SMOOTH ENTRYWISE POSITIVITY PRESERVERS, A HORN–LOEWNER MASTER THEOREM, AND SYMMETRIC FUNCTION IDENTITIES

APOORVA KHARE

To Roger A. Horn and the memory of Charles Loewner, with admiration

ABSTRACT. A special case of a fundamental result of Loewner and Horn [Trans. Amer. Math. Soc. 136 (1969), pp. 269–286] says that given an integer $n \geqslant 1$, if the entrywise application of a smooth function $f : (0,\infty) \to \mathbb{R}$ preserves the set of $n \times n$ positive semidefinite matrices with positive entries, then f and its first $n-1$ derivatives are non-negative on $(0,\infty)$. In a recent joint work with Belton–Guillot–Putinar [J. Eur. Math. Soc., in press], we proved a stronger version, and used it to strengthen the Schoenberg–Rudin characterization of dimension-free positivity preservers [Duke Math. J. 26 (1959), pp. 617–622; Duke Math. J. 9 (1942), pp. 96–108].

In recent works with Belton–Guillot–Putinar [Adv. Math. 298 (2016), pp. 325–368] and with Tao [Amer. J. Math. 143 (2021), pp. 1863-1929] we used local, real-analytic versions at the origin of the Horn–Loewner condition, and discovered unexpected connections between entrywise polynomials preserving positivity and Schur polynomials. In this paper, we unify these two stories via a Master Theorem (Theorem A) which (i) simultaneously unifies and extends all of the aforementioned variants; and (ii) proves the positivity of the first n nonzero Taylor coefficients at individual points rather than on all of $(0,\infty)$.

A key step in the proof is a new determinantal / symmetric function calculation (Theorem B), which shows that Schur polynomials arise naturally from considering arbitrary entrywise maps that are sufficiently differentiable. Of independent interest may be the following application to symmetric function theory: we extend the Schur function expansion of Cauchy's (1841) determinant (whose matrix entries are geometric series $1/(1 - u_j v_k)$), as well as of a determinant of Frobenius [J. Reine Angew. Math. 93 (1882), pp. 53–68] (whose matrix entries are a sum of two geometric series), to arbitrary power series, and over all commutative rings.

1. INTRODUCTION AND MAIN RESULTS

Recently in [1, 2, 14, 15], novel connections were discovered between polynomials acting entrywise that preserve matrix positivity and symmetric functions, specifically Schur polynomials. We explore these connections in greater depth, and prove that Schur polynomials emerge naturally from differentiating determinants involving *arbitrary* sufficiently differentiable functions. To the best of our knowledge, this intriguing connection between analysis and symmetric function theory is not recorded in the literature. This is our Theorem B, and as an application we prove the Master Theorem A, on functions acting entrywise that preserve positive semidefinite matrices. We also apply it to provide a novel symmetric function identity that subsumes modern results as well as classical ones by Cauchy and Frobenius – see Theorem 2.1.

1.1. **Entrywise calculus and positivity: Notation and history.** A real symmetric matrix $A_{n \times n}$ is positive semidefinite if all its eigenvalues are non-negative real numbers – equivalently, the quadratic form $x \mapsto x^T A x$, $x \in \mathbb{R}^n$ is non-negative definite, i.e., takes values in $[0, \infty)$. Given an integer $n \geqslant 1$ and a domain $I \subset \mathbb{R}$, let $\mathbb{P}_n(I)$ comprise the positive semidefinite $n \times n$ matrices with entries in I. We work only with domains $I = (a, b)$ or $[a, b)$ for $0 \leqslant a < b \leqslant \infty$.

A function $f : I \to \mathbb{R}$ acts *entrywise* on a vector or a matrix $A = (a_{jk}) \in I^{m \times n}$ for integers $m, n \geqslant 1$ via: $f[A] := (f(a_{jk}))$. If $f(x) = x^k$ is an integer power function, we write $f[A] = A^{\circ k}$ for the entrywise power – i.e., the k-fold Schur/Hadamard product – of A. Let $\mathbb{1}_{m \times n}$ denote the $m \times n$ matrix with all entries 1.

A question of significant interest in the analysis literature throughout the past century is to understand the entrywise functions[1] that preserve positive semidefiniteness, which we occasionally term *positivity* in the sequel. The first result in this area is the Schur product theorem [25], which asserts that the set $\mathbb{P}_n(I)$ is closed under the entrywise product $A \circ B := (a_{jk} b_{jk})_{j,k=1}^n$ if I is closed under multiplication. Using that $\mathbb{P}_n(\mathbb{R})$ is a closed convex cone, Pólya and Szegö [21] observed the following immediate consequence of the Schur product theorem: if $f(x) = \sum_{k=0}^{\infty} c_k x^k$ is a convergent power series on I and $c_k \geqslant 0$ $\forall k$, then $f[-]$ preserves positivity on $\mathbb{P}_n(I)$. A celebrated result of Schoenberg [24], subsequently improved by Rudin [23], shows that there are no other functions that preserve positivity in all dimensions for $I = (-1, 1)$. These results are motivated by and have connections to metric geometry, positive definite functions, harmonic analysis, and analysis of measures on Euclidean space and on tori.[2] Similar results have been shown for $I = (-\rho, \rho)$ or $(0, \rho)$ for $0 < \rho \leqslant \infty$, as well as for complex domains – in both one and several variables – in the years since Schoenberg's and Rudin's work.

Schoenberg's theorem has a challenging mathematical refinement that is strongly motivated by modern-day applications in high-dimensional covariance estimation. Namely: is it possible to classify the entrywise positivity preservers in a *fixed* dimension n? This problem was resolved in 1979 by Vasudeva [26] for $n = 2$, but remains open for every $n \geqslant 3$. In this paper, we focus on a necessary condition for this to happen.

---

[1]We refer to polynomials – and more generally, functions – that act entrywise on matrices as *entrywise polynomials/functions*, respectively.

[2]See the surveys [3, 4] for more on the classical and modern interest in entrywise positivity preservers.

1.2. **The Horn–Loewner theorem and its variants.** The focus of the present paper is a fundamental result on entrywise preservers in fixed dimension $n \geqslant 3$. This result can be found in Horn's 1967 thesis, and Horn attributes it to Loewner[3]:

Theorem 1.1 (Necessary condition in fixed dimension, see [11])**.** *Suppose* $I = (0, \infty)$ *and* $f : I \to \mathbb{R}$ *is continuous. Fix an integer* $n \geqslant 1$ *and suppose* $f[A] \in \mathbb{P}_n(\mathbb{R}) \ \forall A \in \mathbb{P}_n(I)$*. Then,*

$$f \in C^{n-3}(I), \qquad f^{(k)}(x) \geqslant 0, \forall x \in I, \ 0 \leqslant k \leqslant n - 3,$$

and $f^{(n-3)}$ *is a convex non-decreasing function on* I*. In particular, if* $f \in C^{n-1}(I)$*, then* $f^{(k)}(x) \geqslant 0$ *for all* $x \in I, 0 \leqslant k \leqslant n - 1$*.*

Theorem 1.1 is important for several reasons:

(1) To our knowledge, this 1967 result (together with its refinements, which we will discuss) remains to this day the *only* known necessary condition for a function to be an entrywise positivity preserver in a fixed dimension.

(2) Theorem 1.1 is sharp in the number of non-negative derivatives of f on I. We mention several such settings in Example 3.1, providing in each case positivity preservers that work in dimension n but not $n + 1$.

(3) This fixed dimension result can be used to prove the dimension-free version, i.e. Schoenberg's theorem over $I = (0, \rho)$ for $0 < \rho \leqslant \infty$; the proof uses Bernstein's theorem on absolutely monotonic functions. In turn, the dimension-free version over $(0, \rho)$ (first shown by Vasudeva [26]) can be used to prove Schoenberg's theorem over $I = (-\rho, \rho)$ by using less sophisticated machinery compared to Schoenberg or Rudin's works. In fact this approach has proved even more successful: in recent joint work [5], we first showed a stronger version of Theorem 1.1; then using it, a strengthening of Schoenberg's theorem; and finally, multivariable analogues of these results.

In this paper, we are interested in strengthening Theorem 1.1 in multiple ways. We begin by stating several refinements proved in the analysis literature – our main result simultaneously extends all of these variants.

The first strengthening begins with the observation that the argument in the proof of Theorem 1.1 is entirely local. With this in mind, the domain of f can be generalized to $(0, \rho)$ for any $0 < \rho \leqslant \infty$. Moreover, the continuity assumption can be removed, in the spirit of Rudin's strengthening [23] of Schoenberg's theorem [24]. Finally, Theorem 1.1 uses only a special sub-family of matrices of rank at most 2. Thus, in [5, 10], the following was shown (we state only the stronger of the two results):

Theorem 1.2 (See [5, Theorem 4.2])**.** *Suppose that* $0 < \rho \leqslant \infty$*,* $I = (0, \rho)$*, and* $f : I \to \mathbb{R}$*. Fix* $u_0 \in (0, 1)$ *and an integer* $n \geqslant 1$*, and define* $\mathbf{u} := (1, u_0, \dots, u_0^{n-1})^T$*. Suppose that* $f[A] \in \mathbb{P}_2(\mathbb{R})$ *for all* $A \in \mathbb{P}_2(I)$*, and also that* $f[A] \in \mathbb{P}_n(\mathbb{R})$ *for all Hankel matrices* $A = a \mathbb{1}_{n \times n} + t \mathbf{u} \mathbf{u}^T$*, with* $a \in \overline{I}$ *and* $t \geqslant 0$ *such that* $a + t \in I$*. Then the conclusions of Theorem* 1.1 *hold.*

Theorem 1.2 subsumes Theorem 1.1 and the aforementioned examples (see Example 3.1).

---

[3]As the author has also observed in the Stanford Library archives, the argument in [11] was outlined in a letter to Josephine Mitchell, written by Loewner on October 24, 1967.

2220 APOORVA KHARE

In this paper, we are interested in the case of smooth – i.e., infinitely differentiable – functions f, from which the general case follows using a remarkable 1940 result by Boas–Widder [6] (see Remark 3.7, which fixes a minor typo in their proof). Thus, we henceforth work with smooth functions – in which case the assumption in Theorem 1.2 concerning $\mathbb{P}_2(I)$ is no longer required; see [5] for details.

For smooth functions, one requires weaker assumptions, and reaches stronger conclusions. Indeed, in [1] and [15], we showed:

Lemma 1.3. *Let $n \geqslant 1$ and $0 < \rho \leqslant \infty$. Suppose that $f(x) = \sum_{k \geqslant 0} c_k x^k$ is a convergent power series on $I = [0, \rho)$ that is entrywise positivity preserving[4] on rank-one matrices in $\mathbb{P}_n(I)$. Further assume that $c_{m'} < 0$ for some m'.*

  (1) *If $\rho < \infty$, then $c_m > 0$ for at least n values of $m < m'$. In particular, the first n nonzero Maclaurin coefficients of f, if they exist, must be positive.*
  (2) *If instead $\rho = \infty$, then $c_m > 0$ for at least n values of $m < m'$ and at least n values of $m > m'$. In particular, if f is a polynomial, then the first n nonzero coefficients and the last n nonzero coefficients of f, if they exist, are all positive.*

We further showed in [15] that these conditions are sharp, in that every other nonzero Maclaurin coefficient of f can be negative. The conclusions of Lemma 1.3 are stronger than those of Theorem 1.1 – albeit at $a = 0$ and not at $a > 0$ – and they also cover settings not covered in Examples 3.1: the case of *all* polynomial preservers, not merely ones with the initial Maclaurin coefficients of orders $0, 1, \ldots, n - 1$.

1.3. **The main theorems.** We now propose a stronger version of the Horn–Loewner theorem that addresses the positivity of the first n nonzero Taylor coefficients at a given point; and moreover, one that unifies Theorem 1.1 and Lemma 1.3, which do not imply one another. This is our first main result; in the sequel, we work with f smooth on $[a, a + \epsilon)$, and we refer to $f^{(k)}(a^+)$ as $f^{(k)}(a)$.

Theorem A (Horn–Loewner master theorem). *Let $0 \leqslant a < \infty, \epsilon \in (0, \infty), I = [a, a + \epsilon)$, and let $f : I \to \mathbb{R}$ be smooth. Fix integers $n \geqslant 1$ and $0 \leqslant p \leqslant q \leqslant n$, with $p = 0$ if $a = 0$, and such that $f(x)$ has $q - p$ nonzero derivatives at $x = a$ of order at least p. Now suppose that*

$$p \leqslant m_p < m_{p+1} < \cdots < m_{q-1}$$

are the lowest orders (above p) of the first $q - p$ nonzero derivatives of $f(x)$ at $x = a$.

Also fix pairwise distinct scalars $u_1, \ldots, u_n \in (0, 1)$, and let $\mathbf{u} := (u_1, \ldots, u_n)^T$. If $f[a\mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_n(\mathbb{R})$ for all $t \in [0, \epsilon)$, then $f^{(k)}(a)$ is non-negative for $0 \leqslant k \leqslant m_{q-1}$.

In particular, at $p = 0$ one obtains Corollary 1.4 for any $a \geqslant 0$, which strengthens the conclusions of the Horn–Loewner Theorem 1.1 and of Theorem 1.2 for smooth functions:

Corollary 1.4. *Suppose that $a, \epsilon, I, f, n, \mathbf{u}$ are as in Theorem A. If $f[a\mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_n(\mathbb{R})$ for all $t \in [0, \epsilon)$, then the first n (or fewer) nonzero derivatives of $f(x)$ at $x = a$ are positive.*

---

[4]The results in [15] are stated for $I = (0, \rho)$, but this is equivalent to using $[0, \rho)$ if f is a power series.

We make additional remarks about Theorem A and its special cases in Section 3. For now, we note that it achieves several objectives:

(1) It unifies and further extends all of the above Horn–Loewner type variants – see Proposition 3.4 (hence, the term master theorem).

(2) Theorem A yields more precise information than the theorems stated above over $I = (0, \rho)$: about the derivatives of f at each individual point $a > 0$ in the domain (rather than at all points at once). The hypotheses employed are also local, which clarifies that Theorem 1.1 is not merely local but in fact a pointwise result.

(3) By Corollary 1.4, another strengthening accounts for the zero derivatives at every point $a \geqslant 0$ – and it is this strengthening, whose sharpness (for $a = 0$) we showed in [15] (see the discussion following Lemma 1.3).

(4) An additional strengthening is when $a = 0$ and $\rho < \infty$. In this case, Theorem A holds for all smooth functions, not merely real analytic ones as in Lemma 1.3(1).

Theorem A is proved using a novel connection between analysis and symmetric function theory – an explicit closed-form expression for the derivatives of a certain determinant, which mixes calculus, matrix algebra, and symmetric function theory. This is our second main theorem, and it shows how Schur polynomials arise naturally in the entrywise calculus, from considering (Cauchy-type) determinants for *arbitrary* sufficiently differentiable maps, not just polynomial maps as in [1, 15] – and at all $a \in \mathbb{R}$ (not just $a = 0$ as in [1, 15]):

Theorem B. *Fix integers $n \geqslant 1$ and $0 \leqslant m_0 < m_1 < \cdots < m_{n-1}$, as well as scalars $\epsilon > 0$ and $a \in \mathbb{R}$. Let $M := m_0 + \cdots + m_{n-1}$ and let a function $f : [a, a + \epsilon) \to \mathbb{R}$ be M-times differentiable at a for some fixed $\epsilon > 0$. Fix vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and define $\Delta : [0, \epsilon') \to \mathbb{R}$ via:*

$$\Delta(t) := \det f[a \mathbb{1}_{n \times n} + t \mathbf{u} \mathbf{v}^T],$$

for a sufficiently small $\epsilon' \in (0, \epsilon)$. Then,

$$\Delta^{(M)}(0) = \sum_{\mathbf{m} \vdash M} \binom{M}{m_0, m_1, \ldots, m_{n-1}} V(\mathbf{u}) V(\mathbf{v}) s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{n-1} f^{(m_k)}(a),$$

where the first factor in the summand is the multinomial coefficient, and we sum over all partitions $\mathbf{m} = (m_{n-1}, \ldots, m_0)$ of M, i.e., $M = m_0 + \cdots + m_{n-1}$ and $m_{n-1} \geqslant \cdots \geqslant m_0$.

In particular, $\Delta(0) = \Delta'(0) = \cdots = \Delta^{(N-1)}(0) = 0$, where $N = \binom{n}{2}$.

In Section 2, we explain another application of Theorem B, to a symmetric function master identity – see Theorem 2.1 (in the t-adic topology; followed by Corollary 2.4 in the real topology). We now explain the notation required in Theorem B. Our notation differs from that in the literature [20].

Definition 1.5 (Schur polynomials, Vandermonde determinants)**.**

(1) Given integers $m, N \geqslant 1$ and $0 \leqslant n_0' \leqslant n_1' \leqslant \cdots \leqslant n_{N-1}'$, a *column-strict Young tableau*, with shape $\mathbf{n}' := (n_{N-1}', \ldots, n_0')$ and cell entries $1, 2, \ldots, m$, is a left-aligned two-dimensional rectangular array T of cells, with n_0' cells in the bottom row, n_1' cells in the second lowest row, and so on, such that:

 • Each cell in T has integer entry j for some $j \in \{1, 2, \ldots, m\}$.

APOORVA KHARE

- Entries weakly increase in each row, from left to right.
- Entries strictly increase in each column, from top to bottom.

(2) Given variables $u_1, u_2, \ldots, u_m$ and a column-strict Young tableau T, define its *weight* to be

$$\mathrm{wt}(T) := \prod_{j=1}^{m} u_j^{f_j},$$

where f_j equals the number of cells in T with entry j.

(3) Given an increasing sequence of integers $0 \leqslant n_0 < \cdots < n_{N-1}$, define the partitions/tuples

$$\mathbf{n} := (n_{N-1}, \ldots, n_1, n_0), \qquad \mathbf{n}_{\min} := (N-1, \ldots, 1, 0),$$

and the corresponding *Schur polynomial* over $\mathbf{u} := (u_1, u_2, \ldots, u_m)^T$ to be

$$s_{\mathbf{n}}(\mathbf{u}) := \sum_{T} \mathrm{wt}(T), \tag{1.1}$$

where T runs over all column-strict Young tableaux of shape $\mathbf{n}' := \mathbf{n} - \mathbf{n}_{\min}$ with cell entries $1, 2, \ldots, m$. By convention, we set $s_{\mathbf{n}}(\mathbf{u}) := 0$ if $\mathbf{n}$ does not have pairwise distinct coordinates.

(4) Given a vector $\mathbf{u} = (u_1, \ldots, u_m)^T$ with entries in a commutative ring, we define its *Vandermonde determinant* to be 1 if $m = 1$, and

$$V(\mathbf{u}) := \prod_{1 \leqslant j < k \leqslant m} (u_k - u_j) = \det \begin{pmatrix} 1 & u_1 & \cdots & u_1^{m-1} \\ 1 & u_2 & \cdots & u_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & u_n & \cdots & u_n^{m-1} \end{pmatrix}, \qquad \text{if } m > 1. \tag{1.2}$$

The definition of Schur polynomials is due to Littlewood; notice that it holds over the ground ring $\mathbb{Z}$, and hence over any commutative unital ground ring. If $m = N$, then this definition is equivalent [20] to Cauchy's definition of Schur polynomials:

$$\det(\mathbf{u}^{\circ n_0} \mid \mathbf{u}^{\circ n_1} \mid \ldots \mid \mathbf{u}^{\circ n_{N-1}})_{N \times N} = V(\mathbf{u}) s_{\mathbf{n}}(\mathbf{u}).$$

This is consistent with setting $s_{\mathbf{n}}(\mathbf{u}) = 0$ if $\mathbf{n}$ has two equal coordinates, since the left-hand matrix has two equal columns in that case.

Remark 1.6. From the proof of Theorem B – see the expression (2.1) and the discussion in the subsequent paragraph – it follows that if f is smooth and has at most $n-1$ nonzero derivatives at a, then $\Delta^{(m)}(0) = 0 \ \forall m \geqslant 0$.

Remark 1.7. A curious point is that since one takes derivatives in Theorem B, one might expect one axis to deal with derivatives, and the other to deal with the (anti-)symmetry in one set of variables – but in fact, the final answer reveals the determinant's derivative to be (anti-)symmetric in both sets of variables $\mathbf{u}, \mathbf{v}$.

We close this section by returning to our starting point: the original work of Loewner and Horn. We now explain how Theorem B extends a determinant computation by Loewner (in a 1967 letter, see footnote 3 – and also found in Horn's 1967 thesis and paper [11]) in several ways:

(1) Loewner's computation was for $\mathbf{u} = \mathbf{v}$; Theorem B decouples the two vectors $\mathbf{u}, \mathbf{v}$.

(2) Loewner showed Theorem B only for $M \leqslant \binom{n}{2}$ (this is all that is required to obtain the original conclusions of the Horn–Loewner Theorem 1.1). In particular, Loewner showed that for $M < \binom{n}{2}$ the derivative $\Delta^{(M)}(0) = 0$ vanishes; this can be seen from Theorem B via the pigeonhole principle and since $s_{\mathbf{m}}(\mathbf{u}) = 0$ if $\mathbf{m}$ has two equal coordinates. Moreover, for $M = \binom{n}{2}$, there is a unique partition: $M = 0 + 1 + \cdots + (n - 1)$, and for it the result has a simpler form since the Schur polynomial factor is not manifested: $s_{\mathbf{m}}(\mathbf{u})^2 = 1$.

(3) In Proposition 2.3, we generalize Loewner's computation even further, to work over any commutative ground ring.

2. Theorem B: From any smooth function to Schur polynomials, to symmetric function identities

In this section we prove Theorem B, and use it to extend classical symmetric function identities by Cauchy and Frobenius, as well as modern variants in [1, 15], to a unifying master identity; see Theorem 2.1.

Proof of Theorem B. Let $\mathbf{w}_k$ denote the kth column of $a\mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{v}^T$; thus $\mathbf{w}_k$ has jth entry $a + tu_j v_k$. To differentiate $\Delta(t)$, we use the multilinearity of the determinant and the Laplace expansion of $\Delta(t)$ into a linear combination of $n!$ monomials, each of which is a product of n terms $f(a + tu_j v_k)$. By the product rule, taking the derivative yields n terms from each monomial, and one may rearrange all of these terms into n clusters of terms (grouping by the column that gets differentiated), and regroup using the Laplace expansion to obtain:

$$\Delta'(t) = \sum_{k=1}^{n} \det(f[\mathbf{w}_1] \mid \cdots \mid f[\mathbf{w}_{k-1}] \mid v_k \mathbf{u} \circ f'[\mathbf{w}_k] \mid f[\mathbf{w}_{k+1}] \mid \cdots \mid f[\mathbf{w}_n]).$$

Now apply the derivative repeatedly, using this principle. By the Chain Rule, $\Delta^{(M)}(0)$ is an integer linear combination of terms of the form

$$(2.1) \quad \begin{aligned} &\det(v_1^{m_0} \mathbf{u}^{\circ m_0} \circ f^{(m_0)}[a\mathbb{1}_{n \times 1}] \mid \cdots \mid v_n^{m_{n-1}} \mathbf{u}^{\circ m_{n-1}} \circ f^{(m_{n-1})}[a\mathbb{1}_{n \times 1}]) \\ &= \det(f^{(m_0)}(a) v_1^{m_0} \mathbf{u}^{\circ m_0} \mid \cdots \mid f^{(m_{n-1})}(a) v_n^{m_{n-1}} \mathbf{u}^{\circ m_{n-1}}), \end{aligned}$$

where $\mathbb{1}_{n \times 1} = (1, \ldots, 1)^T \in \mathbb{R}^n$, and $m_0 + \cdots + m_{n-1} = M$ with all $m_k \geqslant 0$.

From each such determinant, one may factor out the product $\prod_{k=0}^{n-1} f^{(m_k)}(a)$. Now $\Delta^{(M)}(0)$ is obtained by summing the determinants corresponding to applying $m_0, m_1, \ldots, m_{n-1}$ derivatives to the columns in some order, for all partitions $\mathbf{m} = (m_{n-1}, \ldots, m_0)$ of M. We first compute the integer multiplicity of each such determinant, noting by symmetry that these multiplicities are all equal. As we are applying M derivatives to Δ (before evaluating at 0), the m_0 derivatives applied to get $f^{(m_0)}$ in some (fixed) column can be any of $\binom{M}{m_0}$; now the m_1 derivatives applied to get $f^{(m_1)}$ in a (different) column can be chosen in $\binom{M-m_0}{m_1}$ ways; and so on. Thus, the multiplicity is

$$\binom{M}{m_0}\binom{M - m_0}{m_1} \cdots \binom{M - m_0 - \cdots - m_{n-2}}{m_{n-1}} = \frac{M!}{\prod_{k=0}^{n-1} m_k!}$$
$$= \binom{M}{m_0, m_1, \ldots, m_{n-1}}.$$

The next step is to compute the sum of all determinant terms. Each term corresponds to a unique permutation of the columns $\sigma \in S_n$, with say $m_{\sigma_k - 1}$ the order of the derivative applied to the kth column $f[\mathbf{w}_k]$. Using (2.1), the determinant corresponding to σ equals

$$\prod_{k=0}^{n-1} f^{(m_k)}(a) v_k^{m_{\sigma_k - 1}} \cdot (-1)^\sigma \cdot \det(\mathbf{u}^{\circ m_0} \mid \mathbf{u}^{\circ m_1} \mid \cdots \mid \mathbf{u}^{\circ m_{n-1}})$$

$$= V(\mathbf{u}) s_{\mathbf{m}}(\mathbf{u}) \prod_{k=0}^{n-1} f^{(m_k)}(a) \cdot (-1)^\sigma \prod_{k=0}^{n-1} v_k^{m_{\sigma_k - 1}}.$$

Summing this term over all $\sigma \in S_n$ yields:

$$V(\mathbf{u}) s_{\mathbf{m}}(\mathbf{u}) \prod_{k=0}^{n-1} f^{(m_k)}(a) \sum_{\sigma \in S_n} (-1)^\sigma \prod_{k=0}^{n-1} v_k^{m_{\sigma_k - 1}}$$

$$= V(\mathbf{u}) s_{\mathbf{m}}(\mathbf{u}) \prod_{k=0}^{n-1} f^{(m_k)}(a) \cdot \det(\mathbf{v}^{\circ m_0} \mid \mathbf{v}^{\circ m_1} \mid \cdots \mid \mathbf{v}^{\circ m_{n-1}})$$

$$= V(\mathbf{u}) s_{\mathbf{m}}(\mathbf{u}) \prod_{k=0}^{n-1} f^{(m_k)}(a) \cdot V(\mathbf{v}) s_{\mathbf{m}}(\mathbf{v}).$$

Now multiply by the (common) integer multiplicity to complete the proof. □

2.1. General power series determinants and Schur polynomials.

We next present a novel application of Theorem B, which unifies and extends classical and modern determinantal identities. Begin by recalling an identity for Cauchy's determinant [7]: if B is the $n \times n$ matrix with entries $(1 - u_j v_k)^{-1} := \sum_{M \geqslant 0} (u_j v_k)^M$ for variables u_j, v_k and $1 \leqslant j, k \leqslant n$, then

$$(2.2) \qquad \det B = V(\mathbf{u}) V(\mathbf{v}) \sum_{\mathbf{m}} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}),$$

where the notation was explained in Definition 1.5, and the sum runs over all partitions $\mathbf{m}$ with at most n parts. See [20, Chapter I.4, Example 6] for a proof. Usually this is written with infinitely many indeterminates u_j, v_k; we work with $u_1, \ldots, u_n; v_1, \ldots, v_n$ by specializing the other variables to zero. See also [18, Section 5] and the references therein, as well as [12, 13, 16, 17, 19, 20] for other such identities involving symmetric functions.

In particular, one can apply to all entries of the matrix $\mathbf{u}\mathbf{v}^T$ power series other than $f(x) = 1/(1 - x) = \sum_{M \geqslant 0} x^M$, and then compute the determinant. For instance, if $f(x)$ has fewer than n monomials then $f[\mathbf{u}\mathbf{v}^T]$ is a sum of fewer than n rank-one matrices, so it is singular. The case when f has precisely n or $n+1$ monomials was crucially used in joint works [1, 15], to study entrywise positivity preservers. Another explicit formula – see (2.3) – was shown by Frobenius [9] in greater generality than (2.2). This formula also appears in Rosengren–Schlosser [22, Corollary 4.7], with $(1 - cx)/(1 - x)$ in place of $1/(1 - x)$ as in (2.2). In this

case, one has:

(2.3)

$$\det \left(\frac{1 - c u_j v_k}{1 - u_j v_k} \right)^n_{j,k=1}$$

$$= V(\mathbf{u})V(\mathbf{v})(1-c)^{n-1} \left(\sum_{\mathbf{m}\,:\,m_0=0} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) + (1-c) \sum_{\mathbf{m}\,:\,m_0>0} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) \right).$$

With this background, we can state the master identity that extends Equations (2.2) and (2.3) from $(1 - cx)/(1 - x)$ for a scalar/parameter c, to all power series – including arbitrary polynomials – and with an additional $\mathbb{Z}^{\geqslant 0}$-grading:

Theorem 2.1. *Fix a commutative unital ring R and let t be an indeterminate. Let $f(t) := \sum_{M\geqslant 0} f_M t^M \in R[[t]]$ be an arbitrary formal power series. Given vectors $\mathbf{u}, \mathbf{v} \in R^n$ for some $n \geqslant 1$, and with the notation in Definition 1.5, we have:*

$$(2.4)\quad \det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{M\geqslant \binom{n}{2}} t^M \sum_{\mathbf{m}=(m_{n-1},\ldots,m_0)\vdash M} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{n-1} f_{m_k}.$$

All of the aforementioned examples are special cases of the $t = 1$ case of Theorem 2.1.

Multiple approaches can be used to show Theorem 2.1; we present one approach and outline another. We begin by formulating Theorem B in greater generality, algebraically. Fix a commutative (unital) ring R and an R-algebra S. The first step is to formalize the notion of the derivative, on a sub-class of S-valued functions. This is more than just the commonly used notion of a *derivation*, so we give it a different name.

Definition 2.2. Given a commutative unital ring R, a commutative R-algebra S (with $R \subset S$), and an R-module X, a *differential calculus* is a pair (A, ∂), where A is an R-subalgebra of functions : $X \to S$ (under pointwise addition and multiplication and R-action) that contains the constant functions, and $\partial : A \to A$ satisfies the following properties:

(1) ∂ is R-linear:
$$\partial \sum_j r_j f_j = \sum_j r_j \partial f_j, \qquad \forall r_j \in R,\ f_j \in A,\ \forall j.$$

(2) ∂ is a derivation (product rule):
$$\partial(fg) = f \cdot (\partial g) + (\partial f) \cdot g, \qquad \forall f, g \in A.$$

(3) ∂ satisfies a variant of the Chain Rule for composing with linear functions: if $x' \in X, r \in R$, and $f \in A$, then the function $g : X \to S$, $g(x) := f(x'+rx)$ also lies in A, and moreover,
$$(\partial g)(x) = r \cdot (\partial f)(x' + rx).$$

For example, the algebra of smooth functions from the real line to itself is a differential calculus, with $R = S = X = \mathbb{R}$ and $\partial = d/dx$.

We can now state an algebraic generalization of Loewner's calculations. The proof is essentially the same as for Theorem B, and is hence omitted.

Proposition 2.3. *Suppose R, S, X are as in Definition 2.2, with an associated differential calculus (A, ∂). Fix an integer $n > 0$, two vectors $\mathbf{u}, \mathbf{v} \in R^n$, a vector $a \in X$, and a function $f \in A$; and $\Delta : X \to R$ via:*

$$\Delta(t) := \det f[a \mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{v}^T], \quad t \in X.$$

Then,

$$(\partial^M \Delta)(0_X) = \sum_{\mathbf{m} \vdash M} \binom{M}{m_0, m_1, \ldots, m_{n-1}} V(\mathbf{u}) V(\mathbf{v}) s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{n-1} (\partial^{m_k} f)(a),$$

where we sum over all partitions $\mathbf{m} = (m_{n-1}, \ldots, m_0)$ of M. In particular,

$$\Delta(0_X) = (\partial \Delta)(0_X) = \cdots = (\partial^{\binom{n}{2}-1} \Delta)(0_X) = 0_R.$$

The algebra A is supposed to remind the reader of smooth functions. One can instead work with an appropriate algebraic notion of M-times differentiable functions in order to generalize Theorem B to a finite degree of differentiability.

Proposition 2.3 helps provide one approach to proving the master identity:

Proof of Theorem 2.1. The idea is to apply Proposition 2.3 to the differential calculus

$$X := t\,R[[t]], \quad S = A := R[[t]],$$

where $f(t) \in A$ acts on $g(t) \in X$ by composition: $f(g(t))$. We set $a = 0$ here, and the composition converges in the t-adic topology by choice of X. Since $a = 0$, we have $\det f[t\mathbf{u}\mathbf{v}^T] = \Delta(t)$. The problem in proceeding thus is that one needs to clear denominators and work with $\Delta^{(M)}(0_R)/M!$ and $f_{m_k} = f^{(m_k)}(0_R)/m_k!$, the Maclaurin coefficients of Δ and f respectively; but to work with these requires R to have characteristic zero.

Thus, we begin by observing that the identity (2.4) is of a universal nature: if it holds for the polynomial ring

$$R = \mathbb{Z}[X_1, \ldots, X_n,\ Y_1, \ldots, Y_n,\ Z_0, Z_1, \ldots]$$

with $u_j = X_j$, $v_k = Y_k$, $f_m = Z_m$ $\forall j, k, m$ algebraically independent elements, then one may specialize to any given ground ring R – or more precisely, to the subring of R generated by $1, u_1, \ldots, u_n,\ v_1, \ldots, v_n,\ f_0, f_1, \ldots$. Hence we assume in the rest of the proof that

$$R = \mathbb{Z}[u_1, \ldots, u_n,\ v_1, \ldots, v_n,\ f_0, f_1, \ldots],$$

with u_j, v_k, f_m being algebraically independent.

We first work over a slightly larger ring:

$$R' := R \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[\{u_j, v_j : 1 \leqslant j \leqslant n,\ f_m : m \geqslant 0\}].$$

In this setting, apply Proposition 2.3 with $X := t\,R'[[t]]$, $S = A := R'[[t]]$, and $a := 0_{R'}$, and define $\partial : A \to A$ to be the usual derivative:

$$\partial \sum_{M \geqslant 0} g_M t^M := \sum_{M \geqslant 1} M g_M t^{M-1} \qquad (g_M \in R,\ \forall M \geqslant 0).$$

One verifies that (A, ∂) is a differential calculus for the data (R', S, X).

We now prove the result over R'. Notice that $\Delta(t) = \det f[t\mathbf{u}\mathbf{v}^T]$ is a linear combination of finite products of elements of $R'[[t]]$, hence lies in $R'[[t]]$. If

$\Delta(t) = \sum_{M \geqslant 0} \delta_M t^M$, then by Proposition 2.3 one can compute each of its Maclaurin coefficients $\delta_M \in R'$ via: $\delta_M = \frac{(\partial^M \Delta)(0_{R'})}{M!}$, and hence each of its Taylor–Maclaurin polynomials as well. Taking limits in the t-adic topology, and recalling from Proposition 2.3 that $\delta_M = 0_{R'}$ for $M < \binom{n}{2}$, we compute:

$$\Delta(t) = \sum_{M \geqslant \binom{n}{2}} t^M \frac{(\partial^M \Delta)(0_{R'})}{M!}$$

$$= \sum_{M \geqslant \binom{n}{2}} t^M \sum_{\mathbf{m}} V(\mathbf{u}) V(\mathbf{v}) s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{n-1} \frac{(\partial^{m_k} f)(0_{R'})}{m_k!},$$

and this concludes the proof for $R' = \mathbb{Q}[u_1, \ldots, u_n, \ v_1, \ldots, v_n, \ f_0, f_1, \ldots]$.

While we just showed the identity (2.4) over R', here both sides of (2.4) belong to $R[[t]]$, where $R = \mathbb{Z}[\{u_j, v_j : 1 \leqslant j \leqslant n, \ f_m : m \geqslant 0\}]$. By our discussion on universality, the result follows for a general commutative unital ring. $\square$

We also sketch an alternative approach to proving Theorem 2.1, via matrix calculus. In the t-adic topology, $f(t) = \lim_{M \to \infty} f_{\leqslant M}(t)$, where $f_{\leqslant M}(t)$ is the Mth Taylor–Maclaurin polynomial of f for $M \geqslant 0$, given by $f_{\leqslant M}(t) := \sum_{m=0}^{M} f_m t^m$. But for $f_{\leqslant M}$ we have an explicit matrix factorization:

$$f_{\leqslant M}[t\mathbf{u}\mathbf{v}^T]$$

$$= \begin{pmatrix} 1 & u_1 & \cdots & u_1^M \\ 1 & u_2 & \cdots & u_2^M \\ \vdots & \vdots & \ddots & \vdots \\ 1 & u_n & \cdots & u_n^M \end{pmatrix} \cdot \begin{pmatrix} f_0 & 0 & \cdots & 0 \\ 0 & f_1 t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_M t^M \end{pmatrix} \cdot \begin{pmatrix} 1 & v_1 & \cdots & v_1^M \\ 1 & v_2 & \cdots & v_2^M \\ \vdots & \vdots & \ddots & \vdots \\ 1 & v_n & \cdots & v_n^M \end{pmatrix}^T$$

and hence one can compute $\det f_{\leqslant M}[t\mathbf{u}\mathbf{v}^T]$ via the Cauchy–Binet formula. Now take the t-adic limit and rearrange terms to deduce via the t-adic continuity of the determinant function:

$$(2.5) \qquad \det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u}) V(\mathbf{v}) \sum_{0 \leqslant m_0 < m_1 < \cdots < m_{n-1}} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{n-1} f_{m_k} t^{m_k},$$

where $\mathbf{m} = (m_{n-1}, \ldots, m_0)$. But this sum equals the right-hand side of (2.4).

We conclude this section by returning to the real topology and working again over $\mathbb{R}$. As Theorem 2.1 holds in the t-adic topology, it is natural to ask about the convergence of the series (2.4) as a real function. This indeed holds, as a consequence of Theorem B:

Corollary 2.4. *Fix scalars $\epsilon > 0$ and $a \in \mathbb{R}$, and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ for some integer $n > 0$. If $f : [a, a + \epsilon) \to \mathbb{R}$ has the power series expansion*

$$f(x) = \sum_{M \geqslant 0} f_M (x - a)^M, \qquad x \in [a, a + \epsilon), \ f_M \in \mathbb{R},$$

and we define $\Delta(t) := \det f[a \mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{v}^T]$ for sufficiently small t, then $\Delta(t)$ equals the right-hand side of Equation (2.4) for sufficiently small t.

Proof. If f has a power series expansion around/near a, then so does Δ near 0, since it is a linear combination of finite products of f-values near a. Thus Δ (real

analytic near 0) can be recovered from its Maclaurin coefficients by repeating the same computation as in the proof of Theorem 2.1. The Maclaurin coefficients of Δ are computed in Theorem B.   □

3. HORN–LOEWNER MASTER THEOREM A: ADDITIONAL REMARKS, AND PROOF

Finally, we return to the Horn–Loewner Theorem 1.1 and its variants. The goal of this section is to prove our Master Theorem A; however, we begin by filling in the details omitted in the Introduction.

One reason why the Horn–Loewner Theorem 1.1 (and its variants) was significant was because it is sharp in the number of non-negative derivatives. We now provide several examples of this phenomenon.

Example 3.1.

(1) If one restricts to the class of power functions $f(x) = x^\alpha$ – with α possibly non-integral – then FitzGerald and Horn [8] showed that such entrywise preservers of positivity on $\mathbb{P}_n((0,\infty))$ correspond precisely to $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-2,\infty)$ – note that the case of $\alpha \in \mathbb{Z}^{\geqslant 0}$ follows immediately from the Schur product theorem [25]. Thus if $\alpha \in (n-2, n-1)$, then $f(x) = x^\alpha$ satisfies Theorem 1.1; moreover, $f^{(n)}$ is negative on $I = (0,\infty)$, showing that Theorem 1.1 is sharp.

(2) Let $I = (-\rho, \rho)$ or $[0, \rho)$ with $0 < \rho < \infty$. If one restricts to polynomial functions $f(x) = \sum_{k \geqslant 0} c_k x^k$ acting on I, then in [1] we showed that for any scalars $c_0, \dots, c_{n-1} > 0$, there exists $c_n < 0$ such that $f(x) = \sum_{k=0}^n c_k x^k$ preserves positivity on $\mathbb{P}_n(I)$; moreover, $f^{(n)}(0) = n!\, c_n < 0$, hence $f^{(n)}(x) < 0$ for $x > 0$ small. This has two consequences: first, it produces the first examples of polynomials / power series with a negative coefficient that preserve positivity in a fixed dimension. Moreover, this produces polynomial functions (the previous example produces power functions) that preserve positivity in dimension n but not $n+1$.

(3) Suppose that $I = (0, \infty)$. In [15], we constructed polynomials of the special form $\sum_{k=0}^{2n} c_k x^k$ with $c_n < 0 < c_k$ for all $k \neq n$, which entrywise preserve positivity on $\mathbb{P}_n((0, \infty))$. These are the first examples with negative coefficients, and in particular, the first polynomial examples that work over $\mathbb{P}_n(I)$ but not over $\mathbb{P}_{n+1}(I)$. Proposition 3.2 provides an explicit family of such polynomials – in fact, the simplest such family – when $n = 2$. This is followed by a remark about all polynomials of the special form above when $n = 2$; the case of general $n \geqslant 2$ is in [15].

Proposition 3.2. *For $\varepsilon > 0$, define $f_\varepsilon(x) = 1 + x - \varepsilon x^2 + x^3 + x^4$. If $0 < \varepsilon \leqslant \frac{1}{10}$, then $f_\varepsilon[-]$ preserves positivity on $\mathbb{P}_2([0, \infty))$.*

In particular, such entrywise polynomials $f_\varepsilon[-]$ preserve positivity on $\mathbb{P}_2((0,\infty))$, but not on $\mathbb{P}_3((0,\rho))$ for any $0 < \rho \leqslant \infty$, by Lemma 1.3.

Proof. We first claim that f_ε is positive on $(0, \infty)$ if $\varepsilon \leqslant 2$. This follows by adding the inequalities $x^2 < 1 + x^4$ and $x^2 < x + x^3$, for any $x > 0$.

Next suppose that $\mathbf{u} = (u_1, u_2)^T \in (0, \infty)^2$ is a column and $A = \mathbf{u}\mathbf{u}^T \in \mathbb{P}_2((0, \infty))$ has rank one. We claim that $f_\varepsilon[A]$ is positive semidefinite if $0 < \varepsilon \leqslant \frac{1}{10}$. By relabelling indices and continuity, it suffices to show this for $0 < u_1 < u_2$. First note that all entries of $f_\varepsilon[\mathbf{u}\mathbf{u}^T]$ are non-negative, by the preceding paragraph.

Next, we apply (2.5) – with $n = 2$, $\mathbf{v} = \mathbf{u}$, $t = 1$, and $f = f_\varepsilon$ – to compute the determinant:

$$(3.1) \qquad \det f_\varepsilon[\mathbf{u}\mathbf{u}^T]$$

$$= V(\mathbf{u})^2 \left[\sum_{0 \leqslant j < k \leqslant 3} s_{(n_k, n_j)}(\mathbf{u})^2 - \varepsilon \sum_{j=0}^{1} s_{(M, n_j)}(\mathbf{u})^2 - \varepsilon \sum_{j=2}^{3} s_{(n_j, M)}(\mathbf{u})^2 \right],$$

where, for ease of exposition, we define and work with

$$(3.2) \qquad\qquad (n_0, n_1, M, n_2, n_3) := (0, 1, 2, 3, 4).$$

To show that $\det f_\varepsilon[\mathbf{u}\mathbf{u}^T]$ is non-negative, we employ bounds on Schur polynomials in two variables. Given integers $0 \leqslant j < k$, notice that

$$s_{(k,j)}(u_1, u_2) = \frac{u_2^k u_1^j - u_2^j u_1^k}{u_2 - u_1} = (u_1 u_2)^j (u_2^{k-j-1} + u_2^{k-j-2} u_1 + \cdots + u_1^{k-j-1}).$$

If $0 < u_1 < u_2$ as above, then the largest among the $k - j$ monomials here is $u_1^j u_2^{k-1}$, so

$$(3.3) \quad u_1^j u_2^{k-1} \leqslant s_{(k,j)}(u_1, u_2) \leqslant (k - j) u_1^j u_2^{k-1}, \qquad \forall 0 < u_1 < u_2, \ \ 0 \leqslant j < k.$$

(See [15, Proposition 3.1] for the corresponding bounds on general Schur polynomials.) In particular, dividing (3.1) by $V(\mathbf{u})^2 > 0$ and using (3.3), we have:

$$\frac{\det f_\varepsilon[\mathbf{u}\mathbf{u}^T]}{V(\mathbf{u})^2} \geqslant \sum_{0 \leqslant j < k \leqslant 3} u_1^{2n_j} u_2^{2(n_k-1)} - \varepsilon \sum_{0 \leqslant j \leqslant 3} (M - n_j)^2 u_1^{2 \min(n_j, M)} u_2^{2 \max(n_j, M) - 2}.$$

We claim that the right-hand side is non-negative when $0 < \varepsilon \leqslant \frac{1}{10}$. This is equivalent to showing that if $0 < u_1 < u_2$, then

$$\varepsilon^{-1} \geqslant 10 = \sum_{j=0}^{3} (M - n_j)^2 \geqslant \frac{\sum_{0 \leqslant j \leqslant 3} (M - n_j)^2 u_1^{2 \min(n_j, M)} u_2^{2 \max(n_j, M) - 2}}{\sum_{0 \leqslant j < k \leqslant 3} u_1^{2n_j} u_2^{2(n_k - 1)}}.$$

For this, it suffices to show that each monomial $u_1^{2 \min(n_j, M)} u_2^{2 \max(n_j, M) - 2}$ in the numerator is bounded above by the sum in the denominator. We claim more strongly that each such monomial is bounded above by one summand in the denominator. This is because for each j, there exist indices $i \neq k$ in $\{0, 1, 2, 3\} \setminus \{j\}$ such that $n_i < M < n_k$. Now for instance if $j = 1$, then

$$u_1^{2n_1} u_2^{2M-2} \leqslant \begin{cases} u_1^{2n_0} u_2^{2(n_1-1)}, & \text{if } u_2 < 1, \\ u_1^{2n_1} u_2^{2(n_3-1)}, & \text{if } u_2 \geqslant 1, \end{cases}$$

and similarly for the other three summands.

This shows the result for all rank-one matrices in $\mathbb{P}_2((0, \infty))$. For the general case, we use the extension principle [15, Theorem 3.5], which says that if $f : (0, \infty) \to \mathbb{R}$ is a continuously differentiable function such that $f[-]$ preserves positivity on rank-one matrices in $\mathbb{P}_n((0, \infty))$ and $f'[-]$ preserves positivity on $\mathbb{P}_{n-1}((0, \infty))$ for some $n \geqslant 2$, then $f[-]$ preserves positivity on $\mathbb{P}_n((0, \infty))$. Via this result, it suffices to show that $f'_\varepsilon[-]$ preserves positivity on $\mathbb{P}_1((0, \infty))$ if $0 < \varepsilon \leqslant \frac{1}{10}$. But this follows by noting that

$$f'_\varepsilon[(x)_{1 \times 1}] = f'_\varepsilon(x) \geqslant f'_{1/10}(x) > \left(1 - \frac{x}{10}\right)^2 \geqslant 0, \qquad \forall x > 0.$$

Thus if $0 < \varepsilon \leqslant \frac{1}{10}$, then $f_\varepsilon[-]$ preserves positivity on $\mathbb{P}_2((0,\infty))$, so on $\mathbb{P}_2([0,\infty))$ by continuity. □

Remark 3.3. In this remark, we consider the more general family of polynomials

$$f_\varepsilon(x) = c_{n_0} x^{n_0} + c_{n_1} x^{n_1} - \varepsilon x^M + c_{n_2} x^{n_2} + c_{n_3} x^{n_3}, \qquad \varepsilon > 0,$$

where $0 \leqslant n_0 < n_1 < M < n_2 < n_3$ are integers, and $c_{n_j} \in (0,\infty)$ are fixed scalars. If

$$(3.4) \qquad 0 < \varepsilon \leqslant \varepsilon_0 := \min\{c_{n_j} : 0 \leqslant j \leqslant 3\} \cdot \left(\sum_{j=0}^3 (M - n_j)^2 \right)^{-1},$$

then we claim that $f_\varepsilon[-]$ preserves positivity on $\mathbb{P}_2([0,\infty))$. Here we outline the argument when it differs from the proof of Proposition 3.2, mainly due to the integers $n_j \geqslant 0$ being arbitrary.

The first reduction is to the case when all $c_{n_j} = 1$. Note that for any positive semidefinite matrix $A_{n \times n}$, one has $f_\varepsilon[A] - g_\varepsilon[A] \in \mathbb{P}_n$, where $g_\varepsilon(x) := \min_j c_{n_j} \cdot \sum_{j=0}^3 x^{n_j} - \varepsilon x^M$. This is because $A^{\circ n_j} \in \mathbb{P}_n$ for all j, by the Schur product theorem [25]. Thus if the result holds for $(\min_j c_{n_j})^{-1} g_\varepsilon(x)$ – with the bound $\varepsilon^{-1} \geqslant \sum_{j=0}^3 (M - n_j)^2$ – then the result holds for f_ε as desired.

Thus, suppose that $c_{n_j} = 1 \,\forall j$ and $f_\varepsilon(x) = x^{n_0} + x^{n_1} - \varepsilon x^M + x^{n_2} + x^{n_3}$, akin to its special case in Proposition 3.2. Now $\varepsilon_0 \leqslant \frac{1}{10}$, so the same arguments as in the preceding proof show: (i) $f_\varepsilon(x) > 0$ if $0 < \varepsilon \leqslant 2$ and $x > 0$; and (ii) $f_\varepsilon[-]$ preserves positivity on rank-one matrices in $\mathbb{P}_2((0,\infty))$, if $0 < \varepsilon \leqslant \varepsilon_0$. The final step – as in the preceding proof – is to use the extension principle, and here we claim the stronger assertion: $h_{\varepsilon_0}(x) := n_1 x^{n_1 - 1} - M \varepsilon_0 x^{M-1} + n_2 x^{n_2 - 1} \geqslant 0 \,\forall x > 0$. To prove this, it suffices to multiply by $x > 0$ and show that $n_1(x^{n_1} + x^{n_2}) \geqslant M \varepsilon_0 x^M$. This would follow from $M \varepsilon_0 \leqslant n_1$, which in turn follows from the sub-claim that

$$2(M - n_1)^2 \geqslant M/n_1, \quad \text{where} \quad M \geqslant n_1 + 1, \; n_1 \geqslant 1.$$

This sub-claim is proved by considering the function $h(x) := 2(x - n_1)^2 - n_1^{-1} x$ on $[n_1, \infty)$. Note that $h'(x) \geqslant 0$ for $x \geqslant n_1 + 1/(4n_1)$, so $h(M) \geqslant h(n_1 + 1) = 1 - n_1^{-1} \geqslant 0$. □

Our next observation explains why we call Theorem A a master theorem: it encompasses all of the previously known versions.

Proposition 3.4. *Theorem* A *specializes to all results stated prior to it, for f smooth.*

Proof. We first show how Theorem A implies Theorem 1.2 (which in turn implies Theorem 1.1). Since f is smooth, by the discussion following Theorem 1.2 we may disregard the hypothesis concerning $\mathbb{P}_2(I)$. Choose any $a \in I = (0,\rho)$, and set

$$\epsilon := \rho - a, \qquad p = q := n, \qquad u_k := u_0^{k-1} \,\forall k \in [1, n].$$

Theorem 1.2 now follows from Theorem A, for smooth f.

Next we show how Theorem A implies a stronger version of Lemma 1.3(1) – for smooth functions, not merely power series. Set $a = 0$ and suppose that f has $N \leqslant \infty$ nonzero derivatives at $a = 0$. Let $l := \min(n, N)$ and denote the smallest l of these orders of derivatives by $m_0, \ldots, m_{l-1}$. Now set $\epsilon := \rho, \; p := 0, \; q := l$. It follows by Theorem A that the first l nonzero Maclaurin coefficients of $f(x)$ at

$x = 0$ are non-negative, hence positive as desired. This shows the result – and with a smaller test set used here than in Lemma 1.3(1).

Finally, we show how Lemma 1.3(2) follows from Theorem A. By Lemma 1.3(1), it suffices to consider only the coefficients of degree $> m'$. Thus, suppose that the assumptions of Lemma 1.3(2) hold, and yet $c_{m'} < 0$ is not followed by n positive coefficients of higher degree. First, if $c_{m'}$ is followed by infinitely many negative coefficients of higher degree, then $f(x) < 0$ for $x \gg 0$. But then $f[x\mathbb{1}_{n \times n}] \notin \mathbb{P}_n(\mathbb{R})$, contradicting the hypotheses.

Thus $c_{m'}$ is followed by finitely many nonzero higher-order coefficients. Without loss of generality, we may redefine m' to be the highest degree coefficient that is negative; thus,

$$f(x) = \sum_{j=0}^{d} c_{n_j} x^{n_j} + c_{m'} x^{m'} + \sum_{k=0}^{l-1} c_{m_k} x^{m_k},$$

where $0 \leqslant n_0 < \cdots < n_d < m' < m_{l-1} < \cdots < m_0$ are integers. Now define

$$g(x) := \sum_{k=0}^{l-1} c_{m_k} c^{m_0 - m_k} + c_{m'} x^{m_0 - m'} + \sum_{j=0}^{d} c_{n_j} x^{m_0 - n_j}.$$

In other words, $g(x) = x^{m_0} f(1/x)$ for $x > 0$. We claim that $g(x)$ entrywise preserves positivity on rank-one matrices $\mathbf{u}\mathbf{u}^T \in \mathbb{P}_n((0, \infty))$. Indeed,

$$g[\mathbf{u}\mathbf{u}^T] = (\mathbf{u}\mathbf{u}^T)^{\circ m_0} \circ f[(\mathbf{u}\mathbf{u}^T)^{\circ -1}] = [\mathbf{u}^{\circ m_0}(\mathbf{u}^{\circ m_0})^T] \circ f[\mathbf{u}^{\circ -1}(\mathbf{u}^{\circ -1})^T],$$

and this is positive semidefinite by the Schur product theorem and since $\mathbf{u}^{\circ -1} \in (0, \infty)^n$ as well. But this reduces us to the previous case of Lemma 1.3(1), which follows from Theorem A and implies that g has at least n positive coefficients of degree lower than $x^{m_0 - m'}$. Therefore $l \geqslant n$, contradicting the assumption. $\square$

Remark 3.5. For completeness, we briefly discuss what happens if one tries to weaken the smoothness hypothesis in Theorem A. The way that Horn/Loewner originally proved Theorem 1.1 was to appeal to a result of Boas and Widder [6] by using mollifiers, that is, convolving f with $\phi(x/\delta)$ for $\delta > 0$ and a certain smooth function $\phi : (-1, 0) \to (0, \infty)$. We now explain why it is not possible to repeat this argument for Theorem A outside of the setting of the Horn–Loewner setting $p = n$.

Indeed, suppose $p < n$, which we may take to mean $m_p > p < n$. To repeat the mollifier argument would at least involve changing the hypothesis

$$f[a\mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_n(\mathbb{R}), \quad \forall t,$$

from each fixed a, to all a belonging to an interval J'. But now if we want $f_\delta^{(p)}(a) = 0$, then assuming that f is nice enough (e.g., if $f, f', \ldots, f^{(p)}$ are bounded on J'), we compute:

$$f_\delta^{(p)}(a) = \int_{-\delta}^{0} f^{(p)}(a - u)\phi\left(\frac{u}{\delta}\right) = 0.$$

From this and since $f^{(p)} \geqslant 0$ on I for $p < n$ (by Theorem 1.1), it follows that $f^{(p)}$ vanishes on some interval J containing a, hence so does $f^{(r)}$ for all $r \geqslant p$. But this does not reconcile with $f^{(m_p)}(a) \neq 0$ for $m_p > p$.

Given Theorem A and the discussion in Remark 3.5, we next observe that the original non-pointwise Theorem 1.1, as well as its strengthening in Theorem 1.2, admits a small generalization: *the domain need not begin at 0.*

Corollary 3.6. *Theorems* 1.1 *and* 1.2 *hold for every open subinterval* $I \subset (0, \infty)$.

This result is *equivalent* to the formulation for $I = (0, \infty)$, and this equivalence is immediate. Indeed, if $I = (r, s)$ with $0 < r < s \leqslant \infty$), then one works instead with the function $g : (0, s - r) \to \mathbb{R}$, $g(x) := f(x + r)$, and this reduces the result to Theorems 1.1 and 1.2 respectively.

Our final remark, before proving the Master Theorem A, addresses a classical result by Boas and Widder:

Remark 3.7. As discussed in [5, 10, 11], the Horn–Loewner theorem for general functions – on all open intervals $(0, \rho)$ for $0 < \rho \leqslant \infty$ – in fact follows from its smooth version, using mollifiers and a 1940 result by Boas and Widder. Here we record a minor typo in Boas and Widder's proof of their (rather remarkable!) main result in [6]. The authors begin the proof of [6, Lemma 13] by claiming that if $I \subset \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is continuous and has non-negative forward differences of order $k \geqslant 3$, then f' is monotonic. However, this is not true as stated: for any $k \geqslant 3$, the function $f(x) = x^3$ satisfies these hypotheses on $I = (-1, 1)$, but f' is not monotone on I.

We now explain how to fix this issue. One has to claim instead that f' is *piecewise* monotone on I. This claim follows by applying in turn [6, Lemmas 9, 4, and 11]. The piecewise monotonicity then suffices to imply the existence of $f'(x\pm)$ at every point in I, and the remainder of the proof of Lemma 13 in [6] goes through verbatim.

3.1. **Proof of the Master Theorem A.** We conclude the paper by using Theorem B to show the Horn–Loewner Master Theorem A. Definition 3.8, which may seem somewhat opaque at first glance, will feature in the proof.

Definition 3.8. Let $a \in \mathbb{R}$ and $\epsilon \in (0, \infty)$. Define $I := [a, a + \epsilon)$ and suppose $f : I \to \mathbb{R}$ is smooth. We say that a tuple of integers

$$0 \leqslant m_0 < \cdots < m_{n-1}$$

is *admissible* for these data if, for all tuples $(l_0, \ldots, l_{n-1})$ of non-negative integers such that $\sum_k l_k \leqslant \sum_k m_k$, at least one of the following three possibilities holds:

(1) The l_k are not pairwise distinct.
(2) There exists k such that $f^{(l_k)}(a) = 0$.
(3) $\{l_0, \ldots, l_{n-1}\} = \{m_0, \ldots, m_{n-1}\}$.

Notice that this definition is independent of $\epsilon > 0$. We now characterize all admissible tuples of each given length:

Lemma 3.9. *Given $a \in \mathbb{R}$, $\epsilon > 0$, an integer $n > 0$, and $f : [a, a + \epsilon) \to \mathbb{R}$ smooth, an integer tuple $0 \leqslant l_0 < \cdots < l_{n-1}$ is admissible for these data if and only if:*

(1) *Either f has at most $n - 1$ nonzero derivatives at a; or*
(2) *If the integers $0 \leqslant m_0 < \cdots < m_{n-1}$ denote the n lowest-order nonzero derivatives of $f(x)$ at $x = a$, then either $l_k = m_k \ \forall k$ or $\sum_k l_k < \sum_k m_k$.*

In particular, given a, ϵ, f, n, there are either finitely many length n admissible tuples, or every length n tuple of pairwise distinct non-negative integers is admissible.

The tuple $m_k = k$, $0 \leqslant k < n$ was used in Loewner's determinant computation and proof of Theorem 1.1, and this tuple is easily seen to be admissible. As discussed in the discussion following Remark 1.6, in this special case the argument is

somewhat less involved and the underlying use of admissibility is not revealed; but this subtlety is made clear in the proof of Theorem A.

Proof. Clearly if f has at most $n - 1$ nonzero derivatives at a, then every integer tuple $0 \leqslant l_0 < \cdots < l_{n-1}$ forms an admissible tuple, by the pigeonhole principle.

Now suppose that $f^{(m_k)}(a) \neq 0$ for all $0 \leqslant k \leqslant n - 1$, and the m_k are minimal with this property, as well as pairwise distinct. One checks that the m_k form an admissible tuple. For any tuple (m_k') such that $\sum_k m_k' \geqslant \sum_k m_k$ but $\{m_0', \ldots, m_{n-1}'\} \neq \{m_0, \ldots, m_{n-1}\}$, we can choose $l_k := m_k$ in Definition 3.8 to verify that (m_k') is not admissible.

Finally, if $\sum_k m_k' < \sum_k m_k$, then we claim that (m_k') is an admissible tuple. Indeed, choose any tuple (l_k) of pairwise distinct integers $l_k \geqslant 0$ with $\sum_k l_k \leqslant \sum_k m_k' < \sum_k m_k$. Then conditions (1), (3) in Definition 3.8 fail to hold, so condition (2) holds by the minimality of the m_k.   □

Finally, we have:

Proof of Theorem A. At the outset, set

$$m_0 := 0, \quad \ldots, \quad m_{p-1} := p - 1.$$

Notice that it suffices to show that $f^{(m_k)}(a) \geqslant 0$ for $0 \leqslant k \leqslant q - 1$. The remaining derivatives at $x = a$ of $f(x)$ of order $\leqslant m_{q-1}$ lie in $[p, m_{q-1}] \setminus \{m_p, \ldots, m_{q-1}\}$, and hence are zero by the choice of the m_k.

The second observation is that the given test set of $n \times n$ matrices contains as principal submatrices a corresponding test set of $q \times q$ matrices. Hence we may restrict to the leading principal $q \times q$ submatrices of the given test set, and work with only this reduced test set. In other words, we may assume without loss of generality that $q = n$.

Having made these reductions, we prove the result. For each $0 \leqslant \delta$ small enough, define $f_\delta(x) := f(x) + \delta x^{p-1}$ with $x \in I$. For the data $a \geqslant 0$, any $\epsilon > 0$, and f_δ with any $\delta > 0$, note by Lemma 3.9(2) that the tuple (m_k) is indeed admissible, since the m_k denote the orders of the first n nonzero derivatives[5] of $f_\delta(x)$ at $x = a$.

Now given a, t and the vector $\mathbf{u}$ as in the theorem, define $\Delta(t) := \det f_\delta[a\mathbb{1}_{n \times n} + t\mathbf{u}\mathbf{u}^T]$ as in Theorem B (i.e., replacing $f, \mathbf{v}$ by $f_\delta, \mathbf{u}$ respectively). Then $\Delta(t) \geqslant 0$ for $t > 0$, by the hypotheses and using the Schur product theorem for x^{p-1}. From this we obtain:

$$0 \leqslant \lim_{t \to 0^+} \frac{\Delta(t)}{t^M}, \qquad \text{where } M := m_0 + \cdots + m_{n-1},$$

provided this limit exists.

We now claim that $\Delta(0) = \Delta'(0) = \cdots = \Delta^{(M-1)}(0) = 0$, so that one can compute the limit using L'Hôpital's rule. Indeed, going through the proof of Theorem B, if we choose any tuple (l_k) of non-negative integers, the determinant (2.1) vanishes if any two l_k are equal, or if $f^{(l_k)}(a) = 0$. Thus if $\sum_k l_k \leqslant \sum_k m_k$, then the admissibility of the tuple (m_k) (shown above) implies that $\{l_0, \ldots, l_{n-1}\} = \{m_0, \ldots, m_{n-1}\}$.

---

[5]In the original proof in [11] for $p = q = n$, Horn/Loewner use $f_\delta(x) := f(x) + \delta x^n$; but for their purposes they could just as well have used any power $\geqslant n - 1$. As the present proof reveals, in order to examine the coefficients of nonzero derivatives of order up to $n - 1$, the optimal power to use in the original Horn–Loewner setting would be $n - 1$.

In particular, $\Delta^{(L)}(0) = 0$ for all $0 \leqslant L < M$, by Theorem B. Continuing the computation,

$$0 \leqslant \lim_{t \to 0^+} \frac{\Delta(t)}{t^M} = \frac{\Delta^{(M)}(0)}{M!} = V(\mathbf{u})^2 s_{\mathbf{m}}(\mathbf{u})^2 \prod_{k=0}^{n-1} \frac{f_\delta^{(m_k)}(a)}{m_k!},$$

where the equalities are by L'Hôpital's rule, and by Theorem B and the admissibility of $\mathbf{m}$. In particular, the right-hand side here is non-negative. Since $\mathbf{u}$ has distinct coordinates, we can cancel all positive factors to conclude that

$$(3.5) \qquad \qquad \prod_{k=0}^{n-1} f_\delta^{(m_k)}(a) \geqslant 0.$$

Notice that for $n = 1$, this proves the inequality, by sending $\delta \to 0^+$.

We now prove the result by induction on $n = q$. For the induction step, we know that $f^{(m_k)}(a) \geqslant 0$ for $0 \leqslant k \leqslant n-2$, since the given test set of $n \times n$ matrices contains as leading principal submatrices a corresponding test set of $(n-1) \times (n-1)$ matrices. There are now two cases. If $a = 0$, then with $\delta = 0$ and $0 \leqslant k \leqslant n - 2$, we have $f_\delta^{(m_k)}(0) > 0$ as was just discussed, so we may divide and obtain $f^{(m_{n-1})}(0) \geqslant 0$, as desired.

If instead $a > 0$ and $\delta > 0$, then

$$f_\delta^{(m_k)}(a) = f^{(m_k)}(a) + \delta\, p(p-1) \cdots (p - m_k + 1) a^{p - m_k},$$

and this is positive for $0 \leqslant k \leqslant n-2$ by the induction hypothesis. (Here we consider separately the cases $0 \leqslant k < p$ and $k \geqslant p$.) Hence by (3.5),

$$f_\delta^{(m_{n-1})}(a) = f^{(m_{n-1})}(a) + \mathbb{1}_{p=n}\, \delta \cdot m_{n-1}! \geqslant 0, \qquad \forall 0 < \delta \ll 1,$$

and it follows that $f^{(m_{n-1})}(a) \geqslant 0$. $\qquad \square$

ACKNOWLEDGMENTS

REFERENCES

[1] Alexander Belton, Dominique Guillot, Apoorva Khare, and Mihai Putinar, *Matrix positivity preservers in fixed dimension. I*, Adv. Math. **298** (2016), 325–368, DOI 10.1016/j.aim.2016.04.016. MR3505743

[2] Alexander Belton, Dominique Guillot, Apoorva Khare, and Mihai Putinar, *Schur polynomials and matrix positivity preservers* (English, with English and French summaries), 28th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2016), Discrete Math. Theor. Comput. Sci. Proc., BC, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, [2016] ©2016, pp. 155–166. MR4111341

[3] Alexander Belton, Dominique Guillot, Apoorva Khare, and Mihai Putinar, *A panorama of positivity. I: Dimension free*, In: Analysis of Operators on Function Spaces (The Serguei Shimorin Memorial Volume; A. Aleman, H. Hedenmalm, D. Khavinson, M. Putinar, Eds.), pp. 117–165, Trends in Math., Birkhauser, 2019.

[4] Alexander Belton, Dominique Guillot, Apoorva Khare, and Mihai Putinar, *A panorama of positivity. II: Fixed dimension*, Complex analysis and spectral theory, Contemp. Math., vol. 743, Amer. Math. Soc., [Providence], RI, [2020] ©2020, pp. 109–150, DOI 10.1090/conm/743/14958. MR4061939

[5] Alexander Belton, Dominique Guillot, Apoorva Khare, and Mihai Putinar, *Moment-sequence transforms*, DOI 10.4171/jems/1145 J. Eur. Math. Soc., Published online.

[6] Ralph Philip Boas Jr. and David Vernon Widder, *Functions with positive differences*, Duke Math. J. **7** (1940), 496–503. MR3436

[7] Augustin-Louis Cauchy, *Mémoire sur les fonctions alternées et sur les sommes alternées*, Exercices Anal. et Phys. Math., 2:151–159, 1841.

[8] Carl Hanson FitzGerald and Roger Alan Horn, *On fractional Hadamard powers of positive definite matrices*, J. Math. Anal. Appl. **61** (1977), no. 3, 633–642, DOI 10.1016/0022-247X(77)90167-6. MR506356

[9] Ferdinand Georg Frobenius, *Ueber die elliptischen Functionen zweiter Art* (German), J. Reine Angew. Math. **93** (1882), 53–68, DOI 10.1515/crll.1882.93.53. MR1579913

[10] Dominique Guillot, Apoorva Khare, and Bala Rajaratnam, *Preserving positivity for rank-constrained matrices*, Trans. Amer. Math. Soc. **369** (2017), no. 9, 6105–6145, DOI 10.1090/tran/6826. MR3660215

[11] Roger Alan Horn, *The theory of infinitely divisible matrices and kernels*, Trans. Amer. Math. Soc. **136** (1969), 269–286, DOI 10.2307/1994714. MR264736

[12] Masao Ishikawa, Masahiko Ito, and Soichi Okada, *A compound determinant identity for rectangular matrices and determinants of Schur functions*, Adv. in Appl. Math. **51** (2013), no. 5, 635–654, DOI 10.1016/j.aam.2013.08.001. MR3118549

[13] Masao Ishikawa, Soichi Okada, Hiroyuki Tagawa, and Jiang Zeng, *Generalizations of Cauchy's determinant and Schur's Pfaffian*, Adv. in Appl. Math. **36** (2006), no. 3, 251–287, DOI 10.1016/j.aam.2005.07.001. MR2219947

[14] Apoorva Khare and Terence Tao, *Schur polynomials, entrywise positivity preservers, and weak majorization*, Sém. Lothar. Combin. **80B** (2018), Art. 14, 12. MR3940589

[15] Apoorva Khare and Terence Tao, *On the sign patterns of entrywise positivity preservers in fixed dimension*, Amer. J. Math. 143 (2021), 1863-1929, DOI 10.1353/ajm.2021.0049

[16] Christian Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** (1999), Art. B42q, 67. The Andrews Festschrift (Maratea, 1998). MR1701596

[17] Christian Krattenthaler, *Advanced determinant calculus: a complement*, Linear Algebra Appl. **411** (2005), 68–166, DOI 10.1016/j.laa.2005.06.042. MR2178686

[18] Greg Kuperberg, *Symmetry classes of alternating-sign matrices under one roof*, Ann. of Math. (2) **156** (2002), no. 3, 835–866, DOI 10.2307/3597283. MR1954236

[19] Dan Laksov, Alain Lascoux, and Anders Thorup, *On Giambelli's theorem on complete correlations*, Acta Math. **162** (1989), no. 3-4, 143–199, DOI 10.1007/BF02392836. MR989395

[20] Ian Grant Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications. MR1354144

[21] Georg Pólya and Gabor Szegő, *Aufgaben und Lehrsätze aus der Analysis. Band II: Funktionentheorie, Nullstellen, Polynome Determinanten, Zahlentheorie* (German), Heidelberger Taschenbücher, Band 74, Springer-Verlag, Berlin-New York, 1971. Vierte Auflage. MR0344041

[22] Hjalmar Rosengren and Michael Schlosser, *Elliptic determinant evaluations and the Macdonald identities for affine root systems*, Compos. Math. **142** (2006), no. 4, 937–961, DOI 10.1112/S0010437X0600203X. MR2249536

[23] Walter Rudin, *Positive definite sequences and absolutely monotonic functions*, Duke Math. J. **26** (1959), 617–622. MR109204

[24] Isaac Jacob Schoenberg, *Positive definite functions on spheres*, Duke Math. J. **9** (1942), 96–108. MR5922

[25] Issai Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen* (German), J. Reine Angew. Math. **140** (1911), 1–28, DOI 10.1515/crll.1911.140.1. MR1580823

[26] Harkrishan Vasudeva, *Positive definite matrices and absolutely monotonic functions*, Indian J. Pure Appl. Math. **10** (1979), no. 7, 854–858. MR537245

APOORVA KHARE

Indian Institute of Science, Bangalore – 560012, India; and Analysis and Probability Research Group, Bangalore – 560012, India

Email address: khare@iisc.ac.in

Licensed to Indian Institute of Science. Prepared on Fri Jul 8 05:28:55 EDT 2022 for download from IP 14.139.128.34.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use