Arch. Math.

https://doi.org/10.1007/s00013-022-01762-w

Archiv der Mathematik

Non-negative divisors and the Grauert metric

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Abstract. Grauert showed that it is possible to construct complete Kähler metrics on the complement of complex analytic sets in a domain of holomorphy. In this note, we study the holomorphic sectional curvatures of such metrics on the complement of a principal divisor in $\mathbb{C}^n$, $n \geq 1$. In addition, we also study how this metric and its holomorphic sectional curvature behave when the corresponding principal divisors vary continuously.

Mathematics Subject Classification. 32T05, 32Q15.

Keywords. Grauert metric, Holomorphic sectional curvature, Non-negative divisors.

1. Introduction. While every domain of holomorphy admits a real analytic, complete Kähler metric according to Grauert [1], the converse is false in general. Indeed, in the same paper, Grauert writes down an explicit example of such a metric on $(\mathbb{C}^n)^* = \mathbb{C}^n \setminus \{0\}$ for $n \geq 1$. Since $(\mathbb{C}^n)^*$ is not a domain of holomorphy for $n \geq 2$, it follows that the property of being a domain of holomorphy is not always equivalent to admitting a real analytic, complete Kähler metric. The construction of such a metric can be carried over to a domain of the form $G \setminus A$, where $G \subset \mathbb{C}^n$ is a domain of holomorphy and $A \subset G$ is a complex anaytic set by pulling back the construction on $(\mathbb{C}^n)^*$ by means of the defining functions for A. A suitable metric can be added to it to remove any degeneracies that may occur in the pull-back metric.

The holomorphic sectional curvature encodes important information about a given hermitian metric and this point was addressed in [3]. We studied the holomorphic sectional curvature of such metrics in two prototype cases, namely, (i) $(\mathbb{C}^n)^*$, $n \geq 2$, and (ii) $\mathbb{B}^N \setminus A$, $N \geq 2$, where $A \subset \mathbb{B}^N$ is a hyperplane of codimension at least 2. We continue our study of this metric here.

SG is supported by CSIR-SPM Ph.D. fellowship.

Published online: 26 July 2022

The purpose of this note is to complete the picture by studying these metrics, which are constructed in the complement of a closed analytic set of codimension 1, i.e., a divisor. First, we study the holomorphic sectional curvatures of the metric that is obtained by this process on the complement Ω of a principal divisor D in $\mathbb{C}^n$, $n \geq 2$. The case $n = 1$ is studied in detail in the last section. Secondly, we show that if $U \subset \mathbb{C}^n$, $n \geq 1$, is a domain of holomorphy (or more generally, admits a complete Kähler metric) with $H^1(U, \mathbb{Z}) = 0$, then Grauert's construction applied to $U \setminus D$, where $D \subset U$ is a principal divisor, possesses an intrinsic continuity property when D varies in an appropriate sense.

To set things in context, recall that the set of divisors on a complex manifold M is

$$\mathcal{D}(M) = \left\{ \sum_{i \in I} n_i V_i : n_i \in \mathbb{Z} \right\}$$

where $\{V_i\}_{i \in I}$ is a locally finite system of irreducible analytic hypersurfaces in M and the support of a divisor $D \in \mathcal{D}(M)$ is the union of the hypersurfaces V_i – this will be denoted by $|D|$. The set of non-negative divisors $\mathcal{D}^+(M) \subset \mathcal{D}(M)$ consists of those divisors wherein $n_i \geq 0$ for all $i \in I$. The natural map

$$\mathsf{Div} : \mathcal{O}(M) \to \mathcal{D}^+(M)$$

defined by

$$\mathsf{Div}(f) = \sum_i (\mathrm{ord}_{V_i} f)\, V_i$$

assigns, to a holomorphic function, the divisor defined by its zero set; here, $\{V_i : i \in I\}$ are the irreducible components of $\mathcal{Z}(f)$, the zero set of a holomorphic function $f \in \mathcal{O}(M)$, and $\mathrm{ord}_{V_i} f$ is the order of $V_i \subset \mathcal{Z}(f)$. The set of *principal divisors* arises as the image $\mathsf{Div}(\mathcal{O}(M)) \subset \mathcal{D}^+(M)$ and this will be denoted by $\mathcal{D}^+_P(M)$.

Let $D \in \mathcal{D}^+_P(\mathbb{C}^n)$ be a principal divisor. Choose $f \in \mathcal{O}(\mathbb{C}^n)$ such that $\mathsf{Div}(f) = D$. Recall that the Grauert metric on $\mathbb{C}^*$ (see [1]) is a conformal metric defined by

$$g = \left(1 + |w|^2\, u^2\left(|w|^2\right)\right) |dw|^2, \tag{1.1}$$

where $u : (0, \infty) \to \mathbb{R}$ is the function $u(t) = (t-1)/(t \log t)$ for $t \neq 1$, and $u(1) = 1$. It was shown in [3] that the curvature of g approaches -4 or 0 according as $|z| \to 0$ or $+\infty$ respectively, and is non-positive everywhere on $\mathbb{C}^*$. Consider the complete Kähler metric (see [4, p. 155]) on $\mathbb{C}^n \setminus |D|$ defined by

$$\phi(z, V) = f^*(g)(z, V) + |V|^2 = \left(1 + |f(z)|^2\, u^2\left(|f(z)|^2\right)\right) |df(z)(V)|^2 + |V|^2, \tag{1.2}$$

where $z \in \mathbb{C}^n \setminus |D|$ and $V \in \mathbb{C}^n$.

For $p \in \mathbb{C}^n \setminus |D|$ and $V \in \mathbb{C}^n$, denote the holomorphic sectional curvature of ϕ at (p, V) by $K(p, V)$. Set $K^+(p)$ to be the supremum of $K(p, V)$ as V varies in $\mathbb{C}^n$. For a non-singular vector field X near $p \in \mathbb{C}^n \setminus |D|$, consider the restriction of ϕ to the leaf of the foliation defined by X passing through p. Thus, we get a conformal metric on the germ of a Riemann surface containing p. Let $\mathcal{K}^X(p)$ denote the Gaussian curvature of this metric at p.

Theorem 1.1. *Let (D, f, ϕ) be as above on $\mathbb{C}^n$, $n \geq 2$.*

(i) *If $p \in \mathbb{C}^n \setminus |D|$ and $V \in \mathrm{Ker}(df(p))$, then $K(p, V) \leq 0$. In particular, if $\mathrm{Rank}(df(p)) = 0$, then $K^+(p) \leq 0$.*

(ii) *If $p \in \mathbb{C}^n \setminus |D|$ and $f_{z_i}(p) = \frac{\partial f}{\partial z_i}(p) \neq 0$ for some $1 \leq i \leq n$, then there exists a non-singular holomorphic vector field X near p such that $X(p) \notin \mathrm{Ker}(df(p))$ and $\mathcal{K}^X(p) \leq 0$.*

(iii) *If $p \in |D|$ and $f_{z_i}(p) = \frac{\partial f}{\partial z_i}(p) \neq 0$, then there exists a non-singular holomorphic vector field X near p such that*

$$\lim_{z \to p} \mathcal{K}^X(z) = -4 < 0.$$

To summarize, for points away from the support of D, (i) shows that the curvature $K(p, X)$ is non-positive along certain directions. On the other hand, though the conclusions of (ii) and (iii) are less precise, they show that the curvature of the restriction of ϕ to the leaves of a certain foliation is again non-positive.

It is natural to identify a non-negative divisor with the current of integration over it, namely:

$$D(\alpha) = \langle \alpha, D \rangle = \sum_j m_j \int_{V_j} \alpha$$

for each smooth compactly supported $(n-1, n-1)-$form α on X where $D = \sum_j m_j V_j$. Thus $\mathcal{D}^+(M)$ can be considered as a subset of the dual space of $D^{(n-1,n-1)}(M)$, the space of compactly supported $(n-1, n-1)-$smooth forms on M. Therefore $\mathcal{D}^+(M)$ inherits two natural topologies, namely, the relative weak* topology and the relative strong topology.

Weak* topology: A net $\{D_\alpha\}_{\alpha \in A}$ in $\mathcal{D}^+(M)$ converges to $D_0 \in \mathcal{D}^+(M)$ iff for every compactly supported smooth $(n-1, n-1)-$form ξ on M,

$$\lim_{\alpha \in A} \int_{D_\alpha} \xi = \int_{D_0} \xi.$$

Strong topology: A net $\{D_\alpha\}_{\alpha \in A}$ in $\mathcal{D}^+(M)$ converges to $D_0 \in \mathcal{D}^+(M)$ iff it converges in the weak* sense and if, moreover, the convergence is uniform on bounded sets in the space $D^{(n-1,n-1)}(M)$.

There is another topology on $\mathcal{D}^+(M)$ introduced by Stoll [6] which is defined in function-theoretic terms:

Stoll's topology: A net $\{D_\alpha\}_{\alpha \in A}$ in $\mathcal{D}^+(M)$ converges to $D_0 \in \mathcal{D}^+(M)$ iff there is an open cover $\mathcal{V} = \{V_j\}_{j \in \mathbb{N}}$ of M such that for each $\alpha \in A$, there is an $f_{j\alpha} \in \mathcal{O}(V_j)$ such that $\mathsf{Div}(f_{j\alpha}) = D_\alpha|_{V_j}$ and $f_{j\alpha}$ converges uniformly on compacts in V_j to $f_{j0} \in \mathcal{O}(V_j)$, where $\mathsf{Div}(f_{j0}) = D_0|_{V_j}$.

Lupacciolu–Stout [5] have shown that on $\mathcal{D}^+(M)$, the above three topologies are equivalent. Using this equivalence they showed: *For a complex-analytic*

manifold M of dimension $n \geq 1$ satisfying $H^1(M,\mathbb{Z}) = 0$ and $H^1(M,\mathcal{O}) = 0$, there exists a continuous map $\psi : \mathcal{D}_P^+(M) \to \mathcal{O}(M)$ such that $\mathsf{Div}(\psi(D)) = D$ *for all $D \in \mathcal{D}_P^+(M)$.* In [5], they also proved: *If M is a domain in a Stein manifold N with the property $H^1(M,\mathbb{Z}) = 0$, then there exists a continuous map $\psi : \mathcal{D}_P^+(M) \to \mathcal{O}(M)$ such that* $\mathsf{Div}(\psi(D)) = D$ for all $D \in \mathcal{D}_P^+(M)$.
Let $U \subset \mathbb{C}^n$ be a domain of holomorphy or more generally a domain that has a complete Kähler metric Φ_U. Let $D \in \mathcal{D}_P^+(U)$ be a principal divisor and $f \in \mathcal{O}(U)$ be a holomorphic function such that $D = \mathsf{Div}(f)$. Define a pseudometric on $U \setminus |D|$ by $\tilde{\phi}_D := f^*(g)$, where g is the Grauert metric on $\mathbb{C}^*$. Therefore, $\phi_D = \tilde{\phi}_D + \Phi_U$ is a complete Kähler metric on $U \setminus |D|$.

The following statements clarify the dependence of the Grauert metric and its curvature as a function of the divisor D.

Theorem 1.2. *Let $U \subset \mathbb{C}^n$, $n \geq 1$, be a domain as above, which also satisfies $H^1(U,\mathbb{Z}) = 0$. If $\{D_j\}_{j\in\mathbb{N}} \subset \mathcal{D}_P^+(U)$ is a sequence of non-negative principal divisors such that D_j converges to $D_0 \in \mathcal{D}_P^+(U)$ with respect to any of the equivalent topologies above, then*

- (i) *there exist a sequence of complete Kähler metrics $\{\phi_j\}$ and ϕ_0 on $U \setminus |D_j|$ and $U \setminus |D_0|$ repectively such that ϕ_j converges to ϕ_0 uniformly on compacts of $U \setminus |D_0|$.*
- (ii) *for a fixed complete Kähler metric on $U \setminus |D_0|$ of the form $\xi = f^*(g) + \Phi_U$ where $D_0 =$* $\mathsf{Div}(f)$*, we can choose a sequence of complete Kähler metrics $\{\xi_j\}$ on $U \setminus |D_j|$ such that ξ_j converges to ξ uniformly on compacts of $U \setminus |D_0|$.*

Fix $p \in U \setminus |D_j|$ and a non-singular holomorphic vector field X near p. For a complete Kähler metric ϕ_j on $U \setminus |D_j|$, denote the Gaussian curvature of the metric ϕ_j restricted to the leaves of foliation induced by X by $\mathcal{K}^X_{\phi_j}(q)$ for points $q \in U \setminus |D_j|$ near p. For $V \in \mathbb{C}^n$, let $K_{\phi_j}(p,V)$ denote the holomorphic sectional curvature of the metric ϕ_j at (p,V).

Theorem 1.3. *Let (U, ϕ_j, ϕ_0) be as in the above theorem and fix $p \in U \setminus |D_0|$. Then*

- (i) *for a non-singular holomorphic vector field X near p, there exists a neighourhood U_p of p in $U \setminus |D_0|$ such that $\mathcal{K}^X_{\phi_j}(q) \to \mathcal{K}^X_{\phi_0}(q)$ for all $q \in U_p$*
- (ii) *$K_{\phi_0}(p,V) \leq \liminf_{j\to\infty} K_{\phi_j}(p,V)$ for all $V \in \mathbb{C}^n$.*

2. Proof of Theorem 1.1. Fix a non-negative principal divisor $D \in \mathcal{D}_P^+(\mathbb{C}^n)$ and $f \in \mathcal{O}(\mathbb{C}^n)$ such that $\mathsf{Div}(f) = D$. The corresponding complete Kähler metric on $\mathbb{C}^n \setminus |D|$ is given by:

$$\phi(z,V) = \Big(1 + |f(z)|^2\, u^2\big(|f(z)|^2\big)\Big)\, |df(z)(V)|^2 + |V|^2$$

where $z \in \mathbb{C}^n \setminus |D|$ and $V \in \mathbb{C}^n$.
For (i), let $p \in \mathbb{C}^n \setminus |D|$ and let X be a non-singular holomorphic vector field near p such that $X_p = X(p) \in \mathrm{Ker}\big(df(p)\big)$. Note that since $n \geq 2$, we have

$\dim\Big(\mathrm{Ker}\big(df(p)\big)\Big) \geq 2-1 = 1$. For a disc $B(0,\epsilon) \subset \mathbb{C}$, let $Z : B(0,\epsilon) \to \mathbb{C}^n \setminus |D|$ be a holomorphic parametrization of a germ of a leaf of X through p, i.e.,

$$Z'(T) = X\big(Z(T)\big)$$

for all $T \in B(0,\epsilon)$ and $Z(0) = p$. We will write Z_T to denote $Z(T)$, and just u, u', u'' instead of $u\big(|f(Z_T)|^2\big), u'\big(|f(Z_T)|^2\big), u''\big(|f(Z_T)|^2\big)$. Therefore

$$Z^*(\phi)(T) = \Big\{\big(1 + |f(Z_T)|^2 u^2\big)\left|df(Z_T)\big(X(Z_T)\big)\right|^2 + |X(Z_T)|^2\Big\}|dT|^2$$

and this can be written as $Z^*(\phi)(T) = h(T)|dT|^2$, where

$$h(T) = \Big(\big(1 + |f(Z_T)|^2 u^2\big)\left|df(Z_T)\big(X(Z_T)\big)\right|^2\Big) + |X(Z_T)|^2.$$

Note that $\partial h(T) = \frac{\partial h(T)}{\partial T} = A_1(T) + A_2(T) + A_3(T)$, where

$$A_1(T) = \big(u^2 + 2|f(Z_T)|^2 uu'\big)\left|df(Z_T)\big(X(Z_T)\big)\right|^2 \left\langle df(Z_T)\big(X(Z_T)\big), f(Z_T)\right\rangle,$$
$$A_2(T) = \big(1 + |f(Z_T)|^2 u^2\big)\Big\langle \partial\Big(df(Z_T)\big(X(Z_T)\big)\Big), df(Z_T)\big(X(Z_T)\big)\Big\rangle,$$
$$A_3(T) = \big\langle dX(Z_T)\big(X(Z_T)\big), X(Z_T)\big\rangle.$$

Using the fact that $\overline{h(T)} = h(T)$, it can be checked that $\bar\partial h(T) = \overline{\partial h(T)}$. Also $A_1(0) = A_2(0) = \bar\partial A_1(0) = 0$, and

$$\bar\partial A_2(T) = P(T) + \Big(\big(1 + |f(Z_T)|^2 u^2\big)\left|\partial\Big(df(Z_T)\big(X(Z_T)\big)\Big)\right|^2\Big),$$
$$\bar\partial A_3(T) = \left|dX(Z_T)\big(X(Z_T)\big)\right|^2,$$

where $P(T) = \bar\partial\Big(1 + |f(Z_T)|^2 u^2\Big)\Big\langle \partial\Big(df(Z_T)\big(X(Z_T)\big)\Big), df(Z_T)\big(X(Z_T)\big)\Big\rangle$ satisfies $P(0) = 0$.

Thus, we get that $h(0) = |X_p|^2$, $\partial h(0) = \big\langle dX(p)\big(X_p\big), X_p\big\rangle$, and $\bar\partial h(0) = \big\langle X_p, dX(p)\big(X_p\big)\big\rangle$, and

$$\begin{aligned}\bar\partial\partial h(0) &= \left|dX(p)\big(X_p\big)\right|^2 + \Big(1 + |f(p)|^2 u^2(|f(p)|^2)\Big)\left|\partial\Big(df(Z_T)\big(X(Z_T)\big)\Big)\Big|_{T=0}\right|^2 \\ &\geq \left|dX(p)\big(X_p\big)\right|^2\end{aligned}$$

and this implies that

$$h(0)\bar\partial\partial h(0) - \partial h(0)\bar\partial h(0) \geq |X_p|^2 |dX(p)(X_p)|^2 - |\langle dX(p)(X_p), X_p\rangle|^2 \geq 0$$

by the Cauchy–Schwarz inequality. Therefore, the Gaussian curvature of the leaves of the foliation induced by X at p is

$$\mathcal{K}^X(p) = -2(h(0))^{-3}\left\{h(0)\bar\partial\partial h(0) - \partial h(0)\bar\partial h(0)\right\} \leq 0.$$

By [7, Lemma 4], there exists a non-singular vector field Y near p such that $Y(p) = V$ and $K(p,V) = \mathcal{K}^Y(p)$. Since the above argument is true for all non-singular holomorphic vector fields near p which equal V at point p, it follows that

$$K(p,V) \leq 0.$$

Now, if $\mathrm{Rank}\big(df(p)\big) = 0$ at some $p \in \mathbb{C}^n \setminus |D|$, then $\mathrm{Ker}\big(df(p)\big) = \mathbb{C}^n$ and therefore $K(p,V) \leq 0$ for all $V \in \mathbb{C}^n$. Thus

$$K^+(p) = \sup_{V \in \mathbb{C}^n} K(p,V) \leq 0.$$

Note: The above proof is true more generally also. Indeed, let M be a complex manifold of dimension $n \geq 2$ equipped with a metric ϕ which has non-positive holomorphic sectional curvature. Let $f : M \to \mathbb{C}$ be a non-constant holomorphic map and assume that $\mathbb{C}^*$ is equipped with a metric ψ. Then the metric $\Phi := f^*(\psi) + \phi$ defined on $M \setminus \{f = 0\}$ satisfies $K(p,V) \leq 0$ for all $p \in M \setminus \{f = 0\}$ and $V \in \mathrm{Ker}\big(df(p)\big)$.

Now let $p \in \mathbb{C}^n$ be such that $\mathrm{Rank}\big(df(p)\big) = 1$ and suppose that $f_{z_1}(p) = \frac{\partial f}{\partial z_1}(p) \neq 0$. Then $W(z) = (f(z), z_2, \ldots, z_n)$ defines a new coordinate system around p with

$$W^{-1}(w) = (\tilde{g}(w), w_2, \ldots, w_n)$$

for a suitable holomorphic function $\tilde{g}$. Since $W \circ W^{-1} \equiv \mathrm{Id}$, we see that $f_{z_1}(z)\tilde{g}_{w_1}(W(z)) \equiv 1$. For $V = (1, 0, \ldots, 0) \in \mathbb{C}^n$,

$$\left((W^{-1})^*\phi\right)(w,V) = (1 + |w_1|^2 u^2(|w_1|^2)) \\ \Big((W^{-1})^* |df(z)(dz)|^2 (w,V)\Big) + (W^{-1})^*(|dz|^2)(w,V)$$

and observe that

$$\left(W^{-1}\right)^* \left(|df(z)(dz)|^2\right)(w,V) = \left|f_{z_1}\left(W^{-1}(w)\right)\right|^2 |\tilde{g}_{w_1}(w)|^2 \equiv 1$$

and $\left(W^{-1}\right)^*(|dz|^2)(w,V) = |\tilde{g}_{w_1}(w)|^2 = \left|f_{z_1}(W^{-1}(w))\right|^{-2}$. Therefore, we get

$$\tilde{\phi}(w,V) := \left((W^{-1})^*\phi\right)(w,V) = \left(1 + |w_1|^2 u^2(|w_1|^2) + |\tilde{g}_{w_1}(w)|^2\right).$$

Consider the constant vector field $\tilde{X}(w) \equiv V$ near the point $W(p)$. Take $X := W^*(\tilde{X})$ near point p and observe that $X_p = W^*(V) \notin \mathrm{Ker}\big(df(p)\big)$. Let $w = (w_1, w_2, \ldots, w_n) = W(z) \in \mathbb{C}^n$ and $Z_T = Z(T) = w + (T, 0, \ldots, 0)$ be a local parametrization of the leaf of $\tilde{X}$ passing through w. We get

$$Z^*(\tilde{\phi})(T) = \left(\left(1 + |w_1 + T|^2 u^2\right) + |\tilde{g}_{w_1}(Z_T)|^2\right)|dT|^2 = \big(A(T) + B(T)\big)|dT|^2$$

where $A(T) = \left(1 + |w_1 + T|^2 u^2(|w_1 + T|^2)\right)$ and $B(T) = |\tilde{g}_{w_1}(Z_T)|^2$.
For (ii), suppose that $p \in \mathbb{C}^n \setminus |D|$. Fix $w = W(p)$ and observe that $A(T)|dT|^2$ is the Grauert metric restricted in a neighbourhood of $0 \neq f(p) \in \mathbb{C}$, therefore the Gaussian curvature is non-positive by [3, p. 8]. It is straightforward to check that

$$\partial B(T) = \tilde{g}_{w_1 w_1}(Z_T)\overline{\tilde{g}_{w_1}(Z_T)}, \ \bar{\partial} B(T) = \tilde{g}_{w_1}(Z_T)\overline{\tilde{g}_{w_1 w_1}(Z_T)}, \text{ and} \\ \bar{\partial}\partial B(T) = |\tilde{g}_{w_1 w_1}(Z_T)|^2,$$

which tells us that the metric $B(T)|dT|^2$ has curvature identically 0 near $T = 0$. Therefore, [2, p. 111] tells us that the Gaussian curvature of the metric $Z^*(\tilde{\phi})$ is non-positive near $0 \in \mathbb{C}$. Thus, we get $\mathcal{K}^X(p) \leq 0$.

For (iii), suppose that $p \in |D|$ and let $w = W(z) \in \mathbb{C}^n$ be such that $w_1 \neq 0$. As we calculated in (ii), it is clear that

$$B(T)\,\bar{\partial}\partial B(T) - \bar{\partial}B(T)\,\partial B(T) \equiv 0 \tag{2.1}$$

and $\tilde{g}_{w_1}, \tilde{g}_{w_1 w_1}$ are bounded near $W(p)$. Since $A(0) = \left(1 + |w_1|^2\, u^2(|w_1|^2)\right)$, it follows that $\lim_{w_1 \to 0} A(0) = +\infty$, and

$$\lim_{w_1 \to 0} \frac{A(0)}{A(0) + B(0)} = 1. \tag{2.2}$$

Now

$$\begin{aligned}
\mathcal{K}^X(z) &= \mathcal{K}^{\tilde{X}}(W(z)) \\
&= -2\left(\frac{(A+B)(0)\bar{\partial}\partial(A+B)(0) - \bar{\partial}(A+B)(0)\partial(A+B)(0)}{(A(0)+B(0))^3}\right) \\
&= -2\frac{N(0)}{D(0)}
\end{aligned}$$

where $D(0) = (A(0)+B(0))^3$ and

$$\begin{aligned}
N(0) = &\left(A\bar{\partial}\partial A - \partial A\bar{\partial}A\right)(0) + \left(B\bar{\partial}\partial B - \bar{\partial}B\partial B\right)(0) \\
&+ \left(A\bar{\partial}\partial B + B\bar{\partial}\partial A - \bar{\partial}A\partial B - \partial A\bar{\partial}B\right)(0).
\end{aligned}$$

Lemma 5.2 in the last section (using $k = 1$) shows that

$$\lim_{w_1 \to 0} \frac{\bar{\partial}\partial A(0)}{(A(0))^3} = \lim_{w_1 \to 0} \frac{\partial A(0)}{(A(0))^3} = \lim_{w_1 \to 0} \frac{\bar{\partial}A(0)}{(A(0))^3} = 0. \tag{2.3}$$

Also, we have

$$-2\frac{\left(A\bar{\partial}\partial A - \partial A\bar{\partial}A\right)(0)}{(A(0))^3} = K_g(w_1). \tag{2.4}$$

Therefore,

$$\lim_{z \to p} \mathcal{K}^X(z) = \lim_{w \to W(p)} \left(-2\frac{N(0)}{D(0)}\right) = \lim_{w \to W(p)} \left(-2\frac{N(0)}{(A(0))^3}\right)\left(\frac{(A(0))^3}{D(0)}\right).$$

Since $w \to W(p)$ is equivalent to $w_1 \to 0$, an application of (2.1), (2.2), (2.3), and (2.4) gives

$$\lim_{z \to p} \mathcal{K}^X(z) = \lim_{w_1 \to 0} K_g(w_1) + 0 = -4.$$

This completes the proof.

3. Proof of Theorem 1.2. Since $H^1(U, \mathbb{Z}) = 0$, [5] shows that there exists a continuous map $\psi : \mathcal{D}_P^+(U) \to \mathcal{O}(U)$ such that $\mathsf{Div}(\psi(D)) = D$ for all $D \in \mathcal{D}_P^+(U)$. Let Φ denote the complete Kähler metric on U. Now for any $D \in \mathcal{D}_P^+(U)$ and $f \in \mathcal{O}(U)$ such that $\mathsf{Div}(f) = D$, we have the following complete Kähler metric on $U \setminus |D|$

$$\phi_D(z) = f^*(g)(z) + \Phi(z) = \left(1 + |f(z)|^2\, u^2(|f(z)|^2)\right) f^*(|dw|^2) + \Phi(z).$$

Observe that $f^*(|dw|^2)(z) = \sum_{i,k=1}^n \frac{\partial f}{\partial z_i}(z)\overline{\frac{\partial f}{\partial z_k}(z)}dz_i d\overline{z_k}$, and therefore

$$\phi_D(z) = \left(\sum_{i,k=1}^n \left(1 + |f(z)|^2 u^2(|f(z)|^2)\right) \frac{\partial f}{\partial z_i}(z)\overline{\frac{\partial f}{\partial z_k}(z)}dz_i d\overline{z_k} \right) + \Phi(z).$$

We are given a sequence of non-negative principal divisors $D_j \in \mathcal{D}_P^+(U)$ such that $D_j \to D_0 \in \mathcal{D}_P^+(U)$.

For (i), define $f_j := \psi(D_j)$ and $f_0 = \psi(D_0)$. Therefore, we get a sequence of holomorphic functions $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{O}(U)$ such that $f_j \to f_0$ uniformly on compacts of U with $\mathsf{Div}(f_j) = D_j$, $\mathsf{Div}(f_0) = D_0$. This gives us corresponding complete Kähler metrics $\phi_j = f_j^*(g) + \Phi$ and $\phi_0 = f_0^*(g) + \Phi$ on $U \setminus |D_j|$ and $U \setminus |D_0|$ respectively. Explicitly

$$\phi_j(z) = \sum_{i,k=1}^n \left(1 + |f_j(z)|^2 u^2(|f_j(z)|^2)\right) \frac{\partial f_j}{\partial z_i}(z)\overline{\frac{\partial f_j}{\partial z_k}(z)}dz_i d\overline{z_k} + \Phi(z),$$

$$\phi_0(z) = \sum_{i,k=1}^n \left(1 + |f_0(z)|^2 u^2(|f_0(z)|^2)\right) \frac{\partial f_0}{\partial z_i}(z)\overline{\frac{\partial f_0}{\partial z_k}(z)}dz_i d\overline{z_k} + \Phi(z).$$

Since $f_j \to f_0$ uniformly on compacts of U, all derivatives of f_j will also converge uniformly to the corresponding derivative of f_0 on compacts of U. Since $u : (0, \infty) \to \mathbb{R}$ is real analytic, it follows that $u(|f_j|^2) \to u(|f_0|^2)$ uniformly on compacts of $U \setminus |D_0|$. Therefore, $\phi_j \to \phi_0$ uniformly on compacts of $U \setminus |D_0|$.

For (ii), suppose we are given a complete Kähler metric on $U \setminus |D_0|$ of the form $\xi = \tilde{f_0}^*(g) + \Phi$ where $\mathsf{Div}(\tilde{f_0}) = D_0$. Let $f_0 = \psi(D_0)$ and observe that $h(z) := \tilde{f_0}(z)/f_0(z) \in \mathcal{O}^*(U)$ since $\mathsf{Div}(f_0) = \mathsf{Div}(\tilde{f_0}) = D_0$. Define

$$\tilde{\psi} : \mathcal{D}_P^+(U) \to \mathcal{O}(U)$$

by $\tilde{\psi}(D)(z) := h(z)[\psi(D)(z)]$. Clearly $\tilde{\psi}$ is continuous and $\tilde{\psi}(D_0) = h\psi(D_0) = hf_0 = \tilde{f_0}$. Also,

$$\mathsf{Div}\left(\tilde{\psi}(D)\right) = \mathsf{Div}\big(h\psi(D)\big) = \mathsf{Div}\big(\psi(D)\big) = D$$

for all $D \in \mathcal{D}_P^+(U)$. So we can define $\tilde{f_j} := \tilde{\psi}(D_j)$. Here $\tilde{f_j} \to \tilde{f_0}$ uniformly on compacts of U and $\mathsf{Div}(\tilde{f_j}) = D_j$. So the corresponding complete Kähler metrics $\xi_j := \tilde{f_j}^*(g) + \Phi$ converge uniformly to the metric $\xi = \tilde{f_0}^*(g) + \Phi$ on compacts of $U \setminus |D_0|$.

4. Proof of Theorem 1.3. We are given complete Kähler metrics $\phi_j = f_j^*(g)+\Phi$ and $\phi_0 = f_0^*(g) + \Phi$ on $U \setminus |D_j|$ and $U \setminus |D_0|$ respectively, where $f_j, f_0 \in \mathcal{O}(U)$ are such that $f_j \to f_0$ uniformly on compacts of U.

(i) Let $p \in U \setminus |D_0|$. Since $D_j \to D_0$, choose a neighbourhood U_p of p relatively compact in $U \setminus |D_0|$ such that $\overline{U}_p \subset U \setminus |D_j|$ for large enough j. Let X be a non-singular holomorphic vector field near p. For $q \in U_p$, consider a parametrization

$Z : B(0,\epsilon) \to U_p$ such that $Z(0) = q$, $Z_T = Z(T)$, and $dZ(T)/dT = X(Z_T)$. Then $Z^*(\phi_j) = h_j(T)|dT|^2$ and $Z^*(\phi_0) = h_0(T)|dT|^2$, where

$$h_j(T) = \sum_{i,k=1}^{n} \left(1 + |f_j(Z_T)|^2 u^2(|f_j(Z_T)|^2)\right) \frac{\partial f_j}{\partial z_i}(Z_T)\overline{\frac{\partial f_j}{\partial z_k}}(Z_T)X_i(Z_T)\overline{X_k(Z_T)} + \tilde{h}(T)$$

and

$$h_0(T) = \sum_{i,k=1}^{n} \left(1 + |f_0(Z_T)|^2 u^2(|f_0(Z_T)|^2)\right) \frac{\partial f_0}{\partial z_i}(Z_T)\overline{\frac{\partial f_0}{\partial z_k}}(Z_T)X_i(Z_T)\overline{X_k(Z_T)} + \tilde{h}(T)$$

with $\tilde{h} = Z^*\Phi$. Further, recall that

$$\mathcal{K}^X_{\phi_j}(q) = -2(h_j(0))^{-3}\left(h_j(0)\partial\bar{\partial}h_j(0) - \partial h_j(0)\bar{\partial}h_j(0)\right).$$

Since $f_j \to f_0$ uniformly on compacts of U, $\partial^i\bar{\partial}^k f_j(Z_T) \to \partial^i\bar{\partial}^k f_0(Z_T)$ for all T in a neighourhood of $0 \in \mathbb{C}$ and for all $i,k \geq 0$. Similarly, $|f_j(Z_T)|^2$, $u(|f_j(Z_T)|^2)$, and their higher derivatives also converge uniformly on a neighourhood of $0 \in \mathbb{C}$. Thus

$$\partial^i\bar{\partial}^k h_j(T) \to \partial^i\bar{\partial}^k h_0(T)$$

for all $0 \leq i,k \leq 1$ and for all T in a neighourhood of $0 \in \mathbb{C}$. Therefore, $\mathcal{K}^X_{\phi_j}(q) \to \mathcal{K}^X_{\phi_0}(q)$ for all $q \in U_p$.

(ii) Let $p \in U \setminus |D_0|$ and $V \in \mathbb{C}^n$. Clearly $p \in U \setminus |D_j|$ for large enough j. Let X be a non-singular vector field near p such that $X(p) = V$ and $K_{\phi_0}(p,V) = \mathcal{K}^X_{\phi_0}(p)$. By (i), $\mathcal{K}^X_{\phi_j}(p) \to \mathcal{K}^X_{\phi_0}(p)$. Therefore, for $\epsilon > 0$, there exists $N \geq 1$ such that

$$\mathcal{K}^X_{\phi_0}(p) \leq \mathcal{K}^X_{\phi_j}(p) + \epsilon \leq K_{\phi_j}(p,V) + \epsilon$$

for all $j \geq N$ and this gives $K_{\phi_0}(p,V) \leq K_{\phi_j}(p,V) + \epsilon$. Hence, $K_{\phi_0}(p,V) \leq \liminf_{j\to\infty} K_{\phi_j}(p,V) + \epsilon$ for all $\epsilon > 0$ and consequently, $K_{\phi_0}(p,V) \leq \liminf_{j\to+\infty} K_{\phi_j}(p,V)$.

5. The case $n = 1$. For a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$, consider the complete Kähler metric on $\mathbb{C} \setminus \{f = 0\}$ given by $\phi(z) = f^*(g) + |dz|^2$. We have

$$\phi(z) = h(z)|dz|^2 = \left(|f'(z)|^2\left(1 + |f(z)|^2 u^2(|f(z)|^2)\right) + 1\right)|dz|^2.$$

Denote the Gaussian curvature of the metric ϕ by K.

Theorem 5.1. *Let (f,ϕ,K) be as above and $p \in \mathbb{C}$.*

(i) *If $p \in \mathbb{C} \setminus \{f = 0\}$, then $K(p) \leq 0$. In addition, if $f'(p) = 0$, then $K(p) = 0$ if and only if $f''(p) = 0$.*

(ii) *If $p \in \{f = 0\}$, then*

$$\lim_{z\to p} K(z) = -4.$$

The proof relies on the following observations.

Lemma 5.2. *For an integer $k \geq 1$, consider the metric*

$$\phi_k(z) = k^2 \left|z\right|^{2(k-1)} \left(1 + \left|z\right|^{2k} u^2(|z|^{2k})\right) |dz|^2 = h_k(z)|dz|^2$$

on $\mathbb{C}^*$. The Gaussian curvature of this metric K_k satisfies $K_k(z) = K_g(z^k)$ for $z \in \mathbb{C}^*$ and

$$\lim_{z \to 0} K_k(z) = -4.$$

Also, the function h_k satisfies $\lim_{z\to 0} h_k(z) = +\infty$ and

$$\frac{\partial h_k(z)}{\left(h_k(z)\right)^3}, \frac{\bar{\partial} h_k(z)}{\left(h_k(z)\right)^3}, \frac{\bar{\partial}\partial h_k(z)}{\left(h_k(z)\right)^3} \to 0$$

as $z \to 0$.

Proof. Note that the conclusion $K_k(z) = K_g(z^k)$ for $z \in \mathbb{C}^*$ can be made by using the fact that $\phi_k(z) = (z^k)^* g$, and the map $z \to z^k$ is a local biholomorphism on $\mathbb{C}^*$. But we will prove it in a different way because we need to make some observations from the calculations.
Since $\phi_k(z) = h_k(z)|dz|^2$, the curvature is given by

$$K_k(z) = -2 \left(\frac{h_k(z)\bar{\partial}\partial h_k(z) - \partial h_k(z)\bar{\partial} h_k(z)}{\left(h_k(z)\right)^3} \right)$$

for $z \in \mathbb{C}^*$. We will write just u instead of $u(|z|^{2k})$. Observe that

$$\begin{aligned}
\partial h_k(z) &= \overline{\bar{\partial} h_k(z)} \\
&= k^2 \bar{z} \left((k-1) \left|z\right|^{2k-4} + (2k-1) \left|z\right|^{4k-4} u^2 + 2k \left|z\right|^{6k-4} uu' \right), \\
\bar{\partial}\partial h_k(z) &= |z|^{2k-4} \big(k^2(k-1)^2 + k^2(2k-1)^2 |z|^{2k} u^2 + 2k^3(5k-2)|z|^{4k} uu' \\
&\quad + 2k^4 |z|^{6k} \big((u')^2 + uu'' \big) \big).
\end{aligned}$$

Upon simplification, we get

$$h_k(z)\bar{\partial}\partial h_k(z) - \partial h_k(z)\bar{\partial} h_k(z) = k^6 \left|z\right|^{6k-6} M(|z|^{2k})$$

where $M : (0, +\infty) \to \mathbb{R}$ is given by

$$M(t) = u^2 + 6tuu' + 2t^2(u')^2 + 2t^2 uu'' + 2t^2 u^3 u' - 2t^3 u^2 (u')^2 + 2t^3 u^3 u''.$$

Now [3, concluding remarks on p. 8] tells us that

$$K_g(z) = -2 \frac{M(|z|^2)}{\left(1 + \left|z\right|^2 u^2(|z|^2)\right)^3}$$

for $z \in \mathbb{C}^*$. Now note that

$$\begin{aligned}
K_k(z) &= -2 \frac{k^6 \left|z\right|^{6k-6} M(|z|^{2k})}{k^6 \left|z\right|^{6k-6} \left(1 + \left|z\right|^{2k} u^2(|z|^{2k})\right)^3} = -2 \frac{M(|z^k|^2)}{\left(1 + \left|z^k\right|^2 u^2(|z^k|^2)\right)^3} \\
&= K_g(z^k).
\end{aligned}$$

This shows that for $k \in \mathbb{N}$, $z \in \mathbb{C}^*$, $K_k(z) = K_g(z^k)$ and

$$\lim_{z\to 0} K_k(z) = \lim_{z\to 0} K_g(z^k) = -4.$$

To focus on the function $h_k(z)$, observe that

$$\begin{aligned}\lim_{z\to 0} h_k(z) &= \lim_{t\to 0^+} k^2 t^{k-1}\left(1 + t^k u^2(t^k)\right) = \lim_{t\to 0^+} \frac{k^2 t^{2k-1}(t^k-1)^2}{t^{2k}(\log t^k)^2} \\ &= \lim_{t\to 0^+} \frac{k^2(t^k-1)^2}{t(\log t^k)^2} = +\infty.\end{aligned}$$

Now substituting for u, u', u'', simplifying the resulting expression, and using the fact that $\lim_{t\to 0^+} t(\log t) = 0$, we get

$$\frac{|z|^{2k-4}}{(h_k(z))^3}, \frac{|z|^{4k-4}\, u^2}{(h_k(z))^3}, \frac{|z|^{6k-4}\, uu'}{(h_k(z))^3}, \frac{|z|^{8k-4}\left((u')^2 + uu''\right)}{(h_k(z))^3} \to 0$$

as $z \to 0$. It is now evident that

$$\lim_{z\to 0} \frac{\partial h_k(z)}{(h_k(z))^3} = \lim_{z\to 0} \frac{\bar\partial h_k(z)}{(h_k(z))^3} = \lim_{z\to 0} \frac{\bar\partial\partial h_k(z)}{(h_k(z))^3} = 0.$$

□

Now we can prove Theorem 5.1.

Proof. For $z \in \mathbb{C}\setminus\{f=0\}$, the curvature is given by

$$K(z) = -2\left(\frac{h(z)\partial\bar\partial h(z) - \partial h(z)\bar\partial h(z)}{(h(z))^3}\right).$$

For (i), suppose that $f'(p) \neq 0$. Let $\hat g$ be a local inverse of f near p, that is $\hat g \circ f(z) = z$ for z in a neighbourhood U_p around p. It can be checked that $\hat g'(w) = \left(f'(\hat g(w))\right)^{-1}$ and $\hat g''(w) = -f''(\hat g(w))\left(f'(\hat g(w))\right)^{-2}$. In these new coordinates, we have

$$\begin{aligned}\hat g^*(\phi)(w) &= \left(1 + |\hat g'(w)|^{-2}\left(1 + |w|^2 u^2(|w|^2)\right)\right)|\hat g'(w)|^2|dw|^2 \\ &= \left(|\hat g'(w)|^2 + \left(1 + |w|^2 u^2(|w|^2)\right)\right)|dw|^2 = \phi_1(w) + \phi_2(w)\end{aligned}$$

where $\phi_2(w)$ is the Grauert metric on $\mathbb{C}^*$ and $\phi_1(w) = |\hat g'(w)|^2 |dw|^2$. By [3], $K_g(w) \le 0$ for all $w \in \mathbb{C}^*$ and note that K_1, the curvature of the metric ϕ_1, satisfies $K_1 \equiv 0$. Now [2] tells us that the curvature of the sum of these two metrics is also non-positive in U_p, that is $K(p) \le 0$.
To continue, note that if $f'(p) = 0$, then $h(p) = 1$. Also

$$\begin{aligned}\partial h(z) &= \left(|f'(z)|^2\, \partial(1 + |f|^2 u^2) + f''(z)\overline{f'(z)}(1 + |f|^2 u^2)\right), \\ \bar\partial\partial h &= \left(\bar\partial\Big(|f'|^2\, \partial(1 + |f|^2 u^2)\Big) + f''\overline{f'}\bar\partial(1 + |f|^2 u^2) + |f''|^2 (1 + |f|^2 u^2)\right)\end{aligned}$$

which gives us $\partial h(p) = \bar\partial h(p) = 0$ and $\bar\partial\partial h(p) = |f''(p)|^2 \left(1 + |f(p)|^2 u^2(|f(p)|^2)\right)$.
Therefore we get

$$K(p) = -2\,|f''(p)|^2\left(1 + |f(p)|^2 u^2(|f(p)|^2)\right),$$

and hence $K(p) = 0$ if and only if $f''(p) = 0$.
For (ii), let $p \in \mathbb{C}$ be such that $f(p) = 0$. By a translation, assume that $p = 0$. Let $k = \mathrm{Ord}_0(f)$. Then there exists a holomorphic map f_1 such that $f_1(0) \neq 0$ and $f(z) = z^k f_1(z)$. Now since $f_1(0) \neq 0$, there exist a holomorphic map f_2 defined near $0 \in \mathbb{C}$ such that $f_2(0) \neq 0$ and $f_1(z) = \big(f_2(z)\big)^k$. So for z in a neighourhood of $0 \in \mathbb{C}$, we get

$$f(z) = \big(z f_2(z)\big)^k.$$

Define $W(z) = z f_2(z)$ for z near the origin and observe that $W'(0) = f_2(0) \neq 0$, that is, W is a biholomorphism on a neighourhood around $0 \in \mathbb{C}$. Therefore there exists a holomorphic function g_1 defined near $0 \in \mathbb{C}$ such that $g_1(0) = 0$ and $W^{-1}(w) = g_1(w)$. Since $f'(z) = k\big(W(z)\big)^{k-1} W'(z)$, it follows that $f'\big(W^{-1}(w)\big) = k w^{k-1} \left((W^{-1})'(w)\right)^{-1}$. Now

$$\begin{aligned}\left((W^{-1})^*\phi\right)(w) &= \Big(1 + k^2 |w|^{2(k-1)} |g_1'(w)|^{-2} \left(1 + |w|^{2k} u^2(|w|^{2k})\right)\Big) |g_1'(w)|^2 |dw|^2 \\ &= \Big(|g_1'(w)|^2 + k^2 |w|^{2(k-1)} \left(1 + |w|^{2k} u^2(|w|^{2k})\right)\Big) |dw|^2.\end{aligned}$$

Therefore, $\left((W^{-1})^*\phi\right)(w) = |g_1'(w)|^2 |dw|^2 + h_k(w)|dw|^2$. Observe that $|g_1'(w)|^2 |dw|^2$ has identically vanishing Gaussian curvature. Now using Lemma 5.2 and the fact that $g_1'(w), g_1''(w)$ are bounded in a neighbourhood of $w = 0 \in \mathbb{C}$, we get

$$\lim_{z \to 0} K(z) = \lim_{w \to 0} -2 \left(\frac{h_k(w)\bar{\partial}\partial h_k(w) - \partial h_k(w)\bar{\partial} h_k(w)}{\left(h_k(w) + |g_1'(w)|^2\right)^3} \right) = \lim_{w \to 0} K_k(w) = -4.$$

This completes the proof. ☐

Acknowledgements. The authors would like to thank the referee for carefully reading the article and giving suggestions to improve the previous version of this manuscript.

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Received: 1 April 2021

Revised: 16 February 2022

Accepted: 29 May 2022.