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RATIONAL CUSPIDAL CURVES ON DEL-PEZZO SURFACES

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Abstract. We obtain an explicit formula for the number of rational cuspidal curves of a given degree on a del-Pezzo surface that pass through an appropriate number of generic points of the surface. This enumerative problem is expressed as an Euler class computation on the moduli space of curves. A topological method is employed in computing the degenerate contribution to the Euler class.

1. Introduction

The Enumerative Geometry of rational curves in $\mathbb{P}^2_{\mathbb{C}}$ is a classical question. However, a formula for the number of degree d rational curves in $\mathbb{P}^2_{\mathbb{C}}$ passing through $3d-1$ generic points was unknown until the early $90^{'s}$ when Ruan–Tian [25] and Kontsevich–Manin [18] obtained a formula for it. More generally, they gave an explicit answer to the following question:

Question 1.1. *Let X be a complex del-Pezzo surface, and let $\beta \in H_2(X;\mathbb{Z})$ be a given homology class. What is the number of rational degree β-curves in X that pass through $\langle c_1(TX),\ \beta\rangle - 1$ generic points?*

Fixing X, the number in Question 1.1 will be denoted by N_β.

A natural generalization to the above question is to ask how many rational curves are there of a given degree, that pass through the right number of generic points and have a specific singularity.

The main result we obtain here is the following:

Theorem 1.2. *Let X be $\mathbb{P}^2$ blown up at k-points with $k \leq 8$, and let*

$$\beta := dL - m_1E_1 - \ldots - m_kE_k \in H_2(X;\mathbb{Z})$$

be a homology class, where L denotes the homology class of a line, $\{E_i\}_{i=1}^k$ are the exceptional divisors and $m_i \geq 0$. Denote

$$x_i := c_i(TX) \qquad \text{and} \qquad \delta_\beta := \langle x_1, \beta\rangle - 1\,,$$

where c_i denotes the i-th Chern class. If $N_{\beta-3L} > 0$, then the number of rational degree β-curves in X that pass through $\delta_\beta - 1$ generic points and have a cusp, is given by

$$C_\beta = \Big(x_2([X]) - \frac{x_1\cdot x_1}{\beta\cdot x_1}\Big)N_\beta + \sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}} \binom{\delta_\beta-1}{\delta_{\beta_1}} N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2)\Big(\frac{(\beta_1\cdot x_1)(\beta_2\cdot x_1)}{2(\beta\cdot x_1)} - 1\Big)\,, \tag{1.1}$$

where "$\cdot$" denotes topological intersection.

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Since the numbers N_β are known using the algorithm described in [18] and [8], the number C_β is computable using (1.1). We have written a C++ program that implements (1.1) and computes C_β for a given β. The program is available on our web page

https://www.sites.google.com/site/ritwik371/home.

We need the condition $N_{\beta-3L} > 0$ in order to prove that the space of rational curves having exactly one genuine cusp is non-empty, and also to prove a transversality result (Section 8). However, based on the numerical evidence we also expect the formula to be valid even when

$$N_{\beta-3L} = 0\,. \tag{1.2}$$

The condition $N_{\beta-3L} > 0$ is sufficient to prove transversality. However, it may not be necessary. For example, (1.2) is sometimes true for a trivial reason if there do not exist any cuspidal curves in the given class $\beta - 3L$. In such a case (1.2) gives us $N_{\beta-3L} = 0$ and $C_{\beta-3L}$ is also equal to zero, which is consistent with the formula (although the hypothesis we imposed does not apply).

When $X := \mathbb{P}^2$, Pandharipande ([22]), and Ran ([23]), obtain a formula for C_β using an algebro-geometric method. Theorem 1.2 is consistent with their results. Furthermore, it is easy to see by direct geometric arguments that

$$C_{dL+\sigma_1 E_1+\sigma_2 E_2+\ldots+\sigma_k E_k} = C_{dL}\,, \tag{1.3}$$

where each of the σ_i is -1 or 0. For the sake of consistency, we verified (1.3) for several cases using the above mentioned program. The reader is invited to use our program to verify these assertions.

In [18], the authors, strictly speaking, give a formula to compute the genus 0 Gromov–Witten invariants of the del-Pezzo surfaces. A priori, these numbers need not be the same as N_β (since the Gromov–Witten invariants are not always enumerative). It is established in [8] that the numbers obtained in [18] are indeed enumerative, meaning they are actually equal to N_β.

Theorem 1.2 is also true when $X := \mathbb{P}^1 \times \mathbb{P}^1$. In [17], Kock obtains a formula for C_β using an algebro-geometric method, which is consistent with (1.1). In order to keep the exposition here more streamlined, we decided to omit working out this case separately. The arguments given in Section 7 go through without any essential change; the arguments in Section 8 need to be modified slightly to address the case of $\mathbb{P}^1 \times \mathbb{P}^1$.

2. Overview of our method and comparison with other methods

We closely adapt the method applied by Zinger in [31] to obtain our formula for del-Pezzo surfaces (Zinger obtains this formula when $X := \mathbb{P}^2$). We express our enumerative number as the number of zeros of a section of an appropriate vector bundle (restricted to an open dense set of an appropriate moduli space). As one typically expects, the Euler class of this vector bundle is our desired enumerative number, *plus* an extra boundary contribution. Calculating this excess boundary contribution (which is also referred to as the *degenerate* contribution to the Euler class) is the most crucial part. The standard algebraic geometric method involves taking a certain blowup of the degenerate locus and then computing the boundary contribution. This is the method which is the subject of Section 9.1 of the famous book by Fulton [7]. The method has been used extensively by algebraic geometers to solve a large number of enumerative geometry questions.

In this paper, we use a different method to compute the degenerate contribution to the Euler class, closer in spirit to the classical approach by "dynamic intersections" (cf. Chapter 11 in [7]). We perturb our section smoothly and count how many zeros are there close to the degenerate locus. In order to do this, we need a very careful understanding of how a neighbourhood of the degenerate locus is described inside the whole moduli space. After that, we study the behaviour of the section in a neighbourhood of the degenerate locus to calculate the multiplicity. This is done in section 7 of our paper. The method describe is local in nature (and hence, it is often called local excess intersection theory; the method developed by Fulton is often called global excess intersection theory). Local intersection theory has been applied by Zinger to solve a large number of enumerative questions; these include counting rational curves with prescribed singularities in $\mathbb{P}^n$ and counting genus g curves with fixed complex structures in $\mathbb{P}^n$ (this has been done in [31] and [32]).

We conclude this brief overview by mentioning that Zinger has also used local excess intersection theory in [30] to enumerate degree d curves in $\mathbb{P}^2$ with up to two nodes and one singularity (till a total codimension three). In [1] and [2], the third author and S. Basu have applied local intersection theory (by extending the ideas in [30]) to enumerate degree d-curves in $\mathbb{P}^2$ with $\delta \leq 1$ nodes and one more singularity of codimension χ, provided the total codimension $(\delta + \chi)$ is less than or equal to seven.

In general, the question of enumerating curves in a linear system with prescribed singularities has been studied extensively by algebraic geometers; these include I. Vainsincher ([28] and [29]), S. Kleiman and R. Piene ([14] and [15]), M. Kool, V. Shende and R. Thomas ([16]), D. Kerner ([11], [12]), J. Li and Y. Tzeng ([19] and [27]), J. Rennemo ([24]) and G. Bérczi ([3]). Finally, a topological approach (very different from the approach in [30], [1] and [2]) has been used by M. Kazarian ([10]) to enumerate curves in a linear system with up to seven singular points (provided the total codimension of the singularities is less than or equal to seven).

We have given here a very brief overview of this subject (namely counting singular curves in a linear system); a more detailed and extensive overview can be found in the excellent survey article by S. Kleiman ([13]).

3. Related results and questions

The formula for C_β (the characteristic curves for rational curves with a cusp) when the target space $X := \mathbb{P}^2$ has been obtained by Pandharipande [22] and also by Ran [23]. The corresponding formula, when the target space is $\mathbb{P} \times \mathbb{P}^1$ has been obtained by Kock [17]. These papers follow an algebro geometric approach.

In [6], the authors express the divisor of cuspidal curves on surfaces in terms of other divisors. In [22], the author shows how to compute the latter on $\mathbb{P}^n$ and hence on any smooth algebraic variety. This should in principle produce our formula (1.1).

The method presented in this paper readily applies in enumerating rational cuspidal curves on complex manifolds of dimension greater than two. Furthermore, the method also applies to enumerating rational curves with more degenerate singularities. In [32], the author further extends the method and obtains a formula for the number of rational curves in $\mathbb{P}^2$ with an E_6 singularity. (In local coordinates an E_6-singularity is given by the equation $y^3 + x^4 = 0$.) Using the results obtained here, we hope that we will be able to extend the results of Zinger in [32] (for E_6 singularities) to del-Pezzo surfaces. We should also be able to extend our results for rational cuspidal curves on higher dimensional complex manifolds (such as products of projective spaces or $\mathbb{P}^n$ blown up at certain number of points).

Finally, we note that the problem of counting rational curves plays an important role in counting genus one curves with a fixed complex structure (which is called the genus one enumerative invariant). This is because it arises as a correction term in the corresponding genus one Symplectic Invariant (which are the number of solutions to the perturbed $\overline{\partial}$ equation). The Symplectic Invariant is computable via the formula given in [25]. By extending the idea in [9], where the author computes the correction term and thereby computes genus one enumerative invariant of $\mathbb{P}^2$, we have been able to compute the genus one enumerative invariant of a del-Pezzo surface in [4]. Similarly, in [31], Zinger computes the genus two Enumerative invariant of $\mathbb{P}^2$ by computing the correction term to the genus two Symplectic Invariant; one of the correction term is the number of rational cuspidal curves in $\mathbb{P}^2$. Hence, one of the applications of the result of this paper is that we have been able to compute the genus two enumerative invariant of a del-Pezzo surface by computing the correction term to the genus two Symplectic Invariant; this is done in [5].

4. Notation

Consider the rational curves on smooth complex del-Pezzo surface X representing $\beta \in H_2(X, \mathbb{Z})$ and equipped with n ordered marked points. Let $\mathcal{M}_{0,n}(X,\beta)$ denote the moduli space of equivalence classes of such curves. In other words,

$$\mathcal{M}_{0,n}(X,\beta) := \{(u, y_1, \cdots, y_n) \in \mathcal{C}^\infty(\mathbb{P}^1, X) \times (\mathbb{P}^1)^n \mid \overline{\partial} u = 0,\ u_*[\mathbb{P}^1] = \beta\}/\mathrm{PSL}(2,\mathbb{C})\,.$$

with $\mathrm{PSL}(2,\mathbb{C})$ acting diagonally on $\mathbb{P}^1 \times (\mathbb{P}^1)^n$. For any $k \leq n$, define the subspace

$$\mathcal{M}_{0,n}(X,\beta; p_1, \cdots, p_k) \subset \mathcal{M}_{0,n}(X,\beta)$$

consisting of n marked points such that the i-th marked point is p_i for all $1 \leq i \leq k$, so,

$$\mathcal{M}_{0,n}(X,\beta; p_1, \cdots, p_k) := \{[u, y_1, \cdots, y_n] \in \mathcal{M}_{0,n}(X,\beta) \mid u(y_i) = p_i \qquad \forall\ i = 1, \cdots, k\}\,.$$

We define $\mathcal{M}^*_{0,n}(X,\beta)$ (respectively, $\mathcal{M}^*_{0,n}(X,\beta; p_1, \cdots, p_k)$) to be the locus in $\mathcal{M}_{0,n}(X,\beta)$ (respectively, $\mathcal{M}_{0,n}(X,\beta; p_1, \cdots, p_k)$) of curves that are not multiply covered.

We will denote $\mathcal{M}^*_{0,\delta_\beta}(X,\beta; p_1, \cdots, p_{\delta_\beta - 1})$ also by $\mathcal{M}^*$.

Let $\overline{\mathcal{M}}_{0,n}(X,\beta)$ denote the stable map compactification of $\mathcal{M}^*_{0,n}(X,\beta)$. Let

$$\mathbb{L}_i \longrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta)$$

be the universal tangent line bundle at the i-th marked point. More precisely, if

$$f_C : \mathcal{C} \longrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta)$$

is the universal curve with $T_{f_C} \longrightarrow \mathcal{C}$ being the relative tangent bundle for f_C and

$$y_i : \overline{\mathcal{M}}_{0,n}(X,\beta) \longrightarrow \mathcal{C}$$

is the section giving the i-th marked point, then $\mathbb{L}_i := y_i^* T_{f_C}$.

Finally, let $\mathcal{L}_i \longrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta)$ denote the universal tangent bundle after dropping all the marked points, except for the i^{th} marked point. More precisely, let

$$\pi_i : \overline{\mathcal{M}}_{0,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)$$

denote the forgetful map, that forgets all but the i^{th} marked point. Let $\mathbb{L} \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)$ denote the universal tangent bundle (in this case there is only one marked point). Then

$$\mathcal{L}_i := \pi_i^* \mathbb{L} \longrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta).$$

Remark 4.1. Throughout the paper, we will be using the line bundle $\mathcal{L}_i$ as opposed to $\mathbb{L}_i$. In particular, Lemma 6.1 applies to the line bundle $\mathcal{L}_{\delta_\beta}$. If instead we considered the line bundle $\mathbb{L}_{\delta_\beta}$, there would be an extra boundary term in the right hand side of (6.1).

5. Euler class computation

Let us now explain how we obtain (1.1). The method employed follows closely the method in [31] to compute C_β when $X = \mathbb{P}^2$.

Evidently, C_β coincides with the cardinality of the following set

$$\{[u; y_1, \cdots, y_{\delta_\beta - 1}; y_{\delta_\beta}] \in \mathcal{M}^*_{0,\delta_\beta}(X, \beta; p_1, \cdots, p_{\delta_\beta - 1}) \mid du|_{y_{\delta_\beta}} = 0\}.$$

Since the above $\delta_\beta - 1$ points are in general position, the curve will have a genuine cusp at the last marked point (as opposed to something more degenerate). Furthermore, the curves will not have any other singular points (aside from the nodes which are just points of self intersections). For any

$$[u; y_1, \cdots, y_{\delta_\beta - 1}; y_{\delta_\beta}] \in \mathcal{M}^*_{0,\delta_\beta}(X, \beta; p_1, \cdots, p_{\delta_\beta - 1}),$$

the differential of u at the last marked point y_{δ_β} produces a section ψ of the rank two vector bundle

$$\mathcal{L}^*_{\delta_\beta} \otimes \mathrm{ev}^*_{\delta_\beta} TX \longrightarrow \mathcal{M}^*_{0,\delta_\beta}(X, \beta),$$

where $\mathrm{ev}_i : \mathcal{M}^*_{0,\delta_\beta}(X, \beta) \longrightarrow X$ is the evaluation at the i-th marked point. This ψ is transverse to the zero section (this is shown in Section 8). Moreover, it has a natural extension to the compactification

$$\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0,\delta_\beta}(X, \beta; p_1, \cdots, p_{\delta_\beta - 1}).$$

Therefore, we have

$$\langle e(\mathcal{L}^*_{\delta_\beta} \otimes \mathrm{ev}^*_{\delta_\beta} TX), [\overline{\mathcal{M}}]\rangle = C_\beta + \mathcal{C}_{\partial\overline{\mathcal{M}}}(du|_{y_{\delta_\beta}}), \tag{5.1}$$

where $\mathcal{C}_{\partial\overline{\mathcal{M}}}(du|_{y_{\delta_\beta}})$ is the contribution of the extended section to the Euler class from the boundary $\overline{\mathcal{M}} \setminus \mathcal{M}^*$.

Let us now take a closer look at (5.1). First, we note that the extended section $du|_{y_{\delta_\beta}}$ (over $\overline{\mathcal{M}}$) vanishes only on a finite set of points of the boundary. It only vanishes when the curve splits as a β_1 curve and a β_2 curve, $\beta_1, \beta_2 \neq 0$, with the last marked point lying on a ghost bubble. It is clear that the section vanishes on this configuration (since taking the derivative of a constant map gives us zero and the map defined on the ghost bubble is a constant map). To see why the section does not vanish on any other configuration we consider all the remaining possible cases. Suppose the curve splits as $\beta = \beta_1 + \beta_2 + \ldots + \beta_k$, with $k \geq 3$ and $\beta_i \neq 0$ for all i. Since $\delta_{\beta_1} + \ldots + \delta_{\beta_k} < \delta_\beta - 1$ as $k \geq 3$, such a configuration can not occur, because it will not pass through $\delta_\beta - 1$ generic points. Next, suppose the curve splits as $\beta = \beta_1 + \beta_2$, with $\beta_1, \beta_2 \neq 0$ and the marked point lying on say the β_1 component. Then the β_1 curve is cuspidal. Hence it can pass through $\delta_{\beta_1} - 1$ general points. Since $\delta_{\beta_1} - 1 + \delta_{\beta_2} < \delta_\beta - 1$, this configuration can not occur. Next, we note that although the section vanishes on curves that have singularities more degenerate than a cusp or curves that have more than one cuspidal point, no such curve will pass through $\delta_\beta - 1$ general points. Finally, we also observe that although the section vanishes on multiply covered curves, such curves will not pass through $\delta_\beta - 1$ generic points. Hence, the only points on which the section vanishes are those that split as a β_1 curve and a β_2 curve, with the last marked point lying on a ghost bubble.

We show in Section 7 that the extended section vanishes with multiplicity 1 on these boundary points. Hence, we gather that

$$\mathcal{C}_{\partial\overline{\mathcal{M}}}(du|_{y_{\delta_\beta}}) = 1 \times \frac{1}{2} \sum_{\substack{\beta_1 + \beta_2 = \beta, \\ \beta_1, \beta_2 \neq 0}} \binom{\delta_\beta - 1}{\delta_{\beta_1}} N_{\beta_1} N_{\beta_2} (\beta_1 \cdot \beta_2). \tag{5.2}$$

Therefore, in order to compute C_β, it remains to evaluate the left–hand side of (5.1). It is easy to see that via the splitting principle,

$$e(\mathcal{L}^*_{\delta_\beta} \otimes \mathrm{ev}^*_{\delta_\beta} TX) \;=\; c_1(\mathcal{L}^*_{\delta_\beta})^2 + c_1(\mathcal{L}^*_{\delta_\beta})\mathrm{ev}^*_{\delta_\beta}(x_1) + \mathrm{ev}^*_{\delta_\beta}(x_2). \tag{5.3}$$

In Section 6 we show that

$$\langle \mathrm{ev}^*_{\delta_\beta}(x_2),\ [\overline{\mathcal{M}}]\rangle \;=\; x_2([X])N_\beta, \tag{5.4}$$

$$\begin{aligned}\langle c_1(\mathcal{L}^*_{\delta_\beta})\mathrm{ev}^*_{\delta_\beta}(x_1),\ [\overline{\mathcal{M}}]\rangle \;=\; &-\frac{(x_1\cdot x_1)}{(\beta\cdot x_1)}N_\beta\\ &+\frac{1}{2(\beta\cdot x_1)}\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}}\binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2)(\beta_1\cdot x_1)(\beta_2\cdot x_1),\end{aligned} \tag{5.5}$$

$$\langle c_1(\mathcal{L}^*_{\delta_\beta})^2,\ [\overline{\mathcal{M}}]\rangle \;=\; -\frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}}\binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2). \tag{5.6}$$

Equations (5.4), (5.5) and (5.6) combined with (5.3), (5.2) and (5.1) yield (1.1).

6. Intersection of Tautological Classes

We will now prove equations (5.4), (5.5) and (5.6).

Lemma 6.1. *On $\overline{\mathcal{M}}$, the following equality of divisors holds:*

$$c_1(\mathcal{L}^*_{\delta_\beta}) \;=\; \frac{1}{(\beta\cdot x_1)^2}\Big((x_1\cdot x_1)\mathcal{H} - 2(\beta\cdot x_1)\mathrm{ev}^*_{\delta_\beta}(x_1) + \sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}}\mathcal{B}_{\beta_1,\beta_2}(\beta_2\cdot x_1)^2\Big), \tag{6.1}$$

where $\mathcal{H}$ is the locus satisfying the extra condition that the curve passes through a given point, $\mathcal{B}_{\beta_1,\beta_2}$ denotes the boundary stratum corresponding to the splitting into a degree β_1 curve and degree β_2 curve with the last marked point lying on the degree β_1 component.

Proof. The proof is similar to the one given in [9]. Let $\mu_1\,,\mu_2\,\in\,X$ be two generic pseudocycles in X that represent the class x_1. Let $\widetilde{\mathcal{M}}$ be a cover of $\overline{\mathcal{M}}$ with two additional marked points with the last two marked points lying on μ_1 and μ_2 respectively. More precisely,

$$\widetilde{\mathcal{M}} \;:=\; \mathrm{ev}^{-1}_{\delta_\beta+1}(\mu_1)\cap \mathrm{ev}^{-1}_{\delta_\beta+2}(\mu_2)\subset \overline{\mathcal{M}}_{0,\delta_\beta+2}(X,\beta)\,.$$

Note that the projection $\pi\,:\,\widetilde{\mathcal{M}}\,\longrightarrow\,\overline{\mathcal{M}}$ that forgets the last two marked points is a $(\beta\cdot x_1)^2$–to–one map.

We now construct a meromorphic section

$$\phi\,:\,\widetilde{\mathcal{M}}\,\longrightarrow\,\mathcal{L}^*_{\delta_\beta}$$

given by

$$\phi([u;y_{_1},\cdots,y_{_{\delta_\beta-1}};y_{_{\delta_\beta}};y_{_{\delta_\beta+1}},y_{_{\delta_\beta+2}}]) := \frac{(y_{_{\delta_\beta+1}}-y_{_{\delta_\beta+2}})dy_{_{\delta_\beta}}}{(y_{_{\delta_\beta}}-y_{_{\delta_\beta+1}})(y_{_{\delta_\beta}}-y_{_{\delta_\beta+2}})}. \tag{6.2}$$

The right–hand side of (6.2) involves an abuse of notation: it is to be interpreted in an affine coordinate chart and then extended as a meromorphic section on the whole of $\mathbb{P}^1$. Note that on $(\mathbb{P}^1)^3$, the holomorphic line bundle

$$\eta \;:=\; q_1^*K_{\mathbb{P}^1}\otimes\mathcal{O}_{(\mathbb{P}^1)^3}(\Delta_{12}+\Delta_{13}-\Delta_{23})$$

is trivial, where $q_1 : (\mathbb{P}^1)^3 \longrightarrow \mathbb{P}^1$ is the projection to the first factor and $\Delta_{jk} \subset (\mathbb{P}^1)^3$ is the divisor consisting of all points (z_i, z_2, z_3) such that $z_j = z_k$. The diagonal action of $\mathrm{PSL}(2,\mathbb{C})$ on $(\mathbb{P}^1)^3$ lifts to η preserving its trivialization. The section ϕ in (6.2) is given by this trivialization of η.

Since $c_1(\mathcal{L}^*_{\delta_\beta})$ is the zero divisor minus the pole divisor of ϕ, we gather that

$$c_1(\mathcal{L}^*_{\delta_\beta}) \,=\, \{y_{\delta_\beta+1} = y_{\delta_\beta+2}\} - \{y_{\delta_\beta} = y_{\delta_\beta+1}\} - \{y_{\delta_\beta} = y_{\delta_\beta+2}\}\,.$$

When projected down to $\overline{\mathcal{M}}$, the divisor $\{y_{\delta_\beta+1} = y_{\delta_\beta+2}\}$ becomes $(x_1\cdot x_1)\mathcal{H} + (\beta_2\cdot x_1)^2\mathcal{B}_{\beta_1,\beta_2}$, while both the divisors $\{y_{\delta_\beta} = y_{\delta_\beta+1}\}$ and $\{y_{\delta_\beta} = y_{\delta_\beta+2}\}$ become $(\beta\cdot x_1)\mathrm{ev}^*_{\delta_\beta}(x_1)$. Since $\widetilde{\mathcal{M}}$ is a $(\beta\cdot x_1)^2$–to–one cover of $\overline{\mathcal{M}}$, we obtain (6.1). $\square$

We are now ready to prove (5.4), (5.5) and (5.6).

Proof of (5.4). Let $s : X \longrightarrow TX$ be a smooth section transverse to the zero set. The number of points at which it vanishes (counted with a sign) is $x_2([X])$. Note that a section $s : X \longrightarrow TX$ induces a section $\mathrm{ev}^*_{\delta_\beta}s$ of the pullback $\mathrm{ev}^*_{\delta_\beta}TX \longrightarrow \overline{\mathcal{M}}$. The zero set of $\mathrm{ev}^*_{\delta_\beta}s$ is a degree β curve through the points $p_1, \cdots, p_{\delta_{\beta-1}}$ and one of the zeros of s. Let us denote one of the zeros of s to be q. Let C_q be one of the curves through $p_1, \cdots, p_{\delta_{\beta-1}}$ and q. If q is a positive zero of s, then C_q is a positive zero of $\mathrm{ev}^*_{\delta_\beta}s$; the reverse is true if q is a negative zero of s. Hence, the number of zeros of $\mathrm{ev}^*_{\delta_\beta}s$ counted with a sign is $x_2([X])N_\beta$, which proves (5.4). $\square$

Proof of (5.5). It is easy to see that

$$\begin{aligned}
\langle \mathrm{ev}^*_{\delta_\beta}(x_1)\mathcal{H},\ [\overline{\mathcal{M}}]\rangle &= (\beta\cdot x_1)N_\beta,\\
\langle \mathrm{ev}^*_{\delta_\beta}(x_1)^2,\ [\overline{\mathcal{M}}]\rangle &= (x_1\cdot x_1)N_\beta \qquad \text{and}\\
\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}} \langle \mathrm{ev}^*_{\delta_\beta}(x_1)\mathcal{B}_{\beta_1,\beta_2},\ [\overline{\mathcal{M}}]\rangle &= \sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}} \binom{\delta_\beta-1}{\delta_{\beta_1}} N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2)(\beta_1\cdot x_1)\\
&= \frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}} \binom{\delta_\beta-1}{\delta_{\beta_1}} N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2)(\beta\cdot x_1). \qquad (6.3)
\end{aligned}$$

Equations (6.3) and (6.1) together imply (5.5). $\square$

Proof of (5.6). First of all, we note that

$$\langle c_1(\mathcal{L}^*_{\delta_\beta})\mathcal{H},\ [\overline{\mathcal{M}}]\rangle \,=\, -2N_\beta\,. \qquad (6.4)$$

Indeed, this follows immediately from (6.1).

We will now show that

$$\langle c_1(\mathcal{L}^*_{\delta_\beta})\mathcal{B}_{\beta_1,\beta_2},\ [\overline{\mathcal{M}}]\rangle \,=\, -\binom{\delta_\beta-1}{\delta_{\beta_1}} N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2)\,. \qquad (6.5)$$

For this, let $\mathcal{B}_{\beta_1,\beta_2}(p_{i_1}, \cdots, p_{i_{\delta_{\beta_1}}})$ denote the component of $\mathcal{B}_{\beta_1,\beta_2}$ where the β_1 curve passes through $p_{i_1}, \cdots, p_{i_{\delta_{\beta_1}}}$. Now define the map

$$\pi\,:\,\mathcal{B}_{\beta_1,\beta_2}(p_{i_1}, \cdots, p_{i_{\delta_{\beta_1}}}) \longrightarrow \overline{\mathcal{M}}_{0,\delta_{\beta_1}+1}(X,\beta_1; p_{i_1}, \cdots, p_{i_{\delta_{\beta_1}}}) \qquad (6.6)$$

which is the projection onto the β_1 component. This map π is $N_{\beta_2}(\beta_1\cdot\beta_2)$–to–one. Let

$$\widetilde{\mathcal{L}}_{i_1,\cdots,i_{\delta_{\beta_1}}} \longrightarrow \overline{\mathcal{M}}_{0,\delta_{\beta_1}+1}(X,\beta_1; p_{i_1}, \cdots, p_{i_{\delta_{\beta_1}}})$$

be the universal tangent bundle line at the last marked point. By (6.4),

$$\langle c_1(\widetilde{\mathcal{L}}^*_{i_1,\cdots,i_{\delta_{\beta_1}}}),\ [\overline{\mathcal{M}}_{0,\delta_{\beta_1}+1}(X,\beta_1;p_{i_1},\cdots,p_{i_{\delta_{\beta_1}}})]\rangle \ = \ -2N_{\beta_1}\,. \tag{6.7}$$

Note that we replaced β by β_1 in (6.4) to obtain the above equation; that is permitted since (6.4) holds for *all* β.

Let

$$\{y\in\mathcal{G}\}$$

denote the divisor inside the space $\mathcal{B}_{\beta_1,\beta_2}$ that corresponds to the marked point lying on a ghost bubble. More precisely, the elements of $\{y\in\mathcal{G}\}$ comprise of maps form a wedge of three spheres into X that is degree β_1 on the first component, degree β_2 on the third component and constant on the middle component, with the marked point lying on the middle component. As stated in [9] (equation 2.10, Page 29), we have the following equality of divisors,

$$c_1(\widetilde{\mathcal{L}}^*_{i_1,\cdots,i_{\delta_{\beta_1}}})\Big|_{\mathcal{B}_{\beta_1,\beta_2}} = \pi^*c_1(\widetilde{\mathcal{L}}^*_{i_1,\cdots,i_{\delta_{\beta_1}}}) + |\{y\in\mathcal{G}\}|.$$

Hence,

$$\begin{aligned}
&\langle c_1(\mathcal{L}^*_{\delta_\beta})\mathcal{B}_{\beta_1,\beta_2},\ [\overline{\mathcal{M}}]\rangle \\
&= \sum_{(i_1,\cdots,i_{\delta_{\beta_1}})\subset\{1,2,\cdots,\delta_\beta-1\}} \langle \pi^*c_1(\widetilde{\mathcal{L}}^*_{i_1,\cdots,i_{\delta_{\beta_1}}}),\ [\overline{\mathcal{M}}_{0,\delta_{\beta_1}+1}(X,\beta_1;p_{i_1},\cdots,p_{i_{\delta_{\beta_1}}})]\rangle + |\{y_{\delta_\beta}\in\mathcal{G}\}| \\
&= \ -2N_{\beta_1}\binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\delta_{\beta_2}}(\beta_1\cdot\beta_2) + \binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\delta_{\beta_1}}N_{\delta_{\beta_2}}(\beta_1\cdot\beta_2) \\
&= \ -\binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2),
\end{aligned} \tag{6.8}$$

which proves (6.5). Equations (6.5), (6.3), (5.5) and (6.1) imply that

$$\begin{aligned}
\langle c_1(\mathcal{L}^*_{\delta_\beta})^2,\ [\overline{\mathcal{M}}]\rangle &= -\frac{1}{(\beta\cdot x_1)^2}\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}}\binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2)\Big((\beta_1\cdot x_1)(\beta_2\cdot x_1)+(\beta_2\cdot x_1)^2\Big) \\
&= -\frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}}\binom{\delta_\beta-1}{\delta_{\beta_1}}N_{\beta_1}N_{\beta_2}(\beta_1\cdot\beta_2).
\end{aligned}$$

This completes the proof. □

7. Degenerate contribution to the Euler class

We start this section by recapitulating a few facts about moduli spaces of curves on del-Pezzo surfaces. As before, let X be a del-Pezzo surface and $\beta\in H_2(X,\mathbb{Z})$ a given homology class. Since X is algebraic, it embeds inside $\mathbb{P}^n$ for some n; fix such an embedding. A map $u:\mathbb{P}^1\longrightarrow X$ also determines a map from $\mathbb{P}^1$ to $\mathbb{P}^n$ obtained by composing with the inclusion map. Let us say that a degree β map into X determines a degree d map into $\mathbb{P}^n$. Given β, this d is uniquely determined. We note that two distinct β can determine the same degree d. This yields an inclusion

$$\mathcal{M}^*_{0,k}(X,\beta)\ \subset\ \mathcal{M}^*_{0,k}(\mathbb{P}^n,d)\,. \tag{7.1}$$

It is well known that $\mathcal{M}^*_{0,k}(\mathbb{P}^n,d)$ is a smooth complex manifold of the expected dimension. It is also known that $\mathcal{M}^*_{0,k}(X,\beta)$ is a smooth manifold of the expected dimension. It also follows from the results of [26] that $\overline{\mathcal{M}}_{0,0}(X,\beta)$ is an irreducible variety of the expected dimension.

For any $u : \mathbb{P}^1 \longrightarrow X$ representing a point of $\mathcal{M}^*_{0,k}(X,\beta)$, let $\widehat{u} : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ be its composition with the embedding of X in $\mathbb{P}^n$. We have $T_u\mathcal{M}^*_{0,k}(X,\beta) = H^0(\mathbb{P}^1, u^*N_{X/\mathbb{P}^1})$ and $T_{\widehat{u}}\mathcal{M}^*_{0,k}(\mathbb{P}^n,d) = H^0(\mathbb{P}^1, \widehat{u}^*N_{\mathbb{P}^n/\mathbb{P}^1})$, where $N_{X/\mathbb{P}^1}$ and $N_{\mathbb{P}^n/\mathbb{P}^1}$ are the normal bundles. Since the homomorphism

$$H^0(\mathbb{P}^1, u^*N_{X/\mathbb{P}^1}) \longrightarrow H^0(\mathbb{P}^1, \widehat{u}^*N_{\mathbb{P}^n/\mathbb{P}^1})$$

induced by the differential of the embedding $X \hookrightarrow \mathbb{P}^n$ is injective, the inclusion map in (7.1) is an immersion. Hence, we conclude that $\mathcal{M}^*_{0,k}(X,\beta)$ is a submanifold of $\mathcal{M}^*_{0,k}(\mathbb{P}^n,d)$.

We are now ready to state our neighborhood construction. Before that let us recapitulate a standard of notation. We will be denoting an element of $\mathbb{P}^n$ as

$$[Z_0, Z_1, \cdots, Z_n],$$

where Z_i are not all zero. The square bracket [ ] is to remind us that we are looking at an equivalence class of $n+1$ tuples. In other words, if λ is a non-zero complex number, then

$$[Z_0, Z_1, \cdots, Z_n] = [\lambda Z_0, \lambda Z_1, \cdots, \lambda Z_n] \in \mathbb{P}^n.$$

Let us also explain one piece of terminology we will be using frequently. Suppose Y is a k-dimensional submanifold of an m-dimensional manifold X. Let p be a point in Y and let $(x_1(p), x_2(p), \cdots, x_m(p))$ be a coordinate chart for X around the point p. Since Y is a submanifold of X there exist $i_1, i_2, \cdots, i_k$ such that $x_{i_1}(p), \cdots, x_{i_k}(p)$ determines a coordinate chart for Y. We will call these coordinates $x_{i_1}(p), \cdots, x_{i_k}(p)$ the *free* coordinates. What this means is the following: suppose $(x_1^t(p), \cdots, x_m^t(p))$ is a point that lies in Y and is close to $(x_1(p), x_2(p), \cdots, x_m(p))$. If we know the coordinates $x_{i_1}(t), \cdots, x_{i_k}(t)$ then we can solve for the remaining coordinates in terms of the free coordinates. We will be using this observation quite frequently henceforth.

7.1. **Neighborhood Construction.** Let $v_{\mathrm{A}} : \mathbb{P}^1 \longrightarrow X$ and $v_{\mathrm{B}} : \mathbb{P}^1 \longrightarrow X$ be two holomorphic curves of degree β_1 and β_2 respectively, such that

$$v_{\mathrm{A}}([1,0]) = v_{\mathrm{B}}([0,1])$$

(we consider $\mathbb{P}^1$ as equivalence classes of points of $\mathbb{C}^2 \setminus \{0\}$). Furthermore, assume that v_{A} and v_{B} are not multiply covered. Let $\beta := \beta_1 + \beta_2$. We will describe a procedure to construct a degree β curve that lies near the degree β bubble map determined by v_{A} and v_{B}.

Since X is projective, it embeds inside $\mathbb{P}^n$ for some n. Suppose v_{A} and v_{B} are explicitly given as

$$v_{\mathrm{A}}([X,Y]) := [\mathrm{A}^0(X,Y), \cdots, \mathrm{A}^n(X,Y)] \qquad \text{and} \qquad v_{\mathrm{B}}([X,Y]) := [\mathrm{B}^0(X,Y), \cdots, \mathrm{B}^n(X,Y)]$$

where $\mathrm{A}^\mu(X,Y)$ and $\mathrm{B}^\nu(X,Y)$ are homogeneous polynomials of degrees d_1 and d_2. Let A^μ_i and B^ν_j be the coefficient of X^i in $\mathrm{A}^\mu(X,Y)$ and $\mathrm{B}^\nu(X,Y)$ respectively. By composing with appropriate Möbius transformations (that fix $[1,0] \in \mathbb{P}^1$ and $[0,1] \in \mathbb{P}^1$ respectively), we can set three of the A^μ_i and three of the B^ν_j to be some specific constant. Now define

$$\mathrm{A}^\mu_t(X,Y) := \sum_{i=0}^{d_1}(\mathrm{A}^\mu_i + t^\mu_i)X^iY^{d_1-i} \qquad \text{and} \qquad \mathrm{B}^\nu_s(X,Y) := \sum_{j=0}^{d_2}(\mathrm{B}^\nu_j + s^\nu_j)X^jY^{d_2-j},$$

where t^μ_i and s^ν_j are small complex numbers. After composing with an automorphism of $\mathbb{P}^n$ if necessary, we may assume that $\mathrm{A}^\mu(0,1), \mathrm{B}^\mu(1,0) \neq 0 \ \forall \ \mu$. Set the three of the t^μ_i and s^ν_j to be zero (the ones that correspond to the six coefficients that were fixed). Next, given an ε that is small, define

$$\begin{aligned}\mathcal{R}^{\mu}_{\varepsilon,t,s}(X,Y) &:= \mathrm{A}^{\mu}_t(X,Y)Y^{d_2} + \frac{\mathrm{A}^{\mu}_t(1,0)}{\mathrm{B}^{\mu}_s(0,1)}\mathrm{B}^{\mu}_s(\varepsilon^2 X,Y)X^{d_1} - \mathrm{A}^{\mu}_t(1,0)X^{d_1}Y^{d_2}\,,\\ \mathcal{J}^{\mu}_{\varepsilon,t,s}(X,Y) &:= \frac{\mathrm{B}^{\mu}_s(0,1)\mathcal{R}^{\mu}_{\varepsilon,t,s}(X,\varepsilon^2 Y)}{\mathrm{A}^{\mu}_t(1,0)\varepsilon^{2d_2}}\,.\end{aligned}$$

If the polynomials $\mathcal{R}^{\mu}_{\varepsilon,t,s}$ induce a well-defined map into $\mathbb{P}^n$, then denote $u^{\mathcal{R}}_{\varepsilon,t,s} : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ to be the degree $d := d_1 + d_2$ map defined by these polynomials. Note that for generic (ε, t, s), the polynomials $\mathcal{R}^{\mu}_{\varepsilon,t,s}$ do induce a well-defined map (i.e., all the coordinates are not zero). Next, we observe that $u^{\mathcal{R}}_{\varepsilon,t,s}$ does not necessarily map into X. In order for the curve to lie in X, only $\delta_{\beta_1}+1$ of the t^{μ}_i and $\delta_{\beta_2}+1$ of the s^{ν}_j are free. Denote the free variables by t^{μ}_i and s^{ν}_j and the remaining ones by $\widehat{t}^{\mu}_i$ and $\widehat{s}^{\nu}_j$ respectively. Let us denote the corresponding polynomials to be $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{J}}$ respectively, and let

$$u^{\widehat{\mathcal{R}}}_{\varepsilon,t,s} : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

be the corresponding degree d map. By definition, now $u^{\widehat{\mathcal{R}}}_{\varepsilon,t,s}$ lies in X.

Next, let

$$\{p_i := [a^0_i, a^1_i, \cdots, a^n_i]\}_{i=1}^{\delta_{\beta_1}} \bigcup \{q_j := [b^0_j, b^2_1, \cdots, b^n_j]\}_{j=1}^{\delta_{\beta_2}}$$

be a collection of $\delta_\beta - 1$ generic points in X. Furthermore, let

$$\{\lambda_i := [x^a_i, y^a_i]\}_{i=1}^{\delta_{\beta_1}} \bigcup \{\gamma_j := [x^b_j, y^b_j]\}_{j=1}^{\delta_{\beta_2}}$$

be a collection of $\delta_\beta - 1$ points in $\mathbb{P}^1$ such that

$$\begin{aligned} v_{\mathrm{A}}([x^a_i, y^a_i]) &= [a^0_i, a^1_i, \cdots, a^n_i] \qquad \forall\, i = 1, \cdots, \delta_{\beta_1} \qquad \text{and}\\ v_{\mathrm{B}}([x^b_j, y^b_j]) &= [b^0_j, b^1_j, \cdots, b^n_j] \qquad \forall\, j = 1, \cdots, \delta_{\beta_2}.\end{aligned}$$

This gives us a set of $2\delta_\beta$ equations

$$\begin{aligned} \mathrm{A}^{\mu}(x^a_i, y^a_i)a^0_i - a^{\mu}_i \mathrm{A}^0(x^a_i, y^a_i) = 0, &\qquad \forall\, \mu = \mu_1,\ \mu_2 \text{ and } i = 1, \cdots, \delta_{\beta_1},\\ \mathrm{B}^{\nu}(x^b_j, y^b_j)b^0_j - b^{\nu}_j \mathrm{B}^0(x^b_j, y^b_j) = 0, &\qquad \forall\, \nu = \nu_1,\ \nu_2 \text{ and } j = 1, \cdots, \delta_{\beta_2},\\ \mathrm{A}^{\mu}(1,0)\mathrm{B}^0(0,1) - \mathrm{B}^{\mu}(0,1)\mathrm{A}^0(1,0) = 0, &\qquad \forall\, \mu = \widetilde{\mu}_1,\ \widetilde{\mu}_2, \end{aligned} \tag{7.2}$$

for some μ_i, ν_i and $\widetilde{\mu}_i \in \{0, 1, 2, \cdots, n\}$. Without loss of generality, we set $y^a_i = 1$ for all $i = 1, \cdots, \delta_{\beta_1}$ and set $x^b_j = 1$ for all $j = 1, \cdots, \delta_{\beta_2}$. Since $\delta_{\beta_1}+1$ of the A^{μ}_i are free and $\delta_{\beta_2}+1$ of the B^{ν}_j are free, it follows that the number of the free unknowns A^{μ}_i, B^{ν}_j, x^a_i and y^b_j is $2\delta_\beta$. Note that the evaluation map

$$\mathrm{ev} : \mathcal{M}^*_{0,\delta_{\beta_1}}(X, \beta_1) \times \mathcal{M}^*_{0,\delta_{\beta_2}}(X, \beta_2) \longrightarrow X^{\delta_{\beta_1}-1} \times X^{\delta_{\beta_2}-1} \times X^2.$$

is transverse to $(p_1, \cdots, p_{\delta_{\beta_1}-1}, q_1, \cdots, q_{\delta_{\beta_2}-1}) \times \Delta_X$ if the points $(p_1, \cdots, p_{\delta_{\beta_1}-1}, q_1, \cdots, q_{\delta_{\beta_2}-1})$ are generic. In other words, for a generic choice of these $\delta_\beta - 1$ points, the equations (7.2) simultaneously vanish transversely at the given value.

Lemma 7.1. *Let ε be a given small nonzero complex number. Then there exists a unique triple $(t(\varepsilon), s(\varepsilon), \theta(\varepsilon))$ that is small and solves the following set of equations:*

$$\begin{aligned} a_i^0\widehat{\mathcal{R}}^{\mu}_{\varepsilon,t,s}(x_i^a+\theta_i^a,1) - a_i^{\mu}\widehat{\mathcal{R}}^{0}_{\varepsilon,t,s}(x_i^a+\theta_i^a,1) &= 0, \qquad \forall\ \mu=\mu_1,\ \mu_2 \text{ and } i = 1,\cdots,\delta_{\beta_1},\\ b_i^0\widehat{\mathcal{J}}^{\nu}_{\varepsilon,t,s}(1,y_j^b+\theta_j^b) - b_i^{\nu}\widehat{\mathcal{J}}^{0}_{\varepsilon,t,s}(1,y_j^b) &= 0, \qquad \forall\ \nu=\nu_1,\ \nu_2 \text{ and } j = 1,\cdots,\delta_{\beta_2},\\ \mathrm{A}_t^{\mu}(1,0)\mathrm{B}_s^0(0,1) - \mathrm{B}_s^{\mu}(0,1)\mathrm{A}_t^1(1,0) &= 0, \qquad \forall\ \mu=\widetilde{\mu}_1,\ \widetilde{\mu}_2. \end{aligned} \tag{7.3}$$

Proof. The equation (7.2) has a transverse solution while equation (7.3) has approximately the same linearization as (7.2) (i.e., to zero-th order in ε, the linearizations are the same). The lemma now follows from the implicit function theorem. $\square$

Corollary 7.2. *Given a nonzero ε that is small, let $t(\varepsilon)$, $s(\varepsilon)$ and $\theta(\varepsilon)$ be the unique solution as given by Lemma 7.1. If $\varepsilon_1 \neq \varepsilon_2$ then the curves $u^{\widehat{\mathcal{R}}}_{\varepsilon_i,t(\varepsilon_1),s(\varepsilon_i)}$ $(i = 1,2)$ both pass through $p_1,\cdots,p_{\delta_{\beta_1}-1}$, $q_1,\cdots,q_{\delta_{\beta_2}-1}$ and they intersect transversally. In particular, the two curves are distinct.*

Proof. Follows immediately from Lemma 7.1. $\square$

Remark 7.3. Note that if the $\delta_\beta - 1$ points are generic, the map $u^{\widehat{\mathcal{R}}}_{\varepsilon,t(\varepsilon),s(\varepsilon)}$ is well-defined and not multiply covered.

Lemma 7.4. *Given a nonzero ε that is small, let $t(\varepsilon)$, $s(\varepsilon)$ and $\theta(\varepsilon)$ be the unique solution as given by Lemma 7.1, and let φ_ε be the Möbius transformation given by*

$$\varphi_\varepsilon([X,Y]) := [X,\varepsilon Y].$$

Let m be some fixed complex number different from 0 and let $\alpha_z := [1,m+z] \in \mathbb{P}^1$.[1] *Let v be a bubble map with δ_β marked points that is defined as follows: it is determined by the maps v_{A} and v_{B}; the first δ_{β_1} marked points lie on the A-bubble and are required to pass through the points p_1 to $p_{\delta_{\beta_1}}$; the next δ_{β_2} marked points lie on the B-bubble and are required to pass through q_1 to $q_{\delta_{\beta_2}}$ and the last marked point α (which is free) lies on a ghost bubble connecting the A-bubble and the B-bubble. Let $\mathrm{B_r}(0)$ be an open ball of radius r around the origin in $\mathbb{C}$. Then, the map*

$$\Phi : (\mathrm{B_r}(0) - 0)\times \mathrm{B_r}(0) \longrightarrow \mathcal{M}^*_{0,\delta_\beta}(X,\beta;p_1,\cdots,p_{\delta_{\beta_1}},q_1,\cdots,q_{\delta_{\beta_2}})$$

given by

$$\begin{aligned}\Phi(\varepsilon,z) \ := [u^{\widehat{\mathcal{R}}}_{\varepsilon,t(\varepsilon),s(\varepsilon)}\circ\varphi_\varepsilon;\ \varphi_\varepsilon^{-1}\circ\lambda_1(\theta(\varepsilon)),\cdots,\varphi_\varepsilon^{-1}\circ\lambda_{\delta_{\beta_1}-1}(\theta(\varepsilon));\\ \varphi_\varepsilon^{-1}\circ\gamma_1(\theta(\varepsilon)),\cdots,\varphi_\varepsilon^{-1}\circ\gamma_{\delta_{\beta_2}-1}(\theta(\varepsilon));\alpha_z]\end{aligned}$$

is a bijective map onto an open neighborhood of v.

Proof. First let us show that $\Phi(\varepsilon,z)$ extends continuously to $\{0\}\times\mathrm{B_r}(0)$ in the Gromov topology. Let (ε_n,z_n) be a sequence that converges to $(0,0)$. By Theorem 4.6.1 and Definition 5.2.1 in [20], we conclude that $\Phi(\varepsilon_n,z_n)$ converges in Gromov topology to $[v]$. Elsewhere Φ is obviously continuous.

Next, we observe that Corollary 7.2 combined with the fact that α_z is first order in z, implies that $\Phi(\varepsilon,z)$ is injective. It remains to show that $\Phi(\varepsilon,z)$ surjects onto an open neighborhood of $[v]$. First, define

$$\pi\ :\ \overline{\mathcal{M}}_{0,\delta_\beta}(X,\beta;p_1,\cdots,p_{\delta_{\beta_1}},q_1,\cdots,q_{\delta_{\beta_2}}) \longrightarrow \overline{\mathcal{M}}_{0,\delta_\beta-1}(X,\beta;p_1,\cdots,p_{\delta_{\beta_1}},q_1,\cdots,q_{\delta_{\beta_2}})$$

[1] The reader can set $m := 1$, but it is instructive to point out that for our arguments to hold it doesn't matter what m is as long as it is not 0. Basically we want the point α_z to be away from $[1,0]$ and $[0,1]$.

to be the map that forgets the last marked point. Also define the following map

$$\Psi : (\mathrm{B_r}(0) - 0) \longrightarrow \mathcal{M}^*_{0,\delta_\beta-1}(X, \beta; p_1, \cdots, p_{\delta_{\beta_1}}, q_1, \cdots, q_{\delta_{\beta_2}}), \quad \varepsilon \longmapsto \pi \circ \Phi(\varepsilon, z).$$

Note that $\pi([v])$ is the bubble map determined by v_{A} and v_{B}, with the $\delta_\beta - 1$ marked points distributed accordingly on the domain, but with no free marked point. In particular there is no ghost bubble. Now, by Theorem 10.1.2 and the arguments presented on Page 384 and 385 in [20], we conclude that there exists a smooth surjection from an open ball in $\mathbb{R}^2$ to an open neighborhood of $\pi([v])$ in

$$\mathcal{M}^*_{0,\delta_\beta-1}(X, \beta; p_1, \cdots, p_{\delta_{\beta_1}}, q_1, \cdots, q_{\delta_{\beta_2}}).$$

Hence an open neighborhood of $\pi([v])$ is one copy of $\mathbb{C}$ (i.e., it has just one branch). Since Ψ (like Φ) also extends as a continuous injection, it has to be a surjection onto an open neighborhood of 0 by invariance of domain. Finally, we need to show that Φ is surjective. First we note that if $[\widetilde{u}_\varepsilon, \widetilde{\lambda}(\varepsilon), \widetilde{\gamma}(\varepsilon), \widetilde{\alpha}_z]$ Gromov converges to $[v]$ then $[\widetilde{u}_\varepsilon, \widetilde{\lambda}(\varepsilon), \widetilde{\gamma}(\varepsilon)]$ Gromov Converges to $\pi([v])$. By the surjectivity of Ψ, we conclude that

$$[\widetilde{u}_\varepsilon, \widetilde{\lambda}(\varepsilon), \widetilde{\gamma}(\varepsilon)] = \Psi(\varepsilon')$$

for some ε'. Finally, we observe that if $[\Psi(\varepsilon'), \widetilde{\alpha}_z]$ converges to $[v]$ then $\widetilde{\alpha}_z = [1, m+z]$. Hence, Φ is a surjection onto an open neighborhood of $[v]$. □

Remark 7.5. In the above proof some care is needed to use Theorem 10.1.2 in [20]. In [20], the Gluing Theorem (i.e., Theorem 10.1.2) holds when we are allowed to vary almost complex structure; however the authors explicitly state that their Theorems do hold for a fixed (almost) complex structure provided we restrict ourselves to non multiply covered curves. Hence, for our arguments to go through, it is essential that the maps v_{A} and v_{B} are non-multiply covered. In fact, if v_{A} or v_{B} were multiply covered, then our assertion that the normal neighborhood has a single branch is false. There are examples of multiply covered curves in the boundary of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ whose normal neighborhood has more than one branch when X is a del-Pezzo surface. However, since the $\delta_\beta - 1$ points are generic, multiply covered curves do not arise and hence do not play a role in our computations.

7.2. **Multiplicity Computation.** We are now ready to compute the multiplicity. Since the map Φ defined in Lemma 7.4 is a bijection, it suffices to count the number of solutions to the set of equations

$$du^{\widehat{\mathcal{R}}}_\varepsilon \circ \varphi_\varepsilon|_{\alpha_z} = \nu$$

for a small perturbation ν. Here $u^{\widehat{\mathcal{R}}}_{\varepsilon,t(\varepsilon),s(\varepsilon)}$ is abbreviated as $u^{\widehat{\mathcal{R}}}_\varepsilon$. Write this equation in local coordinates. Set $w := \frac{Y}{X}$. Assuming that the zeroth and the first coordinates in $\mathbb{P}^n$ are free and the n-th coordinate is non zero, the map $u^{\widehat{\mathcal{R}}}_\varepsilon \circ \varphi_\varepsilon$ in local coordinates is given by $\mathrm{F}_0(\varepsilon, w), \mathrm{F}_1(\varepsilon, w)$ where F_μ are defined to be

$$\mathrm{F}_\mu(\varepsilon, w) := \frac{\mathrm{A}^\mu(1, \varepsilon w) + \frac{\mathrm{A}^\mu(1,0)}{\mathrm{B}^\mu(0,1)} \frac{\mathrm{B}^\mu(\varepsilon,w)}{w^{d_2}} - \mathrm{A}^\mu(1,0)}{\mathrm{A}^n(1, \varepsilon w) + \frac{\mathrm{A}^n(1,0)}{\mathrm{B}^n(0,1)} \frac{\mathrm{B}^n(\varepsilon,w)}{w^{d_2}} - \mathrm{A}^n(1,0)}, \qquad \mu = 0, 1.$$

Let

$$\mathrm{G}_\mu(\varepsilon, w) := \frac{\partial \mathrm{F}_\mu(\varepsilon, w)}{\partial w}, \qquad \mu = 0, 1.$$

It is now easy to see that

$$\mathrm{G}_\mu := -\frac{\mathrm{A}_0^\mu}{\mathrm{A}_0^n}\Big(\frac{1}{\mathrm{B}_{d_2}^\mu} - \frac{\mathrm{A}_1^n}{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}\Big)\varepsilon + \mathrm{H}(\varepsilon, w)\varepsilon^2, \qquad \mu = 0,1, \tag{7.4}$$

where $\mathrm{H}(\varepsilon, w)$ is holomorphic function near $(0, m)$. (Recall that we have taken m to be a fixed nonzero complex number.) Since $\mathrm{G}_\mu(\varepsilon, w)$ is well-defined it follows that A_0^n and $\mathrm{B}_{d_2}^n$ are non-zero. Note that $\mathrm{G}_0(0, w) = 0$ and $\mathrm{G}_1(0, w) = 0$. We need to find the order to which this vanishing takes place near the point $(0, m)$. We now make a change of coordinates near $(0, m)$ given by

$$\widetilde{\varepsilon} := \mathrm{G}_0(\varepsilon, w) \qquad \text{and} \qquad \widetilde{w} := w. \tag{7.5}$$

If

$$-\frac{\mathrm{A}_0^0}{\mathrm{A}_0^n}\Big(\frac{1}{\mathrm{B}_{d_2}^0} - \frac{\mathrm{A}_1^n}{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}\Big) \neq 0, \tag{7.6}$$

then (7.5) defines a change of coordinates near $(0, m)$ (see (7.4)). Let us justify why we can assume (7.6) holds. First of all we note that

$$v_{\mathrm{A}}([0,1]) = [\mathrm{A}_0^0, \mathrm{A}_0^1, \mathrm{A}_0^2, \cdots, \mathrm{A}_0^n] \qquad \text{and} \qquad v_{\mathrm{B}}([1,0]) = [\mathrm{B}_{d_2}^0, \mathrm{B}_{d_2}^1, \mathrm{B}_{d_2}^2, \cdots, \mathrm{B}_{d_2}^n].$$

Consider the following $\mathbb{C}^2$ inside $\mathbb{C}^{n+1}$, namely:

$$L := (*, *, 0, \cdots, 0) \subset \mathbb{C}^{n+1}.$$

We will now consider automorphisms of $\mathbb{P}^n$ induced from an automorphism of $\mathbb{C}^{n+1}$ that acts non trivially on L and acts as identity on

$$(0, 0, *, \cdots, *) \subset \mathbb{C}^{n+1}.$$

We claim that we can find such an automorphism moving $v_{\mathrm{A}}([0,1])$ and $v_{\mathrm{B}}([1,0])$ to two points such that

$$\mathrm{A}_0^0, \mathrm{A}_0^1 \neq 0 \qquad \text{and} \qquad \Big(\frac{1}{\mathrm{B}_{d_2}^0} - \frac{\mathrm{A}_1^n}{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}\Big),\ \Big(\frac{1}{\mathrm{B}_{d_2}^1} - \frac{\mathrm{A}_1^n}{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}\Big) \neq 0. \tag{7.7}$$

To see why this is so, we will consider three cases. Suppose $\mathrm{B}_{d_2}^0 \neq \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$ and $\mathrm{B}_{d_2}^1 \neq \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$. Then we take an automorphism that fixes $\mathrm{B}_{d_2}^0$ and $\mathrm{B}_{d_2}^1$ and takes both A_0^0 and A_0^1 to something non zero. Next, if $\mathrm{B}_{d_2}^0 = \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$, but $\mathrm{B}_{d_2}^1 \neq \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$, then we take an automorphism of $\mathbb{C}^{n+1}$ that takes $\mathrm{B}_{d_2}^0$ to $2\mathrm{B}_{d_2}^0$, takes $\mathrm{B}_{d_2}^1$ to $\mathrm{B}_{d_2}^1$ and takes both A_0^0 and A_0^1 to something non zero. Similar argument holds if $\mathrm{B}_{d_2}^0 \neq \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$, but $\mathrm{B}_{d_2}^1 = \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$. Finally, suppose $\mathrm{B}_{d_2}^0 = \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$ and $\mathrm{B}_{d_2}^1 = \frac{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}{\mathrm{A}_1^n}$. Then we take an automorphism of $\mathbb{C}^{n+1}$ that takes $\mathrm{B}_{d_2}^0$ and $\mathrm{B}_{d_2}^1$ to $2\mathrm{B}_{d_2}^0$ and $2\mathrm{B}_{d_2}^1$ and both A_0^0 and A_0^1 to something non zero. That covers all the cases.

Equation (7.7) implies that (7.6) holds; in addition, it also implies that

$$-\frac{\mathrm{A}_0^0}{\mathrm{A}_0^n}\Big(\frac{1}{\mathrm{B}_{d_2}^1} - \frac{\mathrm{A}_1^n}{\mathrm{A}_0^n\mathrm{B}_{d_2}^n}\Big) \neq 0, \tag{7.8}$$

holds. Note that since our automorphism only acts on L, the initial assumptions we made about the zeroth and first coordinate being free and the n^{th} coordinate being non zero, is still valid. Let

$$\widetilde{\mathrm{G}}_\mu(\widetilde{\varepsilon}, \widetilde{w}) := \mathrm{G}_\mu(\varepsilon, w), \qquad \mu = 0, 1.$$

Hence

$$\mathrm{G}_0(\widetilde{\varepsilon}, \widetilde{w}) := \widetilde{\varepsilon} \qquad \text{and} \qquad \mathrm{G}_1(\widetilde{\varepsilon}, \widetilde{w}) := \widetilde{\varepsilon}\mathrm{K}(\widetilde{\varepsilon}, \widetilde{w}),$$

for some function $\mathrm{K}(\widetilde{\varepsilon}, \widetilde{w})$. It is easy to see that since (7.6) and (7.8) hold, $\mathrm{K}(0, m) \neq 0$ if $m \neq 0$. Next, let $\nu_0(\widetilde{\varepsilon}, w)$ and $\nu_1(\widetilde{\varepsilon}, w)$ be two holomorphic perturbations (they are defined only in a neighborhood of $(0, m)$), whose Taylor expansion near $(0, m)$ is given by

$$\begin{aligned} \nu_0(\widetilde{\varepsilon}, w) &:= a_{00} + a_{10}\widetilde{\varepsilon} + a_{01}(w-m) + \dots \qquad \text{and} \\ \nu_1(\widetilde{\varepsilon}, w) &:= b_{00} + b_{10}\widetilde{\varepsilon} + b_{01}(w-m) + \dots \end{aligned} \tag{7.9}$$

By saying that ν_0 and ν_1 are perturbations, we mean that the the constant terms a_{00} and b_{00} in the Taylor expansion are all small. Now, we need to solve for

$$\widetilde{\varepsilon} = \nu_0(\widetilde{\varepsilon}, w) \qquad \text{and} \tag{7.10}$$

$$\widetilde{\varepsilon}\mathrm{K}(\widetilde{\varepsilon}, \widetilde{w}) = \nu_1(\widetilde{\varepsilon}, w). \tag{7.11}$$

Using (7.11) and (7.9), we conclude that

$$\widetilde{\varepsilon} = \frac{b_{00}}{\mathrm{K}(0, m)} + \mathrm{O}(w-m). \tag{7.12}$$

Using (7.12), (7.10) and (7.9), we conclude that

$$w - m = \frac{\frac{b_{00}(1-a_{10})}{\mathrm{K}(0,m)} - a_{00}}{a_{10}} + \mathrm{O}((w-m)^2). \tag{7.13}$$

Equation (7.13) implies that if the perturbation ν is generic then there exists a unique solution w in a sufficiently small neighborhood of m. Since the ν_i are chosen to be holomorphic, it will be counted with a positive sign. Hence the multiplicity is one.

8. Proof of transversality and general position arguments

It remains to show that the section ψ induced by taking the derivative at a marked point is transverse to the zero set, and that there exists a rational curve in (X, β) that has exactly one genuine cusp and is an immersion otherwise. We prove the latter first. In what follows whenever we count nodes we do so along with their multiplicities.

Proposition 8.1. *Let X be the blowup of $\mathbb{P}^2$ at k generic points q_i (with exceptional divisors E_i) and let $\beta := dL - \sum_i m_i E_i$ be a homology class such that $N_{\beta-3L} > 0$. Then, there exists a non-multiply covered rational curve in the class β having exactly one genuine cusp; furthermore, the curve is an immersion everywhere else.*

Proof. Since $N_{\beta-3L} > 0$, there exists an immersion $v : \mathbb{P}^1 \longrightarrow X$ representing the class $\beta - 3L$ (cf. Theorem 4.1, [8]). Let c represent the homology class of a genuine cuspidal cubic in X; i.e. the homology class of c is $3L$. Choose the cubic to intersect v transversally at $3(d-3)$ non-singular points. We will construct a cuspidal curve in the class $(\beta - 3L) + 3L$ by considering the bubble map formed by $v_A = v$ and $v_B = c$ where we may assume without loss of generality that $v_A([1:0]) = v_B([0:1])$. Now we are in the setting of Section 7. Using the same notation as in that section, consider the map $u^{\widehat{\mathcal{R}}}_{\varepsilon,t,s}$ (whose image is by construction is required to lie in X). If ε is small enough then it is easy to see that the number of nodes of this perturbed degree d curve is $3(d-3) - 1$ more than the number for v_A (which is $\binom{d-4}{2} - t$ with t depending only on m_i) because we "resolve" a node corresponding to the bubble point (which is the point $v_A([1:0]) = v_B([0,1])$).

Let v_B have its cusp at the point $[1:z]$. If we require $u^{\widehat{\mathcal{J}}}_{\varepsilon,t,s}$ to have a cusp at $[1:y]$ then this implies (assuming that the coordinate $\frac{X^{\mu_1}}{X^0}$ and $\frac{X^{\mu_2}}{X^0}$ are free in X) that

$$\begin{aligned} &\left(d\left(\frac{u^{\mu_1}}{u^0}\right), d\left(\frac{u^{\mu_2}}{u^0}\right)\right) = (0,0) \\ \Rightarrow &\left(\frac{Q_1(y)}{Q_0(y)}, \frac{Q_2(y)}{Q_0(y)}\right) = (0,0), \end{aligned} \tag{8.1}$$

where Q_i are polynomials whose coefficients depend rationally on ε, t, s and Q_1, Q_2 have a common root z when $\varepsilon = 0 = s$. Since the variety defined by Q_1, Q_2 (treated as a function of ε, y, s) is non-empty near $0, z, 0$, there exists a small enough $(y-z, \varepsilon, s)$ such that $\varepsilon \neq 0$ so that $du(y) = 0$.

In summary, the perturbed map has one cusp and $3(d-3) - 1 + \binom{d-4}{2} - t$ nodes. By genus considerations the maximum number of nodes/cusps that a rational curve of class β have is $\binom{d-1}{2} - t$. Since that maximum number has been attained, the cusp has to be a genuine one (since anything worse than a cusp would contribute more than 1 to the genus). $\square$

Using a similar construction as in Proposition 8.1 we prove that cuspidal curves form a submanifold by constructing a genuine cuspidal curve at which transversality holds.

Lemma 8.2. *Let* $\psi : \mathcal{M}^*_{0,\delta_\beta - 1}(X, \beta; p_1, p_2, \ldots, p_{\delta_\beta - 1}) \longrightarrow \mathcal{L}_1^* \otimes \mathrm{ev}_1^* TX$ *be the section induced by taking the derivative at the marked point, i.e.,*

$$\psi([u;z]) := du|_z\,.$$

If $N_{\beta - 3L} > 0$*, then* ψ *is transverse to the zero section*

Proof. We will actually construct a (genuine) cuspidal curve in (X, β) such that the section $\psi : \mathcal{M}^*_{0,1}(X,\beta) \longrightarrow \mathcal{L}_1^* \otimes \mathrm{ev}_1^* TX$ (i.e., X with one marked point) is transverse at this curve. This means that curves at which transversality fails form a proper subvariety and therefore, by the requirement of passing through $\delta_\beta - 1$ generic points we may conclude that transversality holds for all such cuspidal curves.

Certainly we may find a cuspidal cubic v_A whose homology class is $3L$ for which transversality holds. We claim that gluing v_A with an immersion representing $\beta - 3L$ (just as in the proof of Proposition 8.1) produces a genuine cuspidal curve for which the section ψ is transverse to the zero section. Indeed, suppose that $(v_{A1}(\tau), z_1(\tau))$ and $(v_{A2}(\tau), z_2(\tau)$ are two paths in $\mathcal{M}^*_{0,1}(X, 3L)$ such that they pass through v_A at $\tau = 0$, and that the tangent vectors to $dv_{Ai}(\tau)(z_i(\tau))$ are linearly independent at $\tau = 0$ (this is true because we assumed transversality for v_A). Then consider the perturbed families of maps $u^{\widehat{\mathcal{R}}}_{1,\varepsilon,t,s,\tau}$ and $u^{\widehat{\mathcal{R}}}_{2,\varepsilon,t,s,\tau}$. It is easy to see that if ε is small enough then for this perturbed family of maps $du_{i,\tau}(z_i(\tau) + y - z_i(0))$ have linearly independent tangent vectors at $\tau = 0$ where y is the location of the cusp of the perturbed map when $\tau = 0$. (Note that y depends on ε, t, s.) $\square$

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