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A mixed variational principle in nonlinear elasticity using Cartan's moving frames and implementation with finite element exterior calculus

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Abstract

This article is an attempt at offering a new perspective for the mechanics of solids using Cartan's moving frames, specifically discussing a mixed variational principle in non-linear elasticity. We treat quantities defined on the co-tangent bundles of reference and deformed configurations as primary unknowns along with deformation. Such a treatment invites compatibility of the fields (defined on the co-tangent bundle) with the base-space (configurations of the solid body) so that the configuration can be realized as a subset of the Euclidean space. Using the moving frame, we rewrite the metric and connection through differential forms. These quantities are further utilized to write the deformation gradient and Cauchy–Green deformation tensor in terms of frame and co-frame fields. The geometric understanding of stress as a co-vector valued 2-form fits squarely within our overall program. We also show that, for a hyperelastic solid, an equation similar to the Doyle–Ericksen formula may be written for the co-vector part of the stress 2-form. Using this kinetic and kinematic understanding, we rewrite a mixed functional in terms of differential forms, whose extremum leads to the compatibility of deformation, constitutive relations, and equations of equilibrium. Finite element exterior calculus is then utilized to construct a finite-dimensional approximation for the differential forms appearing in the variational principle. These approximations are then used to construct a discrete functional which can be numerically extremized. The discretization leads to a mixed model as it involves independent approximations of differential forms related to stress and deformation gradient. The mixed variational principle is then specialized for the 2D case, whose discrete approximation is applied to problems in nonlinear elasticity. From the numerical study, it is found that the present discretization does not suffer from locking and related convergence issues.

Keywords: Non-linear elasticity; Differential forms; Cartan's moving frame; Kinematic closure; Hu–Washizu variational principle; Finite element exterior calculus

1. Introduction

Mixed or complementary variational principles have a long history in solid mechanics [1]. These variational principles are of significant utility in developing approximation schemes where conventional or single field

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approximations perform poorly [2,3]. The Hu–Washizu (HW) principle, for instance, is a three-field variational statement commonly used to construct finite element approximations in nonlinear solid mechanics. The HW variational principle takes deformation gradient and stress as additional inputs along with deformation. From a geometric perspective, deformation gradient and stress are infinitesimal (tangent space based) quantities whose origins can be traced to the geometric hypothesis placed on the configurations of the body. However, scarce little has been achieved in understanding these principles from a geometric standpoint. The method of moving frames developed by Cartan [4,5] is an effective tool to describe geometry using differential forms. Built on the theory of exterior calculus, Cartan's moving frames can encode the connection information on a manifold, an indispensable tool for differentiating vector fields on a manifold. To each point of the configuration, the method of moving frames assigns a bunch of orthonormal vectors called frame fields. The rate at which these frame fields vary across the configuration defines the connection 1-forms. These connection 1-forms must however satisfy the structure equations so that the parallel transport they encode conforms to the underlying manifold structure, which will presently be assumed Euclidean. Apparently, attaching a set of vectors to a material point is nothing new in continuum mechanics. Many such models have been put forth, starting from Cosserat to micro-morphic theories; these theories are sometimes referred to as micro-continuum theories [6]. None of them, however, encode the connection information of the deforming body whilst evolving the frames or the directors. For these models, directors are just additional degrees of freedom to hold energy. A major difficulty with this point of view is that it does not clarify the geometry within which the model is working. An immediate consequence is that it is impossible to give a co-ordinate independent meaning to the derivatives appearing in the equations of motion.

An attempt at utilizing Cartan's moving frames to formulate the equations of elasticity was made by Frankel [7]; however his efforts went largely unnoticed and fell short of offering an appropriate computational implementation. In this work, we formulate the kinematics of an elastic body in the language of differential forms. The advantages of having the kinematics formulated in terms of moving frames are twofold. First, important kinematic quantities are described using differential forms that explicate on issues related to compatibility. Second, the geometric hypothesis behind the kinematics is made explicit. The hypothesis that the geometry of non-linear elasticity is Euclidean [8] conforms well with the tensor fields that typically describe the local state of deformation — the right Cauchy–Green deformation tensor to wit, whose roots can be traced to the (Euclidean) metric tensor. Moreover, compatibility equations for both the deformation gradient and Cauchy–Green deformation tensor [9] depend on the geometry of the configuration. Compatibility in terms of the deformation gradient is related to the vanishing of the torsion tensor while compatibility in terms of the Cauchy–Green deformation tensor is related to the vanishing of the curvature tensor [10]. Both notions of compatibility are however related to the affine connection placed on the configuration. The continuum mechanical definition of stress also has a geometric meaning; Frankel [7] describes Cauchy stress as a bundle valued differential form. The basic idea in his construction is to decompose the stress tensor into a traction (1-form) and an area component (2-form). Such a program was later pursued by Segeve and Rodnay [11] and Kanso et al. [12]. However, both Segeve and Rodnay and Kanso et al. did not pursue a variational principle using this description. From the decomposition of Cauchy stress, it is clear that the constitutive rule needs to be written only for the traction component since the area component is determined kinematically.

Conventionally, numerical techniques for nonlinear elasticity were largely based on single field approximations. It was soon realized that these methods suffered from numerical instabilities, thus affecting the convergence. Displacement based methods with additional stabilization, is a common technique to circumvent these difficulties. Methods like assumed strain and enhanced strain techniques belong to this class; these methods introduce addition terms in the energy function whose origin is purely numerical. Within nonlinear elasticity, techniques based on mixed FE methods are already the preferred choice [13–15] for large deformation problems since they avoid numerical instabilities like locking and preserve important conserved quantities. However, constructing stable mixed finite elements is difficult even in the case of linear elasticity. A few researchers have turned to ideas from differential geometry to construct stable, well performing mixed-FE techniques for nonlinear elasticity; we cite Yavari [16] as an example of one such attempt where tools from discrete differential forms [17] were utilized to construct stable discretization schemes. In case of hyperbolic problems, Yin et al. [18] presented a space–time discontinuous Galerkin formulation which is element-wise conservative. This approach was latter extended in Abedi et al. [19] to linearized elastodynamics, where conservation of linear and angular momenta was ensured element-wise; the authors extensively used exterior calculus to set up the equations of motion. The single-field formulation of Abedi et al. was extended to a three-field space–time discontinuous Galerkin formulation by Miller et al. [20], with independent approximations for displacement, velocity and strain fields. These developments in space–time discontinuous Galerkin approximation were later applied to dynamic brittle fracture problems [21].

Finite element exterior calculus (FEEC) [22] is an FE technique developed by Arnold and co-workers to unify finite elements like Raviart–Thomas, Nédélec [23–25] and other carefully handcrafted elements under a common umbrella. FEEC relies on the theory of differential forms to achieve this unification [22,26–28]. The algebraic and geometric structures brought forth by differential forms are instrumental in achieving this. Recently, Angoshtari et al. [15] and Shojaei and Yavari [13] have brought on ideas from algebraic topology to discretize the equations of nonlinear elasticity. These authors have constructed mixed FE techniques using HW variational principle. The discretization was based on a $\mathbb{R}^3$ valued de Rham complex, as described in Angoshtari and Yavari [29]. These methods still require stabilization terms in the three-dimensional case [14]. Moreover, from the description of the complex given in Angoshtari et al., it is not clear how the HW variational principle is related to the complex and how the operators defining the complex are affected by the connection placed on the configuration.

The goals of this article are twofold. The first is to develop a mixed variational principle for nonlinear elasticity that takes differential forms as its input argument. Towards this, we first reformulate the kinematics and kinetics of an elastic solid using differential forms. The kinematics of deformation is laid out via Cartan's method of moving frames. The structure equations associated with the moving frames establish the important relationship between the geometric hypothesis of the configuration and measures of deformation. The geometric understanding of stress as a bundle valued differential form is then exploited to write the kinetics in terms of differential forms. These two ideas are then used to rewrite the conventional HW variational principle, which now has a bunch of differential forms and deformation as its input arguments. The proposed mixed functional is then extremized with respect to these differential forms to arrive at the equations of mechanical equilibrium, constitutive relation and compatibility constraints. The second goal of this work is to use FEEC to discretize the proposed mixed variational principle. Towards this end, the spaces $\mathcal{P}_r\Lambda^1$ and $\mathcal{P}_r^-\Lambda^1$ are used to discretize the differential forms describing the kinetics and kinematics. Using these approximations, a discrete mixed functional is arrived at, which can then be extremized using numerical optimization techniques.

The rest of the article is organized as follows. A brief introduction to Cartan's moving frames and the associated structure equations are presented in Section 2. The kinematics of an elastic body is then reformulated in Section 3, using the idea of moving frames. In this section, important kinematic quantities like deformation gradient and right Cauchy–Green deformation tensor are described using frame and co-frame fields. This section also contains a discussion on affine-connections using connection 1-forms. In Section 4, we introduce stress as a co-vector valued differential 2-form; this interpretation is originally due to Frankel [7]. We then derive a relationship similar to the Doyle–Ericksen formula relating the stored energy density function to the traction 1-form. In Section 5, we rewrite the mixed variational principle in terms of differential forms describing the kinematics and kinetics of motion. We then show that variation of the mixed functional with respect to different input arguments leads to the compatibility of deformation, constitutive rule and equations of equilibrium. We also remark on the interpretation of stress as a Lagrange multiplier enforcing compatibility of deformation. Section 6 discusses a discrete approximation of differential forms on a simplicial manifold. While these ideas have their roots in the work of Whitney [30], we adopt a description of the FEEC within the framework of Cartan's moving frames. These are utilized to construct a discrete approximation to the mixed variational principle, which is numerically extremized using Newton's method. This FE approximation is then applied to standard benchmark problems in nonlinear elasticity in order to assess the performance of the numerical technique against instabilities like volume and bending locking. Finally, in Section 9, we discuss on the usefulness of the moving frames in formulating other theories in nonlinear solid mechanics (like Kirchhoff shells and dislocation mechanics) and the extension of the present numerical techniques to such nonlinear theories.

1.1. Remarks on notations

We do not use bold face letters to distinguish between a scalar, a vector or a tensor. Indices are used to index objects of the same kind and not the components; for example, if we have three 1-forms, we may denote the ith 1-form by θ^i. A symbol with one index does not mean that it is a component of a vector or a 1-form. The objects featured in the theory are defined wherever they appear first. Often in nonlinear elasticity, lower and upper case indices are used to distinguish objects in the reference and deformed configurations. We do not follow this convention since we adopt separate notations for the same object defined in the reference and deformed configurations. We follow the convention of Einstein summation over repeated indices. In places where the summation convention is not followed, we make it explicit.

2. Cartan's moving frame

In this section, we present a brief introduction to the method of moving frames; our motive being to review some basic results so that the kinematics of an elastic solid can be written in terms of moving frames. For a detailed exposition on moving frames, the reader may consult [4,5]. Cartan introduced the method of moving frames as a tool to study the geometry of surfaces. These techniques were later extended to study the geometry of abstract manifolds. Common examples of moving frames include the Frenet frame for a curve and Durboux frame for a surface.

We denote the reference and deformed configurations of a body by $\mathcal{B}$ and $\mathcal{S}$; the respective tangent bundles are denoted by $T\mathcal{B}$ and $T\mathcal{S}$. Both $\mathcal{B}$ and $\mathcal{S}$ are smooth manifolds with boundaries; their boundaries are denoted by $\partial\mathcal{B}$ and $\partial\mathcal{S}$. These configurations are endowed with a C^∞ chart from which they inherit their smoothness. Positions (placements) of a material point in the reference and deformed configurations are denoted by X and x respectively.

At each tangent space of a configuration, we choose a collection of orthogonal vectors, which we call the frame. The orthogonality of the frame field is with respect to the Euclidean inner product of the respective configuration. The frame fields of the reference and deformed configurations are denoted by E_i and e_i respectively. A collection of frame fields that span the tangent spaces constitutes a moving frame or simply a frame. We denote frames for the reference and deformed configurations by $\mathcal{F}_\mathcal{B} = \{E_1, \dots, E_n\}$ and $\mathcal{F}_\mathcal{S} = \{e_1, \dots, e_n\}$. Given a frame for tangent bundle, the natural (algebraic) duality between tangent and co-tangent spaces induces a co-frame for the co-tangent bundle as well. These co-frames (at a point) constitute a basis for the co-tangent spaces of the respective configurations. We denote the co-frames of the reference and deformed configurations by $\mathcal{F}^*_\mathcal{B} = \{E^1, \dots, E^n\}$ and $\mathcal{F}^*_\mathcal{S} = \{e^1, \dots, e^n\}$ respectively, where E^i and e^i are sections from the cotangent bundles of the reference and deformed configurations. The natural duality between frame and co-frame fields of the reference and deformed configurations may be written as,

$$E^i(E_j) = \delta^i_j; \quad e^i(e_j) = \delta^i_j; \quad E^i \in T^*\mathcal{B}, e^i \in T^*\mathcal{S} \tag{1}$$

For a material point in the reference configuration, the differential of position is denoted by $\mathrm{d}X$. In terms of the frame and co-frame fields, it can be written as,

$$\mathrm{d}X = E_i \otimes E^i \tag{2}$$

Similarly, in terms of the frame and co-frame fields in the deformed configuration, the differential of position is given by,

$$\mathrm{d}x = e_i \otimes e^i \tag{3}$$

From the definition of $\mathrm{d}X$ (2), it follows that a tangent vectors from the reference configuration is mapped to itself under $\mathrm{d}X$. To see this, choose $V \in T_X\mathcal{B}$ with $V = c^i E_i$. Substituting the latter and using the definition of $\mathrm{d}X$, we arrive at $\mathrm{d}X(V) = c^j E_i E^i(E_j)$. Using the duality between the frame and co-frame fields, we conclude that $\mathrm{d}X(V) = V$. Similarly, $\mathrm{d}x$ maps a tangent vector from the deformed configuration to itself. The differential of a position vector in the reference or deformed configuration is thus an identity map on the corresponding tangent space.

Similar to the differential of position, one may also define the differential of a frame, which in the reference configuration is given by,

$$\mathrm{d}E_i = \gamma^j_i \otimes E_j \tag{4}$$

where, γ^i_j is called the connection matrix; it contains 1-forms as its entries. Because of the orthogonality between the frame fields, the connection matrix is skew symmetric, i.e. $\gamma^i_j = -\gamma^j_i$. Similarly, the differential of a frame for the deformed configuration is given by,

$$\mathrm{d}e_i = \bar{\omega}^j_i \otimes e_j \tag{5}$$

$\bar{\omega}^i_j$ is the connection matrix of the deformed frame fields. It is also skew, i.e. $\bar{\omega}^i_j = -\bar{\omega}^j_i$.

For a given choice of connection 1-forms and co-frame fields, there are certain compatibility conditions (Poincaré relations) which guarantee the existence of the placements X and x. These equations are called Cartan's structure equations. In the present context (of all manifolds being Euclidean), the first compatibility condition establishes the torsion-free nature of the configuration. For the reference and deformed configurations, this condition may be written as,

$$\mathrm{d}^2 X = 0; \;\; \mathrm{d}^2 x = 0 \tag{6}$$

Plugging (2) and (3) into the above equation and making use of (4) and (5) leads to,

$$\mathrm{d}E^i = \gamma^i_j \wedge E^j; \;\; \mathrm{d}e^i = \bar{\omega}^i_j \wedge e^j \tag{7}$$

The second compatibility condition presently establishes that the reference and deformed configurations are curvature-free. This leads to the following conditions on the reference and deformed frame fields,

$$\mathrm{d}^2 E_i = 0; \;\;\; \mathrm{d}^2 e_i = 0 \tag{8}$$

Using the differential of the frame fields in the above equations yields,

$$\mathrm{d}\gamma^i_j = \gamma^i_k \wedge \gamma^k_j; \;\; \mathrm{d}\bar{\omega}^i_j = \bar{\omega}^i_k \wedge \bar{\omega}^k_j \tag{9}$$

For a simply connected body, the structure equations (7) and (9) (for the reference and deformed configurations) provide the necessary kinematic closure to ensure that the configurations can be embedded with an Euclidean space. Indeed, without this closure effected by the structure equations, a model cannot in general produce a deformed configuration which is a subset of an Euclidean space.

2.1. Differentials of position and frame

We now present the geometric meaning of the infinitesimal quantities (differentials of position and frame) introduced in the last subsection. This interpretation is valid only when the parallel transport encoded by the frames is path independent or Euclidean. Consider the differential of the position for the reference configuration given in (2). For a given coordinate system, the position X is a smooth function of its coordinates (X^1, X^2, X^3). Let Γ be a curve parametrized by its arc length, $\Gamma : [a, b] \to \mathcal{B}$, $G\left(\frac{d\Gamma}{ds}, \frac{d\Gamma}{ds}\right) = 1$, where $G(.,.)$ denotes the metric tensor of the reference configuration. The frame fields E_i can be constructed by an orthonormalization (Gram–Schmidt procedure) of the tangent vectors to the coordinate curves. Fig. 1 shows the coordinate curves and frame at $X(a)$ and $X(b)$. The tangent vector to the curve Γ is written as $\frac{d\Gamma}{ds} = c^i E_i$, where c^i are real numbers. Now, the differential of position, which is a vector valued 1-form, can be integrated along the curve Γ to produce a vector; this vector translates the position $X(a)$ to $X(b)$. This translation may be formally written as,

$$\begin{aligned} X(b) - X(a) &= \int_a^b \mathrm{d}X(c^i E_i) ds \\ &= \int_a^b c^i E_j E^j(E_i) ds, \\ &= \int_a^b c^i E_j \delta^j_i ds, \\ &= \int_a^b c^i E_i ds. \end{aligned} \tag{10}$$

In the last equation, E_i and c^i can vary along the curve Γ. The above interpretation of $\mathrm{d}X$ is very similar to that of 1-forms as real numbers defined on curves.

We now consider the differential of frame. Integrating (4) along Γ, we have,

$$\int_a^b \mathrm{d}E_i = \int_a^b \mathrm{d}E_i \left(\frac{d\Gamma}{ds}\right) \mathrm{d}s.$$

Evaluating γ^i_j on the tangent vector produces a skew symmetric matrix with real co-efficients. In other words, the above integration can be written as a solution to the ordinary differential equation,

$$\dot{E}_i = \gamma^j_i E_j, \tag{11}$$

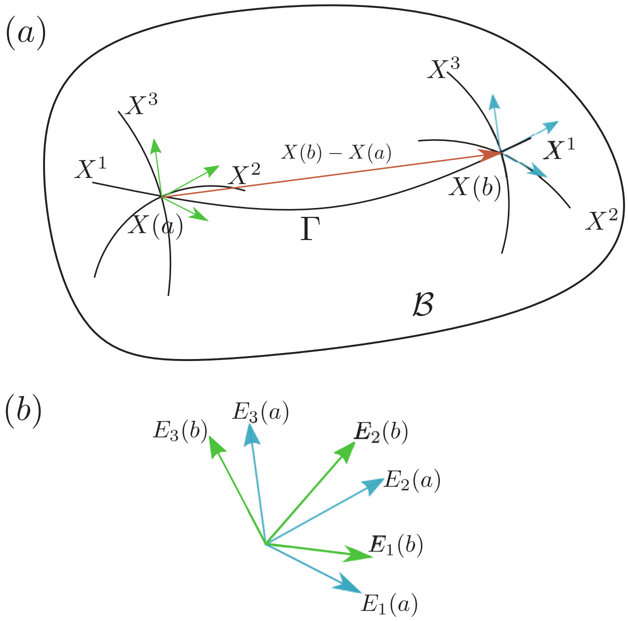


Fig. 1. The coordinate lines and the frame field generated from these coordinate lines are shown in (a). The frame fields at $X(a)$ and $X(b)$ are shown in (b); we have moved the frames to the same point so that it is convenient to interpret the change. We have used the notation $E_i(a)$ and $E_i(b)$ to indicate the frame at the material points $X(a)$ and $X(b)$.

$(\dot{.})$ may be understood as the derivative with respect to the parameter s. Alternatively, one can understand (11), as a restriction of (4) to a curve. Given $E_j(a)$, the solution to (11) is a rotation matrix R which relates the frames between two points on the curve Γ. This is formally written as,

$$E_i(s) = R(s)E_i(a). \tag{12}$$

Fig. 1(b) depicts this idea by overlaying the frames at a and b. From this discussion, it is clear that the vector translating the point $X(a)$ to $X(b)$ is dependent on the frame, the co-frame and the curve chosen for integration. However, the vector $X(b) - X(a)$ is path and frame independent since the body $\mathcal{B}$ is a subset of an Euclidean space. This path independence is exactly what the structure equations enforce.

2.2. Affine connection via frame fields

We now discuss the affine connection and covariant differentiation encoded by the connection 1-forms discussed in the previous subsection. An affine connection on a smooth manifold is a device used to differentiate sections of vector and tensor bundles in a co-ordinate independent manner. Let $w = \sum_{i=1}^{n} w^i e_i$ be an arbitrary section from $T\mathcal{S}$. The covariant derivative of w in the direction of e_i is given by,

$$\nabla_{e_i} w = \mathrm{d}w^j(e_i)e_j + w^j \bar{\omega}_j^k(e_i)e_k \tag{13}$$

It is easy to check that the above definition is a differential satisfying the properties,

$$\begin{aligned} \nabla_{fe_i} w &= f\nabla_{e_i} w; \quad f \in \Lambda^0 \\ \nabla_{e_i}(w+v) &= \nabla_{e_i} w + \nabla_{e_i} v; \;\; w, v \in T\mathcal{S} \\ \nabla_{e_i}(fw) &= e_i[f]w + f\nabla_{e_i} w \end{aligned} \tag{14}$$

Using these properties, it is possible to extend the above definition of covariant differentiation to arbitrary tensor fields [8].

3. Kinematics

The deformation map sends the placement of material points in the reference configuration to their corresponding placements in the deformed configuration. We denote the deformation map by φ so that $x = \varphi(X)$. The differential of the deformation or the deformation gradient, denoted by $\mathrm{d}\varphi$, maps the tangent space of the reference configuration to the corresponding tangent space in the deformed configuration. For an assumed frame field (for both reference and deformed configurations), the differential of deformation can be obtained by pulling the co-vector part of the deformed configuration's differential of position back to the reference configuration.

$$\begin{aligned} \mathrm{d}\varphi &= e_i \otimes \varphi^*(e^i) \\ &= e_i \otimes \theta^i; \quad \theta^i \in T^*\mathcal{B}, e_i \in \mathcal{F}_{\mathcal{S}} \end{aligned} \tag{15}$$

In writing (15), we have introduced the following definition: $\theta^i := \varphi^*(e^i)$. The definition of deformation gradient given in (15) is not new; it had previously appeared in the work of Yavari [16]. In our construction, the 1-forms θ^i contain local information about the deformation map φ. The 1-forms θ^i are a primitive variable in our theory; we call these differential forms, the deformation 1-forms. From (15), we see that the vector leg of the deformation gradient is from the deformed configuration, while the co-vector leg is from the reference configuration, making it a two-point tensor [8]. The action of the deformation gradient on a vector $V(X) \in T_X\mathcal{B}$ is given by,

$$\mathrm{d}\varphi(V) = e_i\theta^i(V) \tag{16}$$

Since $\theta^i(V)$ are real numbers, the above equation is a linear combination of tangent vectors from the deformed configuration. Conventionally, deformation gradient is introduced as the differential of deformation; we have not taken this perspective since our interest is in constructing mixed variational principles for nonlinear elasticity. Another important aspect of the above construction is that, we have written the deformation gradient using a frame and a co-frame. Contrast this with a conventional mixed method, where the deformation gradient is identified with its components in a particular coordinate system.

3.1. Strain and deformation measures

The notion of length is central to continuum mechanics; important kinematic quantities like strain and rate of deformation are derived from it. Indeed, it may not be possible to assess the state of deformation without a metric structure (notions of length and angle) for both reference and deformed configurations. The notion of length is encoded by a symmetric and positive definite tensor, defined on the tangent space of the respective configuration. The metric tensor of the reference and deformed configurations are denoted by G and g respectively; $G : T_X\mathcal{B} \times T_X\mathcal{B} \to \mathbb{R}$ and $g : T_x\mathcal{S} \times T_x\mathcal{S} \to \mathbb{R}$. In this work, we assume the metric structures of both reference and deformed configurations to be Euclidean. In terms of the co-frame field, the metric tensor of the reference configuration is given as,

$$\begin{aligned} G &= \mathrm{d}X \cdot \mathrm{d}X \\ &= (E^j \otimes E_j) \cdot (E_i \otimes E^i) \qquad (17) \\ &= \delta_{ij} E^i \otimes E^j. \qquad (18) \end{aligned}$$

The dot product introduced in the above equation is the inner product between the vector legs of $\mathrm{d}X$, which is computed using the Euclidean inner product. Similarly, the metric tensor in the deformed configurations may be written as,

$$\begin{aligned} g &= \mathrm{d}x \cdot \mathrm{d}x \\ &= (e^j \otimes e_j) \cdot (e_i \otimes e^i) \qquad (19) \\ &= \delta_{ij} e^i \otimes e^j. \qquad (20) \end{aligned}$$

In terms of the frame fields, the inverses of the metric tensors for the reference and deformed configurations are written as,

$$G^{-1} = \delta^{ij} E_i \otimes E_j; \quad g^{-1} = \delta^{ij} e_i \otimes e_j. \tag{21}$$

Now, the right Cauchy–Green deformation tensor may be obtained as the pull-back of the deformed configuration's metric tensor. In terms of the deformation 1-forms, this relationship may be written as,

$$\begin{aligned} C &= \varphi^*(g) \\ &= \varphi^*(\delta_{ij} e^i \otimes e^j) \\ &= \delta_{ij} \theta^i \otimes \theta^j. \end{aligned} \tag{22}$$

An alternative way to compute the C is to use the usual definition in continuum mechanics, $C = \mathrm{d}\varphi^t \mathrm{d}\varphi$. Here, $(.)^t$ is understood to be the adjoint map induced by the metric structure. Using the orthonormality of the frame field we arrive at,

$$\begin{aligned} C &= (\theta^i \otimes e_i) \cdot (e_j \otimes \theta^j) \\ &= \delta_{ij} \theta^i \otimes \theta^j. \end{aligned} \tag{23}$$

The calculations leading to (22) and (23) are exactly the same; only the sequence in which pull-back and inner product are computed differs. The Green–Lagrangian strain tensor may now be written as,

$$\begin{aligned} E &= \frac{1}{2}(C - G) \\ &= \frac{1}{2} \delta_{ij} [(\theta^i \otimes \theta^j) - (E^i \otimes E^j)]. \end{aligned} \tag{24}$$

The first invariant of the right Cauchy–Green tensor is given by,

$$I_1 = \langle \theta^i, \theta^i \rangle_G. \tag{25}$$

Here $\langle ., . \rangle_G$ denotes the inner product induced by G. The area forms induced by the co-frame of the reference configuration are given by,

$$A^1 = E^2 \wedge E^3; \quad A^2 = E^3 \wedge E^1; \quad A^3 = E^1 \wedge E^2, \tag{26}$$

Similarly, the area-forms induced by the co-frame of the deformed configuration are given by,

$$a^1 = e^2 \wedge e^3; \quad a^2 = e^3 \wedge e^1; \quad a^3 = e^1 \wedge e^2, \tag{27}$$

These area forms A^i and a^i serve as a basis for the space of 2-forms defined on their respective configurations. The area forms in the deformed configuration may be pulled back to the reference configuration under the deformation map. These pulled-back area forms are denoted by $\mathsf{A}^i := \varphi^*(a^i)$. In terms of the deformation 1-forms, the pulled back area forms can be written as,

$$\mathsf{A}^1 = \theta^2 \wedge \theta^3; \quad \mathsf{A}^2 = \theta^3 \wedge \theta^1; \quad \mathsf{A}^3 = \theta^1 \wedge \theta^2. \tag{28}$$

In terms of the pulled-back area forms, the second invariant of C may now be written as,

$$I_2 = \langle \mathsf{A}^i, \mathsf{A}^i \rangle_G \tag{29}$$

In terms of the co-frame fields, the volume forms of reference and deformed configurations may be written as,

$$V = E^1 \wedge E^2 \wedge E^3; \quad v = e^1 \wedge e^2 \wedge e^3. \tag{30}$$

The pull back of the volume form in the deformed configuration to the reference configuration is denoted by $\mathsf{V} := \varphi^*(v)$. In terms of the deformation 1-forms, V may be written as,

$$\mathsf{V} = \theta^1 \wedge \theta^2 \wedge \theta^3 \tag{31}$$

Finally, in terms of the pulled-back volume-form, the third invariant of C is given by,

$$I_3 = ({}_\star \mathsf{V})^2 \tag{32}$$

In the above equation ${}_\star(.)$ denotes the Hodge star operator, which establishes an isomorphism between the space of 0-forms and 3-forms. We also define $J := \sqrt{I_3}$ or simply $J = {}_\star \mathsf{V}$.

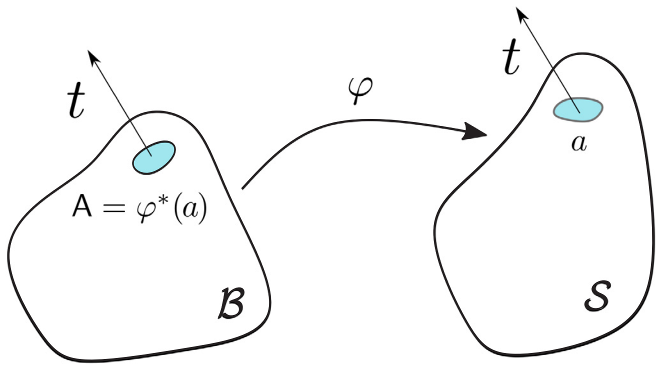


Fig. 2. For an infinitesimal area in the deformed configuration sustaining a traction t, Piola transform applies a pull-back on the area leg, while the traction remains unaltered.

4. Stress as co-vector valued two-form

In the previous section, we reformulated the deformation gradient and right Cauchy–Green deformation tensor in terms of differential forms. We now present a geometric approach to stress, originally due to Frankel [7] and subsequently developed by Segeve and Guy [11] and Kanso et al. [12]. Even though this approach is intuitive and geometric, it was never used to construct a variational principle. This geometric approach to stress has its origins in classical dynamics [31], where force is understood as a co-vector. Identifying force with a co-vector permits us to write power as a pairing between force and velocity (action of a 1-form on a vector) without the use of a metric tensor. Extending this concept to stress in continuum mechanics is non-trivial and requires the machinery of bundle valued differential forms [12]. As in classical dynamics, this interpretation of stress as a bundle valued differential form permits us to write the power expended by stress upon deformation without using a metric.

We denote the Cauchy stress tensor by σ. The traction acting on an infinitesimal area element with unit normal $\hat{n}$ is denoted by t, which is given by the well known formula $t = \sigma\hat{n}$. The traction t is a force which depends on the material point and the area sustaining it. The Cauchy stress theorem establishes a linear relationship between the area's normal and the traction sustained by it. In the language of differential forms, an infinitesimal area is regarded as a 2-form, while from classical dynamics we also know that force is a co-vector or a 1-form. Putting these two ideas together, we are led to a geometric definition of Cauchy stress given by,

$$\sigma = t^i \otimes a^i; \tag{33}$$

Recall that a^i is the area form of the deformed configuration, which sustains the traction vector t^i. The tensor product in the above equation is due to the linearity between traction and area forms. From this equation, it is easy to see that the area form changes orientation if the order of the co-vectors in the area form is reversed. Geometrically, if there are m linearly independent area forms on the manifold, the stress tensor assigns to each area form a 1-form called traction. With this understanding, Cauchy stress may now be identified with a section from the tensor bundle $\Lambda^1 \otimes \Lambda^2$ which has the deformed configuration as its base space.[1]

In contemporary continuum mechanics, Nanson's formula describes the transformation of an infinitesimal area under the deformation map [32]. Geometrically, Nanson's formula is nothing but the pull-back of an area-form in the deformed configuration under the deformation map. These pulled-back area forms are given in (28) (see Fig. 2). The first Piola stress may now be obtained by pulling back the area leg of the Cauchy stress under the deformation map. This partial pull-back of the Cauchy stress is termed the Piola transform. This relationship may be formally written as,

$$\begin{aligned} P &= t^i \otimes \varphi^*(a^i) \\ &= t^i \otimes \mathsf{A}^i \end{aligned} \tag{34}$$

[1] Λ^i denotes the space of differential forms of degree i.

Note that in the definition of Piola transform, traction 1-form is left untouched. Thus, contrary to convention, Cauchy and first Piola stresses are now identified as third order tensors, not the usual second order. This ambiguity can be removed if one applies the Hodge star on the area leg of these two stresses. In three dimensions, the Hodge star in question establishes an isomorphism between differential forms of degree 2 and 1. The usual definition of stress may thus be recovered as,

$$\sigma = t^i \otimes {}_\star(a^i) \tag{35}$$

$$P = t^i \otimes {}_\star(\mathsf{A}^i). \tag{36}$$

Kanso et al. made a distinction between the stress tensors given in (33), (34) and (35), (36). However we do not see a need for it, since both the usual and geometric definitions of stress contain exactly the same information; only the ranks of these tensors are different.

4.1. Traction 1-form via stored energy function

The Doyle–Ericksen formula is an important result in continuum mechanics [33], which relates the Cauchy stress and the metric tensor of the deformed configuration. For a stored energy density function W, Doyle–Ericksen formula gives us the following relationship,

$$\sigma = 2\frac{\partial W}{\partial g}. \tag{37}$$

In writing (37), we have assumed that W is frame-invariant. From the discussion presented so far, it is seen that the area leg of the Cauchy stress tensor is determined by the choice of coordinate system (frame and co-frame fields) for the tangent bundle of the deformed configuration. On the other hand, the area leg of the first Piola stress is determined by both the deformation map and the co-ordinate system for the tangent bundle of the deformed configuration. Clearly, the area leg of a stress tensor does not require a constitutive rule; it is only the traction component that demands a constitutive rule.

We now claim that for a stored energy function W, the traction 1-form has the following constitutive rule,

$$t^i = \frac{1}{J}\frac{\partial W}{\partial e_i} \tag{38}$$

The last equation is in the same spirit as the Doyle–Ericksen formula. To establish this result, we first compute the directional derivative of W along e_i,

$$\frac{\partial W}{\partial e_i} = \frac{\partial W}{\partial(\mathrm{d}\varphi)}\frac{\partial(\mathrm{d}\varphi)}{\partial e_i} \tag{39a}$$

$$= (t^j \otimes {}_\star\mathsf{A}^k)(e^i \otimes e_j \otimes \theta^k) \tag{39b}$$

$$= \langle {}_\star\mathsf{A}^j, \theta^j\rangle t^i \tag{39c}$$

$$= ({}_\star\mathsf{V})t^i \tag{39d}$$

$$= Jt^i \tag{39e}$$

We used the chain rule to arrive at the right hand side of (39a). In (39b), the expression for Piola stress (as a two tensor) in terms of W and the directional derivative of $\mathrm{d}\varphi$ along e_i are used to get the right hand side and performing the required contractions lead to (39c). The claim is finally established by using the definitions of pull-back and Hodge star for volume forms. In three dimensions, constitutive relations have to be supplied to the three traction 1-forms. From these calculation, it is found that the traction 1-forms are conjugate to the frame fields.

If the Cauchy stress (the usual definition) generated by a stored energy function is known, then the expression for the traction 1-form can be computed using the simple relation $t^i = \sigma^{ij}\hat{n}_j$, where the vector fields $\hat{n}_i$ are chosen to be elements from the frame of the deformed configuration.

5. Mixed variational principle

We first present the conventional HW variational principle for a finitely deforming elastic body. As mentioned, the HW variational principle takes the deformation gradient, first Piola stress and deformation map as input arguments.

In the reference configuration, the HW functional for a non-linear elastic solid can be written as,

$$I_{HW} = \int_{\mathcal{B}} [W(C) - P : (F - \mathrm{d}\varphi)]\mathrm{d}V - \int_{\partial\mathcal{B}} \langle t, \varphi \rangle \mathrm{d}A. \tag{40}$$

In the above equation, $t = PN$ is the traction defined on the surface $\partial\mathcal{B}$, whose unit normal is N. The integration in the above equation is with respect to the volume and area forms of the reference configuration. In (40), the deformation gradient is assumed to be independent; this tensor field is denoted by F. On the other hand, the deformation gradient computed as the differential of deformation is denoted by $\mathrm{d}\varphi$. It is worthwhile to note that the second term in the above equation is bilinear in the Piola stress and the deformation gradient. The variation of the HW functional with respect to deformation, deformation gradient and first Piola stress leads to the equilibrium equation, constitutive rule and compatibility of deformation gradient. This form of HW variational principle has been previously exploited to formulate numerical solution procedures for non-linear problems in elasticity; see [13–15].

5.1. Mixed variational principle with geometric definitions of stress and deformation

We now use the definitions of Cauchy and Piola stresses given in (33) and (34) respectively to rewrite the HW variational principle such that it takes deformation 1-forms, traction 1-forms and deformation map as inputs. We also assume that compatible frames for the reference and deformed configurations are given. This assumption permits us to eliminate the frame fields from the list of unknowns. The mixed functional may be now written as,

$$I(\theta^i, t^i, \varphi) = \int_{\mathcal{B}} W(\theta^i)\mathrm{d}V - (t^i \otimes \mathsf{A}^i)\dot{\wedge}(e_i \otimes (\theta^i - \mathrm{d}\varphi^i)) - \int_{\partial\mathcal{B}} \langle t^\sharp, \varphi \rangle dA. \tag{41}$$

In (41), $\dot{\wedge}$ denotes a bilinear map. For $\alpha \in T^*\mathcal{S}$, $v \in T\mathcal{S}$ and $a, b \in \Lambda^n(\mathcal{B})$, $n \geq 1$, the action of this map is given by $(\alpha \otimes a)\dot{\wedge}(v \otimes b) = \alpha(v)a \wedge b$. Note that the definition of $\dot{\wedge}$ given here is a little different from the one in Kanso et al. [12]. Specifically, we do not use the metric tensor. From our definition of $\dot{\wedge}$, it is seen that the work done by stress on deformation is metric independent. This property of our current variational formulation brings the continuum mechanical definition of stress a step closer to the definition of force (as a 1-form) in classical mechanics [31]. We did not write the volume form in the second term on the RHS of (41), since the outcome of $\dot{\wedge}$ is a 3-form which can be integrated over the reference configuration to produce work done by traction 1-forms on deformation. Also note that the second term on the RHS in (41) is equivalent to the second term in (40); however now the relationship between the different arguments is multi-linear. The functional given in (41) can also be discussed within the framework given by Oden and Reddy [1] for the construction of dual and complementary variational principles.

Remark 1. In writing (41), we have postulated that the geometry of the body is Euclidean and it is frozen during the deformation process. Indeed, within the present set-up, this assumption can be relaxed by permitting non-integrability in the connection and deformation 1-forms (i.e. by incorporating source terms in the structure equations).

Remark 2. For a frame to represent Euclidean geometry, it is not required that the connection 1-forms be identically zero. It is only required that the structure equations have a zero source term.

Remark 3. It should be mentioned that in the present calculation of critical points, the reference configuration and the ambient space are assumed to Euclidean.

We now proceed to obtain the Euler–Lagrange equations or the condition that determines the critical points of the functional I. We use the Gateaux derivative for this purpose. Let ϵ denote a small parameter and $\hat{(.)}$ the direction in which the change in the functional I is computed; this change is often referred to as the variation of I.

We first calculate the variation of I with respect to traction 1-forms; $t^i \mapsto t^i + \epsilon\hat{t}^i$, where $\hat{t}^i$ are assumed to be from the tangent space of $T^*\mathcal{S}$. The perturbed functional in the direction of $\hat{t}^i$ can be written as,

$$I(\epsilon) = \int_{\mathcal{B}} W(\theta^i)dV - ((t^i + \epsilon\hat{t}^i) \otimes \mathsf{A}^i)\dot{\wedge}(e_j \otimes (\theta^j - \mathrm{d}\varphi^j)) - \int_{\partial\mathcal{B}} \langle t^\sharp, \varphi \rangle dA. \tag{42}$$

Using the definition of $\dot{\wedge}$ and Gateaux derivative, we get a vector valued 3-form for each i. These three 3-forms have to be equated to zero to get the condition for critical points in the direction of traction 1-forms. These conditions may be formally written as,

$$\begin{bmatrix} (\mathsf{A}^1 \wedge (\theta^1 - d\varphi^1)) & -(\mathsf{A}^1 \wedge d\varphi^2) & -(\mathsf{A}^1 \wedge d\varphi^2) \\ -(\mathsf{A}^2 \wedge d\varphi^1) & (\mathsf{A}^2 \wedge (\theta^2 - d\varphi^2)) & -(\mathsf{A}^2 \wedge d\varphi^2) \\ -(\mathsf{A}^3 \wedge d\varphi^1) & -(\mathsf{A}^3 \wedge d\varphi^2) & (\mathsf{A}^3 \wedge (\theta^3 - d\varphi^3)) \end{bmatrix} \otimes \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{43}$$

Since e_i are orthonormal with respect to a positive definite metric, the above equation can be true only when the coefficient matrix on the LHS is zero, which leads to the following conditions,

$$(\mathsf{A}^1 \wedge (\theta^1 - d\varphi^1)) = 0; \quad (\mathsf{A}^1 \wedge d\varphi^2) = 0; \quad (\mathsf{A}^1 \wedge d\varphi^2) = 0 \tag{44a}$$

$$(\mathsf{A}^2 \wedge d\varphi^1) = 0; \quad (\mathsf{A}^2 \wedge (\theta^2 - d\varphi^2)) = 0; \quad (\mathsf{A}^2 \wedge d\varphi^2) = 0 \tag{44b}$$

$$(\mathsf{A}^3 \wedge d\varphi^1) = 0; \quad (\mathsf{A}^3 \wedge d\varphi^2) = 0; \quad (\mathsf{A}^3 \wedge (\theta^3 - d\varphi^3)) = 0 \tag{44c}$$

Using the definition of A^i, the above equations may be recast as,

$$\begin{bmatrix} \theta^1 - d\varphi^1 & d\varphi^1 & d\varphi^1 \\ d\varphi^2 & \theta^2 - d\varphi^2 & d\varphi^2 \\ d\varphi^3 & d\varphi^3 & d\theta^3 - d\varphi^3 \end{bmatrix} \wedge \begin{bmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{45}$$

For these equations to hold, the following conditions must be met,

$$\theta^1 - d\varphi^1 = 0; \quad \theta^2 - d\varphi^2 = 0; \quad \theta^3 - d\varphi^3 = 0 \quad . \tag{46}$$

The above condition simply states that there exist three 0-forms whose exterior derivatives are the deformation 1-forms; or in other words, the deformation 1-forms are exact and φ^i are the potentials for the corresponding deformation 1-forms.

We now compute the variation of I with respect to the deformation 1-forms. Incremental changes in the deformation 1-forms may be written as, $\theta^i \mapsto \theta^i + \epsilon\hat{\theta}^i$, where $\epsilon\hat{\theta}^i$ is assumed to be an element from the tangent space $T^*\mathcal{B}$. The perturbed functional in the direction of deformation 1-forms may be written as,

$$I(\epsilon) = \int_{\mathcal{B}} W(\theta^i + \epsilon\hat{\theta}^i) dV - (t^i \otimes \mathsf{A}^i(\epsilon))\dot{\wedge}(e_j \otimes (\theta^j(\epsilon) - d\varphi^j)) - \int_{\partial\mathcal{B}} \langle t^\sharp, \varphi\rangle dA. \tag{47}$$

Using the definition of the Gateaux derivative, for each θ^i we have,

$$\begin{aligned} \frac{\partial W}{\partial \theta^1} = {} & [t^1(e_1)_\star(\theta^2 \wedge \theta^3) - t^2(e_1)_\star(d\varphi^1 \wedge \theta^3) + t^2(e_2)_\star((\theta^2 - d\varphi^2) \wedge \theta^3) - t^2(e_3)_\star(d\varphi^3 \wedge \theta^3) \\ & - t^3(e_1)_\star(\theta^2 \wedge d\varphi^1) + t^3(e_2)_\star(\theta^2 \wedge d\varphi^3) + t^3(e_3)_\star(\theta^2 \wedge (\theta^3 - d\varphi^3))]^\sharp \end{aligned} \tag{48a}$$

$$\begin{aligned} \frac{\partial W}{\partial \theta^2} = {} & [t^1(e_1)_\star(\theta^3 \wedge (\theta^1 - d\varphi^1)) - t^1(e_2)_\star(\theta^3 \wedge d\varphi^2) - t^1(e_3)_\star(\theta^3 \wedge d\varphi^3) - t^2(e_2)_\star(\theta^3 \wedge d\theta^1) \\ & - t^3(e_1)_\star(d\varphi^1 \wedge \theta^1) - t^3(e_2)_\star(\theta^2 \wedge \theta^1) + t^3(e_3)_\star((\theta^3 - d\varphi^3) \wedge \theta^1)]^\sharp \end{aligned} \tag{48b}$$

$$\begin{aligned} \frac{\partial W}{\partial \theta^3} = {} & [t^1(e_1)_\star((\theta^1 - d\varphi^1) \wedge \theta^2) - t^1(e_2)_\star(d\varphi^2 \wedge \theta^2) - t^1(e_3)_\star(d\varphi^3 \wedge \theta^2) - t^2(e_1)_\star(\theta^1 \wedge d\varphi^1) \\ & - t^2(e_2)_\star(\theta^1 \wedge (\theta^2 - d\varphi^2)) + t^2(e_3)_\star(\theta^1 \wedge d\varphi^3) + t^3(e_3)_\star(\theta^1 \wedge \theta^2)]^\sharp. \end{aligned} \tag{48c}$$

If we now take into account the compatibility equations previously established in (46), the last equations reduce to,

$$\frac{\partial W}{\partial \theta^1} = [t^1(e_1)_\star(\theta^2 \wedge \theta^3) + t^2(e_1)_\star(\theta^3 \wedge \theta^1) + t^3(e_1)_\star(\theta^2 \wedge \theta^1)]^\sharp \tag{49a}$$

$$\frac{\partial W}{\partial \theta^2} = [t^1(e_2)_\star(\theta^2 \wedge \theta^3) + t^2(e_2)_\star(\theta^3 \wedge \theta^1) + t^3(e_2)_\star(\theta^1 \wedge \theta^2)]^\sharp \tag{49b}$$

$$\frac{\partial W}{\partial \theta^3} = [t^1(e_3)_\star(\theta^2 \wedge \theta^3) + t^2(e_3)_\star(\theta^3 \wedge \theta^1) + t^3(e_3)_\star(\theta^1 \wedge \theta^2)]^\sharp. \tag{49c}$$

From these equations, we see that a 2-form accompanies the components of traction 1-forms; this is indeed true since we use a Piola transform to write the constitutive rule in the reference configuration. From a comparison of

(48) and (49), we note that the expressions for traction in the former have additional terms. These additional terms may be related to incompatibilities created by the emergence of defects (such as dislocations) as the deformation evolves.

Finally, we compute the variation of I with respect to deformation; $\varphi^i \mapsto \varphi^i + \epsilon\hat{\varphi}^i$, where $\hat{\varphi}$ belongs to $T\mathcal{B}$. Using the definition of the superimposed incremental deformation in I and upon computing the Gateaux derivative, we have the following equation,

$$\int_{\mathcal{B}} (t^i \otimes \mathsf{A}^i)\dot{\wedge}(e_j \otimes \mathrm{d}\hat{\varphi}^i) = 0 \tag{50}$$

To complete the variation, we need to shift the differential from $\hat{\varphi}$. We first calculate the following,

$$\mathrm{d}(\varphi^k t^j(e_k)\mathsf{A}^j) = \mathrm{d}\varphi^k \wedge t^j(e_k)\mathsf{A}^j + \varphi^k \mathrm{d}(t^j(e_k)) \wedge \mathsf{A}^j + \varphi^k t^j(e_k)\mathrm{d}\mathsf{A}^j \tag{51}$$

This equation invites a few comments. The first is that we are calculating the exterior derivative of a 2-form, with $\varphi^k t^j(e_k)$ being a scalar. Using the product rule of differentiation, we have expanded the right hand side of (51). The second term in (51) should be evaluated using the connection 1-forms since it involves the exterior derivative of a vector. This term is relevant when one works with a frame field whose connection 1-forms are different from zero. If we invoke the compatibility of deformation, we have $\mathrm{d}\mathsf{A}^i = 0$, which leaves (51) in the following form,

$$\mathrm{d}(\varphi^k t^j(e_k)\mathsf{A}^j) = \mathrm{d}\varphi^k \wedge t^j(e_k)\mathsf{A}^j + \varphi^k \mathrm{d}(t^j(e_k)) \wedge \mathsf{A}^j \tag{52}$$

An expression similar to (52) was utilized by Kanso et al. [12] to define the mechanical equilibrium. The expression for the exterior derivative defined in (52) involves the connection 1-forms of the manifold, which is similar to the covariant exterior derivatives used in gauge theories of physics [34]. Using (51) in (50) leads to,

$$\int_{\mathcal{B}} \mathrm{d}(\hat{\varphi}^k t^j(e_k)\mathsf{A}^j) - \hat{\varphi}^k \mathrm{d}(t^j(e_k)) \wedge \mathsf{A}^j = 0 \tag{53}$$

The first term in the above equation may be converted to a boundary term via Stokes' theorem leading to,

$$\int_{\partial\mathcal{B}} \hat{\varphi}^k t^j(e_k)\mathsf{A}^j - \int_{\mathcal{B}} \hat{\varphi}^k \mathrm{d}(t^j(e_k)) \wedge \mathsf{A}^j = 0 \tag{54}$$

Using the arbitrariness of $\hat{\varphi}^k$, we conclude that,

$$\mathrm{d}(t^j(e_k)) \wedge \mathsf{A}^j = 0 \tag{55}$$

This is the condition for the critical point of the mixed functional in the direction of deformation, which represents the balance of forces. Note that the connection 1-forms of the deformed configuration appear through the exterior derivatives of the frame fields of the deformed configuration.

5.2. Stress a Lagrange multiplier

For a hyper-elastic solid, stress is derived from the stored energy which may be written as a function of the deformation gradient. This assumption permits us to write the equations of equilibrium as the Euler–Lagrange equation of the stored energy functional. In a certain sense, the stress generated in a hyper-elastic solid should satisfy certain integrability condition (i.e. the existence of the stored energy function). Moreover, if we assume the stored energy function to be translation and rotation invariant, it implies equilibrium of forces and moments. Thus for the hyper-elastic solid, balances of forces and moments are consequences of translation and rotation invariance; stress is only a secondary variable introduced for writing the equations of equilibrium in a convenient way.

When formulated as a mixed problem, the stress tensor or more specifically the traction 1-form has a completely different role. Our mixed functional has deformation, deformation 1-forms and stress 2-forms as inputs. For the stored energy, viewed as a function of deformation 1-forms, translation and rotation invariance cannot be discussed directly, since nothing about the geometry of the co-tangent bundle from which the deformation 1-forms were pulled back is known. In other words, there is nothing in the stored energy function that requires the base space for the deformed configuration to be Euclidean. The second term in (41) is introduced to impose this constraint. Observe that, in (41), the second term is multilinear in the input arguments, i.e., stress 2-form, differential of deformation and deformation 1-form. The traction 1-form may now be thought of as a Lagrange multiplier introduced to impose the equality between the differential of deformation and deformation 1-forms. Alternatively, the equality between the differential of deformation and deformation 1-forms implies that the deformed configuration is Euclidean.

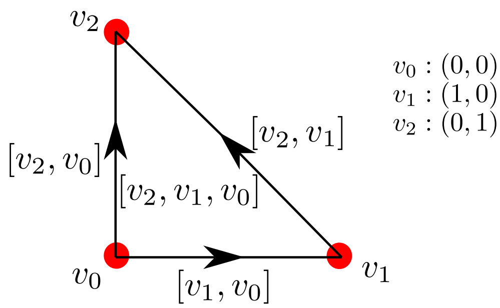


Fig. 3. Reference triangle (2-simplex); a red dot indicates a vertex, while the arrow along an edge indicates its orientation.

6. Discretization of differential forms

In this section, we consider local finite element spaces which are suitable for approximating differential forms over a simplicial complex. A simplicial complex in $\mathbb{R}^m$, denoted by K, is a set with simplices as its elements. By an n-simplex, we mean the convex hull of $n+1$ points in $\mathbb{R}^m$. We denote a general n-simplex by K^n and a specific n-simplex by $[v_0, \dots, v_n]$, where $v_i \in \mathbb{R}^m$; these v_i are referred to as the vertices of the simplex. We may also call a simplex $K^n \in K$, $n \leq m$ a n-face of K. We expect every face of K to be an element in K and if two faces $K_1, K_2 \in K$ intersect, then their intersection is also a face in K. The dimension of K is defined as the largest dimension of the simplex contained in K. Finite element mesh created by triangulating a two dimensional domain is a good example of a two dimensional simplicial complex. Such a finite element mesh has nodes, edges and faces, which are simplices of dimensions 0, 1 and 2 respectively.

From now on we restrict ourselves to two spatial dimensions; techniques discussed here can be extended to any spatial dimensions. The just stated assumption also permits us to work with a simplicial complex of dimension 2. We denote the barycentric coordinates of a 2-simplex (triangle) by λ^i, $i = 0, \dots, 2$. These coordinates satisfy the relation $\sum_{i=0}^{n} \lambda^i = 1$. In terms of the Cartesian coordinates, the barycentric coordinates of the reference triangle are given by,

$$\lambda^0 = 1 - x^1 - x^2; \quad \lambda^1 = x^1; \quad \lambda^2 = x^2, \tag{56}$$

where, x^1 and x^2 are the Cartesian coordinates of a point within the triangle. The reference triangle, along with the vertices and orientation of edges is presented in Fig. 3; we denote this reference triangle by $\hat{K}^2$ or simply by $\hat{K}$.

6.1. Spaces $\mathcal{P}_r \Lambda^n$ and $\mathcal{P}_r^- \Lambda^n$

We denote the space of m variable polynomials of degree r by $\mathcal{P}_r(\mathbb{R}^m)$. The space of polynomial differential forms with form degree n and polynomial degree r is denoted by $\mathcal{P}_r \Lambda^n(\mathbb{R}^m)$, $n \leq m$. Often we suppress $\mathbb{R}^m$ from our notation and simply denote these spaces by $\mathcal{P}_r$ and $\mathcal{P}_r \Lambda^n$. For vector fields, $v_1, \dots, v_n \in T\mathbb{R}^m$,

$$\mathcal{P}_r \Lambda^n = \{\omega \in \Lambda^n(\mathbb{R}^m) | \omega(v_1, \dots, v_n) \in \mathcal{P}_r\}. \tag{57}$$

In other words, for a polynomial differential form ω, the coefficient functions are polynomials of degree r. From the definition of the spaces $\mathcal{P}_r$ and $\mathcal{P}_r \Lambda^n$, their dimensions can be computed as $\binom{r+m}{m}$ and $\binom{n+r}{n}\binom{n}{k}$ respectively. For a differential form ω of degree n, the interior product of ω with a vector $v|_x$, $x \in \mathbb{R}^m$, is given as,

$$\kappa_v \omega = \omega(v, v^1, \dots, v^{n-1}) \tag{58}$$

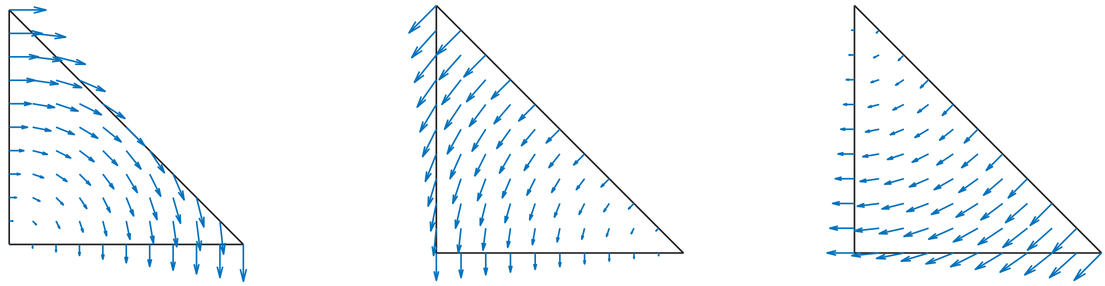
for any vectors, $v^1, \dots, v^{n-1}$. From the above definition it is easy to see that $\kappa_v \omega$ is a differential form of degree $n-1$.

For any point $x \in \mathbb{R}^m$, the vector field $X \in T\mathbb{R}^m$ translates the origin to $x \in \mathbb{R}^m$. Using this vector field X, a Koszul type operator on the space of polynomial differential forms with form degree n can be defined. This operator is given as,

$$\kappa_X \omega = \omega(X, v^1, \dots, v^{n-1}) \tag{59}$$

Table 1
Basis functions for different FE spaces.

FE space	Node $[i]$	Edge $[i, j]$	Face $[i, j, k]$
$\mathcal{P}_1\Lambda^0$	λ^i	–	–
$\mathcal{P}_1^-\Lambda^1$	–	$\lambda^i d\lambda^j - \lambda^j d\lambda^i$	–
$\mathcal{P}_1\Lambda^1$	–	$\lambda^i d\lambda^j, \lambda^j d\lambda^i$	–



(a) ϕ^1 (b) ϕ^2 (c) ϕ^3

Fig. 4. Vector plot of the basis functions from the space $\mathcal{P}_1^-\Lambda^1$, the 1-form basis functions are converted to vector fields by using the Euclidean metric.

where, $v^1, \ldots, v^{n-1}$ are arbitrary vector fields form $T\mathbb{R}^m$. From the definition, it is easy to see that κ_X decreases the degree of a differential form by 1 and increases the polynomial degree by 1. An important property of κ_X is $\kappa_X^2 = 0$. The operator κ_X also commutes with affine pull-back. If T is an affine linear map, $T : \mathbb{R}^m \to \mathbb{R}^m$, then $T^*\kappa_X\omega = \kappa_X T^*\omega$. The polynomial spaces $\mathcal{P}_r^-\Lambda^n$, can be defined as,

$$\mathcal{P}_r^-\Lambda^k = \{\omega \in \mathcal{P}_r\Lambda^k | \kappa_X(\omega) \in \mathcal{P}_r\Lambda^{k-1}\} \tag{60}$$

The dimension of the space $\mathcal{P}_r^-\Lambda(\mathbb{R}^m)$ can be computed as $\binom{r+k-1}{k}\binom{m+r}{m-k}$ which is larger than that of $\mathcal{P}_{r-1}\Lambda^n$ but smaller than that of $\mathcal{P}_r\Lambda^n$. In the case of polynomial differential forms with form degree 0, we have $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$; these spaces can be identified with the Lagrange family of finite element spaces. The spaces $\mathcal{P}_r\Lambda^n$ and $\mathcal{P}_r^-\Lambda^n$ constitute a large family of finite elements. Well known members of this family include the Raviart–Thomas [23] and Nédélec [25] type vector finite elements. These subspaces of polynomial differential forms were proposed by Arnold and co-workers [22] to unify the vector finite elements used in the construction of mixed finite element techniques. These finite dimensional polynomial spaces form the cornerstone for the finite element techniques developed under the umbrella of finite element exterior calculus.

6.2. Degrees of freedom and finite element bases

In the previous subsection, we introduced the polynomial spaces $\mathcal{P}_r^-\Lambda^n$ and $\mathcal{P}_r\Lambda^n$. We now restrict these polynomial spaces to the reference triangle and construct basis functions suitable for computation. The description of the computational basis functions for the spaces $\mathcal{P}_r^-\Lambda^n$ and $\mathcal{P}_r\Lambda^n$ closely follows the work of Arnold et al. [35] where a geometric decomposition was utilized to construct the computational basis function on a simplex. The idea of the geometric decomposition is to index the DoF's of a finite element (FE) space using the sub-simplices of the simplex on which the FE space is constructed. For Lagrange finite elements over simplices, this amounts to assigning DoF's to the vertices of the simplex. For finite elements in the family $\mathcal{P}_r\Lambda^n$ and $\mathcal{P}_r^-\Lambda^n$, DoF's may be indexed with edges, faces and volume. Table 1 gives the basis functions for the spaces $\mathcal{P}_1\Lambda^0$, $\mathcal{P}_1\Lambda^1$ and $\mathcal{P}_1^-\Lambda^1$. From Table 1, it can be seen that $\mathcal{P}_1\Lambda^0$ has DoF only on the vertex, while $\mathcal{P}_1^-\Lambda^1$ has one DoF on each edge and $\mathcal{P}_1\Lambda^1$ has two DoFs per edge. A vector plot of the basis functions for the spaces $\mathcal{P}_1^-\Lambda^1$ and $\mathcal{P}_1\Lambda^1$ are show in Fig. 4 and Fig. 5 respectively.

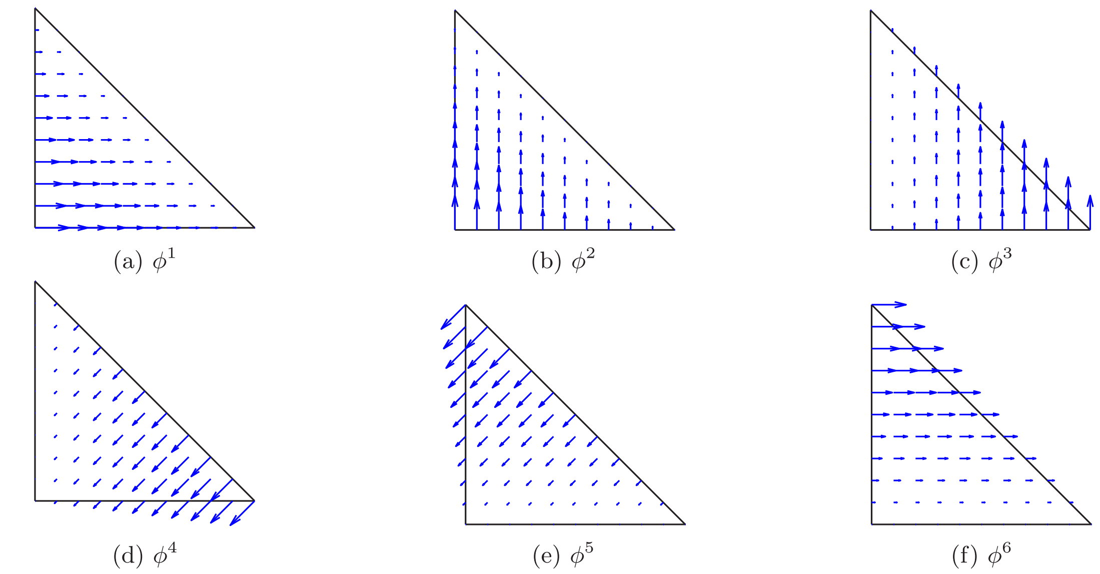


Fig. 5. Vector plot of the basis functions from the space $\mathcal{P}_1\Lambda^1$; the 1-form basis functions are converted to vector fields by using the Euclidean metric.

6.3. Whitney forms and $\mathcal{P}_1^-\Lambda^n$

Differential forms of degree n are functions that take n vectors and produce real numbers. Alternatively, one can also view an n-form as a real number defined on a plane spanned by n tangent vectors. The latter definition is more suitable for the construction of discrete differential forms on a simplicial complex. On a k-simplex, a discrete differential form of degree n can be thought as a real number defined on the k-subsimplex. Using this idea, Hirani [17] constructs a discrete analog of the exterior calculus. For such a discrete approximation, Whitney [30] gave a formula for interpolating these differential forms on a k-simplex. By producing basis functions whose degrees of freedom are the real numbers defined on subsimplices, Whitney was able to interpolate these differential forms within the simplex. The basis functions, constructed by Whitney to approximate differential forms over a simplex are now-a-days referred to as Whitney forms. In terms of the barycentric coordinates, these polynomial differential forms are given by the formula,

$$\phi_i^n = r! \sum_{\epsilon_i \in C(k,r)} (-1)^i \lambda_{\epsilon_i} \left(\mathrm{d}\lambda_{\epsilon_0} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda}_{\epsilon_i} \wedge \cdots \wedge \mathrm{d}\lambda_{\epsilon_r}\right). \tag{61}$$

In the above expression $C(k,r)$ is the r combination of k-element sets from $\{1,\ldots,k\}$. The superscript n in ϕ_i^n indicates the degree of the differential form and i indicates the ith basis function. The symbol $\widehat{(.)}$ means that the term should be removed. Arnold showed that the space spanned by Whitney forms is exactly $\mathcal{P}_1^-\Lambda^n$ [22].

Using Whitney's forms, we now produce basis functions for the space $\mathcal{P}_1^-\Lambda^n$ for different values $0 \leq n \leq 2$. These basis functions can be used to interpolate differential forms of degrees $0,\ldots,2$ defined on a 2-simplex. Whitney forms reduce to the usual linear Lagrange basis functions on a triangle for the case $n = 0$. For $n = 1$, Whitney forms produce three basis functions; they span the space $\mathcal{P}_1^-\Lambda^1$. These basis functions are given by,

$$\phi_1^1 = \lambda_1 \mathrm{d}\lambda_2 - \lambda_2 \mathrm{d}\lambda_1; \quad \phi_2^1 = \lambda_2 \mathrm{d}\lambda_3 - \lambda_3 \mathrm{d}\lambda_2; \quad \phi_3^1 = \lambda_3 \mathrm{d}\lambda_1 - \lambda_1 \mathrm{d}\lambda_3. \tag{62}$$

For differential forms of degree 2, the space of Whitney forms produces one basis function given by,

$$\phi_1^2 = 2\left(-\lambda_1(\mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3) + \lambda_2(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3) - \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2)\right). \tag{63}$$

Table 2 gives a summary of the dimensions and locations of the DoF's for the spaces $\mathcal{P}_1^-\Lambda^n$, for different values of n on a 2-simplex.

Table 2
Dimensions of spaces $\mathcal{P}_1^- \Lambda^n$ on a 2-simplex.

Form degree	Location of DoF	Dimension of FE space
0-form	Nodes	3
1-form	Edges	3
2-form	Face	1

6.4. Coordinate transformation for the FE basis

The FE spaces we have just discussed are affine invariant [27]. This property allows us to construct the finite element basis functions in terms of the barycentric coordinates and then use an affine transformation to write the shape functions in terms of the Cartesian coordinates. The coordinate transformation between Cartesian coordinates and barycentric coordinates of a triangle is given by,

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \lambda^1 P_1 + \lambda^2 P_2 + \lambda^3 P_3 \tag{64}$$

Here P_i are the vertices of the triangle and x^i denote the Cartesian coordinates of the triangle given by the vertices $[P_1, P_2, P_3]$. From the above equation, the relationship between the differentials of the Cartesian and barycentric coordinates can be expressed as,

$$\begin{bmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \end{bmatrix} = (P_1 - P_3)\mathrm{d}\lambda^1 + (P_2 - P_3)\mathrm{d}\lambda^2. \tag{65}$$

In writing the above equation, we have eliminated λ^3 using $\lambda^1 + \lambda^2 + \lambda^3 = 1$. In matrix form, the above relation can be written as,

$$\begin{bmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \end{bmatrix} = T \begin{bmatrix} \mathrm{d}\lambda^1 \\ \mathrm{d}\lambda^2 \end{bmatrix}, \tag{66}$$

where the matrix T has $P_1 - P_3$ and $P_2 - P_3$ as its columns. From the constraint involving the barycentric coordinates, we have,

$$\mathrm{d}\lambda^1 + \mathrm{d}\lambda^2 + \mathrm{d}\lambda^3 = 0 \tag{67}$$

In Table 1, the shape functions for the 1-form spaces, $\mathcal{P}_1^1\lambda^1$ and $\mathcal{P}_1^{1-}\lambda^1$ are presented in the barycentric coordinate system. These shape functions can be transformed to the Cartesian coordinate system using (64) and (66). These calculations used to transform the 1-form basis functions are consistent with the laws of transformation of 1-forms [36].

7. Discretization of modified mixed functional

We now undertake a discrete approximation for the mixed functional discussed in Section 5. To achieve this, we first construct discrete approximations to the configurations and fields defined on it. The configurations of the body are discretized using a simplicial approximation, with $K(\mathcal{B})$ and $K(\mathcal{S})$ denoting the simplicial approximation for $\mathcal{B}$ and $\mathcal{S}$ respectively. In a 2-dimensional case, a simplicial approximation amounts for placing a triangular finite element mesh on a configuration. The objective now is to find a simplicial map, which can produce $K(\mathcal{S})$ given $K(\mathcal{B})$, material and boundary data. By a simplicial map we mean a function which sends the vertices of $K(\mathcal{B})$ to the vertices of $K(\mathcal{S})$. In the present study, we require the simplicial map to preserve the topology of $K(\mathcal{B})$; that is, the connectivity of the edges and faces should be preserved by the deformation.

The next step in discretizing the new functional is to get a discrete approximation for the fields defined on the two configurations of the body. An important point here is that, the proposed variational principle has differential forms (not functions) as input. In a conventional (scalar) FE method, Lagrange basis functions may be used to approximate the trial solution on a 2-simplex, DoF's are often associated with the vertices. In the present context, we understand scalar valued functions as 0-forms and a conventional FE approximation (using Lagrange basis functions) as a finite dimensional approximation of differential forms of degree zero. As in the conventional finite element

Table 3
FE spaces used to approximate different fields.

Field	FE space	# DoF per element
Displacement	$\mathcal{P}_1\Lambda^0$	3
Deformation 1-form	$\mathcal{P}_1\Lambda^1$	6
Traction 1-form	$\mathcal{P}_1^-\Lambda^1$	3

method, constructing an FE approximation for a differential form amounts to identifying a suitable finite dimensional approximation space and enumerating a basis set for this space. Basis functions and DoF's associated with the finite dimensional approximation of differential forms defined on an n-simplex have been discussed in Section 6. We use these finite dimensional spaces of differential forms to discretize our variation principle in Section 5. Since we are dealing with two dimensional problems, we expect the approximation for deformation to be in a suitable finite dimensional subspace of $\Lambda^0 \times \Lambda^0$. The approximate deformation and traction 1-forms should be from a subspace contained in Λ^1. Yavari and co-workers [13–15] have presented an alternative approach to the discretization of an HW-type variational principle; they construct tensorial analogs of Raviart–Thomas and Nédélec finite elements to discretize the Piola stress and displacement gradient. In contrast, the variational principle considered in this work permits us to use FE approximations for scalar valued differential forms (Raviart–Thomas, Nédélec and other well known finite elements) directly without the need for constructing the so-called approximation spaces for vector valued differential forms.

In the following, we use FE spaces with polynomial degree 1 to approximate the mixed variational principle. A summary of different finite element spaces used to approximate the different fields is presented in Table 3. A detailed study discussing the stability of various combinations of FE spaces used to approximate displacement, displacement gradient and Piola stress was under taken by Angoshtari *at al.* [15]; it was found that the not all combinations of FE spaces introduced in CSMFEM were stable. Using the FE approximation for a 1-form, the deformation 1-forms θ^i may be written as,

$$\theta_h^i = \sum_{j=1}^{n} \hat{\theta}_j^i \psi^j, \tag{68}$$

where, ψ_j span the space $\mathcal{P}_1\Lambda^1$. A subscript h is utilized in the above equation to indicate that the right hand side is only an approximation to θ^i. This finite dimensional approximation may be conveniently written as,

$$\theta_h^i = \boldsymbol{\psi}\boldsymbol{\theta}^i. \tag{69}$$

where, $\boldsymbol{\psi}$ is a matrix with ψ^i as its columns, $\boldsymbol{\theta}^i$ is a vector containing the DoF's of θ^i as its components. Similarly, the finite dimensional approximation for the traction 1-forms may be written as,

$$t_h^i = \sum_{j=1}^{n} \hat{t}_j^i \phi^j, \tag{70}$$

Here, ϕ_j are 1-forms spanning the space $\mathcal{P}_r^-\Lambda^1$ and $\hat{t}_j^i$ are the DoF's associated with t^i. In matrix form, the above approximation becomes,

$$t_h^i = \boldsymbol{\phi}\mathbf{t}^i, \tag{71}$$

where, $\boldsymbol{\phi}$ is a matrix with the basis of $\mathcal{P}_r^-\Lambda^1$ as its columns. The finite dimensional approximation for deformation and its differential can be written as,

$$\varphi_h^i = \sum_{j=1}^{n} \hat{\varphi}_j^i \lambda^j; \quad \mathrm{d}\varphi_h^i = \sum_{j=1}^{n} \hat{\varphi}_j^i \mathrm{d}\lambda^j, \tag{72}$$

where, λ^j denotes the Lagrange basis functions and $\hat{\varphi}_j^i$ denotes the DoF associated with the ith deformation component. In matrix form, the finite dimensional approximation of the differential of deformation can be written as,

$$\mathrm{d}\varphi_h^i = \mathrm{d}\mathbf{N}\boldsymbol{\varphi}^i, \tag{73}$$

Here, d$\mathbf{N}$ denotes the matrix with exterior derivatives of the Lagrange shape functions as its columns and $\boldsymbol{\varphi}^i$ is the DoF vector associated with φ^i. Having introduced the discrete approximation for configuration and fields, we may now write the discrete approximation for the mixed functional as,

$$I^h = \int_{K(\mathcal{B})} W^h(\theta^i_h)dV - (t^i_h \otimes \mathrm{d}\mathsf{A}^i_h)\dot{\wedge}(E_i \otimes (\theta^i_h - \mathrm{d}\phi^i_h)) - \int_{K(\partial\mathcal{B})} \langle t^\sharp, \boldsymbol{\varphi}\rangle \mathrm{d}A. \tag{74}$$

In the last equation t is the traction impressed on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$. Also notice that in (74), we have assumed the frame fields for both deformed configuration to be same as that of the reference configuration. Our aim here is to construct numerical approximations for the 2-dimensional case. Recall that for an n-dimensional body, stress is a co-vector-valued differential form of degree $n-1$. Hence the Piola stress becomes a co-vector valued 1-form, the area-forms in the Piola stress can be identified as $d\mathsf{A}^1 = \theta^2$ and $d\mathsf{A}^2 = \theta^1$. Incorporating these details in (74), the mixed functional for the 2-dimensional case can be written as,

$$I^h = \int_{K(\mathcal{B})} W^h(\theta^1_h, \theta^2_h)dA - \int_{K(\mathcal{B})} (t^1_h \otimes \theta^2_h + t^2_h \otimes \theta^1_h)\dot{\wedge}(E_1 \otimes (\theta^1_h - \mathrm{d}\phi^1_h) + E_2 \otimes (\theta^2_h - \mathrm{d}\phi^2_h)) - \int_{K(\partial\mathcal{B})} \langle t^\sharp, \boldsymbol{\varphi}\rangle \mathrm{d}L. \tag{75}$$

In the above equation, dL denotes the infinitesimal line element of the boundary curve $\partial\mathcal{B}$.

7.1. Residue and tangent operator

We first recall the definition of Gateaux derivative. Let $I^h : \mathcal{V}_n \times \cdots \times \mathcal{V}_n \to \mathbb{R}$ be a real valued function defined on the product space $\mathcal{V}_1 \times \cdots . \times \mathcal{V}_n$, each $\mathcal{V}_i$ being a vector space of finite or infinite dimension. If we pick elements $(w_1, \ldots, w_i, \ldots, w_n) \in \mathcal{V}_1 \times \cdots \times \mathcal{V}_i \times \cdots \times \mathcal{V}_n$, the Gateaux derivative of I in the direction $v_i \in \mathcal{V}_i$ is denoted by $\mathsf{D}_{w_i} I[v_i]$ and is given by the following limit,

$$\begin{aligned} \mathsf{D}_{w_i} I[v_i] &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}[I(w_1, \ldots, (w_i + \epsilon v_i), \ldots, w_n) - I(w_1, \ldots, w_i, \ldots, w_n)] \\ &= \frac{d}{d\epsilon}[I(w_1, \ldots, (w_i + \epsilon v_i), \ldots, w_n) - I(w_1, \ldots, w_i, \ldots, w_n)] \end{aligned} \tag{76}$$

The derivative is thus evaluated at $(w_i, \ldots, w_n)$ and it produces an element from $\mathcal{V}_i^*$. Similarly, one may also define the second derivative of I by applying the above definition twice. The second derivative of I^h is thus denoted by $\mathsf{D}_{w_i}\mathsf{D}_{w_j} I^h$. We now apply the definition of Gateaux derivative to compute the first and the second derivatives of the discrete functional. As discussed earlier, the discrete functional is obtained by restricting the spaces $\mathcal{V}_i$ to a suitable finite dimensional subspace; we denote these spaces by $\mathcal{V}_i^h$. Formally, the discrete variational functional I^h can be written as, $I^h : \mathcal{V}^h_{\theta^1} \times \mathcal{V}^h_{\theta^2} \times \mathcal{V}^h_{t^1} \times \mathcal{V}^h_{t^2} \times \mathcal{V}^h_{\varphi^1} \times \mathcal{V}^h_{\varphi^2} \to \mathbb{R}$; here $\mathcal{V}^h_{\theta^i}$, $\mathcal{V}^h_{t^i}$ and $\mathcal{V}^h_{\varphi^i}$ denote the finite dimensional approximation spaces for deformation 1-form, traction 1-form and deformation respectively. In the discrete functional, we do not discretize the frame fields E^i since we choose a fixed Cartesian frame for the reference configuration. The dimension of the product space on which the extremization problem is posed largely depends on the FE mesh and the finite dimensional approximation space utilized to discretize different fields. Strain energy functions associated with non-linear elasticity is non-quadratic; thus finding extremizers for the discrete functional now falls within the realm of numerical optimization. We utilize Newton's method to numerically find the extremizers of the discrete functional. Newton's method involves the computation of the first and second derivatives associated with the discrete functional given in (75). In the finite element literature, the first and second derivatives of the discrete energy functional are often called the residual vector and tangent operator respectively. The components of a residual vector are the generalized forces acting on the respective DoF. Thus the global residual vector is obtained by stacking the first derivatives of the discrete functional with respect to different DoF one above the other. Formally, the residue vector can be written as,

$$\mathcal{R} = \left[\mathsf{D}_{\boldsymbol{\theta}^1} I^h,\ \mathsf{D}_{\boldsymbol{\theta}^2} I^h,\ \mathsf{D}_{\mathbf{t}^1} I^h,\ \mathsf{D}_{\mathbf{t}^2} I^h,\ \mathsf{D}_{\boldsymbol{\varphi}^1} I^h,\ \mathsf{D}_{\boldsymbol{\varphi}^2} I^h\right]^t \tag{77}$$

The matrix form of the tangent operator may now be written as,

$$\mathcal{K} = \begin{bmatrix} \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\boldsymbol{\theta}^1}I^h & \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\boldsymbol{\theta}^2}I^h & \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\mathbf{t}^1}I^h & \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\mathbf{t}^2}I^h & \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\boldsymbol{\varphi}^1}I^h & \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h \\ & \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\theta}^2}I^h & \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\mathbf{t}^1}I^h & \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\mathbf{t}^2}I^h & \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\varphi}}I^h & \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h \\ & & \mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\mathbf{t}^1}I^h & \mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\mathbf{t}^2}I^h & \mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\boldsymbol{\varphi}^1}I^h & \mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h \\ & \text{Symmetric} & & \mathsf{D}_{\mathbf{t}^2}\mathsf{D}_{\mathbf{t}^2}I^h & \mathsf{D}_{\mathbf{t}^2}\mathsf{D}_{\boldsymbol{\varphi}^1}I^h & \mathsf{D}_{\mathbf{t}^2}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h \\ & & & & \mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{\boldsymbol{\varphi}^1}I^h & \mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h \\ & & & & & \mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h \end{bmatrix} \tag{78}$$

It should be noted that the tangent operator is symmetric since it is obtained as the Hessian of an energy functional. We now present the expressions for the first and second derivatives of the discrete functional.

$$\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} \mathrm{D}_{\boldsymbol{\theta}^1}W^h - t_h^1(E_2)\boldsymbol{\psi}\wedge\theta_h^2 + t_h^2(E_1)\boldsymbol{\psi}\wedge\theta_h^2 + t_h^1(E_1)\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^1 + t_h^1(E_2)\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^2 \tag{79}$$

$$\mathsf{D}_{\boldsymbol{\theta}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} \mathrm{D}_{\boldsymbol{\theta}^2}W^h + t_h^1(E_2)\boldsymbol{\psi}\wedge\theta_h^1 - t_h^2(E_1)\boldsymbol{\psi}\wedge\theta_h^1 + t_h^2(E_1)\boldsymbol{\psi}\wedge \mathrm{d}\phi_h^1 + t_h^2(E_2)\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^2 \tag{80}$$

$$\mathsf{D}_{\mathbf{t}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} \theta_h^1\wedge(\mathrm{d}\varphi_h^2 - \theta_h^2)\boldsymbol{\phi}^t E_2 + (\theta_h^1\wedge \mathrm{d}\varphi_h^1)\boldsymbol{\phi}^t E_1 \tag{81}$$

$$\mathsf{D}_{\mathbf{t}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} \theta_h^2\wedge(\mathrm{d}\varphi_h^1 - \theta_h^1)\boldsymbol{\phi}^t E_1 + (\theta_h^2\wedge \mathrm{d}\varphi_h^2)\boldsymbol{\phi}^t E_2 \tag{82}$$

$$\mathsf{D}_{\boldsymbol{\varphi}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} -t_h^1(E_1)\mathrm{d}\mathbf{N}\wedge\theta_h^1 - t_h^2(E_1)\mathrm{d}\mathbf{N}\wedge\theta_h^2 \tag{83}$$

$$\mathsf{D}_{\boldsymbol{\varphi}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} -t_h^1(E_2)\mathrm{d}\mathbf{N}\wedge\theta_h^1 - t_h^2(E_2)\mathrm{d}\mathbf{N}\wedge\theta_h^2 \tag{84}$$

The second derivative of the discrete functional may be computed as,

$$\mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} \mathsf{D}_{\boldsymbol{\theta}^1}\mathsf{D}_{\boldsymbol{\theta}^1}W^h; \quad \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\theta}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\theta}^2}W^h \tag{85}$$

$$\mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} \mathsf{D}_{\boldsymbol{\theta}^2}\mathsf{D}_{\boldsymbol{\theta}^1}W^h + (t_h^2(E_1) - t_h^1(E_2))(\boldsymbol{\phi}\wedge\boldsymbol{\phi}) \tag{86}$$

$$\mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\mathbf{t}^1}I^h = 0; \quad \mathsf{D}_{\mathbf{t}^2}\mathsf{D}_{\mathbf{t}^2}I^h = 0; \quad \mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\mathbf{t}^2}I^h = 0 \tag{87}$$

$$\mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{\boldsymbol{\varphi}^1}I^h = 0; \quad \mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h = 0; \quad \mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{\boldsymbol{\varphi}^2}I^h = 0 \tag{88}$$

$$\mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} -(\boldsymbol{\psi}\wedge\theta_h^2)\otimes\boldsymbol{\phi}^t E_2 + (\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^1)\otimes\boldsymbol{\phi}^t E_1 + (\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^2)\otimes\boldsymbol{\phi}^t E_2 \tag{89}$$

$$\mathsf{D}_{\mathbf{t}^2}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} (\boldsymbol{\psi}\wedge\theta_h^2)\otimes\boldsymbol{\phi}E_1; \quad \mathsf{D}_{\mathbf{t}^1}\mathsf{D}_{\boldsymbol{\theta}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} (\boldsymbol{\psi}\wedge\theta_h^1)\otimes\boldsymbol{\psi}^t E_2 \tag{90}$$

$$\mathsf{D}_{\mathbf{t}^2}\mathsf{D}_{\boldsymbol{\theta}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} -(\boldsymbol{\psi}\wedge\theta_h^1)\otimes\boldsymbol{\phi}^t E_1 + (\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^1)\otimes\boldsymbol{\phi}^t E_1 + (\boldsymbol{\psi}\wedge \mathrm{d}\varphi_h^2)\otimes\boldsymbol{\phi}^t E_2 \tag{91}$$

$$\mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} t_h^1(E_1)(\boldsymbol{\psi}\wedge \mathrm{d}\mathbf{N}); \quad \mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} t_h^1(E_2)(\boldsymbol{\psi}\wedge \mathrm{d}\mathbf{N}) \tag{92}$$

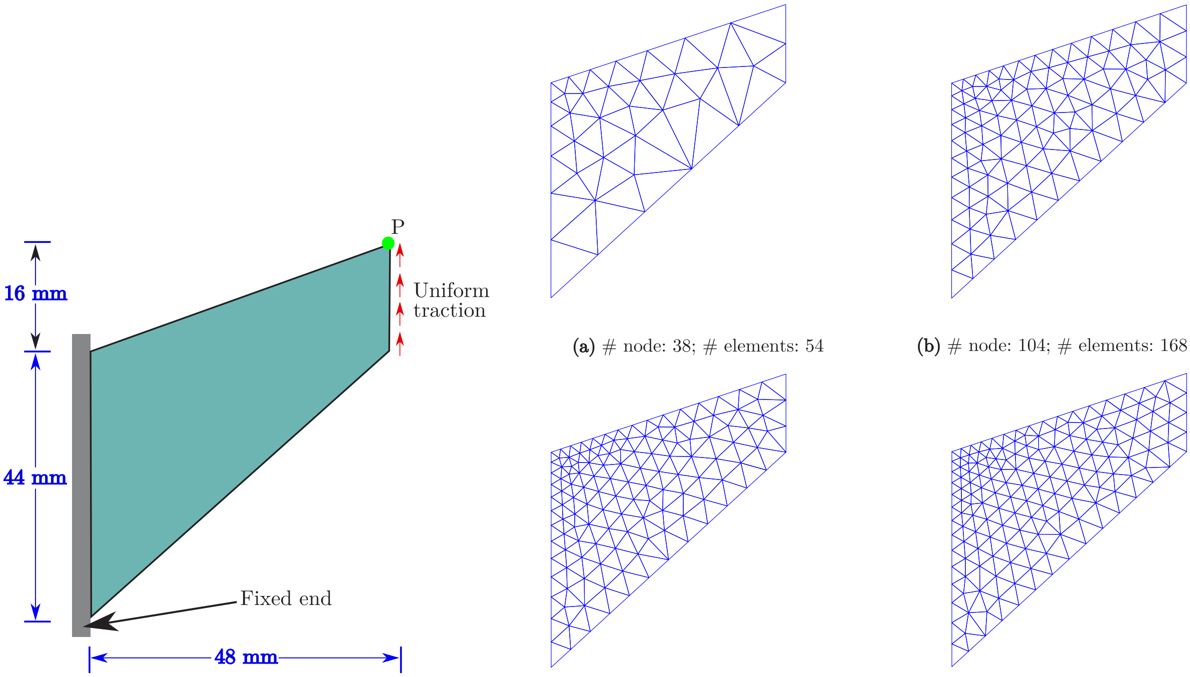
$$\mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{\boldsymbol{\theta}^1}I^h = \int_{\mathcal{T}_\mathcal{B}} t_h^2(E_1)(\boldsymbol{\psi}\wedge \mathrm{d}\mathbf{N}); \quad \mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{\boldsymbol{\theta}^2}I^h = \int_{\mathcal{T}_\mathcal{B}} t_h^2(E_2)(\boldsymbol{\psi}\wedge \mathrm{d}\mathbf{N}) \tag{93}$$

$$\mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{t^1}I^h = \int_{\mathcal{T}_\mathcal{B}} -\boldsymbol{\phi}E_1\otimes(\mathrm{d}\mathbf{N}\wedge\theta_h^1); \quad \mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{t^1}I^h = \int_{\mathcal{T}_\mathcal{B}} -\boldsymbol{\phi}E_2\otimes(\mathrm{d}\mathbf{N}\wedge\theta_h^1) \tag{94}$$

$$\mathsf{D}_{\boldsymbol{\varphi}^1}\mathsf{D}_{t^2}I^h = \int_{\mathcal{T}_\mathcal{B}} -\boldsymbol{\phi}E_1\otimes(\mathrm{d}\mathbf{N}\wedge\theta_h^2); \quad \mathsf{D}_{\boldsymbol{\varphi}^2}\mathsf{D}_{t^2}I^h = \int_{\mathcal{T}_\mathcal{B}} -\boldsymbol{\phi}E_2\otimes(\mathrm{d}\mathbf{N}\wedge\theta_h^2) \tag{95}$$

8. Numerical results

We apply the mixed FE approximation based on our variational principle to numerically study solutions of a few benchmark problems; the objective is to demonstrate its efficacy against numerical instabilities such as volume and



(a) # node: 38; # elements: 54

(b) # node: 104; # elements: 168

(c) # node: 136; # elements: 227

(d) # node: 164; # elements: 278

(a) Cook's membrane: Dimensions of the domain along with boundary conditions; displacement is measured at the point P.

(b) FE mesh used to study convergence

Fig. 6. (a) Boundary conditions used to study the Cook's membrane; (b) representative sequence of mesh used in the convergence study.

bending locking. We consider only two-dimensional problems. The material response in the simulations is calculated using a neo-Hookean type stored energy function with the density given by [37],

$$W(\theta^1, \theta^2) = \frac{\mu}{2}(I_1 - 2) - \mu \ln J + \frac{\kappa}{2}(\ln J)^2. \tag{96}$$

A discrete approximation to the stored energy density is obtained by replacing the deformation 1-forms by their finite dimensional approximations. Using the discrete stored energy functional, the contribution of the stored energy density to the residue and tangent may be computed as the first and second derivatives of (96). These derivatives are computed using the definition of Gateaux derivative in (76). The first derivative of the stored energy function can be evaluated as,

$$\mathsf{D}_{\boldsymbol{\theta}^i} W^h = \frac{\mu}{2}\mathsf{D}_{\boldsymbol{\theta}^i} I_1^h + \left(\frac{\kappa \ln J^h}{J} - \frac{\mu}{J^h}\right)\mathsf{D}_{\boldsymbol{\theta}^i} J^h. \tag{97}$$

Here, I_1^h and J^h denote the finite dimensional approximations for I_1 and J. The first derivative of I_1^h with respect to $\boldsymbol{\theta}^i$ may be computed as,

$$\mathsf{D}_{\boldsymbol{\theta}^i} I_1^h = 2\boldsymbol{\psi}\boldsymbol{\theta}^i. \tag{98}$$

Similarly, the first derivative of J^h with respect to $\boldsymbol{\theta}^i$ may be computed as,

$$\mathsf{D}_{\boldsymbol{\theta}^1} J^h = \boldsymbol{\psi} \wedge \theta^2; \quad \mathsf{D}_{\boldsymbol{\theta}^2} J^h = -\boldsymbol{\psi} \wedge \theta^1 \tag{99}$$

The stiffness associated with 1-form degrees of freedom may be evaluated as,

$$\mathsf{D}_{\boldsymbol{\theta}^j}\mathsf{D}_{\boldsymbol{\theta}^i} W^h = \frac{\mu}{2}\mathsf{D}_{\boldsymbol{\theta}^j}\mathsf{D}_{\boldsymbol{\theta}^i} I_1^h + \left[\frac{\mu}{(J^h)^2} + \frac{\kappa}{(J^h)^2} - \frac{\kappa \ln J^h}{(J^h)^2}\right]\mathsf{D}_{\boldsymbol{\theta}^j} J^h \otimes \mathsf{D}_{\boldsymbol{\theta}^i} J^h + \left[\frac{-\mu}{J^h} + \frac{\kappa \ln J^h}{J^h}\right]\mathsf{D}_{\boldsymbol{\theta}^j}\mathsf{D}_{\boldsymbol{\theta}^i} J^h. \tag{100}$$

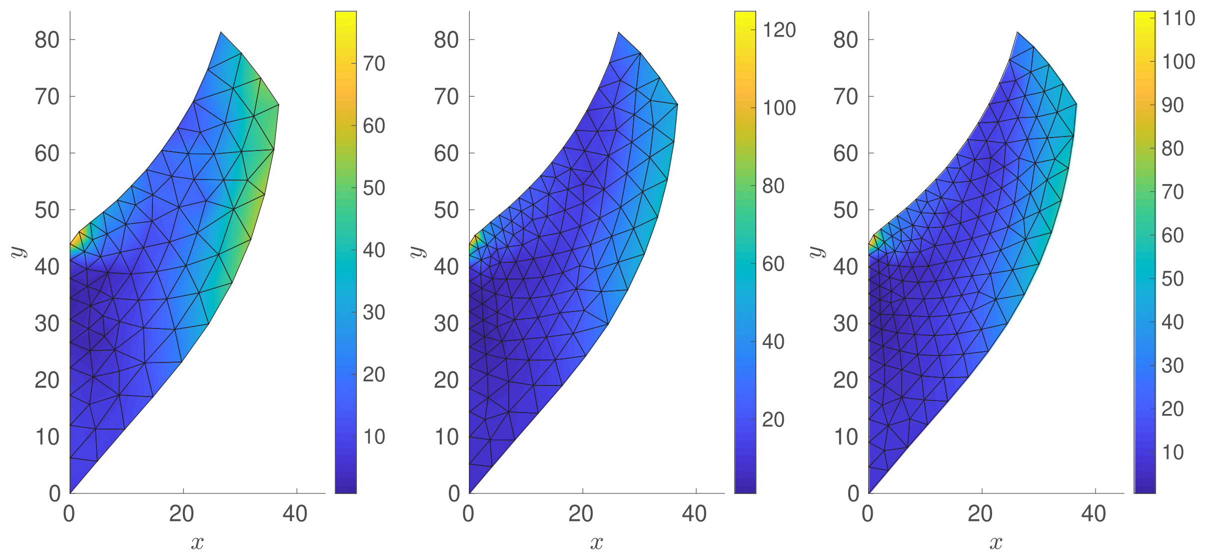


Fig. 7. Deformed configuration of Cook's membrane for a load level of 32 N/mm^2. The colors indicate the norm of the first Piola stress. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Explicit expressions for the second derivatives of I_1^h may be computed as,

$$\mathsf{D}_{\boldsymbol{\theta}^i}\mathsf{D}_{\boldsymbol{\theta}^j} I_1^h = \begin{cases} 2\boldsymbol{\psi}^t\boldsymbol{\psi}; & i=j \\ 0; & i \neq j. \end{cases} \tag{101}$$

Similarly, the second derivatives of J^h may be computed as,

$$\mathsf{D}_{\boldsymbol{\theta}^i}\mathsf{D}_{\boldsymbol{\theta}^j} J^h = \begin{cases} 0; & i=j \\ \boldsymbol{\psi}\wedge\boldsymbol{\psi}; & i \neq j. \end{cases} \tag{102}$$

The convergence of deformation 1-forms is assessed through the integral $\int_{\mathcal{B}} \left\|\theta^1\right\|^2 + \left\|\theta^2\right\|^2 dV$. Here $\|.\|$ is given by the metric tensor of the reference configuration. The convergence of stress is evaluated using $\int_{\mathcal{B}} \|P\|\, dV$, where P is the first Piola stress in the conventional sense, which can be constructed using the traction and deformation 1-forms in the following manner,

$$P = t^1 \otimes \theta^2 + t^2 \otimes \theta^1. \tag{103}$$

In writing the above equation, use has been made of the definition of stress as a co-vector valued 1-form.

8.1. Cook's membrane problem

We first study the performance of our finite element model against bending locking at the incompressible limit. Cook's membrane is a standard benchmark problem used to test the efficiency of a finite element model against bending induced numerical instabilities. Cook's membrane is a trapezoidal cantilever beam; the domain and boundary conditions used in the simulations are shown in Fig. 6(a). A representative sequence of the finite element mesh used to numerically study the convergence is shown in Fig. 6(b). We choose the material constants as $\mu = 80.194\ N/mm^2$, $\kappa = 400889.8\ N/mm^2$; these parameters correspond to a quasi-incompressible material. The convergence of the tip displacement (at point P) is shown in Fig. 8. The convergence plot obtained by Reese [37] and Angoshtari et al. [15] is also plotted alongside. Results on convergence of the deformation 1-form and first Piola stress are given in Fig. 9. In Fig. 7, the deformed configuration predicted by our mixed FE method along with the norm of the first Piola stress is presented.

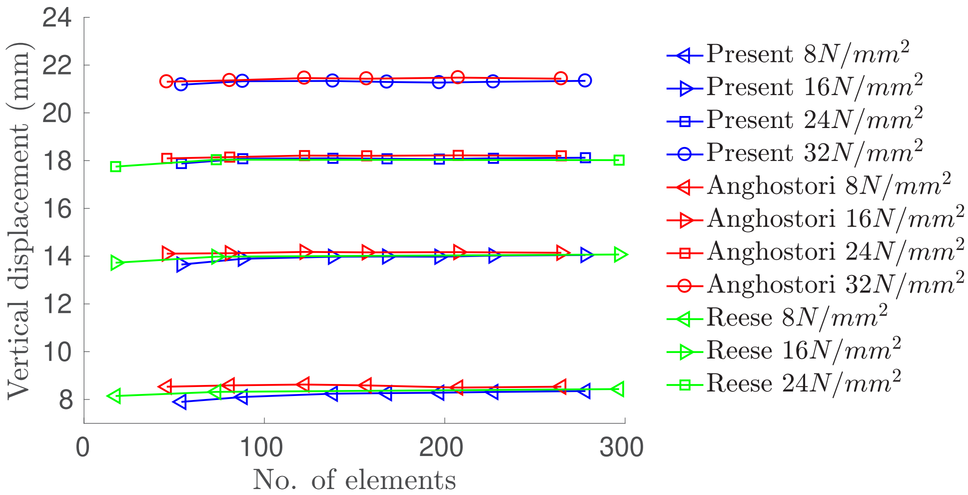
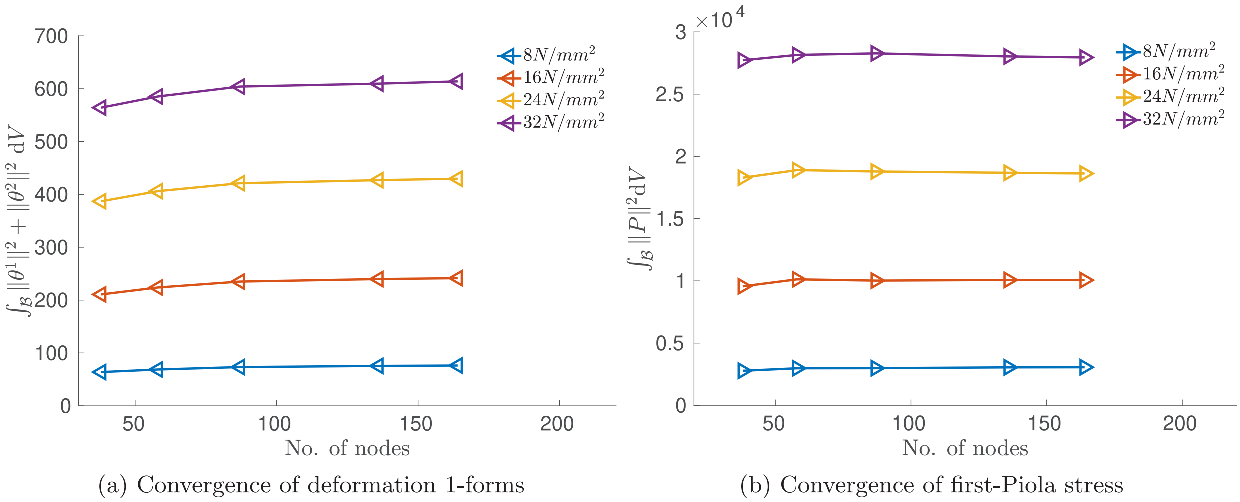


Fig. 8. Convergence of displacement at point P for different load levels and mesh refinements. For comparison, we have plotted the convergence curves given in Reese and Wriggers [38] (in green) and Angoshtari et al. [15](in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

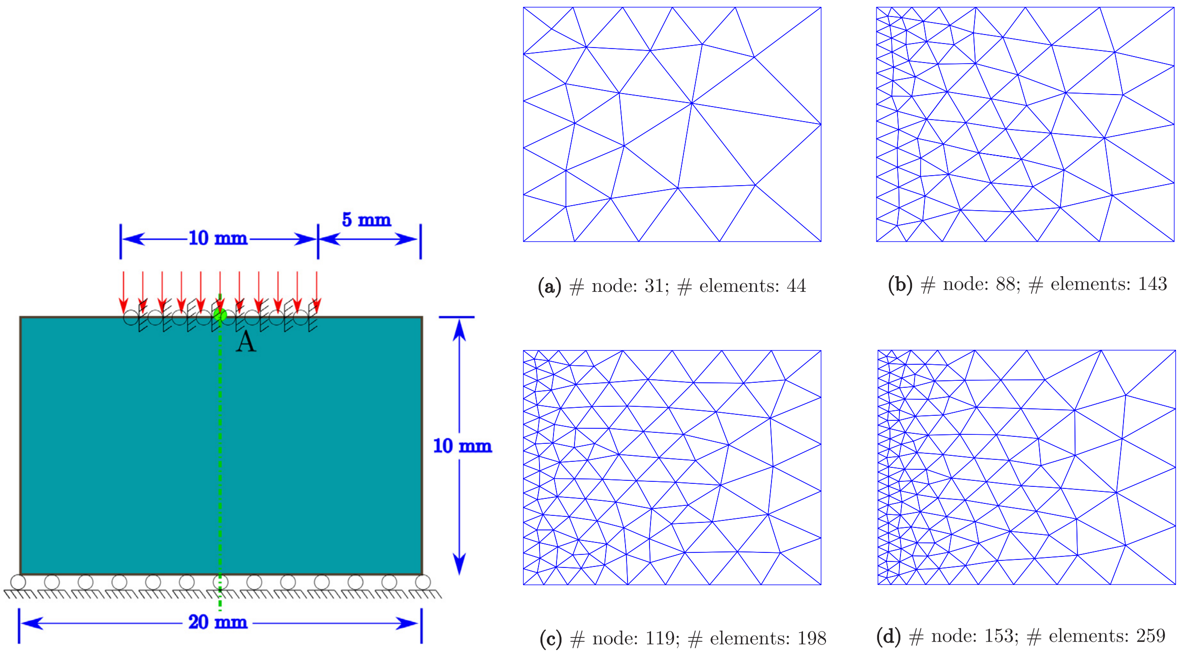


(a) Convergence of deformation 1-forms (b) Convergence of first-Piola stress

Fig. 9. Convergence of deformation 1-form and first-Piola stress for different load levels.

8.2. Compression of a rectangular block

We now use the finite element approximation to study the compression of a rectangular block under quasi-incompressible conditions. The neo-Hookean material model with the same material constants as in Cook's membrane problem is used in this simulation. The domain and boundary conditions are shown in Fig. 10. The convergence of displacement at the point A for different finite element meshes is shown in Fig. 12. For comparison, the displacement convergence plot obtained using the present FE approximation is reported along with the corresponding convergence plots obtained by Reese [37] and Angoshtari et al. [15]. The deformed configuration predicted by our finite element model is also shown in Fig. 11. It should be mentioned that our present FE approximation, which is of polynomial degree 1, performs comparable to the FE approximation of Angoshtari et al. which uses polynomial degree 2. The convergence of deformation 1-forms and Piola stress are also computed and



(a) Boundary conditions

(b) FE mesh used in the simulation

Fig. 10. (a) Boundary conditions used to study the compression of the rectangular block; (b) representative sequence of mesh used in the convergence study.

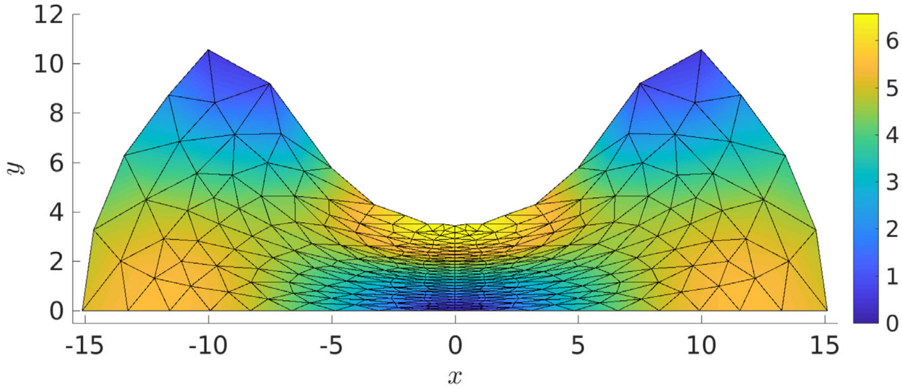


Fig. 11. Deformed shape predicted by our FE simulation; the deformation is mirrored about the y-axis for clarity. Colors on the deformed configuration indicate the norm of displacement. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

presented in Fig. 13. From these convergence curves, it is amply clear that the field quantities of interest converge well even with a relatively coarse mesh.

8.3. Extension of plate with hole

Here, we numerically study the extension of a square plate with a circular hole under applied displacement field. The plate has a side length of 2 cm and a hole of radius 0.5 cm placed at the centroid. neo-Hookean material model (96) with the material parameters $\kappa = 1000$ and $\mu = 10$ is used to study the response of the plate; the assumed material parameters correspond to a compressible hyperelastic material. The simulation is performed under

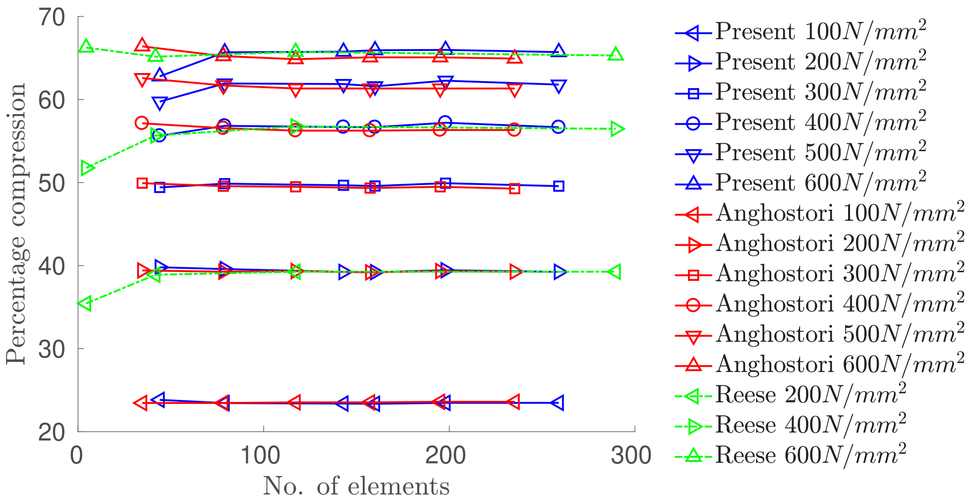
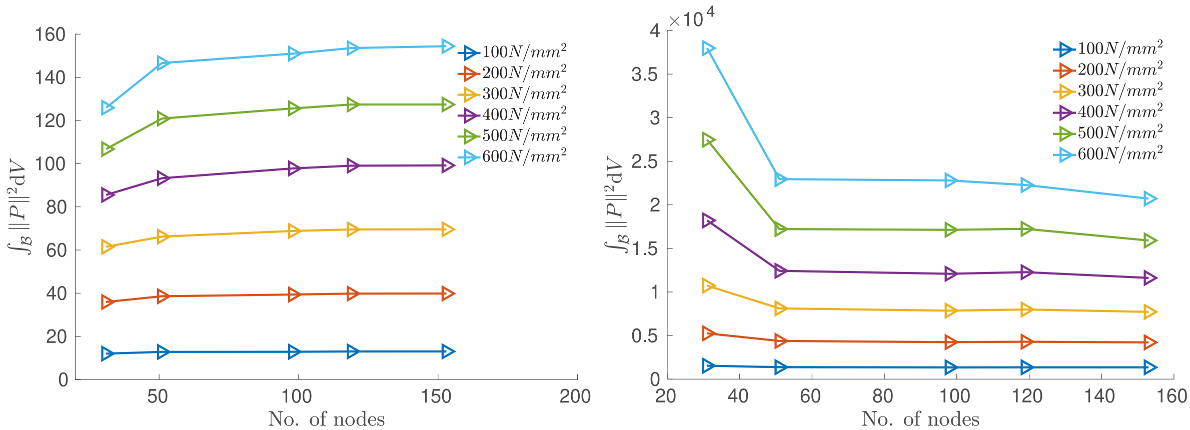


Fig. 12. Convergence of displacement at the point P for different load levels. For comparison we have plotted the convergence results of Reese and Wriggers [38] and Angoshtari et al. [15].



(a) Convergence of deformation 1-forms

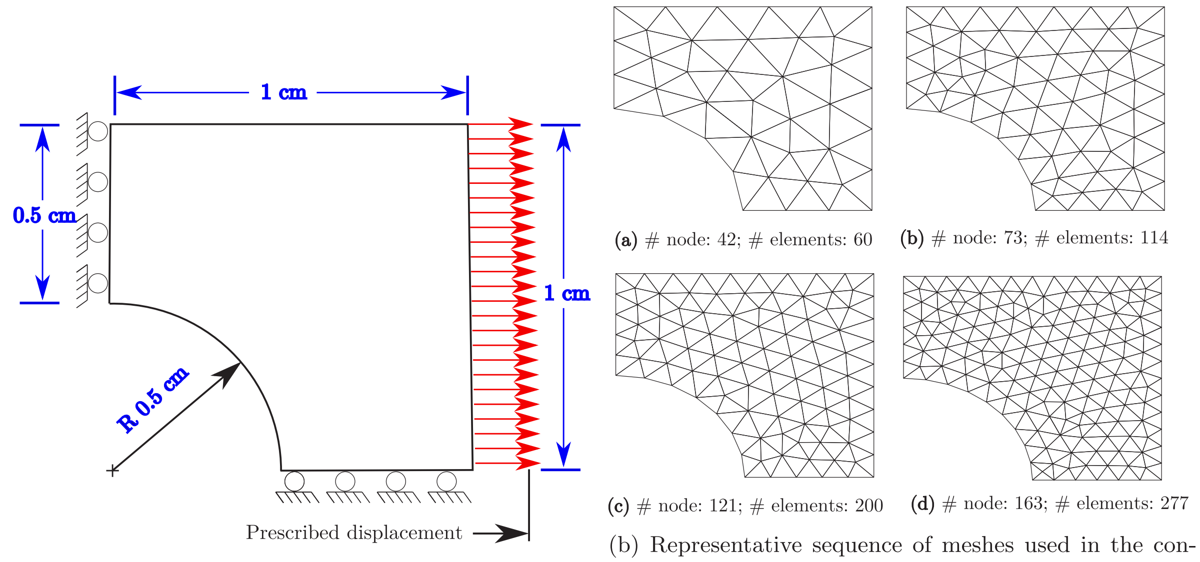
(b) Convergence of first-Piola stress

Fig. 13. Convergence plots for the deformation 1-form and first-Piola stress for different load levels.

a displacement controlled condition, with displacements prescribed along the face whose normal is along the positive x direction. Taking into account the symmetry of the problem, only a quarter of the domain is modeled. Boundary conditions used to simulate the deformation of the plate are presented in Fig. 14. The convergence of the deformation 1-form and Piola stress is shown in Fig. 15. The deformed configuration along with the norm of first Piola stress computed using different finite element meshes is shown in Fig. 16.

9. Conclusions

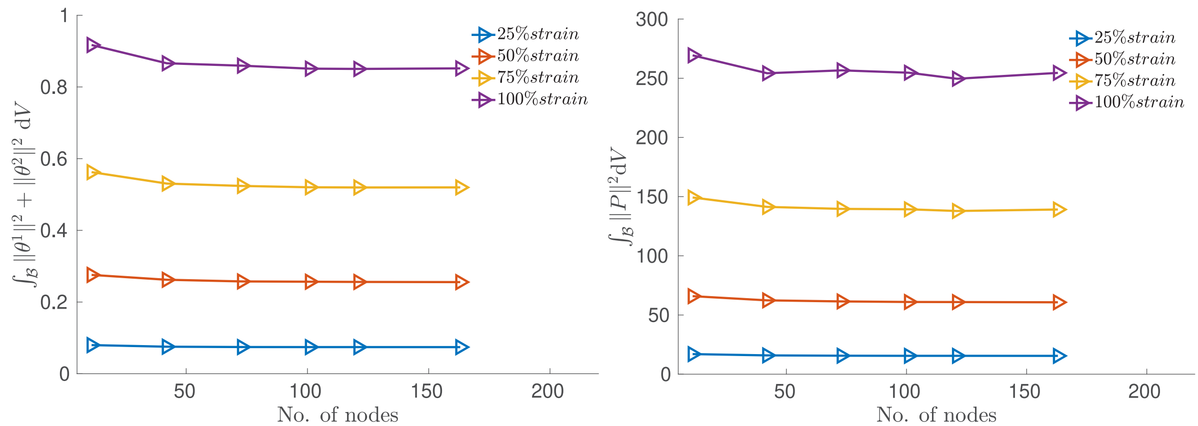
In this study, an entirely new perspective to mixed variational principle in nonlinear elasticity using Cartan's moving frames is presented. This approach to the mixed variational principle facilitates a reformulation of the mixed functional in terms of differential forms. It is also demonstrated that the equations of mechanical equilibrium, constitutive rule, and compatibility could be obtained as conditions for the critical point of the reformulated mixed functional. A closer examination of the critical points reveals that in the absence of compatibility, additional stresses



(a) Quarter model of a plate with a hole

(b) Representative sequence of meshes used in the convergence study

Fig. 14. Boundary conditions and FE mesh used in the convergence study.



(a) Convergence of deformation 1-forms

(b) Convergence of first-Piola stress

Fig. 15. Convergence plots of deformation 1-form and first-Piola stress for different load levels.

could appear in the body, without altering the constitutive rule. These additional stresses, though spurious in a strictly Euclidean setup, offer a pointer to an extension of the present variational principle to problems with incompatibilities like dislocations and disclinations. Our present approach to the kinematics of continua using Cartan's moving frame may play a pivotal role in such an endeavor, since the setup readily permits the incorporation of torsion and curvature into the formulation. These aspects of the proposed variational approach require further study and should be pursued.

The mixed variational principle based on differential forms is utilized to construct novel mixed FE model to solve 2D problems in nonlinear elasticity. This finite element approach uses finite element exterior calculus to discretize the differential forms appearing in the variational principle. The present finite element model does not require additional stabilization terms, since it strictly adheres to the algebraic and geometric structures defined by differential forms, even at the discrete level. The numerical studies using our finite element model clearly demonstrate its superior convergence characteristics. The extension of the present numerical approach to 3D problems in nonlinear elasticity is the immediate work that is awaiting.

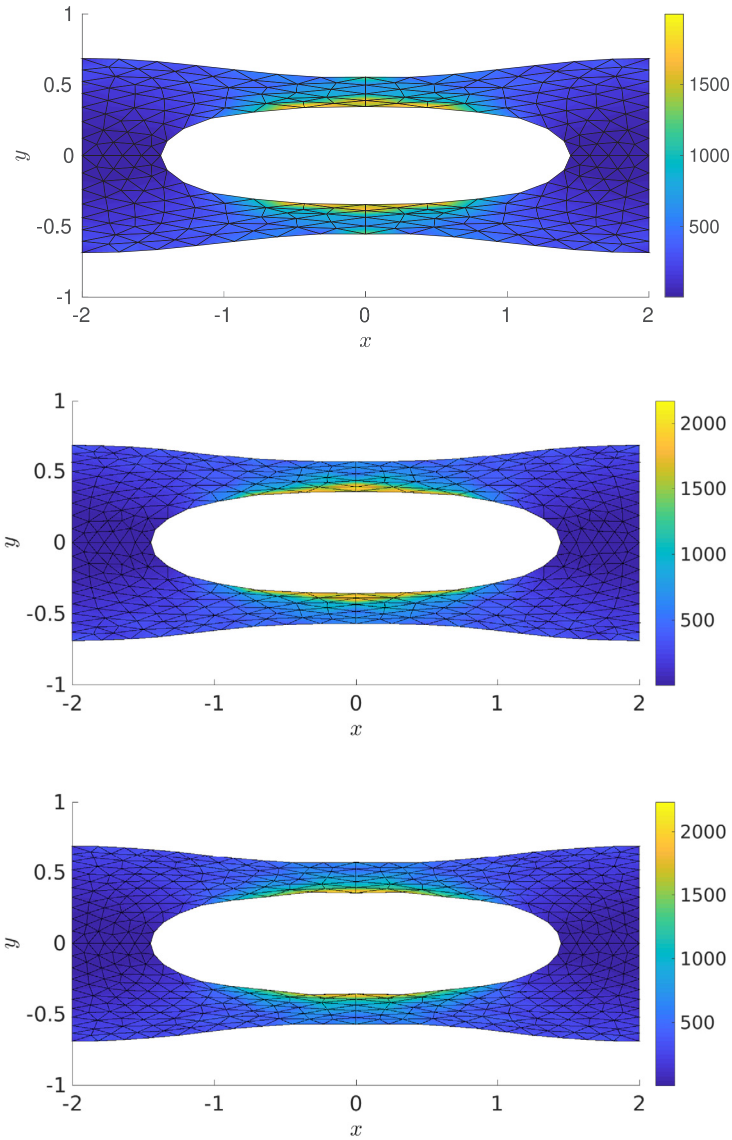


Fig. 16. Deformed configuration of the square plate with a circular hole for an extension of 1 cm. Color profile indicates the norm of the first Piola stress. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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