On Stabilizability of Switched Linear Systems Under Restricted Switching

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***Abstract*—This article deals with the stability of discrete-time switched linear systems whose all subsystems are unstable and the set of admissible switching signals obeys prespecified restrictions on switches between the subsystems and dwell times on the subsystems. We derive sufficient conditions on the subsystems matrices such that a switched system is globally exponentially stable under a set of purely time-dependent switching signals that obeys the given restrictions. The main apparatuses for our analysis are (matrix) commutation relations between certain products of the subsystems matrices and graph-theoretic arguments.**

***Index Terms*—Graph theory, matrix commutators, stabilizability, switched systems.**

I. Introduction

A *switched system* has two ingredients—a family of systems and a switching signal. The *switching signal* selects an *active subsystem* at every instant of time, i.e., the system from the family that is currently being followed [15, Section 1.1.2]. In this article we focus on discrete-time switched linear systems whose all subsystems are unstable and the set of switching signals obeys prespecified restrictions on admissible switches between the subsystems and admissible dwell times on the subsystems. We call a switched system *stabilizable* if there exists a switching signal under which the system is globally exponentially stable (GES). Given the set of admissible switches between the subsystems and the admissible minimum and maximum dwell times on the subsystems, we find sufficient conditions on the subsystems matrices such that a switched system under consideration is stabilizable.

Given a set of linear unstable subsystems, the problem of deciding whether (or not) there exists a switching signal that stabilizes the resulting switched system, in general, belongs to the class of NP-hard problems; see [18] and [20] for results and discussions. In the setting of unrestricted switching, necessary and sufficient conditions for this problem are proposed only recently in [4].[1] However, verifying the condition of [4] involves checking the containment of a set in the union of other sets and hence, possesses inherent computational complexity. The existing sufficient conditions primarily rely on the so-called min-switching signals [15]. The subsystems matrices are required to satisfy a set of bilinear matrix inequalities (BMIs), e.g., Lyapunov–Metzler inequalities [6] or S-procedure characterization [7] or their generalized versions [3], [14], for stability under these switching signals. In [19] the existence of min-switching signals is claimed to be necessary and sufficient for exponential stabilizability of a switched system. While the above class of results depends on state-dependent switching signals, in [3] stability of a switched system is also addressed under purely time-dependent periodic switching signals based on the satisfaction of a set of linear matrix inequalities (LMIs). The proposed LMI conditions are equivalent to the generalized versions of Lyapunov–Metzler inequalities and is implied by the classical Lyapunov–Metzler inequalities [3]. Stabilization of a switched linear system whose switching signals are restricted by the language of a nondeterministic finite automaton [9] is addressed recently in [5]. The authors present an algorithm to design stabilizing state-feedback switching signals and show using geometry of certain sets that the termination of this algorithm is a necessary and sufficient condition for recurrent stabilizability, which in turn is a sufficient condition for stabilizability of a switched system.

In this article we focus on stabilizability of switched systems under purely time-dependent switching signals that are not restricted to periodic constructions and transcend beyond the regime of Lyapunov functions and/or matrix inequalities based conditions. We assume that there exist two subsystems that form a Schur stable combination including values from the admissible dwell time interval. If switches between these two subsystems are unrestricted, then the switched system under consideration admits a stabilizing periodic switching signal. We address the general setting where switches between the two subsystems forming a stable combination may be restricted. Toward this end, we associate a directed graph with the switched system: the vertices of this graph are the indices of the subsystems and the directed edges correspond to the admissible switches between the subsystems. Clearly, a path from a source vertex to a destination vertex on this graph implies that the set of admissible switches allows us to reach the destination subsystem from the source subsystem. We show that if the underlying directed graph of a switched system admits paths between those subsystems that satisfy certain conditions involving the following components: (i) the rate of decay of the Schur stable combination, (ii) upper bounds on the Euclidean norms of the (matrix) commutators of certain products of the subsystems matrices that appear in the paths, and (iii) certain scalars capturing the properties of the subsystems matrices, then the switched system under consideration is stabilizable under restricted switching. In case of multiple favorable paths, our construction of stabilizing switching signals is nonperiodic, while with unique choice of these paths, we construct stabilizing periodic switching signals. In both cases, the stabilizing switching signals activate the subsystems that appear in the paths under consideration, in different patterns, and dwell on them for admissible durations of time. The settings where switches between the subsystems forming a Schur stable combination are partially restricted or unrestricted fall as special cases of our results.

Our stability conditions involve scalar inequalities and are, therefore, numerically easier to verify compared to the existing matrix inequalities

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[1]By "unrestricted switching," we mean that the switches between the subsystems and the dwell times on the subsystems are unrestricted, unlike the setting of this article.

based conditions. The use of upper bounds on Euclidean norm of matrix commutators adds an inherent robustness to our stabilizability conditions in the sense that if the elements of the subsystems matrices are only partially known and/or they evolve over time in a manner such that the matrices are not "too far" from a set of matrices for which the products under consideration commute, then the switched system continues to be stable under our switching signals. To the best of our knowledge, this is the first instance where a blend of directed graphs and matrix commutators is employed to address stability of switched systems with all unstable subsystems under restricted switching.

The remainder of this article is organized as follows. In Section II, we formulate the problem under consideration. A set of preliminaries for our results are described in Section III. Our results appear in Section IV. We present a numerical example in Section V and conclude in Section VI with a brief discussion of future research directions.

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\|\cdot\|$ denotes the Euclidean norm (respectively, induced matrix norm) of a vector (respectively, a matrix). For a matrix $\mathcal{M}$, given by a product of matrices M_k's, $|\mathcal{M}|$ denotes the length of the product, i.e., the number of matrices that appear in $\mathcal{M}$, counting repetitions. The d-dimensional 0-matrix is represented as $0_{d\times d}$.

II. Problem Statement

We consider a family of systems

$$x(t+1) = A_\ell x(t),\ \ x(0) = x_0,\ \ \ell \in \mathcal{Q},\ \ t \in \mathbb{N}_0, \tag{1}$$

where $x(t) \in \mathbb{R}^d$ is the vector of states at time t, $\mathcal{Q} = \{1, 2, \dots, N\}$ is an index set, and $A_\ell \in \mathbb{R}^{d\times d}$, $\ell \in \mathcal{Q}$ are constant unstable matrices.[2] Let $\sigma : \mathbb{N}_0 \to \mathcal{Q}$ be a switching signal that specifies at every time t, the index of the active subsystem, i.e., the dynamics from (1) that is being followed at t. A switched system [15, Section 1.1.2] generated by the family of systems (1) and a switching signal σ is given by

$$x(t+1) = A_{\sigma(t)} x(t),\ \ x(0) = x_0,\ \ t \in \mathbb{N}_0. \tag{2}$$

Let $E(\mathcal{Q})$ be the set of all ordered pairs (k, ℓ) such that a switch from subsystem k to subsystem ℓ is admissible, $k, \ell \in \mathcal{Q}$, $k \neq \ell$, and δ_ℓ and Δ_ℓ denote the admissible minimum and maximum dwell times on each subsystem $\ell \in \mathcal{Q}$, respectively, $0 < \delta_\ell < \Delta_\ell$, $\delta_\ell, \Delta_\ell \in \mathbb{N}$. Let $0 =: \tau_0 < \tau_1 < \cdots$ be the *switching instants*; these are the points in time where σ jumps. We call a switching signal σ *admissible* if it satisfies the following two conditions: $(\sigma(\tau_h), \sigma(\tau_{h+1})) \in E(\mathcal{Q})$ and $\delta_{\sigma(\tau_h)} \le \tau_{h+1} - \tau_h \le \Delta_{\sigma(\tau_h)}$, $h = 0, 1, 2, \dots$. Let $\mathcal{S}$ denote the set of all admissible switching signals. The solution to (2) is given by $x(t) = A_{\sigma(t-1)} \dots A_{\sigma(1)} A_{\sigma(0)} x_0,\ \ t \in \mathbb{N}$ where we have suppressed the dependence of x on $\sigma \in \mathcal{S}$ for notational simplicity.

We consider the set of admissible switches, $E(\mathcal{Q})$, and the admissible dwell times on the subsystems, δ_ℓ and Δ_ℓ, $\ell \in \mathcal{Q}$ to be "given," and our objective is to "identify" conditions on the unstable subsystems matrices, A_ℓ, $\ell \in \mathcal{Q}$, such that $\mathcal{S}$ admits a stabilizing switching signal.

Definition 1 [1, Section 2]: The switched system (2) is *GES under a switching signal* $\sigma \in \mathcal{S}$ if there exist positive numbers c and λ such that for arbitrary choices of the initial condition x_0, the following inequality holds:

$$\|x(t)\| \le c e^{-\lambda t} \|x_0\| \ \text{ for all } \ t \in \mathbb{N}, \tag{3}$$

where $\|v\|$ denotes the Euclidean norm of a vector v.

Definition 2: The switched system (2) is called *stabilizable* if there exists a switching signal $\sigma \in \mathcal{S}$ under which (2) is GES.

[2] A matrix $\mathcal{M} \in \mathbb{R}^{d\times d}$ is Schur stable if all its eigenvalues are inside the open unit disk. $\mathcal{M}$ is unstable if it is not Schur stable.

We will solve the following problem.

Problem 1: Given the set of admissible switches between the subsystems, $E(\mathcal{Q})$, and the admissible minimum and maximum dwell times on the subsystems, δ_ℓ and Δ_ℓ, $\ell \in \mathcal{Q}$, find conditions on the matrices A_ℓ, $\ell \in \mathcal{Q}$, such that the switched system (2) is stabilizable.

Remark 1: Note that our problem setting differs from the classical problem of stability under dwell time switching [15, Ch. 3]. Indeed, instead of identifying switches between the subsystems and minimum (respectively, maximum) dwell times on subsystems such that a switched system is stable, we consider the admissible switches and admissible minimum and maximum dwell times on the subsystems to be "given" and aim to identify families of systems that admit stabilizing switching signals obeying the given restrictions. The restrictions on a switching signal considered in this article are inherent to many engineering applications. A distinction between admissible and inadmissible transitions captures situations where switches between certain subsystems may be prohibited. For instance, in automobile gear switching, certain switching orders (e.g., from first gear to second gear, etc.) are followed [16, Section III]. A restriction on minimum dwell time on subsystems arises in situations where actuator saturations may prevent switching frequency beyond a certain limit. Also, in order to switch from one component to another, a system may undergo certain operations of non-negligible durations [8, Section I] resulting in a minimum dwell time restriction on subsystems. Restricted maximum dwell time is natural to systems whose components need regular maintenance or replacements, e.g., aircraft carriers, MEMS systems, etc. Moreover, systems dependent on diurnal or seasonal changes, e.g., components of an electricity grid, have inherent restrictions on admissible dwell time.

At this point, it is worth highlighting that Problem 1 does not admit a trivial solution even when (1) admits stable subsystems. Indeed, in the presence of restrictions on admissible maximum dwell times, a constant switching signal on a stable subsystem is not admissible, and switching between stable subsystems does not necessarily preserve stability of a switched system. Prior to presenting our solution to Problem 1, we catalog a set of preliminaries.

III. Preliminaries

We will consider the following.

Assumption 1: There exist $i, j \in \mathcal{Q}$, $p \in \{\delta_i, \delta_i + 1, \dots, \Delta_i\}$ and $q \in \{\delta_j, \delta_j + 1, \dots, \Delta_j\}$ such that the matrix $\mathcal{A} := A_i^p A_j^q$ is Schur stable.

It follows from the properties of Schur stable matrices that

Fact 1: There exists $m \in \mathbb{N}$ such that the following condition holds:

$$\|\mathcal{A}^m\| \le \rho < 1. \tag{4}$$

The motivation behind Assumption 1 is the use of purely time-dependent switching signals for the stabilization of (2). Since we do not utilize (or assume access to) information about system state, $x(t)$, $t \in \mathbb{N}_0$, the existence of a stable combination formed by two subsystems matrices is useful in our analysis. Let switches between the subsystems i and j be unrestricted, i.e., (i, j) and $(j, i) \in E(\mathcal{Q})$. Consider a switching signal σ that activates the subsystems j and i alternatively and dwells on them for q and p units of time, respectively, i.e., σ satisfies $\sigma(\tau_0) = j,\ \ \sigma(\tau_1) = i,\ \ \sigma(\tau_2) = j,\ \ \sigma(\tau_3) = i, \dots$ with

$$\tau_{h+1} - \tau_h = \begin{cases} q, & \text{if} \quad \sigma(\tau_h) = j, \\ p, & \text{if} \quad \sigma(\tau_h) = i. \end{cases} \quad h = 0, 1, 2, \dots.$$

Since (j, i) and $(i, j) \in E(\mathcal{Q})$ and $p \in \{\delta_i, \delta_i + 1, \dots, \Delta_i\}$, $q \in \{\delta_j, \delta_j + 1, \dots, \Delta_j\}$, it follows that $\sigma \in \mathcal{S}$. GES of (2) under σ is immediate from Fact 1. Indeed, we are dealing with stability of the difference equation $x(\bar{t}+1) = \mathcal{A}x(\bar{t}),\ \ x(0) = x_0$, where $\mathcal{A}$ is Schur stable and $\bar{t} : t = (p+q) : 1$.

However, since admissible switches between the subsystems are restricted, Assumption 1 does not lead to a trivial solution to Problem 1 if either (i,j) or (j,i) or both (i,j) and (j,i) are not elements of $E(\mathcal{Q})$. In the sequel, we aim for a general solution to Problem 1, which works as long as it is possible to reach subsystem i from subsystem j and vice versa, through a set of favorable subsystems. Clearly, we need to take into account the properties of the subsystems matrices that appear between i and j in this case. The settings where the switches between the subsystems i and j are partially restricted (either (i,j) or $(j,i) \in E(\mathcal{Q})$) and unrestricted (both (i,j) and $(j,i) \in E(\mathcal{Q})$) are special cases of our results.

Let $\overline{m} \in \mathbb{N}$ be such that

$$\overline{m} = \begin{cases} 1, & \text{if } m \in \{1,2\}, \\ 2, & \text{if } m \in \{3,4,5\}, \\ 3, & \text{if } m \in \{6,7,8,9\}, \\ 4, & \text{if } m \in \{10,11,12,13,14\}, \\ \vdots, & \end{cases} \tag{5}$$

where $m \in \mathbb{N}$ is as described in (4). Let

$$M := \max_{\ell \in \mathcal{Q}} \|A_\ell\| . \tag{6}$$

We associate a directed graph $G(\mathcal{Q}, E(\mathcal{Q}))$ with the family of systems (1) in the following manner: the set of vertices of G is the index set $\mathcal{Q}$ and the set of edges of G is the set of admissible switches $E(\mathcal{Q})$.

Definition 3: Fix $u, v \in \mathcal{Q}$. A $u \to v$ *path on* $G(\mathcal{Q}, E(\mathcal{Q}))$ is a finite alternating sequence, $P_G^{u\to v} = w_0, (w_0, w_1), w_1, \dots, w_{n-1}, (w_{n-1}, w_n), w_n$, where $w_k \in \mathcal{Q}$, $k = 0,1,\dots,n$, $(w_k, w_{k+1}) \in E(\mathcal{Q})$, $k = 0,1,\dots,n-1$, $w_0 = u$, $w_n = v$, $w_k \neq u$, $w_k \neq v$, $k = 1,2,\dots,n-1$.

In the sequel, we will employ $\gamma_{u\to v}$ to denote the sum $\sum_{k=1}^{n-1} \delta_{w_k}$, where δ_{w_k} is the admissible minimum dwell time on the subsystem (vertex) $w_k \in \mathcal{Q}$, $k = 1,2,\dots,n-1$.

Definition 4: Two paths, $P_G^{u\overset{1}{\to}v}$ and $P_G^{u\overset{2}{\to}v}$, on $G(\mathcal{Q}, E(\mathcal{Q}))$ are *distinct*, if there exists $w \in \mathcal{Q}$ such that w appears either in $P_G^{u\overset{1}{\to}v}$ or in $P_G^{u\overset{2}{\to}v}$ but not in both.[3] We use $P_G^{u\overset{1}{\to}v} \neq P_G^{u\overset{2}{\to}v}$ to denote that they are distinct and $P_G^{u\overset{1}{\to}v} = P_G^{u\overset{2}{\to}v}$ to denote that they are not.

Definition 5: For a $u \to v$ path on $G(\mathcal{Q}, E(\mathcal{Q}))$, we define the following set of (matrix) commutators of products of matrices:

$$F^a_{u\to v,\ell} := A^a_\ell \left(A^{\delta_{w_{n-1}}}_{w_{n-1}} \dots A^{\delta_{w_1}}_{w_1} \right) - \left(A^{\delta_{w_{n-1}}}_{w_{n-1}} \dots A^{\delta_{w_1}}_{w_1} \right) A^a_\ell, \tag{7}$$

where $\ell \in \mathcal{Q}$, $w_k \neq \ell$, $k = 1,2,\dots,n-1$, and $a \in \mathbb{N}$.

We note that for a $u \to v$ path, $P_G^{u\to v} = u, (u,v), v$, on $G(\mathcal{Q}, E(\mathcal{Q}))$, $F^a_{u\to v,\ell} = 0_{d\times d}$ for any $\ell \in \mathcal{Q}$ and $a \in \mathbb{N}$.

We are now in a position to present our solutions to Problem 1.

IV. Results

Let $i, j \in \mathcal{Q}$ satisfy Assumption 1 and $(P_G^{j\overset{r}{\to}i}, P_G^{i\overset{r}{\to}j})$, $r = 1,2$, be two pairs of $j \to i$ and $i \to j$ paths on $G(\mathcal{Q}, E(\mathcal{Q}))$. We define

$$\xi_{j\overset{1}{\to}i,i} = \gamma_{j\overset{1}{\to}i}(\overline{m}-1) + \gamma_{i\overset{1}{\to}j}\overline{m} + \left(\gamma_{j\overset{2}{\to}i} + \gamma_{i\overset{2}{\to}j}\right)(m-\overline{m}) + p(m-1) + qm,$$

$$\xi_{j\overset{2}{\to}i,i} = \left(\gamma_{j\overset{1}{\to}i} + \gamma_{i\overset{1}{\to}j}\right)\overline{m} + \gamma_{j\overset{2}{\to}i}(m-\overline{m}-1) + \gamma_{i\overset{2}{\to}j}(m-\overline{m}) + p(m-1) + qm,$$

$$\xi_{j\overset{1}{\to}i,j} = \gamma_{j\overset{1}{\to}i}(\overline{m}-1) + \gamma_{i\overset{1}{\to}j}\overline{m} + \left(\gamma_{j\overset{2}{\to}i} + \gamma_{i\overset{2}{\to}j}\right)(m-\overline{m}) + pm + q(m-1),$$

$$\xi_{j\overset{2}{\to}i,j} = \left(\gamma_{j\overset{1}{\to}i} + \gamma_{i\overset{1}{\to}j}\right)\overline{m} + \gamma_{j\overset{2}{\to}i}(m-\overline{m}-1) + \gamma_{i\overset{2}{\to}j}(m-\overline{m}) + pm + q(m-1),$$

$$\xi_{i\overset{1}{\to}j,i} = \gamma_{j\overset{1}{\to}i}\overline{m} + \gamma_{i\overset{1}{\to}j}(\overline{m}-1) + \left(\gamma_{j\overset{2}{\to}i} + \gamma_{i\overset{2}{\to}j}\right)(m-\overline{m}) + p(m-1) + qm,$$

$$\xi_{i\overset{2}{\to}j,i} = \left(\gamma_{j\overset{1}{\to}i} + \gamma_{i\overset{1}{\to}j}\right)\overline{m} + \gamma_{j\overset{2}{\to}i}(m-\overline{m}) + \gamma_{i\overset{2}{\to}j}(m-\overline{m}-1) + p(m-1) + qm,$$

$$\xi_{i\overset{1}{\to}j,j} = \gamma_{j\overset{1}{\to}i}\overline{m} + \gamma_{i\overset{1}{\to}j}(\overline{m}-1) + \left(\gamma_{j\overset{2}{\to}i} + \gamma_{i\overset{2}{\to}j}\right)(m-\overline{m}) + pm + q(m-1),$$

$$\xi_{i\overset{2}{\to}j,j} = \left(\gamma_{j\overset{1}{\to}i} + \gamma_{i\overset{1}{\to}j}\right)\overline{m} + \gamma_{j\overset{2}{\to}i}(m-\overline{m}) + \gamma_{i\overset{2}{\to}j}(m-\overline{m}-1) + pm + q(m-1),$$

$$\xi_{j\overset{1}{\to}i\overset{1}{\to}j} = \left(\left(\gamma_{j\overset{1}{\to}i} + \gamma_{i\overset{1}{\to}j}\right) + p + q\right)\overline{m},$$

$$\xi_{j\overset{2}{\to}i\overset{2}{\to}j} = \left(\left(\gamma_{j\overset{2}{\to}i} + \gamma_{i\overset{2}{\to}j}\right) + p + q\right)(m-\overline{m}),$$

where m, $\overline{m}$, p, and q are as described in (4), (5), and Assumption 1, respectively.

Theorem 1: Let $i, j \in \mathcal{Q}$ satisfy Assumption 1 and λ be an arbitrary positive number satisfying

$$\rho e^{\lambda m} < 1, \tag{8}$$

where ρ and m are as described in (4). Suppose that $G(\mathcal{Q}, E(\mathcal{Q}))$ admits two pairs of $j \to i$ and $i \to j$ paths, $(P_G^{j\overset{r}{\to}i}, P_G^{i\overset{r}{\to}j})$, $r = 1,2$, such that the following conditions hold:

i) either a) $P_G^{j\overset{1}{\to}i} \neq P_G^{j\overset{2}{\to}i}$ or b) $P_G^{i\overset{1}{\to}j} \neq P_G^{i\overset{2}{\to}j}$ or c) $P_G^{j\overset{1}{\to}i} \neq P_G^{j\overset{2}{\to}i}$ and $P_G^{i\overset{1}{\to}j} \neq P_G^{i\overset{2}{\to}j}$, and

ii) there exist scalars $\varepsilon_{j\overset{r}{\to}i,i}$, $\varepsilon_{j\overset{r}{\to}i,j}$, $\varepsilon_{i\overset{r}{\to}j,i}$, $\varepsilon_{i\overset{r}{\to}j,j}$, $r = 1,2$, small enough, such that

$$\left\| F^p_{j\overset{r}{\to}i,i} \right\| \le \varepsilon_{j\overset{r}{\to}i,i}, \left\| F^q_{j\overset{r}{\to}i,j} \right\| \le \varepsilon_{j\overset{r}{\to}i,j}, \ \ r = 1,2, \tag{9}$$

$$\left\| F^p_{i\overset{r}{\to}j,i} \right\| \le \varepsilon_{i\overset{r}{\to}j,i}, \left\| F^q_{i\overset{r}{\to}j,j} \right\| \le \varepsilon_{i\overset{r}{\to}j,j}, \ \ r = 1,2, \tag{10}$$

[3] Note that two distinct paths on $G(\mathcal{Q}, E(\mathcal{Q}))$ may or may not have the same number of vertices that appear in the paths excluding the initial and the final vertices.

and

$$\rho e^{\lambda m} + \Bigg(\frac{\overline{m}(\overline{m}-1)}{2} M^{\xi_{j\xrightarrow{1}i,i}} \varepsilon_{j\xrightarrow{1}i,i} + \frac{\overline{m}(\overline{m}+1)}{2} \times \Big(M^{\xi_{j\xrightarrow{1}i,j}}\varepsilon_{j\xrightarrow{1}i,j} + M^{\xi_{i\xrightarrow{1}j,i}}\varepsilon_{i\xrightarrow{1}j,i} + M^{\xi_{i\xrightarrow{1}j,j}}\varepsilon_{i\xrightarrow{1}j,j}\Big) + \frac{m(m-1)-\overline{m}(\overline{m}-1)}{2} M^{\xi_{j\xrightarrow{2}i,i}}\varepsilon_{j\xrightarrow{2}i,i} + \frac{m(m+1)-\overline{m}(\overline{m}+1)}{2}\Big(M^{\xi_{j\xrightarrow{2}i,j}}\varepsilon_{j\xrightarrow{2}i,j} + M^{\xi_{i\xrightarrow{2}j,i}}\varepsilon_{i\xrightarrow{2}j,i} + M^{\xi_{i\xrightarrow{2}j,j}}\varepsilon_{i\xrightarrow{2}j,j}\Big)\Bigg) \times e^{\lambda\left(\xi_{j\xrightarrow{1}i\xrightarrow{1}j} + \xi_{j\xrightarrow{2}i\xrightarrow{2}j}\right)} \le 1. \tag{11}$$

Then there exists a nonperiodic switching signal $\sigma \in \mathcal{S}$ under which the switched system (2) is GES.

Proof: Let

$$P_G^{j\xrightarrow{1}i} = w_0^{(1)}, (w_0^{(1)}, w_1^{(1)}), w_1^{(1)}, \ldots, w_{\ell_1-1}^{(1)}, (w_{\ell_1-1}^{(1)}, w_{\ell_1}^{(1)}), w_{\ell_1}^{(1)},$$

$$P_G^{j\xrightarrow{2}i} = w_0^{(2)}, (w_0^{(2)}, w_1^{(2)}), w_1^{(2)}, \ldots, w_{\ell_2-1}^{(2)}, (w_{\ell_2-1}^{(2)}, w_{\ell_2}^{(2)}), w_{\ell_2}^{(2)},$$

$$P_G^{i\xrightarrow{1}j} = w_{\ell_1}^{(1)}, (w_{\ell_1}^{(1)}, w_{\ell_1+1}^{(1)}), w_{\ell_1+1}^{(1)}, \ldots, w_{n_1-1}^{(1)}, (w_{n_1-1}^{(1)}, w_{n_1}^{(1)}), w_{n_1}^{(1)},$$

$$P_G^{i\xrightarrow{2}j} = w_{\ell_2}^{(2)}, (w_{\ell_2}^{(2)}, w_{\ell_2+1}^{(2)}), w_{\ell_2+1}^{(2)}, \ldots, w_{n_2-1}^{(2)}, (w_{n_2-1}^{(2)}, w_{n_2}^{(2)}), w_{n_2}^{(2)}$$

satisfy conditions 1) and 2). Here, $w_0^{(1)} = w_0^{(2)} = w_{n_1}^{(1)} = w_{n_1}^{(2)} = j$ and $w_{\ell_1}^{(1)} = w_{\ell_2}^{(2)} = i$.

Consider a switching signal σ that activates the sequence of subsystems $w_0^{(1)}, w_1^{(1)}, \ldots, w_{\ell_1}^{(1)}, w_{\ell_1+1}^{(1)}, \ldots, w_{n_1-1}^{(1)}$ followed by s-many instances of the sequence of subsystems $w_0^{(2)}, w_1^{(2)}, \ldots, w_{\ell_2}^{(2)}, w_{\ell_2+1}^{(2)}, \ldots, w_{n_2-1}^{(2)}$ repeatedly, $s = 1, 2, 3, \ldots$ with dwell times p, q, and δ_k units of time on the subsystems i, j, and $k \in \mathcal{Q} \setminus \{i, j\}$, respectively.

We first show that $\sigma \in \mathcal{S}$. By construction of σ, its values at two consecutive switching instants τ_h and τ_{h+1} are two vertices of $G(\mathcal{Q}, E(\mathcal{Q}))$ such that there is a directed edge from vertex (subsystem) $\sigma(\tau_h)$ to vertex (subsystem) $\sigma(\tau_{h+1})$, $h = 0, 1, 2, \ldots$. Moreover,

$$\tau_{h+1} - \tau_h = \begin{cases} p \in \{\delta_i, \delta_i + 1, \ldots, \Delta_i\}, & \text{if } \sigma(\tau_h) = i, \\ q \in \{\delta_j, \delta_j + 1, \ldots, \Delta_j\}, & \text{if } \sigma(\tau_h) = j, \\ \delta_{\sigma(\tau_h)}, & \text{if } \sigma(\tau_h) \in \mathcal{Q} \setminus \{i, j\}, \end{cases}$$

$h = 0, 1, 2, \ldots$. Thus, $\sigma \in \mathcal{S}$. Also, by construction, σ is nonperiodic.

We next show that (2) is GES under σ. Let W be the matrix product corresponding to σ, defined as $W = \cdots A_{\sigma(2)} A_{\sigma(1)} A_{\sigma(0)}$. We let $\overline{W}$ denote an initial segment of W. The length of $\overline{W}$ is denoted by $|\overline{W}|$. Then the condition (3) for GES of (2) under σ can be written equivalently as [1, Section 2]: for every initial segment $\overline{W}$ of W, we have

$$\|\overline{W}\| \le c e^{-\lambda |\overline{W}|}. \tag{12}$$

We apply mathematical induction on the length of an initial segment $\overline{W}$ to establish (12).

1) Induction basis: Pick c large enough so that (12) holds for $\overline{W}$ satisfying $|\overline{W}| \ge \xi_{j\xrightarrow{1}i\xrightarrow{1}j} + \xi_{j\xrightarrow{2}i\xrightarrow{2}j}$.

2) Induction hypothesis: Let $|\overline{W}| \ge \xi_{j\xrightarrow{1}i\xrightarrow{1}j} + \xi_{j\xrightarrow{2}i\xrightarrow{2}j} + 1$ and assume that (12) is proved for all products of length less than $|\overline{W}|$.

3) Induction step: Let $\overline{W} = LR$, where $|L| = \xi_{j\xrightarrow{1}i\xrightarrow{1}j} + \xi_{j\xrightarrow{2}i\xrightarrow{2}j}$. It follows by construction of σ that L contains exactly m-many A_i^p and m-many A_j^q. Let us rewrite L as $L = \mathcal{A}^m L_1 + L_2$, where

$$|L_1| = \left(\gamma_{j\xrightarrow{1}i} + \gamma_{i\xrightarrow{1}j}\right)\overline{m} + \left(\gamma_{j\xrightarrow{2}i} + \gamma_{i\xrightarrow{2}j}\right)(m - \overline{m})$$

and L_2 contains the following:

a) $\frac{\overline{m}(\overline{m}-1)}{2}$-many terms of length $\xi_{j\xrightarrow{1}i,i} + 1$ with $\xi_{j\xrightarrow{1}i,i}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^p_{j\xrightarrow{1}i,i}$,

b) $\frac{m(m-1)-\overline{m}(\overline{m}-1)}{2}$-many terms of length $\xi_{j\xrightarrow{2}i,i} + 1$ with $\xi_{j\xrightarrow{2}i,i}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^p_{j\xrightarrow{2}i,i}$,

c) $\frac{\overline{m}(\overline{m}+1)}{2}$-many terms of length $\xi_{j\xrightarrow{1}i,j} + 1$ with $\xi_{j\xrightarrow{1}i,j}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^q_{j\xrightarrow{1}i,j}$,

d) $\frac{m(m+1)-\overline{m}(\overline{m}+1)}{2}$-many terms of length $\xi_{j\xrightarrow{2}i,j} + 1$ with $\xi_{j\xrightarrow{2}i,j}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^q_{j\xrightarrow{2}i,j}$,

e) $\frac{\overline{m}(\overline{m}+1)}{2}$-many terms of length $\xi_{i\xrightarrow{1}j,i} + 1$ with $\xi_{i\xrightarrow{1}j,i}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^p_{i\xrightarrow{1}j,i}$,

f) $\frac{m(m+1)-\overline{m}(\overline{m}+1)}{2}$-many terms of length $\xi_{i\xrightarrow{2}j,i} + 1$ with $\xi_{i\xrightarrow{2}j,i}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^p_{i\xrightarrow{2}j,i}$,

g) $\frac{\overline{m}(\overline{m}+1)}{2}$-many terms of length $\xi_{i\xrightarrow{1}j,j} + 1$ with $\xi_{i\xrightarrow{1}j,j}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^q_{i\xrightarrow{1}j,j}$,

h) $\frac{m(m+1)-\overline{m}(\overline{m}+1)}{2}$-many terms of length $\xi_{i\xrightarrow{2}j,j} + 1$ with $\xi_{i\xrightarrow{2}j,j}$-many A_ℓ, $\ell \in \mathcal{Q}$, and 1 $F^q_{i\xrightarrow{2}j,j}$.

(Consider, for example,

$$P_G^{j\xrightarrow{1}i} = j, (j,3), 3, (3,i), i, \quad P_G^{j\xrightarrow{2}i} = j, (j,4), 4, (4,5), 5, (5,i), i,$$

$$P_G^{i\xrightarrow{1}j} = i, (i,1), 1, (1,2), 2, (2,j), j, \quad P_G^{i\xrightarrow{2}j} = i, (i,6), 6, (6,j), j.$$

Suppose that $m = 2$. Let $L = A_6^{\delta_6} A_i^p A_5^{\delta_5} A_4^{\delta_4} A_j^q A_2^{\delta_2} A_1^{\delta_1} A_i^p A_3^{\delta_3} A_j^q$. It can be rewritten as

$$\underbrace{A_i^p A_j^q A_i^p A_j^q}_{\mathcal{A}^m} A_6^{\delta_6} A_5^{\delta_5} A_4^{\delta_4} A_2^{\delta_2} A_1^{\delta_1} A_3^{\delta_3}$$
$$- A_i^p A_j^q A_i^p F^q_{i\xrightarrow{2}j,j} A_5^{\delta_5} A_4^{\delta_4} A_2^{\delta_2} A_1^{\delta_1} A_3^{\delta_3}$$
$$- A_i^p A_j^q A_i^p A_6^{\delta_6} F^q_{j\xrightarrow{2}i,j} A_2^{\delta_2} A_1^{\delta_1} A_3^{\delta_3}$$
$$- A_i^p A_j^q F^p_{j\xrightarrow{2}i,i} A_5^{\delta_5} A_4^{\delta_4} A_2^{\delta_2} A_1^{\delta_1} A_3^{\delta_3} A_j^q$$
$$- A_i^p A_j^q A_i^p A_6^{\delta_6} A_5^{\delta_5} A_4^{\delta_4} F^q_{i\xrightarrow{1}j,j} A_3^{\delta_3}$$
$$- A_i^p A_j^q A_i^p A_6^{\delta_6} A_5^{\delta_5} A_4^{\delta_4} A_2^{\delta_2} A_1^{\delta_1} F^q_{j\xrightarrow{1}i,j}$$
$$- A_i^p A_j^q A_6^{\delta_6} F^p_{j\xrightarrow{2}i,i} A_2^{\delta_2} A_1^{\delta_1} A_3^{\delta_3} A_j^q$$
$$- A_i^p A_j^q A_6^{\delta_6} A_5^{\delta_5} A_4^{\delta_4} F^p_{i\xrightarrow{1}j,i} A_3^{\delta_3} A_j^q$$
$$- A_i^p F^p_{i\xrightarrow{2}j,j} A_5^{\delta_5} A_4^{\delta_4} A_2^{\delta_2} A_1^{\delta_1} A_i^p A_3^{\delta_3} A_j^q$$
$$- A_i^p A_6^{\delta_6} F^p_{j\xrightarrow{2}i,j} A_2^{\delta_2} A_1^{\delta_1} A_i^p A_3^{\delta_3} A_j^q$$

$$- F^p_{i\xrightarrow{2}j,i} A_5^{\delta_5} A_4^{\delta_4} A_j^q A_2^{\delta_2} A_1^{\delta_1} A_i^p A_3^{\delta_3} A_j^{\delta_j}.\Big)$$

Now, from the sub-multiplicativity and sub-additivity properties of the induced norm, we have

$$\begin{aligned}
\|\overline{W}\| &= \|(\mathcal{A}^m L_1 + L_2)R\| \leq \|\mathcal{A}^m\| \, \|L_1 R\| + \|L_2\| \, \|R\| \\
&\leq \rho c e^{-\lambda(|\overline{W}|-m)} + \Bigg(\frac{\overline{m}(\overline{m}-1)}{2} M^{\xi_{j\xrightarrow{1}i,i}} \varepsilon_{j\xrightarrow{1}i,i} \\
&+ \frac{m(m-1) - \overline{m}(\overline{m}-1)}{2} M^{\xi_{j\xrightarrow{2}i,i}} \varepsilon_{j\xrightarrow{2}i,i} \\
&+ \frac{\overline{m}(\overline{m}+1)}{2} M^{\xi_{j\xrightarrow{1}i,j}} \varepsilon_{j\xrightarrow{1}i,j} \\
&+ \frac{m(m+1) - \overline{m}(\overline{m}+1)}{2} M^{\xi_{j\xrightarrow{2}i,j}} \varepsilon_{j\xrightarrow{2}i,j} \\
&+ \frac{\overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{1}j,i}} \varepsilon_{i\xrightarrow{1}j,i} \\
&+ \frac{m(m+1) - \overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{2}j,i}} \varepsilon_{i\xrightarrow{2}j,i} \\
&+ \frac{\overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{1}j,j}} \varepsilon_{i\xrightarrow{1}j,j} \\
&+ \frac{m(m+1) - \overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{2}j,j}} \varepsilon_{i\xrightarrow{2}j,j} \Bigg) \\
&\times c e^{-\lambda\left(|\overline{W}| - \left(\xi_{j\xrightarrow{1}i\xrightarrow{1}j} + \xi_{j\xrightarrow{2}i\xrightarrow{2}j}\right)\right)} \\
&= c e^{-\lambda|\overline{W}|} \Bigg(\rho e^{\lambda m} + \Bigg(\frac{\overline{m}(\overline{m}-1)}{2} M^{\xi_{j\xrightarrow{1}i,i}} \varepsilon_{j\xrightarrow{1}i,i} \\
&+ \frac{m(m-1) - \overline{m}(\overline{m}-1)}{2} M^{\xi_{j\xrightarrow{2}i,i}} \varepsilon_{j\xrightarrow{2}i,i} \\
&+ \frac{\overline{m}(\overline{m}+1)}{2} M^{\xi_{j\xrightarrow{1}i,j}} \varepsilon_{j\xrightarrow{1}i,j} \\
&+ \frac{m(m+1) - \overline{m}(\overline{m}+1)}{2} M^{\xi_{j\xrightarrow{2}i,j}} \varepsilon_{j\xrightarrow{2}i,j} \\
&+ \frac{\overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{1}j,i}} \varepsilon_{i\xrightarrow{1}j,i} \\
&+ \frac{m(m+1) - \overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{2}j,i}} \varepsilon_{i\xrightarrow{2}j,i} \\
&+ \frac{\overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{1}j,j}} \varepsilon_{i\xrightarrow{1}j,j} \\
&+ \frac{m(m+1) - \overline{m}(\overline{m}+1)}{2} M^{\xi_{i\xrightarrow{2}j,j}} \varepsilon_{i\xrightarrow{2}j,j} \Bigg) \times e^{\lambda\left(\xi_{j\xrightarrow{1}i\xrightarrow{1}j} + \xi_{j\xrightarrow{2}i\xrightarrow{2}j}\right)} \Bigg).
\end{aligned} \tag{13}$$

In the above inequality, the upper bounds on $\|L_1 R\|$ and $\|R\|$ are obtained from the relations $|\overline{W}| = |\mathcal{A}^m| + |L_1 R|$ and $|\overline{W}| = |L| + |R|$, respectively. Applying (11) to (13) leads to (12). Consequently, (2) is GES under σ.

This completes our proof of Theorem 1. ■

Given the admissible switches between the subsystems, $E(\mathcal{Q})$, and the admissible minimum and maximum dwell times on the subsystems, δ_ℓ and Δ_ℓ, $\ell \in \mathcal{Q}$, Theorem 1 provides sufficient conditions on the subsystems matrices, A_ℓ, $\ell \in \mathcal{Q}$, such that there exists a nonperiodic switching signal $\sigma \in \mathcal{S}$ under which the switched system (2) is GES. Since $\rho < 1$, it is always possible to find a $\lambda > 0$ (could be small) such that condition (8) holds. Suppose that the underlying directed graph $G(\mathcal{Q}, E(\mathcal{Q}))$ of the switched system (2) admits two pairs of $j \to i$ and $i \to j$ paths, $(P_G^{j\xrightarrow{r}i}, P_G^{i\xrightarrow{r}j})$, $r = 1, 2$, $i, j \in \mathcal{Q}$, where i and j are as described in Assumption 1, for which the Euclidean norms of the commutators $F^p_{j\xrightarrow{r}i,i}$, $F^q_{j\xrightarrow{r}i,j}$, $F^p_{i\xrightarrow{r}j,i}$, and $F^q_{i\xrightarrow{r}j,j}$ are bounded above by small enough scalars $\varepsilon_{j\xrightarrow{r}i,i}$, $\varepsilon_{j\xrightarrow{r}i,j}$, $\varepsilon_{i\xrightarrow{r}j,i}$, and $\varepsilon_{i\xrightarrow{r}j,j}$, respectively, $r = 1, 2$, such that condition (11) holds. Then there exists a $\sigma \in \mathcal{S}$ under which (2) is GES. This switching signal activates the sequence of subsystems $w_0^{(1)}, w_1^{(1)}, \ldots, w_{\ell_1}^{(1)}, w_{\ell_1+1}^{(1)}, \ldots, w_{n_1-1}^{(1)}$ followed by s-many instances of the sequence of subsystems $w_0^{(2)}, w_1^{(2)}, \ldots, w_{\ell_2}^{(2)}, w_{\ell_2+1}^{(2)}, \ldots, w_{n_2-1}^{(2)}$ repeatedly, $s = 1, 2, 3, \ldots$ with dwell times p, q, and δ_k units of time on the subsystems i, j, and $k \in \mathcal{Q} \setminus \{i, j\}$, respectively. Note that Theorem 1 does not require (i, j) and/or $(j, i) \in E(\mathcal{Q})$ for the utilization of stability of the matrix product, $A_i^p A_j^q$, and works as long as it is possible to reach from subsystem j to subsystem i and vice versa through multiple paths that satisfy conditions (9)–(11). Moreover, the settings of partially restricted switching between subsystems i and j, i.e., either $(i, j) \in E(\mathcal{Q})$ or $(j, i) \in E(\mathcal{Q})$ but not both, falls as a special case of Theorem 1 with $P_G^{i\xrightarrow{1}j} = P_G^{i\xrightarrow{2}j} = i, (i, j), j$ and $P_G^{j\xrightarrow{1}i} \neq P_G^{j\xrightarrow{2}i}$, $P_G^{j\xrightarrow{1}i} = P_G^{j\xrightarrow{2}i} = j, (j, i), i$ and $P_G^{i\xrightarrow{1}j} \neq P_G^{i\xrightarrow{2}j}$, respectively.

We now show that it is possible to stabilize (2) even if $G(\mathcal{Q}, E(\mathcal{Q}))$ admits exactly one pair containing a $j \to i$ and an $i \to j$ path, which satisfies a set of scalar inequalities similar to (8)–(10). For a pair of $j \to i$ and $i \to j$ paths, $(P_G^{j\to i}, P_G^{i\to j})$ on $G(\mathcal{Q}, E(\mathcal{Q}))$, we define

$$\begin{aligned}
\zeta_{j\to i,i} &= \gamma_{j\to i}(m-1) + \gamma_{i\to j} m + p(m-1) + qm, \\
\zeta_{j\to i,j} &= \gamma_{j\to i}(m-1) + \gamma_{i\to j} m + pm + q(m-1), \\
\zeta_{i\to j,i} &= \gamma_{j\to i} m + \gamma_{i\to j}(m-1) + p(m-1) + qm, \\
\zeta_{i\to j,j} &= \gamma_{j\to i} m + \gamma_{i\to j}(m-1) + pm + q(m-1), \\
\zeta_{j\to i\to j} &= \Big(\big(\gamma_{j\to i} + \gamma_{i\to j} \big) + p + q \Big) m.
\end{aligned}$$

Theorem 2: Let $i, j \in \mathcal{Q}$ satisfy Assumption 1 and λ be an arbitrary positive number satisfying (8). Suppose that $G(\mathcal{Q}, E(\mathcal{Q}))$ admits a pair of $j \to i$ and $i \to j$ paths, $(P_G^{j\to i}, P_G^{i\to j})$, that satisfies the following condition: There exist scalars $\varepsilon_{j\to i,i}$, $\varepsilon_{j\to i,j}$, $\varepsilon_{i\to j,i}$ and $\varepsilon_{i\to j,j}$, small enough such that

$$\left\|F^p_{j\to i,i}\right\| \leq \varepsilon_{j\to i,i} \quad \text{and} \quad \left\|F^q_{j\to i,j}\right\| \leq \varepsilon_{j\to i,j}, \tag{14}$$

$$\left\|F^p_{i\to j,i}\right\| \leq \varepsilon_{i\to j,i} \quad \text{and} \quad \left\|F^q_{i\to j,j}\right\| \leq \varepsilon_{i\to j,j}, \tag{15}$$

and

$$\rho e^{\lambda m} + \Bigg(\frac{m(m-1)}{2} M^{\zeta_{j\to i,i}} \varepsilon_{j\to i,i} + \frac{m(m+1)}{2} \Big(M^{\zeta_{j\to i,j}} \varepsilon_{j\to i,j} + M^{\zeta_{i\to j,i}} \varepsilon_{i\to j,i} + M^{\zeta_{i\to j,j}} \varepsilon_{i\to j,j} \Big) \Bigg) \times e^{\lambda \zeta_{j\to i\to j}} \leq 1. \tag{16}$$

Then there exists a periodic switching signal $\sigma \in \mathcal{S}$ under which the switched system (2) is GES.

Proof: Let

$$P_G^{j\to i} = w_0, (w_0, w_1), w_1, \ldots, w_{\ell-1}, (w_{\ell-1}, w_\ell), w_\ell$$

$$P_G^{i\to j} = w_\ell, (w_\ell, w_{\ell+1}), w_{\ell+1}, \ldots, w_{n-1}, (w_{n-1}, w_n), w_n$$

satisfy conditions (14)–(16). Here, $w_0 = w_n = j$ and $w_\ell = i$. Consider the switching signal $\sigma \in \mathcal{S}$ constructed in our proof of Theorem 1 with $P_G^{j\overset{1}{\to}i} = P_G^{j\overset{2}{\to}i} = P_G^{j\to i}$ and $P_G^{i\overset{1}{\to}j} = P_G^{i\overset{2}{\to}j} = P_G^{i\to j}$. Clearly, σ is periodic now. GES of (2) under σ follows the set of arguments employed in our proof of Theorem 1. ■

Theorem 2 asserts that if the underlying directed graph, $G(\mathcal{Q}, E(\mathcal{Q}))$, of the switched system (2) admits a pair of paths, $(P_G^{j\to i}, P_G^{i\to j})$ with $i, j \in \mathcal{Q}$ satisfying Assumption 1, for which the Euclidean norms of (matrix) commutators $F^p_{j\to i,i}$, $F^q_{j\to i,j}$, $F^p_{i\to j,i}$, and $F^q_{i\to j,j}$ are bounded above by small enough scalars $\varepsilon_{j\to i,i}, \varepsilon_{j\to i,j}, \varepsilon_{i\to j,i}$, and $\varepsilon_{i\to j,j}$, respectively, such that condition (16) holds, then (2) is GES under a periodic switching signal $\sigma \in \mathcal{S}$. The setting of unrestricted switching between subsystems i and j, satisfying Assumption 1 (i.e., both (i,j) and $(j,i) \in E(\mathcal{Q})$), falls as a special case of Theorem 2 with $P_G^{j\overset{1}{\to}i} = P_G^{j\overset{2}{\to}i} = j, (j,i), i$ and $P_G^{i\overset{1}{\to}j} = P_G^{i\overset{2}{\to}j} = i, (i,j), j$.

Example 1: Suppose that $\mathcal{Q} = \{1,2\}$ and $E(\mathcal{Q}) = \{(1,2),(2,1)\}$. Let $i \in \mathcal{Q}$ and $j \in \mathcal{Q} \setminus \{i\}$ satisfy Assumption 1. Then there exists a periodic switching signal $\sigma \in \mathcal{S}$ under which the switched system (2) is GES. Indeed, we have $P_G^{j\to i} = j, (j,i), i$ and $P_G^{i\to j} = i, (i,j), j$ on $G(\mathcal{Q}, E(\mathcal{Q}))$, and hence, $F^p_{j\to i,i} = F^q_{j\to i,j} = F^p_{i\to j,i} = F^q_{i\to j,j} = 0_{d\times d}$. Choose $\varepsilon_{j\to i,i} = \varepsilon_{j\to i,j} = \varepsilon_{i\to j,i} = \varepsilon_{i\to j,j} = 0$. Condition (16) holds by the choice of λ. Now, suppose that $\mathcal{Q} = \{1,2,3\}$ and $E(\mathcal{Q}) = \{(1,2),(2,3),(3,1)\}$. Let $i \in \mathcal{Q}$ and $j \in \mathcal{Q} \setminus \{i\}$ satisfy Assumption 1. Let $k = \mathcal{Q} \setminus \{i,j\}$. If a pair of paths $(P_G^{j\to i}, P_G^{i\to j})$ on $G(\mathcal{Q}, E(\mathcal{Q}))$, with $P_G^{j\to i} = j, (j,i), i$ and $P_G^{i\to j} = i, (i,k), k, (k,j), j$ or, $P_G^{j\to i} = j, (j,k), k, (k,i), i$ and $P_G^{i\to j} = i, (i,j), j$ satisfy conditions (14)–(16), then there exists a periodic switching signal $\sigma \in \mathcal{S}$ under which the switched system (2) is GES. In both the cases above, the connectivity of $G(\mathcal{Q}, E(\mathcal{Q}))$ does not allow us to aim for a nonperiodic construction of stabilizing σ. One can, however, design a nonperiodic σ that preserves GES of (2) with, for example, $N = 3$, $i = 2$, $j = 1$, and $E(\mathcal{Q}) = \{(1,2),(1,3),(2,3),(3,1),(3,2)\}$ provided that the pairs of paths, $(P_G^{j\overset{r}{\to}i}, P_G^{i\overset{r}{\to}j})$, $r = 1,2$, with $P_G^{j\overset{1}{\to}i} = j, (j,i), i$, $P_G^{j\overset{2}{\to}i} = j, (j,k), k, (k,i), i$ and $P_G^{i\overset{1}{\to}j} = P_G^{i\overset{2}{\to}j} = i, (i,k), k, (k,j), j$ satisfy the desired conditions.

Remark 2: Our stability conditions involve the rate of decay of the Schur stable matrix, $A_i^p A_j^q$, upper bounds on the Euclidean norms of the commutators of certain products of the matrices, $A_k^{\delta_k}$, $k \in \mathcal{Q} \setminus \{i,j\}$, that appear in the paths $P_G^{j\overset{r}{\to}i}$ (respectively, $P_G^{i\overset{r}{\to}j}$), $r = 1,2$ and the matrix product A_j^q (respectively, A_i^p), and a set of scalars capturing the properties of the matrices, A_ℓ, $\ell \in \mathcal{Q}$. In the simplest case when the matrix products $A_{w^{(r)}_{\ell_r-1}}^{\delta_{w^{(r)}_{\ell_r-1}}} \dots A_{w^{(r)}_1}^{\delta_{w^{(r)}_1}}$ and A_j^q and the matrix products $A_{w^{(r)}_{n_r-1}}^{\delta_{w^{(r)}_{n_r-1}}} \dots A_{w^{(r)}_{\ell_r+1}}^{\delta_{w^{(r)}_{\ell_r+1}}}$ and A_i^p commute, $r = 1,2$, the condition (11) [respectively, (16)] reduces to (8). Theorems 1 and 2, therefore, accommodate subsystems matrices, A_ℓ, $\ell \in \mathcal{Q}$, for which the above matrix products do not necessarily commute but are close to matrices, $\overline{A}_\ell$, $\ell \in \mathcal{Q}$, for which they commute. This feature associates an inherent robustness with our stabilizability conditions. Indeed, if we are relying on approximate models of A_ℓ, $\ell \in \mathcal{Q}$, or the elements of A_ℓ, $\ell \in \mathcal{Q}$, are prone to evolve over time, then GES of (2) is preserved under our set of switching signals as long as Assumption 1 and conditions (9)–(11) (respectively, conditions (14)–(16)) continue to hold.

Remark 3: A vast body of the existing literature on stabilization of switched systems relies on state-dependent switching signals. The corresponding stability conditions involve set containment conditions (see, e.g., [5]) and/or BMIs (see, e.g., [3],[6], and[7]). In case of purely time-dependent switching signals, the work [3] presents (among others) a class of stabilizing periodic switching signals. The corresponding stability conditions involve LMIs. Our article differs from the existing literature in the following aspects.

1) We focus on purely time-dependent switching signals that are not restricted to periodic constructions.
2) Our stability conditions rely on scalar inequalities that are numerically easier to verify than the set inclusions and matrix inequalities based conditions available in the literature.
3) Our stability conditions are only sufficient and their nonsatisfaction does not imply the nonexistence of a stabilizing switching signal $\sigma \in \mathcal{S}$.

Remark 4: Commutation relations between the subsystems matrices or certain products of these matrices have been employed to study stability of the switched system (2) under arbitrary switching [1], [17], minimum dwell time switching [11], restrictions on admissible switches between the subsystems [12], and restrictions on admissible minimum and maximum dwell times on the subsystems [13] earlier in the literature. The problem setting considered in this article differs from and extends existing works in the following ways.

1) The works [1], [11], and [17] consider all subsystems to be stable, while we deal with all unstable subsystems.
2) The works [12] and [13] assume the existence of at least one stable subsystem. Even though we assume the existence of a Schur stable combination formed by two of the unstable subsystems matrices, our results do not rely on unrestricted switches between them. Consequently, the results of [12] and [13] do not extend to our setting with a Schur stable subsystem replaced by a Schur stable combination of subsystems.
3) The work [13] considers common admissible minimum and maximum dwell times on all the subsystems and does not allow consecutive activation of unstable subsystems. In this article, we allow admissible dwell times on each subsystem to be different and do not restrict our switching signals to any maximum number of unstable subsystems that can be activated between the two subsystems forming a Schur stable combination as long as a favorable path between them is chosen.
4) The type of matrix commutators used in our analysis differs from the existing works. Indeed, the works [1], [12], and [17] consider commutators between individual subsystems matrices, while the works [11] and [13] use commutators between products of individual subsystems matrices. In contrast, we employ commutators of products of different subsystems matrices and products of individual subsystems matrices. The choice of the commutator under consideration is, however, not unique. The crux of the (matrix) commutation relations based stability analysis for switched systems lies in splitting matrix products into sums and applying combinatorial arguments on them (see [1]), where this analysis technique was introduced. One may employ different choices of commutators to split a matrix product into sums, thereby leading to different sets of sufficient solutions to Problem 1.

Remark 5: Stability of switched nonlinear systems under restricted switching was addressed earlier in [10]. The proposed technique assumes the existence of at least one stable subsystem and involves constructing negative weight cycles on the underlying weighted directed graph of a switched system. The existence of these cycles depends on the existence of Lyapunov-like functions [2] corresponding to the subsystems that satisfy certain conditions individually and among themselves. Given a family of systems, designing such functions is, in general, a numerically difficult problem. In contrast, in this article, we consider all subsystems to be unstable and do not rely on verifying if

TABLE I
DESCRIPTION OF THE SUBSYSTEMS MATRICES

ℓ	A_ℓ	eigenvalues of A_ℓ
1	$\begin{pmatrix} 0.796323 & -0.9122466 \\ -0.7126696 & 0.1040671 \end{pmatrix}$	1.3276544, −0.4272643
2	$\begin{pmatrix} 0.9660338 & -0.972049 \\ -0.6582197 & -0.94077 \end{pmatrix}$	1.2571386, −1.2318748
3	$\begin{pmatrix} -0.5085495 & -0.6519882 \\ -0.7370684 & -0.5013346 \end{pmatrix}$	−1.1981757, 0.1882916
4	$\begin{pmatrix} -0.990773 & 0.8742857 \\ 0.780567 & 0.9401844 \end{pmatrix}$	−1.2959586, 1.24537

suitable Lyapunov-like functions exist for a given family of systems. Our stability conditions are, however, limited to the setting of linear subsystems.

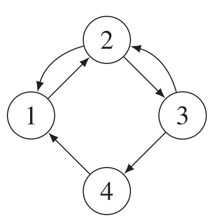
V. NUMERICAL EXPERIMENT

Consider a family of systems (1) with $N = 4$. We generate the matrices $A_\ell \in \mathbb{R}^{2\times 2}$, $\ell \in \mathcal{Q}$, by selecting elements from the interval $[-1, 1]$ uniformly at random. It is ensured that all the matrices are unstable. The numerical values of A_ℓ, $\ell \in \mathcal{Q}$ along with their eigenvalues are furnished in Table I. We have $M = 1.41$.

Let $E(\mathcal{Q}) = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,1)$, $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 2$ and $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 3$. Note that unrestricted switches between the subsystems 1 and 2 and between 2 and 3 are allowed. However, they do not form a stable combination in the sense of Assumption 1.

We compute that Assumption 1 holds with $i = 1, j = 3$, and $p = q = 2$.[4] Indeed, the matrix $\mathcal{A} = A_1^2 A_3^2 = \begin{pmatrix} 0.3379075 & 0.2444357 \\ 0.0176692 & 0.061251 \end{pmatrix}$ has eigenvalues 0.3527252 and 0.0464333. We obtain $m = 1$ and $\rho = 0.42$. Let $\lambda = 0.0001$. It follows that $\overline{m} = 1$ and $\rho e^{\lambda m} = 0.42 < 1$. However, both (i, j) and $(j, i) \notin E(\mathcal{Q})$. So, our results are useful to cater to this setting.

We construct a directed graph, $G(\mathcal{Q}, E(\mathcal{Q}))$, as shown below



Fix $P_G^{j \xrightarrow{1} i} = 3, (3,2), 2, (2,1), 1$, $P_G^{j \xrightarrow{2} i} = 3, (3,4), 4, (4,1), 1$, and $P_G^{i \xrightarrow{1} j} = P_G^{i \xrightarrow{2} j} = 1, (1,2), 2, (2,3), 3$. We have

$$\left\| F^p_{j \xrightarrow{1} i, i} \right\| = \left\| A_1^2 A_2^2 - A_2^2 A_1^2 \right\| = 0.02,$$

$$\left\| F^p_{j \xrightarrow{2} i, i} \right\| = \left\| A_1^2 A_4^2 - A_4^2 A_1^2 \right\| = 0.05,$$

$$\left\| F^q_{j \xrightarrow{1} i, j} \right\| = \left\| A_3^2 A_2^2 - A_2^2 A_3^2 \right\| = 0.04,$$

[4]We have a finite number of choices for i, j, p, and q. In addition, the dimension of the subsystems matrices is small. We have used a trial-and-error method with eigenvalue checks to find $\mathcal{A}$. To avoid eigenvalue checks for large matrices, one may apply the set of LMIs [3, condition (20)] on subsets of $\mathcal{Q}$ with size 2 and the given bounds on p and q.

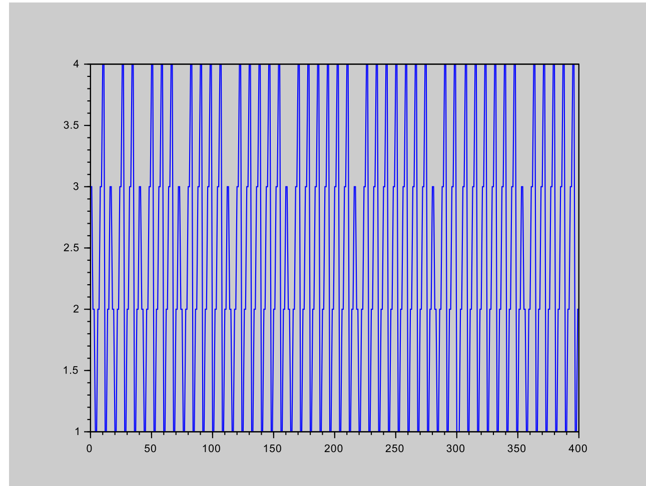


Fig. 1. $(\sigma(t))_{t\in\mathbb{N}_0}$ obtained from our experiment.

$$\left\| F^q_{j \xrightarrow{2} i, j} \right\| = \left\| A_3^2 A_4^2 - A_4^2 A_3^2 \right\| = 0.08,$$

$$\left\| F^p_{i \xrightarrow{1} j, i} \right\| = \left\| F^p_{i \xrightarrow{2} j, i} \right\| = \left\| A_1^2 A_2^2 - A_2^2 A_1^2 \right\| = 0.02,$$

$$\left\| F^q_{i \xrightarrow{1} j, j} \right\| = \left\| F^q_{i \xrightarrow{2} j, j} \right\| = \left\| A_3^2 A_2^2 - A_2^2 A_3^2 \right\| = 0.04,$$

$$\xi_{j \xrightarrow{r} i, i} = \xi_{j \xrightarrow{r} i, j} = \xi_{i \xrightarrow{r} j, i} = \xi_{i \xrightarrow{r} j, j} = 4, \ \ r = 1, 2,$$

$$\xi_{j \xrightarrow{1} i \xrightarrow{1} j} = 8, \ \ \xi_{j \xrightarrow{2} i \xrightarrow{2} j} = 0.$$

Consequently, the left-hand side of condition (11) is

$$0.42 + \Big(0 + 1 \times 1.41^4 \times (0.04 + 0.02 + 0.04)\Big) e^{0.0008}$$
$$= \ 0.82 < 1.$$

The assertion of Theorem 1 holds and the switched system (2) is GES under a switching signal $\sigma \in \mathcal{S}$ that activates the sequence of subsystems $3, 2, 1, 2$, followed by s-many instances of the sequence of subsystems $3, 4, 1, 2$, $s = 1, 2, 3, \ldots$, repeatedly with dwell time 2 units of time on all subsystems. σ is illustrated in Fig. 1. We generate 100 different initial conditions x_0 from the interval $[-1, 1]^2$ uniformly at random. The corresponding $(\|x(t)\|)_{t\in\mathbb{N}_0}$ for the switched system (2) with σ as described above is plotted in Fig. 2. GES of (2) is observed.

Remark 6: Note that one may employ the results of [5] to design stabilizing state-dependent switching signals for the setting considered in our experiment. Indeed, an automaton can be constructed with Δ_ℓ-many instances of each vertex $\ell \in \mathcal{Q}$ with appropriate connectivity to accommodate $E(\mathcal{Q})$ and the minimum and maximum dwell time constraints. Applying the results of [5], however, requires checking certain set containment conditions, which is a numerically difficult task. In contrast, we have verified scalar inequalities and designed a stabilizing purely time-dependent switching signal. It is important to note that both for the results of [5] and our results, nonsatisfaction of the stability conditions does not ensure nonexistence of a stabilizing switching signal that obeys the given restrictions.

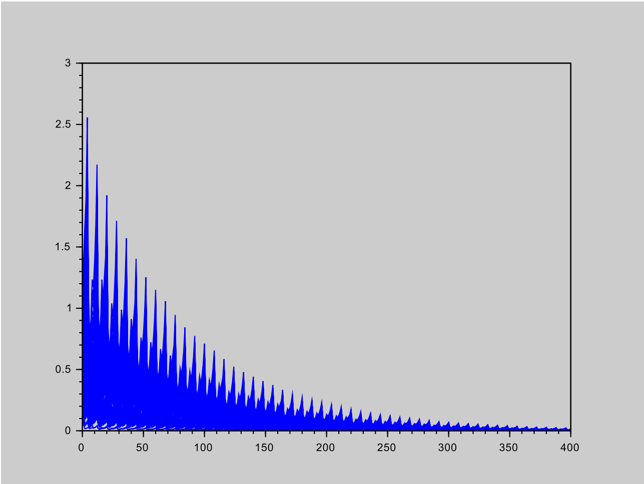


Fig. 2. $(\|x(t)\|)_{t\in\mathbb{N}_0}$ for (2) under the σ in Fig. 1.

VI. Conclusion

In this article, we studied stabilizability of discrete-time switched linear systems under restricted switching. Our switching signals are purely time-dependent. The proposed stability conditions involve scalar inequalities and are, therefore, easy to verify. Our results are derived in the premise of the existence of a suitable Schur stable combination formed by any two of the subsystems matrices. A next natural question is to address stabilizability of a switched system under purely time-dependent restricted switching signals when the subsystems matrices may not admit Schur stable combinations. This matter is currently under investigation and will be reported elsewhere.

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