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Corrigendum to: An extension problem and trace Hardy inequality for the sublaplacian on H-type groups

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Recently we have found a couple of errors in our paper entitled *An extension problem and trace Hardy inequality for the sub-Laplacian on H-type groups*, Int. Math. Res. Not. IMRN (2020), no. 14, 4238–4294. They concern Propositions 3.12–3.13, and Theorem 1.5, Corollary 1.6 and Remark 4.10. The purpose of this corrigendum is to point out the errors and supply necessary modifications where it is applicable.

1

In the proof of [1, Proposition 3.12], we have used a wrong asymptotic property of the function $G(\rho)$ defined by

$$G(\xi) = \int_{\mathbb{R}^n} (1 + |x|^2)^{-(n+s)/2} e^{-ix\cdot\xi}\, dx.$$

The asymptotic properties of G proved in [3, p. 132, (29), (30)] cannot be used here as we have $n + s > n$ owing to the fact that $s > 0$. Hence the integral $\int_0^\infty G(\rho)^2 \rho^{-s-1} ds$ is not finite as claimed, which renders the result of the proposition invalid. This also makes the result of [1, Proposition 3.13] wrong, as its proof depends on the result of [1, Proposition 3.12].

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Fortunately, these two propositions are independent observations, not playing any role in the proofs of main results proved in the paper.

2

The proof of [1, Theorem 1.5] presented in [1, Section 4, pp. 4287–4289] needs a modification. We have claimed that the identity [1, (4.12)] can be obtained from [1, (4.11)] by appealing to Vitali's theorem [2, p. 133]. Similar arguments are used to compute the limit under the integral for δ approaching 0 and η approaching the constant function 1. Unfortunately, by oversight we missed the fact that one of the hypotheses of Vitali's theorem, namely the uniform integrability, is not satisfied. Consequently, we do not know how to obtain the identity [1, (4.12)] from [1, (4.11)] and we cannot ensure the legitimacy of taking the limit inside the integral, preserving the equality.

However, from [1, (4.11)] we trivially have the inequality

$$\int_0^\infty \int_N \left|\nabla u(x,\rho)\right|^2 \rho^{1-2s}\, dx\, d\rho \geq \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)} \int_N \frac{\mathcal{L}_s\varphi(x)}{\varphi(x)} u^2(x,0)\, dx.$$

By taking $\varphi = \eta\varphi_s * \varphi_{-s,\delta}$, as it is done in the paper, and using Fatou's lemma, in place of the identity [1, (4.12)] we can arrive at

$$\int_0^\infty \int_N \left|\nabla u(x,\rho)\right|^2 \rho^{1-2s}\, dx\, d\rho \geq \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)} C_2(n,m,s) \int_N u^2(v,z,0) w_s(v,z)\, dv\, dz,$$

where $w_s(v,z)$ is defined in [1, (1.12)], namely

$$w_s(v,z) = \varphi_s(v,z)\psi_s(v,z)^{-1},$$

with $\varphi_s(v,z) = |(v,z)|^{-(n+m+s)}$ and $\psi_s(v,z) = C_1(n,m,s)(\varphi_s * |\cdot|^{-Q+2s})(v,z)$. This proves the trace Hardy inequality stated in [1, Theorem 1.5].

Nevertheless, our claim about the sharpness of the constant in [1, Theorem 1.5] remains to be proved, since it is inferred from the identity [1, (4.12)]. And consequently the sharpness of the constant in [1, Corollary 1.6] also remains open. Further, as [1, Remark 4.10, p. 4291] also depends on the identity [1, (4.12)], we take it back. Though we have not been able to prove the sharpness, we strongly believe that our original claims are true, see the remarks below.

Remark 2.1. We can still say something about the sharp constant in the trace Hardy and Hardy inequalities in [1, Theorem 1.5] and [1, Corollary 1.6]. Consider the inequality

$$\int_0^\infty \int_N \left|\nabla u(x,\rho)\right|^2 \rho^{1-2s}\,dx\,d\rho \geq \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)} C \int_N u^2(v,z,0) w_s(v,z)\,dv\,dz, \tag{2.1}$$

where $C > 0$. As in the proof of [1, Corollary 4.7], let us take $u(v,z,\rho) = C_1(n,m,s)\,\rho^{2s}\,\varphi_{-s,\delta} * \varphi_{s,\rho}(v,z)$. Then it follows that

$$C_2(n,m,s)\delta^{2s} \int_N \left((\delta^2+|v|^2)^2+16|z|^2\right)^{-n-m} dv\,dz$$
$$\geq C \int_N \left((\delta^2+|v|^2)^2+16|z|^2\right)^{-n-m+s} w_s(v,z)\,dv\,dz.$$

By making the change of variables $v/\delta \mapsto v$ and $z/\delta^2 \mapsto z$ and recalling that $w_s(v,z)$ is homogeneous of degree $-2s$ we obtain

$$C_2(n,m,s) \int_N \left((1+|v|^2)^2+16|z|^2\right)^{-n-m} dv\,dz$$
$$\geq C \int_N \left((1+|v|^2)^2+16|z|^2\right)^{-n-m+s} w_s(v,z)\,dv\,dz.$$

This gives an upper bound for the best constant C in the trace Hardy inequality (2.1), namely

$$C \leq C_2(n,m,s) \frac{\int_N \left((1+|v|^2)^2+16|z|^2\right)^{-n-m} dv\,dz}{\int_N \left((1+|v|^2)^2+16|z|^2\right)^{-n-m+s} w_s(v,z)\,dv\,dz}.$$

The same remark applies to the constant C in the Hardy's inequality

$$(\mathcal{L}_s f, f) \geq C \int_N f^2(v,z) w_s(v,z)\,dv\,dz \tag{2.2}$$

stated in [1, Corollary 1.6]. Thus the sharpness of Hardy and trace Hardy inequalities will be proved once we show that

$$\int_N \left((1+|v|^2)^2+16|z|^2\right)^{-n-m} dv\,dz \leq \int_N \left((1+|v|^2)^2+16|z|^2\right)^{-n-m+s} w_s(v,z)\,dv\,dz.$$

Since $w_s(v,z)$ is not explicit (see the definition [1, (1.12)]) and [1, Remark 4.9]), at present we do not know how to check if the above is true or not.

Remark 2.2. Instead of the above Hardy inequality (2.2) (which we have proved with $C = C_2(n,m,s)$) consider now the ideal inequality

$$(\mathcal{L}_s f, f) \geq C \int_N f^2(v,z)|(v,z)|^{-2s}\, dv\, dz \tag{2.3}$$

which indeed was our original goal in [1]. Unlike the case of (2.2) we can now say something about the best constant in the inequality (2.3). Proceeding as above we arrive at the inequality

$$C \leq C_2(n,m,s) \frac{\int_N \left((1+|v|^2)^2 + 16|z|^2\right)^{-n-m} dv\, dz}{\int_N \left((1+|v|^2)^2 + 16|z|^2\right)^{-n-m+s} |(v,z)|^{-2s}\, dv\, dz}.$$

Since $|(v,z)|^{2s} = (|v|^4 + 16|z|^2)^{s/2} \leq \left((1+|v|^2)^2 + 16|z|^2\right)^{s/2} \leq \left((1+|v|^2)^2 + 16|z|^2\right)^s$ it follows that

$$\int_N \left((1+|v|^2)^2 + 16|z|^2\right)^{-n-m} dv\, dz \leq \int_N \left((1+|v|^2)^2 + 16|z|^2\right)^{-n-m+s} |(v,z)|^{-2s}\, dv\, dz.$$

Consequently, the inequality (2.3) cannot hold for any $C > C_2(n,m,s)$. We conjecture that

$$(\mathcal{L}_s f, f) \geq C_2(n,m,s) \int_N f^2(v,z)|(v,z)|^{-2s} dv dz \tag{2.4}$$

whose sharpness we have just proved. Since we have established the above inequality with $w_s(v,z)$ in place of $|(v,z)|^{-2s}$, the conjecture will be immediate once we show that $w_s(v,z) \geq |(v,z)|^{-2s}$. As explained in [1, Remark 4.9] this seems to be difficult at present.

Hence, we are able to prove inequalities (2.1) and (2.2), but we cannot assert the sharpness of the constant and, on the other hand, we have the following conditional statement: if (2.3) holds, then the constant $C = C_2(n,m,s)$ is sharp.

Remark 2.3. Combining Remarks 2.1 and 2.2 we see that the sharpness of the inequality

$$(\mathcal{L}_s f, f) \geq C_2(n,m,s) \int_N f^2(v,z) w_s(v,z)\, dv\, dz \tag{2.5}$$

will be a consequence of $w_s(v,z) \geq |(v,z)|^{-2s}$. In view of the definition of $w_s(v,z)$, this will follow once we can prove that $\psi_s(v',z') \leq 1$ where $(v',z') = (v,z)|(v,z)|^{-1}$. In the case of the Heisenberg group, this reduces to the following inequality (see [1, Remark 4.9]):

$$C_3(n,1,s)\int_{S^{2n+1}} |1-\zeta\cdot\bar{\eta}|^{-\lambda/2}|(1-\zeta_{n+1})(1+\zeta_{n+1})|^{-\gamma/2}d\zeta \leq ((1+|z'|^2)^2+16t'^2)^{\lambda/4} \quad (2.6)$$

where $\lambda = 2(n+1)-2s, \gamma = n+1+s$, η is the Cayley transform of (z',t') and $C_3(n,1,s)$ is an explicit constant. Thus the optimal Hardy inequality (2.4) along with the sharpness of the constant will be established once we prove (2.6).

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