 1) + \lambda(e^{-\theta} - 1) + \frac{e^{\theta} h_1(i, u_1) + e^{-\theta} h_2(i, u_2) - h_1(i, u_1) - h_2(i, u_2)}{i}\right] \\
&= i\mathcal{V}(i)[\tilde{\mu}(e^{\theta} - 1) + \lambda(e^{-\theta} - 1)] + \Big[u_1(e^{\theta} - 1) + u_2(e^{-\theta} - 1)\Big] I_{\mathscr{K}}(i) \\
&\leq i\mathcal{V}(i)[\tilde{\mu}(e^{\theta} - 1) + \lambda(e^{-\theta} - 1)] + L(e^{\theta} - 1) I_{\mathscr{K}}(i).
\end{aligned} \tag{4.3}$$

Now for every $\theta > 0$, we know

$$\lambda(e^{-\theta} - 1) + \tilde{\mu}(e^{\theta} - 1) < 0 \Leftrightarrow \tilde{\mu} < \lambda e^{-\theta}.$$

Let $[\tilde{\mu}(e^{\theta} - 1) + \lambda(e^{-\theta} - 1)] = -\alpha$ for some $\alpha > 0$. Also, let $\ell(i) = i\alpha$ and $C_4 = \max\left\{L(e^{\theta} - 1), \frac{e^{-2\theta}}{1 - e^{-\theta}}\right\}$ (see (4.5)). Then for $i \geq 2$,

$$\sup_{(u_1,u_2) \in U_1(i) \times U_2(i)} \sum_{j \in S} \mathcal{V}(j)\bar{\pi}_{ij}(u_1, u_2) \leq C_4 I_{\mathscr{K}}(i) - \ell(i)\mathcal{V}(i) \; \forall i \in S. \tag{4.4}$$

Also, we have

$$\sum_{j \in S} \bar{\pi}_{1j}(u_1, u_2)\mathcal{V}(j) < \bar{\pi}_{11}(u_1, u_2)e^{\theta} + \sum_{j \geq 2} e^{-2\theta j} e^{\theta j} \leq \bar{\pi}_{11}(u_1, u_2)e^{\theta} + \frac{e^{-2\theta}}{1 - e^{-\theta}} < \infty. \tag{4.5}$$

Since $-\ell(i) < 1$ for all $i \in S$. Hence from (4.4) and (4.5), for $i \geq 1$, we have

$$\sum_{j \in S} \bar{\pi}_{ij}(u_1, u_2)\mathcal{V}(j) \leq C_1\mathcal{V}(i) + C_2, \quad \text{where } C_1 = 1 \text{ and } C_2 = C_4. \tag{4.6}$$

For $i \geq 2$,

$$\begin{aligned} -\bar{\pi}_{ii}(u_1, u_2) &= \tilde{\mu} i + \lambda i + h_1(i, u_1) + h_2(i, u_2) \\ &\leq i(\tilde{\mu} + \lambda) + 2L \\ &\leq \frac{1}{\theta}(\tilde{\mu} + \lambda)\mathcal{V}(i) + 2L\mathcal{V}(i) \\ &= [2L + (\tilde{\mu} + \lambda)\frac{1}{\theta}]\mathcal{V}(i) \\ &= C_3\mathcal{V}(i). \end{aligned} \tag{4.7}$$

Take $W = \tilde{W} = \mathcal{V}$. Now for $k = 1, 2$

$$\begin{aligned} \ell(i) - \sup_{(u_1,u_2) \in U_1(i) \times U_2(i)} \bar{c}_k(i, u_1, u_2) &= \alpha i - i p_k + \inf_{u_k \in U_k(i)} r_k(i, u_k) \\ &= i\beta_k + \inf_{u_k \in U_k(i)} r_k(i, u_k). \end{aligned} \tag{4.8}$$

We see that from condition (IV), that $\beta_k = \alpha - p_k \geq 0$. So, $\ell(i) - \sup_{(u_1,u_2) \in U_1(i) \times U_2(i)} \bar{c}_k(i, u_1, u_2)$ is norm-like function for $k = 1, 2$. Now by (4.6), we say Assumption 2.1 (i) holds. Also by (4.1) and (4.7), Assumption 2.1 (ii) is verified.
Now we verify Assumption 2.2. By (4.4), (4.5) and (4.8), it is easy to see that Assumption 2.2 is satisfied.
Now by condition (III) and (4.2), we say $\bar{c}_k(i, u_1, u_2)$ and $\bar{\pi}_{ij}(u_1, u_2)$ are continuous in $(u_1, u_2) \in U_1(i) \times U_2(i)$ for each fixed $i, j \in S$ and for $k = 1, 2$. So, Assumption 2.3 (i) is verified. By (4.3) and (4.5) and condition (III), we say that Assumption 2.3 (ii) is verified. Also, from (4.1) it is easy to see that Assumption 2.3 (iii) is satisfied.
Hence by Theorem 3.2 there exists a Nash equilibrium for this controlled process.

□

Remark 4.1 It should be noted that, here we assume when the number of stock in the shop is one and the players independently choose action according to some strategies $(\xi_1, \xi_2) \in \mathcal{U}_1^{SM} \times \mathcal{U}_2^{SM}$, respectively, then with a positive probability first player may buy any number of stocks of the item, i.e., $\pi_{1j}(\xi_1(1), \xi_2(1)) > 0$ for all $j \in S$. In view of Remark 3.1, one can weaken this Assumption. For any $i \in \mathcal{D}_n$ (for n large enough) $i >> 1$, player may increase the number of stock in the shop from 1 to i in m number of steps in $\mathcal{D}_n$, i.e., there exists a finite sequence of states $i_0, i_1, \ldots, i_m$ connecting $i_0 = 1$ to $i_m = i$, satisfying

$$\pi_{i_0 i_1}(\xi_1(i_0), \xi_2(i_0))\pi_{i_1 i_2}(\xi_1(i_1), \xi_2(i_1)) \cdots \pi_{i_{m-1} i}(\xi_1(i_{m-1}), \xi_2(i_{m-1})) > 0\,,$$

where $i_0, i_1, \ldots, i_m \in \mathcal{D}_n$.

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