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An extension problem and Hardy's inequality for the fractional Laplace-Beltrami operator on Riemannian symmetric spaces of noncompact type

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A B S T R A C T

In this paper we study an extension problem for the Laplace-Beltrami operator on Riemannian symmetric spaces of noncompact type and use the solution to prove Hardy-type inequalities for fractional powers of the Laplace-Beltrami operator. Next, we study the mapping properties of the extension operator. In the last part we prove Poincaré-Sobolev inequalities on these spaces.

1. Introduction

In recent years there has been intensive research on various kinds of inequalities for fractional order operators because of their applications to many areas of analysis (see for instance [8,19,40] and the references therein). The classical definitions of the fractional

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operator in terms of the Fourier analysis involve functional analysis and singular integrals. They are nonlocal objects. This fact does not allow to apply local PDE techniques to treat nonlinear problems for the fractional operators. To overcome this difficulty, in the Euclidean case, Caffarelli and Silvestre [11] studied the extension problem associated with the Laplacian and realized the fractional power as the map taking Dirichlet data to the Neumann data. On a certain class of noncompact manifolds, this definition of the fractional Laplacian through an extension problem has been studied by Banika et al. [6].

In the first part of this article we will concern with the Hardy-type inequalities for the fractional operators. Let $\Delta_{\mathbb{R}^n} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ denote the Euclidean Laplacian on $\mathbb{R}^n$. For $0 < s < n/2$ and $f \in C_c^\infty(\mathbb{R}^n)$, the Hardy's inequality for fractional powers of the Laplacian states the following

$$\int\limits_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^{2s}}\, dx \leq 4^{-s} \frac{\Gamma\left(\frac{n-2s}{4}\right)^2}{\Gamma\left(\frac{n+2s}{4}\right)^2} \langle (-\Delta_{\mathbb{R}^n})^s f, f\rangle. \tag{1.1}$$

This is a generalization of the original Hardy's inequality proved for the gradient $\nabla_{\mathbb{R}^n}$ of f: for $n \geq 3$,

$$\frac{(n-2)^2}{4} \int\limits_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx \leq \int\limits_{\mathbb{R}^n} |\nabla_{\mathbb{R}^n} f(x)|^2\, dx, \quad \textit{for } f \in C_c(\mathbb{R}^n). \tag{1.2}$$

The constant appearing in the equation (1.1) is sharp [7,28,42]. It is also known that the equality is not obtained in the class of functions for which both sides of the inequality (1.1) are finite. Using a ground state representation, Frank, Lieb, and Seiringer gave a different proof of the inequality (1.1) when $0 < s < \min\{1, n/2\}$ which improved the previous results [19]. There is another version of Hardy's inequality where the homogeneous weight function $|x|^{-2s}$ is replaced by non-homogeneous one:

$$\int\limits_{\mathbb{R}^n} \frac{|f(x)|^2}{(\delta^2 + |x|^2)^{2s}}\, dx \leq 4^{-s} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\, \delta^{-2s}\, \langle (-\Delta_{\mathbb{R}^n})^s f, f\rangle,\ \ \delta > 0. \tag{1.3}$$

Here also the constant is sharp and equality is achieved for the functions $(\delta^2 + |x|^2)^{-(n-2s)/2}$ and their translates [10].

Generalization of the classical Hardy's inequality (1.2) to Riemannian manifolds was intensively pursued after the seminal work of Carron [12], see for instance [9,18,29–31,43]. In [12], the following weighted Hardy's inequality was obtained on a complete noncompact Riemannian manifold M:

$$\int\limits_M \eta^\alpha |\nabla_g \phi|^2\, dv_g \geq \left(\frac{C+\alpha-1}{2}\right)^2 \int\limits_M \eta^\alpha \frac{\phi^2}{\eta^2}\, dv_g,$$

where $\phi \in C_c^\infty(M - \eta^{-1}\{0\})$, $\alpha \in \mathbb{R}$, $C > 1$, $C + \alpha - 1 > 0$ and the weight function η satisfies $|\nabla_M \eta| = 1$ and $|\Delta_M \eta| \geq C/\eta$ in the sense of distribution. Here ∇_g, dv_g denote respectively the Riemannian gradient and Riemannian measure on M. In the case of Cartan-Hadamard manifold M of dimension N (namely, a manifold which is complete, simply-connected, and has everywhere non-positive sectional curvature), the geodesic distance function $d(x, x_0)$, where $x_0 \in M$, satisfies all the assumptions of the weight η and the above inequality holds with the best constant $(N-2)^2/4$, see [31]. Analogues of Hardy-type inequalities for fractional powers of the sublaplacian are also known, for instance, the work by P. Ciatti, M. Cowling and F. Ricci for stratified Lie groups [14]. There the authors have not paid attention to the sharpness of the constants. Recently, in [38], Roncal and Thangavelu have proved analogues of Hardy-type inequalities with sharp constants for fractional powers of the sublaplacian on the Heisenberg group. For recent results on the Hardy-type inequalities for the fractional operators we refer [10,21,37,39].

Our first aim in this article is to prove analogues of Hardy's inequalities (1.1) and (1.3) for fractional powers of the Laplace-Beltrami operator Δ on Riemannian symmetric space X of noncompact type of arbitrary rank. We have the following analogue of Hardy's inequality in the non-homogeneous case.

Theorem 1.1. *Let $0 < \sigma < 1$ and $y > 0$. Then there exists a constant $C_\sigma > 0$ such that for $F \in H^\sigma(X)$*

$$\langle (-\Delta)^\sigma F, F \rangle \geq C_\sigma\, y^{2\sigma} \left(\int\limits_{\{x:|x|^2+y^2<1\}} \frac{|F(x)|^2}{(y^2+|x|^2)^{2\sigma}}\, dx + \int\limits_{\{x:|x|^2+y^2\geq 1\}} \frac{|F(x)|^2}{(y^2+|x|^2)^{\sigma}}\, dx \right).$$

Remark 1.2. In contrast with the inequality (1.3) for the Euclidean space, we get an improvement in the theorem above. This comes as a consequence of the geometry of the symmetric space. In the following theorem also we get similar improvement.

For the homogeneous weight function, we prove the following analogue of Hardy's inequality on X.

Theorem 1.3. *Let $0 < \sigma < 1$. Then there exists a constant $C'_\sigma > 0$ such that for $F \in C_c^\infty(X)$*

$$\langle (-\Delta)^\sigma F, F \rangle \geq C'_\sigma \left(\int\limits_{\{x:|x|<1\}} \frac{|F(x)|^2}{|x|^{2\sigma}}\, dx + \int\limits_{\{x:|x|\geq 1\}} \frac{|F(x)|^2}{|x|^{\sigma}}\, dx \right).$$

Remark 1.4. When the underlying Lie group is complex we have obtained the sharp constant for the inequality in Theorem 1.1 with an explicit weight and the equality is achieved for a particular function (see Theorem 4.8). Similarly, we have an explicit constant corresponding to Theorem 1.3 (see Theorem 4.9).

In [10,37] solutions of the extension problem were used to prove a trace Hardy inequality, from which Hardy's inequality follows. The operators treated there are of the form $L = \sum_{j=1}^m X_j^2$ where the vector fields X_j satisfy Hörmander's condition. But in our case the operator Δ is not of the form. Therefore, we could not use these results to prove Hardy's inequality for Δ. Instead, we use solutions of the extension problem in combination with ground state representation method to prove our result [19,38].

Given $\sigma \in (0,1)$, the fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\sigma$ on $\mathbb{R}^n$ is defined as a pseudo-differential operator by

$$\mathcal{F}\left((-\Delta_{\mathbb{R}^n})^\sigma f\right)(\xi) = |\xi|^{2\sigma}\mathcal{F}f(\xi), \;\; \xi \in \mathbb{R}^n,$$

where $\mathcal{F}f$ is the Fourier transform of f given by

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx, \;\; \xi \in \mathbb{R}^n.$$

It can also be written as the singular integral

$$(-\Delta_{\mathbb{R}^n})^\sigma f(x) = c_{n,\sigma} P.V. \int\limits_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\sigma}}\, dy,$$

where $c_{n,\sigma}$ is a positive constant. Caffarelli and Silvestre have developed in [11] an equivalent definition of the fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\sigma, \sigma \in (0,1)$, using an extension problem to the upper half-space $\mathbb{R}^{n+1}_+$. For a function $f : \mathbb{R}^n \to \mathbb{R}$, consider the solution $u : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ of the following differential equation

$$\begin{aligned} &\Delta_{\mathbb{R}^n} u + \frac{(1-2\sigma)}{y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} = 0, \;\; y > 0; \\ &u(x,0) = f(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.4}$$

Then the fractional Laplacian of f can be computed as

$$(-\Delta_{\mathbb{R}^n})^\sigma f = -2^{2\sigma-1}\frac{\Gamma(\sigma)}{\Gamma(1-\sigma)} \lim_{y\to 0^+} y^{1-2\sigma}\frac{\partial u}{\partial y}.$$

The Poisson kernel for the fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\sigma$ in $\mathbb{R}^n$ is

$$K_\sigma(x,y) = c_{n,\sigma}\frac{y^{2\sigma}}{(|x|^2+y^2)^{\sigma+\frac{n}{2}}},$$

and then $u(x,y) = f *_{\mathbb{R}^n} K_\sigma$. Therefore

$$(-\Delta_{\mathbb{R}^n})^\sigma f = -2^{2\sigma-1}\frac{\Gamma(\sigma)}{\Gamma(1-\sigma)} \lim_{y\to 0^+} y^{1-2\sigma}\frac{\partial}{\partial y}(f *_{\mathbb{R}^n} K_\sigma)(x).$$

Later, Stinga and Torrea [40] showed that one can define the fractional Laplacian on a domain $\Omega \subset \mathbb{R}^n$ through the extension (1.4) using the heat-diffusion semigroup generated by the Laplacian Δ_Ω provided that the heat kernel associated with Δ_Ω exists and it satisfies some decay properties. Since the heat kernel on general noncompact manifolds has been extensively studied depending on the underlying geometry, Banica et al. in [6] take this approach to define the fractional Laplace-Beltrami operator on some noncompact manifolds which in particular, include the Riemannian symmetric spaces of noncompact type. Let $\mathbf{d}$ be a Riemannian metric on a Riemannian symmetric space X and let Δ be the corresponding Laplace-Beltrami operator on X. Also, let $\mathbf{g}$ be the product metric on $X \times \mathbb{R}^+$ given by $\mathbf{g} = \mathbf{d} + dy^2$. For $\sigma > 0$, let $H^\sigma(X)$ denote the Sobolev space on X (defined in Section 2). In [6, Theorem 1.1], the following result is proved.

Theorem 1.5 *(Banica; Gonźalez; Sáez). Let $\sigma \in (0,1)$. Then for any given $f \in H^\sigma(X)$, there exists a unique solution of the extension problem*

$$\begin{aligned} &\Delta u + \frac{(1-2\sigma)}{y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} = 0, \;\; y > 0; \\ &u(x,0) = f(x), \quad x \in X. \end{aligned} \tag{1.5}$$

Moreover, the fractional Laplace-Beltrami operator on X can be recovered through

$$(-\Delta)^\sigma f(x) = -2^{2\sigma-1}\frac{\Gamma(\sigma)}{\Gamma(1-\sigma)} \lim_{y \to 0^+} y^{1-2\sigma}\frac{\partial u}{\partial y}(x,y). \tag{1.6}$$

The following theorem gives an alternative expression of a solution of the extension problem (1.5), which will be useful for us. The proof is similar to [6, Theorem 3.1], [40, Theorem 1.1].

Theorem 1.6. *Let $f \in Dom(-\Delta)^\sigma$. A solution of (1.5) is given by*

$$u(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{t\Delta}(-\Delta)^\sigma f(x) e^{-y^2/4t}\,\frac{dt}{t^{1-\sigma}}, \tag{1.7}$$

and u is related to $(-\Delta)^\sigma f$ by the equation (1.6). Moreover, the following Poisson formula for u holds:

$$u(x,y) = \int_X f(\zeta) P_y^\sigma(\zeta^{-1}x)\, d\zeta = (f * P_y^\sigma)(x), \tag{1.8}$$

where

$$P_y^\sigma(x) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)}\int_0^\infty h_t(x)\, e^{-y^2/4t}\,\frac{dt}{t^{1+\sigma}}. \tag{1.9}$$

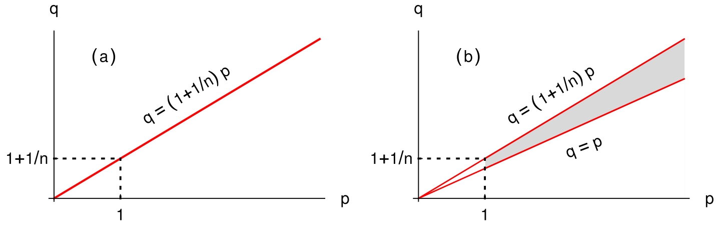

Fig. 1. (a) Euclidean (b) Symmetric spaces.

All these identities in the theorem above are to be understood in the L^2 sense. The mapping properties of the Poisson operator P_σ on $\mathbb{R}^n$ which maps boundary value f to the solution u of the extension problem (1.4) were studied by Möllers et al. [35]. In the same paper, the authors have also obtained a similar result for Heisenberg groups. On the Euclidean spaces, they proved the following

Theorem 1.7 *(Möllers; Ørsted; Zhang). Let* $0 < \sigma < \frac{n}{2}$. *Then*

(1) $P_\sigma : H^\sigma(\mathbb{R}^n) \to H^{\sigma+1/2}(\mathbb{R}^n \times \mathbb{R}^+)$ *is isometric up to a constant.*
(2) P_σ *extends to a bounded operator from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n \times \mathbb{R}_+)$, *for* $1 < p \leq \infty$ *and* $q = \frac{n+1}{n}p$ *(Fig. 1, (a)).*

In [13], Chen proved that for particular values $p = \frac{2n}{n-2\sigma}$ and $q = \frac{2n+2}{n-2\sigma}$, there exists a sharp constant C such that

$$\|P_\sigma f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad \textit{for } f \in L^p(\mathbb{R}^n),$$

and the optimizer of this inequality are translations, dilations and multiples of the function

$$f(x) = \left(1 + |x|^2\right)^{-\frac{n}{2}+\sigma}.$$

Our second main aim in this article is to study the mapping properties of the "Poisson operator" T_σ given by

$$T_\sigma f(x, y) = f * P_y^\sigma, \;\; x \in X, \; y > 0, \tag{1.10}$$

which maps f to the solution u of the extension problem (1.5) related to the Laplace-Beltrami operator on Riemannian symmetric spaces of noncompact type. The following analogue of Theorem 1.7 is our main result in this direction.

Theorem 1.8. *Let* $\dim X = n$ *and* $0 < \sigma < 1$. *Then*

(1) $T_\sigma : H^\sigma(X) \to H^{\sigma+1/2}(X \times \mathbb{R}_+)$ *is isometric up to a constant.*

(2) *T_σ extends to a bounded operator from $L^p(X)$ to $L^q(X \times \mathbb{R}_+)$, for $1 < p < \infty$ and $p < q \leq \frac{n+1}{n}p$; and from $L^1(X)$ to $L^q(X)$, for $1 < q < \frac{n+1}{n}$ (Fig. 1, (b)).*

Remark 1.9. In contrast with Theorem 1.7 on Euclidean space, the exponents p, q in Theorem 1.8 on X can vary over a much larger region (see Fig. 1). This striking phenomenon comes as a consequence of the Kunze-Stein phenomenon. The Kunze-Stein phenomenon, proved by Cowling [16] on connected semi-simple Lie groups G with finite center, says that the convolution inequality

$$L^2(G) * L^p(G) \subset L^2(G),$$

holds for $p \in [1, 2)$. We use the following generalized version [17, Theorem 2.2, (ii)]: let $k \in L^q(X)$, for $1 < q \leq 2$ and let $1 \leq p < q$. Then the map $f \mapsto f * k$ is bounded from $L^p(X)$ to $L^q(X)$. We note that the above inequalities on Euclidean space are only valid for $p = 1$.

An explicit expression of the heat kernel is known for certain symmetric spaces. Using this we write the precise expression of the kernel P_y^σ in the case of complex and rank one symmetric spaces.

The final topic we shall deal with here is analogues of the Poincaré-Sobolev inequalities for the fractional Laplace-Beltrami operator on X. In [34], Mancini and Sandeep proved the following optimal Poincaré-Sobolev inequalities for the Laplace-Beltrami operator $\Delta_{\mathbb{H}^n}$ on the real hyperbolic space $\mathbb{H}^n$ of dimension $n \geq 3$.

Theorem 1.10 *(Mancini; Sandeep). Let $n \geq 3$. Then for $2 < p \leq \frac{2n}{n-2}$, there exists $S = S_{n,p} > 0$ such that for all $u \in C_c^\infty(\mathbb{H}^n)$,*

$$\| \left(-\Delta_{\mathbb{H}^n} - (n-1)^2/4\right)^{1/2} u\|^2_{L^2(\mathbb{H}^n)} \geq S\|u\|^2_{L^p(\mathbb{H}^n)}.$$

In case of real hyperbolic space $\mathbb{H}^3$ of dimension three, Benguria, Frank and Loss [8] proved that the best constant S_3 in the theorem above is the same as the best sharp Sobolev constant for the first order Sobolev inequality on $\mathbb{H}^3$. Recently, using Green kernel estimates Li, Lu, Yang [32, Theorem 6.2] proved the following Poincaré-Sobolev inequalities for the fractional Laplace-Beltrami operator $\Delta_{\mathbb{H}^n}$ on $\mathbb{H}^n$.

Theorem 1.11 *(Li; Lu; Yang). Let $n \geq 3$ and $1 \leq \sigma < 3$. Then there exists a constant $C = C_{n,\sigma,p} > 0$ such that*

$$\| \left(-\Delta_{\mathbb{H}^n} - (n-1)^2/4\right)^{\frac{\sigma}{4}} u\|^2_{L^2(\mathbb{H}^n)} \geq C\|u\|^2_{L^{\frac{2n}{n-\sigma}}(\mathbb{H}^n)}, \quad \textit{for } u \in H^{\frac{\sigma}{2}}(\mathbb{H}^n).$$

For related results and their sharpness, we refer the reader to [33,41]. Our aim in the final section is to prove an analogue of the Poincaré-Sobolev inequality for the fractional

Laplace-Beltrami operator Δ on X which generalizes the above mentioned theorems. The idea of the proof is to use the estimate of the Bassel-Green-Riesz kernel due to Anker-Ji [4]. Since we are working on general Riemannian symmetric spaces of noncompact type, it is difficult to get the explicit values of the constants involve and we do not make attempt to get the optimal constant. Here is our final result. We refer the reader to the next section for the unexplained notation used in the theorem below.

Theorem 1.12. *Let* $\dim X = n \geq 3$ *and* $0 < \sigma < \min\{l+2|\Sigma_0^+|, n\}$. *Then for* $2 < p \leq \frac{2n}{n-\sigma}$ *there exists* $S = S_{n,\sigma,p} > 0$ *such that for all* $u \in H^{\frac{\sigma}{2}}(X)$,

$$\|(-\Delta - |\rho|^2)^{\sigma/4} u\|^2_{L^2(X)} \geq S \|u\|^2_{L^p(X)}.$$

2. Preliminaries

In this section, we describe the necessary preliminaries regarding semisimple Lie groups and harmonic analysis on Riemannian symmetric spaces. These are standard and can be found, for example, in [20,25–27]. To make the article self-contained, we shall gather only those results which will be used throughout this paper.

2.1. Notations

Let G be a connected, noncompact, real semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. We fix a Cartan involution θ of $\mathfrak{g}$ and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of θ respectively. Then $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a linear subspace of $\mathfrak{g}$. The Cartan involution θ induces an automorphism Θ of the group G and $K = \{g \in G \mid \Theta(g) = g\}$ is a maximal compact subgroup of G. Let B denote the Cartan Killing form of $\mathfrak{g}$. It is known that $B\mid_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and hence induces an inner product and a norm $|\cdot|$ on $\mathfrak{p}$. The homogeneous space $X = G/K$ is a smooth manifold. The tangent space of X at the point $o = eK$ can be naturally identified to $\mathfrak{p}$ and the restriction of B on $\mathfrak{p}$ then induces a G-invariant Riemannian metric $\mathbf{d}$ on X. For $x \in X$ and $r > 0$, we denote $\mathbf{B}(x, r)$ to be the ball of radius r centered at x in this metric.

Let $\mathfrak{a}$ be a maximal subalgebra in $\mathfrak{p}$; then $\mathfrak{a}$ is abelian. We assume that $\dim \mathfrak{a} = l$, called the real rank of G. We can identify $\mathfrak{a}$ endowed with the inner product induced from $\mathfrak{p}$ with $\mathbb{R}^l$ and let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$. The set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ. It consists of all $\alpha \in \mathfrak{a}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X, \quad \text{for all } Y \in \mathfrak{a}\}$$

is nonzero with $m_\alpha = \dim(\mathfrak{g}_\alpha)$. We choose a system of positive roots Σ^+ and with respect to Σ^+, the positive Weyl chamber $\mathfrak{a}_+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \quad \text{for all } \alpha \in \Sigma^+\}$. We also let Σ_0^+ be the set of positive indivisible roots. We denote by

$$\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and we obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$ then N is a Nilpotent Lie group and A normalizes N. For the group G, we now have the Iwasawa decomposition $G = KAN$, that is, every $g \in G$ can be uniquely written as

$$g = \kappa(g) \exp H(g) \eta(g), \quad \kappa(g) \in K, H(g) \in \mathfrak{a}, \eta(g) \in N,$$

and the map

$$(k, a, n) \mapsto kan$$

is a global diffeomorphism of $K \times A \times N$ onto G. Let n be the dimension of X then

$$n = l + \sum_{\alpha \in \Sigma^+} m_\alpha.$$

We always assume that $n \geq 2$. Let ρ denote the half sum of all positive roots counted with their multiplicities:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \, \alpha.$$

It is known that the L^2-spectrum of the Laplace-Beltrami operator Δ on X is the half-line $(-\infty, -|\rho|^2]$. Let M' and M be the normalizer and centralizer of $\mathfrak{a}$ in K respectively. Then M is a normal subgroup of M' and normalizes N. The quotient group $W = M'/M$ is a finite group, called the Weyl group of the pair $(\mathfrak{g}, \mathfrak{k})$. The Weyl group W acts on $\mathfrak{a}$ by the adjoint action. It is known that W acts as a group of orthogonal transformations (preserving the Cartan-Killing form) on $\mathfrak{a}$. Each $w \in W$ permutes the Weyl chambers and the action of W on the Weyl chambers is simply transitive. Let $A_+ = \exp \mathfrak{a}_+$. Since $\exp : \mathfrak{a} \to A$ is an isomorphism we can identify A with $\mathbb{R}^l$. If $\overline{A_+}$ denotes the closure of A_+ in G, then one has the polar decomposition $G = KAK$, that is, each $g \in G$ can be written as

$$g = k_1 (\exp Y) k_2, \;\; k_1, k_2 \in K, Y \in \mathfrak{a}.$$

In the above decomposition, the A component of $\mathfrak{g}$ is uniquely determined modulo W. In particular, it is well defined in $\overline{A_+}$. The map $(k_1, a, k_2) \mapsto k_1 a k_2$ of $K \times A \times K$ into G induces a diffeomorphism of $K/M \times A_+ \times K$ onto an open dense subset of G. We extend the inner product on $\mathfrak{a}$ induced by B to $\mathfrak{a}^*$ by duality, that is, we set

$$\langle \lambda, \mu \rangle = B(Y_\lambda, Y_\mu), \quad \lambda, \mu \in \mathfrak{a}^*, \;\; Y_\lambda, Y_\mu \in \mathfrak{a},$$

where Y_λ is the unique element in $\mathfrak{a}$ such that

$$\lambda(Y) = B(Y_\lambda, Y), \quad \text{for all } Y \in \mathfrak{a}.$$

This inner product induces a norm, again denoted by $|\cdot|$, on $\mathfrak{a}^*$,

$$|\lambda| = \langle \lambda, \lambda \rangle^{\frac{1}{2}}, \quad \lambda \in \mathfrak{a}^*.$$

The elements of the Weyl group W act on $\mathfrak{a}^*$ by the formula

$$sY_\lambda = Y_{s\lambda}, \quad s \in W, \ \lambda \in \mathfrak{a}^*.$$

Let $\mathfrak{a}^*_{\mathbb{C}}$ denote the complexification of $\mathfrak{a}^*$, that is, the set of all complex-valued real linear functionals on $\mathfrak{a}$. The usual extension of B to $\mathfrak{a}^*_{\mathbb{C}}$, using conjugate linearity is also denoted by B. Through the identification of A with $\mathbb{R}^l$, we use the Lebesgue measure on $\mathbb{R}^l$ as the Haar measure da on A. As usual on the compact group K, we fix the normalized Haar measure dk and dn denotes a Haar measure on N. The following integral formulae describe the Haar measure of G corresponding to the Iwasawa and polar decomposition respectively. For any $f \in C_c(G)$,

$$\begin{aligned} \int\limits_G f(g)dg &= \int\limits_K \int\limits_{\mathfrak{a}} \int\limits_N f(k \exp Y n) \, e^{2\rho(Y)} \, dn \, dY \, dk \\ &= \int\limits_K \int\limits_{\overline{A_+}} \int\limits_K f(k_1 a k_2) \, J(a) \, dk_1 \, da \, dk_2, \end{aligned}$$

where dY is the Lebesgue measure on $\mathbb{R}^l$ and for $H \in \overline{\mathfrak{a}_+}$

$$J(\exp H) = c \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} \asymp \left\{ \prod_{\alpha \in \Sigma^+} \left(\frac{\alpha(H)}{1 + \alpha(H)} \right)^{m_\alpha} \right\} e^{2\rho(H)}, \tag{2.1}$$

where c (in the equality above) is a normalizing constant. If f is a function on $X = G/K$ then f can be thought of as a function on G which is right invariant under the action of K. It follows that on X we have a G invariant measure dx such that

$$\int\limits_X f(x) \, dx = \int\limits_{K/M} \int\limits_{\mathfrak{a}_+} f(k \exp Y) \, J(\exp Y) \, dY \, dk_M, \tag{2.2}$$

where dk_M is the K-invariant measure on K/M.

2.2. Fourier analysis on X

For a sufficiently nice function f on X, its Fourier transform $\widetilde{f}$ is a function defined on $\mathfrak{a}_{\mathbb{C}}^* \times K$ given by

$$\widetilde{f}(\lambda, k) = \int_G f(g) e^{(i\lambda - \rho)H(g^{-1}k)} dg, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \ k \in K,$$

whenever the integral exists [27, P. 199]. As M normalizes N the function $k \mapsto \widetilde{f}(\lambda, k)$ is right M-invariant. It is known that if $f \in L^1(X)$ then $\widetilde{f}(\lambda, k)$ is a continuous function of $\lambda \in \mathfrak{a}^*$, for almost every $k \in K$ (in fact, holomorphic in λ on a domain containing $\mathfrak{a}^*$). If in addition, $\widetilde{f} \in L^1(\mathfrak{a}^* \times K, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dk)$ then the following Fourier inversion holds,

$$f(gK) = |W|^{-1} \int_{\mathfrak{a}^* \times K} \widetilde{f}(\lambda, k)\ e^{-(i\lambda + \rho)H(g^{-1}k)}\ |\mathbf{c}(\lambda)|^{-2} d\lambda\, dk,$$

for almost every $gK \in X$ [27, Chapter III, Theorem 1.8, Theorem 1.9]. Here $\mathbf{c}(\lambda)$ denotes Harish Chandra's $\mathbf{c}$-function. Moreover, $f \mapsto \widetilde{f}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times K, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dk)$ [27, Chapter III, Theorem 1.5]:

$$\int_X |f(x)|^2\, dx = |W|^{-1} \int_{\mathfrak{a}^* \times K} |\widetilde{f}(\lambda, k)|^2\ |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dk.$$

It is known [25, Ch. IV, prop 7.2] that there exists a positive number C and $d \in \mathbb{N}$ such that for all $\lambda \in \mathfrak{a}_+^*$

$$\begin{aligned} |\mathbf{c}(\lambda)|^{-2} &\leq C(1 + |\lambda|)^{n-l}, \quad \textit{for } |\lambda| \geq 1; \\ &\leq C(1 + |\lambda|)^{d}, \quad \textit{for } |\lambda| < 1. \end{aligned} \tag{2.3}$$

We now specialize in the case of K-biinvariant function f on G. Using the polar decomposition of G we may view a K-biinvariant integrable function f on G as a function on A_+, or by using the inverse exponential map we may also view f as a function on $\mathfrak{a}$ solely determined by its values on $\mathfrak{a}_+$. Henceforth, we shall denote the set of K-biinvariant functions in $L^1(G)$ by $L^1(K\backslash G/K)$. If $f \in L^1(K\backslash G/K)$ then the Fourier transform $\widetilde{f}$ reduces to the spherical Fourier transform $\widehat{f}(\lambda)$ which is given by the integral

$$\widetilde{f}(\lambda, k) = \widehat{f}(\lambda) := \int_G f(g)\phi_{-\lambda}(g)\, dg, \tag{2.4}$$

for all $k \in K$ where

$$\phi_\lambda(g) = \int_K e^{-(i\lambda + \rho)(H(g^{-1}k))}\, dk, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \tag{2.5}$$

is Harish Chandra's elementary spherical function. We now list down some well-known properties of the elementary spherical functions which are important for us ([4, Prop. 2.2.12], [20, Prop. 3.1.4], [27, Lemma 1.18, P. 221]).

Theorem 2.1.

(1) *$\phi_\lambda(g)$ is K-biinvariant in $g \in G$ and W-invariant in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*
(2) *$\phi_\lambda(g)$ is C^∞ in $g \in G$ and holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*
(3) *The elementary spherical function ϕ_0 satisfies the following global estimate:*

$$\phi_0(\exp H) \asymp \left\{ \prod_{\alpha \in \Sigma_0^+} (1 + \alpha(H)) \right\} e^{-\rho(H)}, \quad \text{for all } H \in \overline{\mathfrak{a}^+}. \tag{2.6}$$

(4) *For all $\lambda \in \overline{\mathfrak{a}^*_+}$ we have*

$$|\phi_\lambda(g)| \le \phi_0(g) \le 1. \tag{2.7}$$

2.3. Function spaces on X

For $1 \le p < \infty$ we define

$$L^p(X \times \mathbb{R}) = \left\{ u \mid \|u\|^p_{L^p(X\times\mathbb{R})} := \int_{X\times\mathbb{R}} |u(x,y)|^p \, dx \, dy < \infty \right\},$$

and $L^p(X \times \mathbb{R}_+)$ to be the subspace of $L^p(X \times \mathbb{R})$ consisting of all functions $u(x,y)$ which are even in the y-variable. We also define $L^\infty(X \times \mathbb{R}^+)$ analogously. For $\sigma > 0$, the Sobolev space of order σ on X is defined by

$$H^\sigma(X) = \left\{ f \in L^2(X) \mid \|f\|^2_{H^\sigma(X)} := \int_{\mathfrak{a}^* \times K} |\tilde{f}(\lambda,k)|^2 \, (|\lambda|^2 + |\rho|^2)^\sigma \, |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk < \infty \right\}.$$

Similarly, for $\sigma > 0$ we define $H^\sigma(X \times \mathbb{R})$ as the space of all functions $u \in L^2(X \times \mathbb{R})$ such that

$$\|u\|^2_{H^\sigma(X\times\mathbb{R})} := \int_{\mathbb{R}} \int_{\mathfrak{a}^* \times K} |\mathcal{F}\left(\tilde{u}(\lambda,k,\cdot)(\xi)\right)|^2 \, \left(|\lambda|^2 + |\rho|^2 + \xi^2\right)^\sigma \, |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk \, d\xi < \infty,$$

where $\mathcal{F}\tilde{u}(\lambda,k,\cdot)(\xi)$ denotes the Euclidean Fourier transform of the function $y \mapsto \tilde{u}(\lambda,k,y)$ at the point $\xi \in \mathbb{R}$, for almost every $(\lambda,k) \in \mathfrak{a}^* \times K$. Let $H^\sigma(X \times \mathbb{R}_+)$ be the subspace of $H^\sigma(X \times \mathbb{R})$ consisting of all elements $u(x,y)$ which are even in the y-variable.

2.4. Heat kernel on X

For the details of the heat kernel h_t on $X = G/K$ we refer [3,4]. It is a family $\{h_t : t > 0\}$ of smooth functions with the following properties:

(a) $h_t \in L^p(K\backslash G/K),\ p \in [1, \infty]$, for each $t > 0$.
(b) For each $t > 0$, h_t is positive with

$$\int_G h_t(g)\, dg = 1. \tag{2.8}$$

(c) $h_{t+s} = h_t * h_s,\ t, s > 0$.
(d) For each $f \in L^p(G/K),\ p \in [1, \infty)$ the function $u(x, t) = f * h_t(x)$, for $x \in X$ solves the heat equation

$$\begin{aligned} \Delta_x u(x,t) &= \frac{\partial}{\partial t} u(x,t) \\ u(\cdot, t) &\to f \text{ in } L^p(X), \text{ as } t \to 0. \end{aligned}$$

(e) The spherical Fourier transform of h_t is given by

$$\widehat{h_t}(\lambda) = e^{-t(|\lambda|^2+|\rho|^2)}, \quad \lambda \in \mathfrak{a}^*. \tag{2.9}$$

We need the following both side estimates of the heat kernel [4, Theorem 3.7].

Theorem 2.2. *Let κ be an arbitrary positive number. Then there exist positive constants C_1, C_2 (depending on κ) such that*

$$C_1 \leq \frac{h_t(\exp H)}{t^{-\frac{n}{2}}(1+t)^{\frac{n-l}{2}-|\Sigma_0^+|}\left\{\prod_{\alpha\in\Sigma_0^+}(1+\alpha(H)\right\}e^{-|\rho|^2 t-\rho(H)-\frac{|H|^2}{4t}}} \leq C_2,$$

for all $t > 0$, and $H \in \overline{\mathfrak{a}^+}$, with $|H| \leq \kappa(1 + t)$.

For $H \in \overline{\mathfrak{a}^+}$ with $t \ll H$, we will use the following global upper bound [3, Theorem 3.1]

$$|h_t(\exp H)| \leq t^{-d_1}(1 + |H|)^{d_2} e^{-|\rho|^2 t-\rho(H)-|H|^2/(4t)}, \tag{2.10}$$

where d_1 and d_2 are positive constants depending on the position of $H \in \overline{\mathfrak{a}^+}$ with respect to the walls and on the relative size of $t > 0$ and $1 + |H|$.

3. Extension problem and kernel estimates

In this section, we study the required properties of the Poisson kernel P_y^σ (defined in (1.9)). In particular, we prove an intertwining property of P_y^σ and $P_y^{-\sigma}$ and establish asymptotic estimates of P_y^σ.

Let us recall that for $-1 < \sigma < 1$ and $y > 0$, the function P_y^σ is given by

$$P_y^\sigma(x) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty h_t(x)\ e^{-y^2/4t}\ \frac{dt}{t^{1+\sigma}}, \quad \textit{for } x \in X.$$

By the estimate of the heat kernel (Theorem 2.2), it follows that P_y^σ is well defined. For $0 < \sigma < 1$, we observe that $\Gamma(-\sigma) := \frac{\Gamma(1-\sigma)}{-\sigma} < 0$ and hence $P_y^{-\sigma} \leq 0$. Since the heat kernel h_t is K-biinvariant so is the function $P_y^{-\sigma}$. By (2.4) the spherical Fourier transform is given by

$$\widehat{P_y^{-\sigma}}(\lambda) = \int_X P_y^{-\sigma}(x)\ \phi_{-\lambda}(x)\ dx = \frac{y^{-2\sigma}}{4^{-\sigma}\Gamma(-\sigma)} \int_0^\infty \widehat{h_t}(\lambda)\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}}, \quad \textit{for } \lambda \in \mathfrak{a}^*. \tag{3.1}$$

Interchange of the integration is possible by the Fubini's theorem. Indeed, by (2.7) and (2.9)

$$\begin{aligned}\int_0^\infty \int_X h_t(x)\ |\phi_{-\lambda}(x)|\ dx\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}} &\leq \int_0^\infty \int_X h_t(x)\ \phi_0(x)\ dx\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}} \\ &= \int_0^\infty e^{-t|\rho|^2}\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}} < \infty.\end{aligned}$$

Moreover, $P_y^{-\sigma}$ is contained in the Sobolev space $H^\sigma(X)$. Indeed, by using (3.1), (2.9) and Minkowski's integral inequality we get that

$$\begin{aligned}\|P_y^{-\sigma}\|_{H^\sigma(X)} &= \left(\int_{\mathfrak{a}^*} |\widehat{P_y^{-\sigma}}(\lambda)|^2\ \left(|\lambda|^2 + |\rho|^2\right)^\sigma\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda \right)^{\frac{1}{2}} \\ &\leq \frac{y^{-2\sigma}}{4^{-\sigma}|\Gamma(-\sigma)|} \int_0^\infty \left(\int_{\mathfrak{a}^*} |\widehat{h_t}(\lambda)|^2\ (|\lambda|^2 + |\rho|^2)^\sigma\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda \right)^{\frac{1}{2}} e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}} \\ &\leq \frac{y^{-2\sigma}}{4^{-\sigma}|\Gamma(-\sigma)|} \int_0^\infty \left(\int_{\mathfrak{a}^*} e^{-t(|\lambda|^2+|\rho|^2)}\ (|\lambda|^2 + |\rho|^2)^\sigma\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda \right)^{\frac{1}{2}} e^{-\frac{|\rho|^2}{2}t}\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}} \\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \frac{y^{-2\sigma}}{4^{-\sigma}|\Gamma(-\sigma)|}\int_0^1 \left(\int_{\mathfrak{a}^*} e^{-t(|\lambda|^2+|\rho|^2)}\ (|\lambda|^2+|\rho|^2)^{\sigma}\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda\right)^{\frac{1}{2}} e^{-\frac{|\rho|^2}{2}t}\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}}, \tag{3.2}$$

and I_2 is defined as above with the integration in the t-variable over the interval $[1,\infty)$. It is enough to show that both I_1 and I_2 are finite. We consider I_1 first. Using the property (2.3) of $|\mathbf{c}(\lambda)|^{-2}$, we estimate the inner integral in the equation above as follows

$$\begin{aligned}
&\int_{\{\lambda\in\mathfrak{a}^*:|\lambda|<1\}} e^{-t|\lambda|^2}\ (|\lambda|^2+|\rho|^2)^{\sigma+d}\ d\lambda + \int_{\{\lambda\in\mathfrak{a}^*:|\lambda|\geq 1\}} e^{-t|\lambda|^2}\ (|\lambda|^2+|\rho|^2)^{\sigma+n-l}\ d\lambda\\
&\leq C_1 + C_2\int_1^\infty e^{-tr^2}\ r^{2(\sigma+n-l)}\ r^{l-1}\ dr\\
&\leq C_1 + C_2\ t^{-(\sigma+n-l/2)}.
\end{aligned}$$

It now follows from (3.2) that

$$I_1 \leq C\int_0^1 t^{-\frac{1}{2}(\sigma+n-l/2)}\ e^{-\frac{|\rho|^2}{2}t}e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}} < \infty.$$

On the other hand

$$\begin{aligned}
I_2 &\leq C\int_1^\infty \left(\int_{\mathfrak{a}^*} e^{-1(|\lambda|^2+|\rho|^2)}\ (|\lambda|^2+|\rho|^2)^{\sigma}\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda\right)^{\frac{1}{2}} e^{-\frac{|\rho|^2}{2}t}\ e^{-y^2/4t}\ \frac{dt}{t^{1-\sigma}}\\
&\leq C\|h_{1/2}\|_{H^\sigma(X)}.
\end{aligned}$$

This completes the proof that $P_y^{-\sigma}\in H^\sigma(X)$.

The following intertwining property of the Poisson kernel $P_y^{-\sigma}$ and P_y^{σ} will crucially be used in the proof of Hardy's inequality. Using (3.1) and (2.9) the proof of the lemma below follows similarly to that of [10, Lemma 2.3].

Lemma 3.1. *For $0<\sigma<1$ and $y>0$ we have, $(-\Delta)^\sigma P_y^{-\sigma}(x) = \frac{4^\sigma\Gamma(\sigma)}{y^{2\sigma}\Gamma(-\sigma)}\ P_y^\sigma(x)$.*

We will now compute the asymptotic behavior of the Poisson kernel P_y^σ on X. We use this estimate crucially for the remaining part of this article.

Theorem 3.2. *For $-1<\sigma<1, \sigma\neq 0$ and $y>0$ we have*

$$\Gamma(\sigma)\,P_y^\sigma(x) \asymp \frac{y^{2\sigma}}{4^\sigma}\sqrt{|x|^2+y^2}^{\,-l/2-1/2-\sigma-|\Sigma_0^+|}\phi_0(x)e^{-|\rho|\sqrt{|x|^2+y^2}},\ \textit{for } |x|^2+y^2\geq 1,$$

$$\asymp y^{2\sigma} \ \left(|x|^2+y^2\right)^{-n/2-\sigma}, \ \textit{for} \ \ |x|^2+y^2<1.$$

Proof. We first assume that $|x|^2+y^2<1$. In this case, we will use the following local expansion of the heat kernel $h_t(x)$

$$h_t(x) = e^{-|x|^2/4t} t^{-n/2} v_0(x) + e^{-c|x|^2/t} \mathcal{O}\left(t^{-n/2+1}\right), \tag{3.3}$$

where $v_0(x) = (4\pi)^{-n/2} + \mathcal{O}(|x|^2)$ and $c < 1/4$ ([3, (3.9), p. 278]). Using this we have

$$\begin{aligned}
\Gamma(\sigma) P_y^\sigma(x) &= \frac{y^{2\sigma}}{4^\sigma} \int_0^1 \left(e^{-|x|^2/4t} t^{-n/2} v_0(x) + e^{-c|x|^2/t} \mathcal{O}\left(t^{-n/2+1}\right)\right) e^{-y^2/4t} \ \frac{dt}{t^{1+\sigma}} \\
&\quad + \frac{y^{2\sigma}}{4^\sigma} \int_1^\infty h_t(x) \ e^{-y^2/4t} \ \frac{dt}{t^{1+\sigma}} \\
&= \frac{y^{2\sigma}}{4^\sigma} v_0(x) \int_0^1 e^{-(|x|^2+y^2)/4t} \ t^{-n/2-1-\sigma} \ dt \\
&\quad + \frac{y^{2\sigma}}{4^\sigma} \int_0^1 e^{-(c|x|^2+y^2/4)/t} \ \mathcal{O}(t^{-n/2+1}) \ t^{-1-\sigma} \ dt + \frac{y^{2\sigma}}{4^\sigma} \int_1^\infty h_t(x) \ e^{-y^2/4t} \ \frac{dt}{t^{1+\sigma}}.
\end{aligned}$$

We write the right-hand side of the equation above as $I_1 + I_2 + I_3$, where I_1, I_2 and I_3 are the first, second and third term respectively. Then applying change of variable $s = \left(|x|^2+y^2\right)/(4t)$, we have

$$I_1 = 4^{\frac{n}{2}} y^{2\sigma} \ \left(|x|^2+y^2\right)^{-n/2-\sigma} \ v_0(x) \int_{(|x|^2+y^2)/4}^\infty e^{-s} \ s^{n/2+\sigma-1} \ ds.$$

As $|x|^2+y^2<1$,

$$\int_1^\infty e^{-s} \ s^{n/2+\sigma-1} \ ds \leq \int_{(|x|^2+y^2)/4}^\infty e^{-s} \ s^{n/2+\sigma-1} \ ds \leq \int_0^\infty e^{-s} \ s^{n/2+\sigma-1} \ ds.$$

This implies that for $|x|^2+y^2<1$

$$I_1 \asymp y^{2\sigma} \left(|x|^2+y^2\right)^{-n/2-\sigma},$$

as $v_0(x) = (4\pi)^{-n/2} + \mathcal{O}(|x|^2)$. For I_2, using $c < 1/4$ we have that

$$\begin{aligned}I_2 &\leq C\frac{y^{2\sigma}}{4^\sigma}\int\limits_0^1 e^{-c(|x|^2+y^2)/t}\ \mathcal{O}(t^{-n/2+1})\ t^{-1-\sigma}\ dt\\ &\leq C\,y^{2\sigma}\ \left(|x|^2+y^2\right)^{-n/2-\sigma+1}\int\limits_{c(|x|^2+y^2)}^{\infty} e^{-s}\ s^{n/2+\sigma-2}\ ds\\ &\leq C\,y^{2\sigma}\ \left(|x|^2+y^2\right)^{-n/2-\sigma}\int\limits_0^\infty e^{-s}\ s^{n/2+\sigma-1}\ ds\\ &\leq C\,y^{2\sigma}\ \left(|x|^2+y^2\right)^{-n/2-\sigma}.\end{aligned}$$

For the integral I_3, using Theorem 2.2 we get that for $|x|^2+y^2<1$,

$$I_3=\frac{y^{2\sigma}}{4^\sigma}\int\limits_1^\infty h_t(x)\ e^{-y^2/4t}\ \frac{dt}{t^{1+\sigma}}\leq Cy^{2\sigma}\leq Cy^{2\sigma}\left(|x|^2+y^2\right)^{-n/2-\sigma}.$$

This proves that for $|x|^2+y^2<1$,

$$\Gamma(\sigma)P_y^\sigma(x)\asymp y^{2\sigma}\ (|x|^2+y^2)^{-n/2-\sigma}.$$

We will now assume that $|x|^2+y^2\geq 1$. Let us fix a positive number $\kappa>4$. We proceed as in the proof of [4, Theorem 4.3.1].

$$\begin{aligned}\Gamma(\sigma)P_y^\sigma(x)&=\frac{y^{2\sigma}}{4^\sigma}\int\limits_0^\infty h_t(x)\ e^{-y^2/4t}\ \frac{dt}{t^{1+\sigma}}\\ &=\frac{y^{2\sigma}}{4^\sigma}\{I_4+I_5+I_6\},\end{aligned}$$

where the quantities I_4, I_5 and I_6 are defined by the integration of the above integrand $h_t(x)\ e^{-y^2/4t}\ t^{-1-\sigma}$ over the intervals $\left[0,\kappa^{-1}b\right)$, $\left[\kappa^{-1}b,\kappa b\right)$ and $\left[\kappa b,\infty\right)$ with $b=\sqrt{|x|^2+y^2}/(2|\rho|)$ respectively. For the integral I_5, using Theorem 2.2 and the asymptotic of ϕ_0 in Theorem 2.1 (3) we get the following:

$$\begin{aligned}I_5\asymp&\int\limits_{\frac{\kappa^{-1}\sqrt{|x|^2+y^2}}{2|\rho|}}^{\frac{\kappa\sqrt{|x|^2+y^2}}{2|\rho|}} t^{-n/2}(1+t)^{(n-l)/2-|\Sigma_0^+|}\left\{\prod_{\alpha\in\Sigma_0^+}(1+\alpha(x))\right\}\\ &\times e^{-|\rho|^2t-\rho(\log x)-|x|^2/4t}e^{-y^2/4t}\frac{dt}{t^{1+\sigma}}\end{aligned}$$

$$\asymp \int_{\frac{\kappa^{-1}\sqrt{|x|^2+y^2}}{2|\rho|}}^{\frac{\kappa\sqrt{|x|^2+y^2}}{2|\rho|}} t^{-l/2-|\Sigma_0^+|}\phi_0(x)e^{-|\rho|^2 t}e^{-(|x|^2+y^2)/4t}\frac{dt}{t^{1+\sigma}}$$

$$= \int_{\kappa^{-1}}^{\kappa} (s\sqrt{|x|^2+y^2}/2|\rho|)^{-l/2-\sigma-1-|\Sigma_0^+|}\phi_0(x)e^{-s|\rho|\sqrt{|x|^2+y^2}/2}$$

$$\times e^{-|\rho|\sqrt{|x|^2+y^2}/(2s)}\left(\frac{\sqrt{|x|^2+y^2}}{2|\rho|}\right)ds$$

$$\asymp \left(\frac{\sqrt{|x|^2+y^2}}{2|\rho|}\right)^{-l/2-\sigma-|\Sigma_0^+|}\phi_0(x)\int_{\kappa^{-1}}^{\kappa} e^{-\sqrt{|x|^2+y^2}|\rho|(s+1/s)/2}ds.$$

The last both sides estimate follows because

$$\kappa^{-(l/2+\sigma+1+|\Sigma_0^+|)} \le s^{-(l/2+\sigma+1+|\Sigma_0^+|)} \le \kappa^{(l/2+\sigma+1+|\Sigma_0^+|)}.$$

Now, using the fact that

$$\int_{\kappa^{-1}}^{\kappa} e^{-|\rho|\sqrt{|x|^2+y^2}(s+1/s)/2}ds \asymp |\rho|^{-1/2}(|x|^2+y^2)^{-1/4}e^{-|\rho|\sqrt{|x|^2+y^2}},$$

(this follows by the Laplace method [15, Ch 5]) we get from the above equation that

$$I_5 \asymp \left(\sqrt{|x|^2+y^2}\right)^{-l/2-1/2-\sigma-|\Sigma_0^+|}\phi_0(x)e^{-|\rho|\sqrt{|x|^2+y^2}}.$$

For the third integral I_6, we will use the fact that $\kappa > 4$. Using Theorem 2.2, we get

$$I_6 \le \phi_0(x)\int_{\kappa\sqrt{|x|^2+y^2}/(2|\rho|)}^{\infty} t^{-l/2-|\Sigma_0^+|-1-\sigma}e^{-|\rho|^2 t}e^{-(|x|^2+y^2)/4t}dt$$

$$\le \phi_0(x)\left(\kappa\sqrt{|x|^2+y^2}/(2|\rho|)\right)^{-l/2-|\Sigma_0^+|-1}\int_{\kappa\sqrt{|x^2|+y^2}/(2|\rho|)}^{\infty} t^{-\sigma}e^{-|\rho|^2 t}e^{-(|x|^2+y^2)/(4t)}dt$$

$$\le C\phi_0(x)\left(\sqrt{|x|^2+y^2}\right)^{-l/2-|\Sigma_0^+|-1} e^{-|\rho|^2 k\sqrt{|x|^2+y^2}/(4|\rho|)}$$

$$\times \int_{\kappa\sqrt{|x^2|+y^2}/(2|\rho|)}^{\infty} t^{-\sigma}e^{-|\rho|^2 t/2}e^{-(|x|^2+y^2)/(4t)}dt$$

$$\le C\left(\sqrt{|x|^2+y^2}\right)^{-l/2-|\Sigma_0^+|-1/2}\phi_0(x)e^{-(|\rho|+\eta)\sqrt{|x|^2+y^2}},$$

where $\eta = |\rho|\kappa/4 - |\rho| > 0$. For the first integral I_4, we use heat kernel Gaussian estimate (2.10) and the estimate of ϕ_0 in Theorem 2.1 to obtain the following

$$\begin{aligned}
I_4 &\leq \int_0^{\kappa^{-1}\sqrt{|x|^2+y^2}/(2|\rho|)} t^{-d_1}(1+|x|)^{d_2} e^{-|\rho|^2 t - \rho(\log x)} e^{-(|x|^2+y^2)/(4t)} \frac{dt}{t^{1+\sigma}} \\
&\leq (1+|x|)^{d_2 - |\Sigma_0^+|}\phi_0(x) \int_0^{\kappa^{-1}\sqrt{|x|^2+y^2}/(2|\rho|)} e^{-(|x|^2+y^2)/(4t)} t^{-1-\sigma-d_1}\, dt \\
&= (1+|x|)^{d_2 - |\Sigma_0^+|}\phi_0(x) \int_0^{\kappa^{-1}\sqrt{|x|^2+y^2}/(2|\rho|)} e^{-(|x|^2+y^2)/(8t)} e^{-(|x|^2+y^2)/(8t)} t^{-1-\sigma-d_1}\, dt \\
&\leq C(1+|x|)^{d_2 - |\Sigma_0^+|}\phi_0(x)\, e^{-|\rho|\kappa\frac{\sqrt{x^2+y^2}}{4}} \int_0^{\kappa^{-1}\sqrt{|x|^2+y^2}/(2|\rho|)} e^{-(|x|^2+y^2)/(8t)} t^{-1-\sigma-d_1}\, dt \\
&\leq C(1+|x|)^{d_2 - |\Sigma_0^+|}\phi_0(x)\, e^{-(|\rho|+\epsilon)\sqrt{|x|^2+y^2}} (|x|^2+y^2)^{-\sigma-d_1},
\end{aligned}$$

for some $\epsilon > 0$, as $\kappa > 4$.

This completes the proof. $\square$

To prove Hardy's inequalities we use an integral representation for the operator $(-\Delta)^\sigma$. The following function

$$P_0^\sigma(x) = \int_0^\infty h_t(x) \frac{dt}{t^{1+\sigma}}, \tag{3.4}$$

serves as the kernel of the integral representation. We state both sides estimate of P_0^σ, whose proof is exactly the same as of Theorem 3.2.

Theorem 3.3. *For any $\alpha > -n/2$ the following asymptotic estimates holds:*

$$\begin{aligned}
P_0^\alpha(x) &\asymp |x|^{-l/2-1/2-\alpha-|\Sigma_0^+|}\phi_0(x) e^{-|\rho||x|}, \ \textit{for } |x| \geq 1, \\
&\asymp |x|^{-n-2\alpha}, \ \textit{for } \ |x| < 1.
\end{aligned}$$

Corollary 3.4. *Let χ be the characteristic function of the unit ball $\mathbf{B}(\mathbf{o}, \mathbf{1})$ in X and $\alpha > 0$. Then the function $(1-\chi)P_0^\alpha$ is in $L^p(X)$ for $1 \leq p \leq \infty$.*

Proof. For $1 < p \leq \infty$, the result follows trivially from the asymptotic formula in Theorem 3.3. We prove the case $p = 1$. We recall from (2.2) that

$$\int\limits_{\{x\in X:|x|>1\}} P_0^\alpha(x)\ dx \leq C \int\limits_{\{H\in\overline{\mathfrak{a}_+}:|H|>1\}} P_0^\alpha(\exp H)\ e^{2\rho(H)}\ dH.$$

Let Γ be a small circular cone in $\overline{\mathfrak{a}_+}$ around the ρ-axis. By introducing polar coordinates in Γ (as in [3, p. 293]) and using (2.6) we get

$$\begin{aligned}
&\int\limits_{\{H\in\Gamma:|H|>1\}} P_0^\alpha(\exp H)\ e^{2\rho(H)}\ dH \\
&\leq C \int\limits_{\{H\in\Gamma:|H|>1\}} |H|^{-l/2-1/2-\alpha}\ e^{\rho(H)-|\rho||H|}\ dH \\
&\leq C \int\limits_1^\infty r^{-l/2-1/2-\alpha} r^{l-1} \int\limits_0^\nu (\sin\xi)^{l-2}\ e^{-r(1-\cos\xi)}\ d\xi\ dr.
\end{aligned}$$

Since $\sin\xi \sim \xi$ and $1-\cos\xi \sim \xi^2$, the inner integral behaves like $r^{1/2-l/2}$. Consequently, the integral above is finite. On the other hand, $e^{\rho(H)-|\rho||H|}$ decreases exponentially outside Γ, and therefore

$$\int\limits_{\{H\in\overline{\mathfrak{a}_+}\setminus\Gamma:|H|>1\}} P_0^\alpha(\exp H)\, e^{2\rho(H)}\ dH = \int\limits_{\{H\in\overline{\mathfrak{a}_+}\setminus\Gamma:|H|>1\}} H^{-l/2-1/2-\alpha}\ e^{\rho(H)-|\rho||H|}\ dH < \infty.$$

This completes the proof. $\square$

4. Fractional Hardy inequalities

This section aims to prove two versions of Hardy's inequalities for fractional powers of the Laplace-Beltrami operator on X, namely Theorem 1.1 and Theorem 1.3 with homogeneous and non-homogeneous weight functions respectively. In order to prove these inequalities, we will follow similar ideas used in [19,38]. Therefore, we need to obtain a ground state representation for the operator $(-\Delta)^\sigma$. We start with the following integral representations of $(-\Delta)^\sigma$ on X. For the cases of real hyperbolic spaces, analogues integral representations were proved in [6, Theorem 2.5].

Lemma 4.1. *Let $0<\sigma<1/2$. Then for all $f\in C_c^\infty(X)$ we have*

$$(-\Delta)^\sigma f(x) = \frac{1}{|\Gamma(-\sigma)|} \int\limits_X (f(x)-f(z))\, P_0^\sigma(z^{-1}x)\ dz,$$

where P_0^σ is defined in (3.4).

Proof. Let $f \in C_c^\infty(X)$. Using the numerical identity

$$\lambda^\sigma = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \left(1 - e^{-t\lambda}\right) \frac{dt}{t^{1+\sigma}}, \ \ \lambda > 0,$$

and the spectral theorem we have

$$(-\Delta)^\sigma f(x) = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \left(f(x) - e^{t\Delta} f(x)\right) \frac{dt}{t^{1+\sigma}}.$$

By (2.8) it follows that

$$f(x) - e^{t\Delta} f(x) = f(x) - f * h_t(x) = \int_X (f(x) - f(xz^{-1}))\, h_t(z)\, dz. \tag{4.1}$$

Thus, we have the following representation

$$(-\Delta)^\sigma f(x) = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \int_X \left(f(x) - f(xz^{-1})\right) h_t(z)\, dz\, \frac{dt}{t^{1+\sigma}}.$$

We now show that the right-hand side is absolutely integrable and hence, interchange of the order of integral is possible. Then the result follows by the change of variable $z \mapsto z^{-1}x$. To show absolute integrability let us define

$$I_1 = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \int_{\{z\in X:|z|<1\}} \left|f(x) - f(xz^{-1})\right| h_t(z)\, dz\, \frac{dt}{t^{1+\sigma}},$$

$$I_2 = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \int_{\{z\in X:|z|\geq 1\}} \left|f(x) - f(xz^{-1})\right| h_t(z)\, dz\, \frac{dt}{t^{1+\sigma}}.$$

For the integral I_2, we use the fact that $P_0^\sigma \in L^1(X)$ away from the origin (Corollary 3.4). Indeed, we have that

$$\int_{\{z\in X:|z|\geq 1\}} \int_0^\infty |f(x) - f(xz^{-1})|\, h_t(z)\, \frac{dt}{t^{1+\sigma}}\, dz \leq \|f\|_{L^\infty(X)} \int_{\{z\in X:|z|\geq 1\}} P_0^\sigma(z)\, dz < \infty.$$

Therefore, by Fubini's theorem I_2 is also finite. For I_1 we first observe by the fundamental theorem of calculus (see [2, eqn. 34]) that

$$|f(x) - f(x(\exp H))| \leq |H| \int_0^1 |\nabla f(x \exp(sH))| \, ds \leq |H| \, \|\nabla f\|_{L^\infty(X)}, \tag{4.2}$$

for $x \in X, H \in \mathfrak{a}$. Using the above estimate and the fact that $P_0^\sigma(x) \asymp |x|^{-n-2\sigma}$ around the origin (Theorem 3.3) it follows that

$$\begin{aligned}
&\int_{|z|<1} \int_0^\infty |f(x) - f(xz^{-1})| \, h_t(z) \, \frac{dt}{t^{1+\sigma}} \, dz \\
&\leq C\|\nabla f\|_{L^\infty(X)} \int_{\{H \in \overline{\mathfrak{a}_+} : |H|<1\}} |H| \, |H|^{-n-2\sigma} \, J(\exp H) \, dH \\
&\leq C\|\nabla f\|_{L^\infty(X)} \int_0^1 r^{1-n-2\sigma} \, r^{n-1} \, dr,
\end{aligned}$$

and the right-hand side is finite if $0 < \sigma < 1/2$. This completes the proof. □

Remark 4.2. If $rank(X) = 1$, then for $1/2 \leq \sigma < 1$ the integral formula in Lemma 4.1 exists in principal value sense. To see this, let $\mathfrak{a} = span\{H_0\}$ with $|H_0| = 1$. Clearly, for $\sigma > 0$, the integral I_2 is absolutely convergent and we can interchange the order of the integral. On the other hand the formula (2.2) yields

$$I_1 = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \int_{-1}^1 \big(f(x) - f(x\exp(-sH_0))\big) \, h_t(\exp(sH_0)) \; J(\exp(sH_0)) \; ds \frac{dt}{t^{1+\sigma}}.$$

We now define $F(s) := f(x\exp(sH_0))$, for $s \in \mathbb{R}$. Since $f \in C_c^\infty(X)$, it follows that for each $x \in X$, the function $F \in C_c^\infty(\mathbb{R})$. By using the Taylor development of F, we get that

$$I_1 = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \int_{-1}^1 \left(sF'(x) + \frac{s^2 F''(x)}{2!} + \mathcal{O}(s^3)\right) \, h_t(\exp(sH_0)) \; J(\exp(sH_0)) \; ds \frac{dt}{t^{1+\sigma}}.$$

Since the heat kernel h_t and the Jacobian J is even, the first order term vanishes. Hence, using the fact that $P_0^\sigma(x) \sim |x|^{-n-2\sigma}$, around the origin (Theorem 3.3), it follows that

$$I_1 \leq C_f \int_0^\infty \int_0^1 s^2 \; h_t(\exp(sH_0)) \; s^{n-1} \; ds \; \frac{dt}{t^{1+\sigma}} = C_f \int_0^1 s^{n+1} \; s^{-n-2\sigma} \; ds,$$

which is finite if $0 < \sigma < 1$. Hence, the required integral formula exists as a principal value sense. For the case of higher rank symmetric spaces, neither the heat kernel $h_t(\exp(\cdot))$

nor the Jacobian $J(\exp(\cdot))$ is, in general, radial function on $\mathfrak{a}$. They are only Weyl group invariant. This is the main difficulty that we could not prove the integral formula in the lemma above for $1/2 \le \sigma < 1$ in case of $rank(X) > 1$.

Lemma 4.3. *Let $0 < \sigma < 1$. Then, for all $f \in H^\sigma(X)$*

$$\langle (-\Delta)^\sigma f, f\rangle = \frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X |f(z) - f(x)|^2 \, P_0^\sigma(z^{-1}x) \; dz \; dx.$$

Proof. We first prove that for $0 < \sigma < 1$ and $f \in C_c^\infty(X)$ the quantity

$$\frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X |f(z) - f(x)|^2 \, P_0^\sigma(z^{-1}x) \; dz \; dx < \infty. \tag{4.3}$$

To show this let us assume $supp\ f \subset \mathbf{B}(o, m)$ for some $m > 1$ and define

$$I_1 = \frac{1}{2|\Gamma(-\sigma)|} \int\limits_{\mathbf{B}(o,2m)} \int\limits_X |f(z) - f(x)|^2 \, P_0^\sigma(z^{-1}x) \; dz \; dx,$$

$$I_2 = \frac{1}{2|\Gamma(-\sigma)|} \int\limits_{X \setminus \mathbf{B}(o,2m)} \int\limits_X |f(z) - f(x)|^2 \; P_0^\sigma(z^{-1}x) \; dz \; dx.$$

Since $supp\ f \subset \mathbf{B}(o, m)$ it follows that

$$\begin{aligned} I_2 &= \frac{1}{2|\Gamma(-\sigma)|} \int\limits_{X \setminus \mathbf{B}(o,2m)} \int\limits_{\mathbf{B}(o,m)} |f(z)|^2 \; P_0^\sigma(z^{-1}x) \; dz \; dx \\ &\le \frac{\|f\|^2_{L^\infty(X)}}{2|\Gamma(-\sigma)|} \int\limits_{\mathbf{B}(o,m)} \int\limits_{X \setminus \mathbf{B}(o,2m)} P_0^\sigma(z^{-1}x) \; dx \; dz \\ &\le \frac{\|f\|^2_{L^\infty(X)}}{2|\Gamma(-\sigma)|} |\mathbf{B}(o, m)| \int\limits_{X \setminus \mathbf{B}(o,m)} P_0^\sigma(x) \; dx < \infty. \end{aligned}$$

The last term is finite because of the fact that P_0^σ is integrable away from the origin (Corollary 3.4). To show that I_1 is finite we write it as follows

$$\begin{aligned} I_1 = &\int\limits_{\mathbf{B}(o,2m)} \int\limits_{\mathbf{B}(0,3m)} |f(z) - f(x)|^2 \, P_0^\sigma(z^{-1}x) \; dz \; dx \\ &+ \int\limits_{\mathbf{B}(o,2m)} \int\limits_{X \setminus \mathbf{B}(0,3m)} |f(z) - f(x)|^2 \, P_0^\sigma(z^{-1}x) \; dz \; dx. \end{aligned}$$

Using change of variable $z \mapsto xz^{-1}$ in the first integral, the estimate (4.2) and the asymptotic estimates of P_0^σ in Theorem 3.3 it follows that

$$\begin{aligned}
I_1 \leq & \int\limits_{\mathbf{B}(o,2m)} \int\limits_{\mathbf{B}(o,5m)} \left|f(xz^{-1}) - f(x)\right|^2 P_0^\sigma(z)\, dz\, dx \\
& + \int\limits_{\mathbf{B}(o,2m)} \int\limits_{X \setminus \mathbf{B}(o,3m)} |f(z) - f(x)|^2 P_0^\sigma(z^{-1}x)\, dz\, dx \\
\leq & C\|\nabla f\|^2_{L^\infty(X)} \int\limits_{\mathbf{B}(o,2m)} dx \int\limits_{\{H \in \overline{\mathfrak{a}_+} : |H| < 5m\}} |H|^2\ H^{-n-2\sigma}\ J(\exp H)\, dH \\
& + C \int\limits_{\mathbf{B}(o,2m)} \|f\|^2_{L^\infty(X)} \int\limits_{X \setminus \mathbf{B}(o,3m)} P_0^\sigma(z^{-1}x)\, dz\, dx \\
\leq & C\|\nabla f\|^2_{L^\infty(X)} \int_0^{5m} r^{2-n-2\sigma}\ r^{n-1}\, dr + C\|f\|^2_{L^\infty(X)} \int\limits_{X \setminus \mathbf{B}(o,m)} P_0^\sigma(z)\, dz.
\end{aligned}$$

The first term of the above quantity is finite provided $\sigma < 1$ and the second one finite by Corollary 3.4. This completes the proof of the fact the quantity in (4.3) is finite.

Let $0 < \sigma < 1/2$ and $f \in C_c^\infty(X)$. By the integral representation in Lemma 4.1 it follows that

$$\langle (-\Delta)^\sigma f, f\rangle = \frac{1}{|\Gamma(-\sigma)|} \int\limits_X \int\limits_X (f(x) - f(z))\, P_0^\sigma(z^{-1}x)\ \overline{f(x)}\, dz\, dx.$$

As the kernel P_0^σ is symmetric, that is $P_0^\sigma(x) = P_0^\sigma(x^{-1})$, the above quantity is also equals to

$$\frac{1}{|\Gamma(-\sigma)|} \int\limits_X \int\limits_X (f(z) - f(x))\, P_0^\sigma(z^{-1}x)\ \overline{f(z)}\, dx\, dz.$$

By adding them up we get that

$$\langle (-\Delta)^\sigma f, f\rangle = \frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X |f(z) - f(x)|^2\, P_0^\sigma(z^{-1}x)\, dz\, dx.$$

The justification of the change of order of integration follows from (4.3). By the analytic continuation, we extend the range of σ to $0 < \sigma < 1$ provided $f \in C_c^\infty(X)$. Indeed, the functions $\sigma \mapsto -\Gamma(-\sigma)$ and $\sigma \mapsto \langle (-\Delta)^\sigma f, f\rangle$ are holomorphic on $S = \{w \in \mathbb{C} : 0 < \Re w < 1\}$. Hence their product $F(\sigma) = -\Gamma(-\sigma)\ \langle (-\Delta)^\sigma f, f\rangle$ is also holomorphic on S. On the other hand, since right-hand side of (4.3) is finite for $0 < \sigma < 1$, by the Morera's theorem it follows that the function G defined by

$$G(\sigma) = \frac{1}{2} \int\limits_X \int\limits_X |f(z) - f(x)|^2 \, P_0^\sigma(z^{-1}x) \, dz \, dx,$$

is holomorphic on S. Since $F(\sigma) = G(\sigma)$ for $0 < \sigma < 1/2$ we get that $F(\sigma) = G(\sigma)$ for all $\sigma \in S$, in particular, for $0 < \sigma < 1$.

By approximating any function $f \in H^\sigma(X)$ by a sequence of functions $f_k \in C_c^\infty(X)$, we complete the proof. This uses the fact that $P_0^\sigma(x) \asymp |x|^{-n-2\sigma}$ around the origin and the rest follows as in the proof of Lemma 5.1 in [38]. $\square$

We now establish ground state representation for the operator $(-\Delta)^\sigma$ as a consequence of the integral representation proved in Lemma 4.3. As in the Euclidean case, we define the following error term. For $0 < \sigma < 1$ and $y > 0$ we let,

$$H_y^\sigma[F] = \langle (-\Delta)^\sigma F, F \rangle - \frac{4^\sigma \Gamma(\sigma)}{y^{2\sigma}\Gamma(-\sigma)} \int\limits_X |F(x)|^2 \left(\frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)} \right) dx.$$

Theorem 4.4. *Let $0 < \sigma < 1$ and $y > 0$. If $F \in C_c^\infty(X)$ and $G(x) = F(x) \ \left(P_y^{-\sigma}(x)\right)^{-1}$ then*

$$H_y^\sigma[F] = \frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X |G(x) - G(z)|^2 \ P_y^{-\sigma}(x) P_y^{-\sigma}(z) \ P_0^{-\sigma}(z^{-1}x) \ dx \ dz.$$

Proof. Let $f, g \in H^\sigma(X)$. From Lemma 4.3 we get that

$$\langle (-\Delta)^\sigma f, g \rangle = \frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X (f(z) - f(x)) \ \overline{(g(z) - g(x))} \ P_0^\sigma(z^{-1}x) \ dz \ dx. \tag{4.4}$$

Let us assume $g = P_y^{-\sigma}$, and $f(x) = |F(x)|^2 \ g(x)^{-1}$. Then the right-hand side of (4.4) reduces to

$$\frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X \left(\frac{|F(z)|^2}{g(z)} - \frac{|F(x)|^2}{g(x)} \right) \overline{(g(z) - g(x))} \ P_0^\sigma(z^{-1}x) \ dz \ dx \tag{4.5}$$

$$= \frac{1}{2|\Gamma(-\sigma)|} \int\limits_X \int\limits_X \left(|F(x) - F(z)|^2 - \left| \frac{F(x)}{g(x)} - \frac{F(z)}{g(z)} \right|^2 \ g(x) g(z) \right) \ P_0^\sigma(z^{-1}x) \ dz \ dx.$$

Also, using Lemma 3.1 the left-hand side of (4.4) reduces to

$$\begin{aligned} \langle (-\Delta)^\sigma f, g \rangle &= \left\langle (-\Delta)^\sigma (|F(x)|^2 / g(x)), g(x) \right\rangle \\ &= \left\langle \left(|F(x)|^2 / g(x) \right), (-\Delta)^\sigma P_y^{-\sigma} \right\rangle \\ &= \frac{4^\sigma \Gamma(\sigma)}{y^{2\sigma}\Gamma(-\sigma)} \left\langle (|F(x)|^2 / g(x)), P_y^\sigma \right\rangle \end{aligned}$$

$$= \frac{4^\sigma \Gamma(\sigma)}{y^{2\sigma}\Gamma(-\sigma)} \int_X |F(x)|^2 \frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)}\, dx.$$

Therefore, equating the left-hand and right-hand sides of the equation (4.4) we have

$$\frac{4^\sigma \Gamma(\sigma)}{y^{2\sigma}\Gamma(-\sigma)} \int_X |F(x)|^2 \frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)}\, dx = \frac{1}{2|\Gamma(-\sigma)|} \int_X \int_X |F(x) - F(z)|^2\, P_0^\sigma(z^{-1}x)\, dx\, dz$$
$$- \frac{1}{2|\Gamma(-\sigma)|} \int_X \int_X \left| \frac{F(x)}{g(x)} - \frac{F(z)}{g(z)} \right|^2 g(x)g(z)\, P_0^\sigma(z^{-1}x)\, dz\, dx.$$

By Lemma 4.3 the first term in the right-hand side of the above equation is equals to $\langle (-\Delta)^\sigma F, F\rangle$. Hence, it follows that

$$\langle (-\Delta)^\sigma F, F\rangle - \frac{4^\sigma \Gamma(\sigma)}{y^{2\sigma}\Gamma(-\sigma)} \int_X |F(x)|^2 \frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)}\, dx$$
$$= \frac{1}{2|\Gamma(-\sigma)|} \int_X \int_X |G(x) - G(z)|^2\, P_y^{-\sigma}(x) P_y^{-\sigma}(x)\, P_0^{-\sigma}(z^{-1}x)\, dx\, dz,$$

where $G(x) = F(x)P_y^{-\sigma}(x)^{-1}$. This completes the proof. $\square$

We have already observed that for $0 < \sigma < 1$, $\Gamma(-\sigma) < 0$ and hence $P_y^{-\sigma} \leq 0$. Therefore, as a corollary of Theorem 4.4 we get the following result.

Corollary 4.5. *For a fixed $y > 0$ and $0 < \sigma < 1$ we have*

$$\langle (-\Delta)^\sigma F, F\rangle \geq \frac{4^\sigma}{y^{2\sigma}} \int_X |F(x)|^2 \left(\frac{\Gamma(\sigma)}{\Gamma(-\sigma)} \frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)} \right) dx, \quad \text{for } F \in H^\sigma(X).$$

Remark 4.6. By Lemma 3.1 it follows that the equality in the expression above is achieved for the function $F = P_y^{-\sigma}$. Therefore, the constant $4^\sigma \Gamma(\sigma)/y^{2\sigma}|\Gamma(-\sigma)|$ appeared in the corollary above is sharp.

Now, using the estimate of P_y^σ (Theorem 3.2) in Corollary 4.5 we get Theorem 1.1.

Proof of Theorem 1.1. From Theorem 3.2 we have

$$\frac{\Gamma(\sigma)}{\Gamma(-\sigma)} \frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)} \asymp \begin{cases} \frac{y^{4\sigma}}{(|x|^2+y^2)^\sigma} & \text{if } |x|^2 + y^2 \geq 1 \\ \frac{y^{4\sigma}}{(|x|^2+y^2)^{2\sigma}} & \text{if } |x|^2 + y^2 < 1. \end{cases}$$

Therefore, from Corollary 4.5 we have

$$\langle(-\Delta)^\sigma F, F\rangle$$
$$\geq C_\sigma\, y^{2\sigma}\left(\int\limits_{\{x:|x|^2+y^2<1\}} \frac{|F(x)|^2}{(y^2+|x|^2)^{2\sigma}}\,dx + \int\limits_{\{x:|x|^2+y^2\geq 1\}} \frac{|F(x)|^2}{(y^2+|x|^2)^{\sigma}}\,dx\right). \quad \square$$

We now prove Hardy's inequality corresponding to the homogeneous weight function (Theorem 1.3). To prove this theorem we need the following expression of the error term.

Theorem 4.7. *Let $0<\sigma<1$ and $\alpha>(2\sigma+n)/4$. Then for $F\in C_c^\infty(X)$ and $G(x)=F(x)\ \left(P_0^{-\alpha}(x)\right)^{-1}$ we have*

$$\langle(-\Delta)^\sigma F, F\rangle - \frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\int\limits_X |F(x)|^2\left(\frac{P_0^{\sigma-\alpha}(x)}{P_0^{-\alpha}(x)}\right)dx$$
$$= \frac{1}{2|\Gamma(-\sigma)|}\int\limits_X\int\limits_X |G(x)-G(z)|^2\ P_0^{-\alpha}(x)P_0^{-\alpha}(z)\ P_0^\sigma(z^{-1}x)\ dx\ dz, \tag{4.6}$$

where the function $P_0^{-\alpha}$ is defined by (3.4).

Proof. Since $\alpha>n/4$, we observe from Theorem 3.3 that $P_0^{-\alpha}\in L^2(X)$. As before by Fubini theorem the spherical Fourier transform of $P_0^{-\alpha}$ is given by

$$\widehat{P_0^{-\alpha}}(\lambda) = \int\limits_0^\infty e^{-t(|\lambda|^2+|\rho|^2)}\frac{dt}{t^{1-\alpha}} = \Gamma(\alpha)\ \left(|\lambda|^2+|\rho|^2\right)^{-\alpha},\ \ \lambda\in\mathfrak{a}^*.$$

Since $\alpha>(2\sigma+n)/4$, it follows that $P_0^{-\alpha}\in H^\sigma(X)$. Indeed, using (2.3) we get that

$$\int\limits_{\mathfrak{a}^*}|\widehat{P_0^{-\alpha}}(\lambda)|^2\ (|\lambda|^2+|\rho|^2)^\sigma\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda$$
$$\leq C + C'\int\limits_{\{\mathfrak{a}^*:|\lambda|\geq 1\}} (|\lambda|^2+|\rho|^2)^{-2\alpha+\sigma}\ (1+|\lambda|)^{n-l}\ d\lambda,$$

which is finite. We recall from (4.4) that for $f,g\in H^\sigma(X)$

$$\langle(-\Delta)^\sigma f, g\rangle = \frac{1}{2|\Gamma(-\sigma)|}\int\limits_X\int\limits_X (f(z)-f(x))\ \overline{(g(z)-g(x))}\ P_0^\sigma(z^{-1}x)\ dz\ dx. \tag{4.7}$$

If we put $g(x)=P_0^{-\alpha}(x)$ and $f(x)=|F(x)|^2(P_0^{-\alpha}(x))^{-1}$ in the equation above, then the left-hand side reduces to

$$\begin{aligned}\langle(-\Delta)^\sigma f, g\rangle &= \int_{\mathfrak{a}^*} \left(|\lambda|^2+|\rho|^2\right)^\sigma \widehat{f}(\lambda)\,\widehat{g}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\ &= \Gamma(\alpha)\int_{\mathfrak{a}^*} \left(|\lambda|^2+|\rho|^2\right)^{\sigma-\alpha} \widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\int_{\mathfrak{a}^*} \widehat{P_0^{\sigma-\alpha}}(\lambda)\ \widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\int_X |F(x)|^2\frac{P_0^{\sigma-\alpha}(x)}{P_0^{-\alpha}(x)}\,dx.\end{aligned}$$

The right-hand side of the equation (4.7) becomes (see (4.5))

$$\frac{1}{2|\Gamma(-\sigma)|}\int_X\int_X \left(|F(x)-F(z)|^2 - \left|\frac{F(x)}{g(x)}-\frac{F(z)}{g(z)}\right|^2 g(x)g(z)\right) P_0^\sigma(z^{-1}x)\ dz\ dx. \tag{4.8}$$

Hence, equating both sides of the equation (4.7) we have

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\int_X |F(x)|^2\frac{P_0^{\sigma-\alpha}(x)}{P_0^{-\alpha}(x)}\,dx = \frac{1}{2|\Gamma(-\sigma)|}\int_X\int_X |F(x)-F(z)|^2\ P_0^\sigma(z^{-1}x)\ dz\ dx$$
$$-\frac{1}{2|\Gamma(-\sigma)|}\int_X\int_X \left|\frac{F(x)}{g(x)}-\frac{F(z)}{g(z)}\right|^2 g(x)g(z)\ P_0^\sigma(z^{-1}x)\ dz\ dx.$$

By Lemma 4.3 the first term in the right-hand side of the above equation is equals to $\langle(-\Delta)^\sigma F, F\rangle$ and hence the required identity follows. $\square$

Proof of Theorem 1.3. Since $\sigma < 1$ and $n \geq 2$, we can choose a positive α such that $2\sigma + n/4 < \alpha < n/2$. From Theorem 3.3 above it follows that

$$\begin{aligned}\frac{P_0^{\sigma-\alpha}(x)}{P_0^{-\alpha}(x)} &\asymp |x|^{-2\sigma}, \quad \textit{for } |x|<1; \\ &\asymp |x|^{-\sigma}, \quad \textit{for } |x|\geq 1.\end{aligned}$$

Therefore, it follows from Theorem 4.7 that

$$\langle(-\Delta)^\sigma F, F\rangle \geq C_\sigma \left(\int_{|x|<1} \frac{|F(x)|^2}{|x|^{2\sigma}}\,dx + \int_{|x|\geq 1}\frac{|F(x)|^2}{|x|^{\sigma}}\,dx\right). \quad \square$$

We now find the optimal constants (in Theorem 1.1 and Theorem 1.3) for the case when the group G is complex (cf. [21, Theorem 1.2] for Ornstein-Uhlenbeck operator on $\mathbb{R}^n$). In this case, we have the following formula for the heat kernel [5]

$$h_t(\exp H) = (4\pi t)^{-n/2}\ e^{-|\rho|^2 t} \left(\prod_{\alpha\in\Sigma^+} \frac{\alpha(H)}{\sinh \alpha(H)} \right) e^{-H^2/4t},\ \ t>0,\ \ H \in \mathfrak{a}.$$

It now follows from the definition (1.9) of P_y^σ that

$$\begin{aligned} P_y^\sigma(\exp H) &= \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} (4\pi)^{-n/2} \left(\prod_{\alpha\in\Sigma^+} \frac{\alpha(H)}{\sinh \alpha(H)} \right) \int_0^\infty t^{-n/2}\ e^{-|\rho|^2 t}\ e^{-(|H|^2+y^2)/4t}\ \frac{dt}{t^{1+\sigma}} \\ &= \frac{y^{2\sigma}}{\Gamma(\sigma)}\ 2^{1-n/2-\sigma}\ \pi^{-n/2} \left(\prod_{\alpha\in\Sigma^+} \frac{\alpha(H)}{\sinh \alpha(H)} \right) \left(\frac{\sqrt{|H|^2+y^2}}{|\rho|} \right)^{-(n+2\sigma)/2} \\ &\quad \times K_{-n/2-\sigma}(\sqrt{|H|^2+y^2}|\rho|). \end{aligned}$$

Here the last equality follows from the formula [23, 3.471(9), p. 368], and $K_{-n/2-\sigma}$ is the modified Bessel function (defined in [23, 8.407 (1), p. 911]). Therefore, using the above expression we have the following result.

Theorem 4.8. *Let $0 < \sigma < 1$ and $F \in H^\sigma(X)$. Then for every $y > 0$ we have*

$$\langle (-\Delta)^\sigma F, F\rangle \geq |\rho|^{2\sigma} y^{2\sigma} \int_X \frac{|F(x)|^2}{(y^2+|x|^2)^\sigma}\ w_\sigma(\sqrt{|x|^2+y^2}|\rho|)\, dx,$$

for an explicit $w_\sigma(t) \geq 1$. The inequality is sharp and equality is attained for $F(x) = P_y^{-\sigma}(x)$.

Proof. Using the fact that $K_{-\nu} = K_\nu$, it follows from the above expression of P_y^σ that

$$\frac{P_y^\sigma(x)}{P_y^{-\sigma}(x)} = \frac{y^{4\sigma}\Gamma(-\sigma)}{4^\sigma\ \Gamma(\sigma)}\ |\rho|^{2\sigma}\ (|x|^2+y^2)^{-\sigma}\ \frac{K_{n/2+\sigma}(\sqrt{|x|^2+y^2}|\rho|)}{K_{n/2-\sigma}(\sqrt{|x|^2+y^2}|\rho|)}.$$

Let $w_\sigma(t) = \frac{K_{n/2+\sigma}(t)}{K_{n/2-\sigma}(t)}$, for $t > 0$. Now using the fact that for $t > 0$, $K_\nu(t)$ is increasing function of ν we note that $w_\sigma(t) \geq 1$, for all $t > 0$. Hence the required inequality follows from Corollary 4.5. The sharpness of the constant follows from Remark 4.6. □

In the case when G is complex, we also have

$$\begin{aligned} P_0^\sigma(x) &= \int_0^\infty h_t(x)\ \frac{dt}{t^{1+\sigma}} \\ &= (4\pi)^{-n/2} \left(\prod_{\alpha\in\Sigma^+} \frac{\alpha(\log x)}{\sinh \alpha(\log x)} \right) \int_0^\infty t^{-n/2}\ \ e^{-|\rho|^2 t}\ e^{-|x|^2/4t}\ \frac{dt}{t^{1+\sigma}} \end{aligned}$$

$$= (4\pi)^{-n/2} \left(\prod_{\alpha\in\Sigma^+} \frac{\alpha(\log x)}{\sinh \alpha(\log x)} \right) 2 \left(\frac{|x|}{2|\rho|} \right)^{-(n/2+\sigma)} K_{-n/2-\sigma}\left(|x||\rho|\right).$$

Therefore, using $K_{-\nu} = K_\nu$ it follows that

$$\frac{P_0^{\sigma-\alpha}(x)}{P_0^{-\alpha}(x)} = 2^\sigma \ |\rho|^\sigma \ |x|^{-\sigma} \ \frac{K_{n/2+\sigma}(|x| \ |\rho|)}{K_{n/2-\sigma}(|x| \ |\rho|)}.$$

Now applying Theorem 4.7 and taking $\alpha \to (2\sigma + n)/4$ we have

Theorem 4.9. *Let $0 < \sigma < 1$. Then for $F \in C_c^\infty(X)$ we have*

$$\langle (-\Delta)^\sigma F, F\rangle \geq |\rho|^\sigma \ 2^\sigma \ \frac{\Gamma\left((n+2\sigma)/4\right)}{\Gamma\left((n-2\sigma)/4\right)} \int_X \frac{|F(x)|^2}{|x|^\sigma} \ w_\sigma(|x||\rho|)\, dx,$$

for the explicit $w_\sigma(t) \geq 1$ given above.

In the case of $\mathbb{R}^n$ and of the Heisenberg groups the function P_y^σ is the classical Poisson kernel. For symmetric spaces, we only have the integral expression (1.9) and the both-sides estimates (Theorem 3.2) for P_y^σ. We have obtained a precise expression of P_y^σ for the case when G is complex. We now write the expression of P_y^σ for rank one symmetric spaces using the expression of the heat kernel. Let $F = \mathbb{R}, \mathbb{C}, H$, or O be the real numbers, the complex numbers, the quaternions or the Cayley octonions respectively. The rank one symmetric spaces can be realized as the hyperbolic space $\mathbb{H}^n(F)$. Here the subscript n denotes the dimension over the base field F. Using the expression of the heat kernel [5,22] we have the following results.

(1) $X = \mathbb{H}^n(\mathbb{R})$, and $n \geq 3$ odd. Using the formula [23, 3.471(9), p. 368] we get

$$\begin{aligned} P_y^\sigma(x) &= c \int_0^\infty t^{-1/2} \ e^{-\rho^2 t} \ e^{-y^2/4t} \left(-\frac{1}{\sinh x} \ \frac{\partial}{\partial x} \right)^{(n-1)/2} e^{-|x|^2/4t} \ \frac{dt}{t^{1+\sigma}} \\ &= c \left(-\frac{1}{\sinh x} \ \frac{\partial}{\partial x} \right)^{(n-1)/2} \int_0^\infty t^{-3/2-\sigma} \ e^{-\rho^2 t} \ e^{-(|x|^2+y^2)/4t} \ dt \\ &= c \left(-\frac{1}{\sinh x} \ \frac{\partial}{\partial x} \right)^{(n-1)/2} \left(\frac{\sqrt{|x|^2+y^2}}{\rho} \right)^{-\sigma-1/2} K_{-\sigma-1/2}(\rho\sqrt{|x|^2+y^2}). \end{aligned}$$

(2) $X = \mathbb{H}^n(\mathbb{R})$, and $n \geq 2$ even. In this case

$$P_y^\sigma(x) = c \int_0^\infty t^{-1/2} \ e^{-\rho^2 t} \ e^{-y^2/4t} \int_x^\infty \frac{\sinh z}{\sqrt{\cosh^2 z - \cosh^2 x}} \left(-\frac{1}{\sinh z} \ \frac{\partial}{\partial z} \right)^{n/2}$$

$$\times e^{-|z|^2/4t}\ dz\ \frac{dt}{t^{1+\sigma}}$$

$$= c\int_x^\infty \frac{\sinh z}{\sqrt{\cosh^2 z-\cosh^2 x}}\left(-\frac{1}{\sinh z}\ \frac{\partial}{\partial z}\right)^{n/2}$$

$$\times \int_0^\infty t^{-3/2-\sigma}\ e^{-\rho^2 t}\ e^{-(|z|^2+y^2)/4t}\ dt\ dz$$

$$= c\int_x^\infty \frac{\sinh z}{\sqrt{\cosh^2 z-\cosh^2 x}}\left(-\frac{1}{\sinh z}\ \frac{\partial}{\partial z}\right)^{n/2}\left(\frac{\sqrt{|z|^2+y^2}}{\rho}\right)^{-\sigma-1/2}$$

$$\times K_{-\sigma-1/2}(\rho\sqrt{|z|^2+y^2})\ dz.$$

(3) $X=\mathbb{H}^n(F)$ where $F=\mathbb{C},H$ or O. Then there exist constants $c_1,c_2,\cdots,c_{n/2}$ such that

$$P_y^\sigma(x)=\int_0^\infty t^{-1/2}\ e^{-\rho^2 t}\sum_{j=1}^{n/2}c_j\int_x^\infty \frac{\sinh z}{\sqrt{\cosh^2 z-\sinh^2 x}}\ (\cosh z)^{j+1-d}$$

$$\times\left(-\frac{1}{2\pi\sinh z}\ \frac{\partial}{\partial z}\right)^{j+m_\alpha/2} e^{-|z|^2/4t}\ dz\ \frac{dt}{t^{1+\sigma}}$$

$$= c_\sigma\sum_{j=1}^{d/2}c_j\int_x^\infty \frac{\sinh z}{\sqrt{\cosh^2 z-\sinh^2 x}}\ (\cosh z)^{j+1-d}\ \rho^{1+2\sigma}$$

$$\times\left(-\frac{1}{2\pi\sinh z}\ \frac{\partial}{\partial z}\right)^{j+m_\alpha/2} 2\left(\frac{2}{\sqrt{|z|^2+y^2}\rho}\right)^{\sigma+1/2}$$

$$\times K_{-\sigma-1/2}(\rho\sqrt{|z|^2+y^2})\ dz,$$

where the constant c_σ depends only on σ.

5. Mapping properties of Poisson operator

In this section we prove Theorem 1.8. We start with the following lemma.

Lemma 5.1. *For $0<\sigma<1$ and $1<q<\frac{n+1}{n}$, the function $(x,y)\mapsto P_y^\sigma(x)\in L^q(X\times\mathbb{R}_+)$.*

Proof. We first observe from (2.1) that for $H\in\mathfrak{a}$ with $|H|<1$, the Jacobian $J(\exp H)$ corresponding to the polar decomposition is of order $|H|^{n-l}$. From Theorem 3.2 it follows that

$$\int_{|x|^2+y^2<1}|P_y^\sigma(x)|^q\ dx\ dy\le C\int_{|x|^2+y^2<1} y^{2\sigma q}(|x|^2+y^2)^{-nq/2-\sigma q}\ dx\ dy$$

$$\leq C \int_{y=0}^{1} \int_{\{H\in\overline{\mathfrak{a}^+}:|H|<1\}} y^{2\sigma q}(|H|^2+y^2)^{-nq/2-\sigma q}\, |H|^{n-l}\, dH\, dy$$

$$= \int_0^1 \int_0^1 y^{2\sigma q}(r^2+y^2)^{-nq/2-\sigma q}\, r^{n-l}\, r^{l-1}\, dr\, dy$$

$$\leq \int_0^1 \left(\int_0^\infty (1+s^2)^{-nq/2-\sigma q} s^{n-1}\, ds \right) y^{n-nq}\, dy.$$

We now use the following fact from [23, 3.251, (2); p. 324]

$$\int_0^\infty x^{\mu-1}(1+x^2)^{\nu-1}\, dx = \frac{1}{2}B\left(\mu/2, (1-\nu-\mu/2)\right),\ \text{if } \Re\mu > 0,\ \text{and } \Re(\nu+\mu/2) < 1. \tag{5.1}$$

In our case, $\mu = n$ and $\nu = -nq/2 - \sigma q + 1$. Hence, $\nu + \mu/2 < 1$ if and only if $q > n/(n+2\sigma)$. Therefore, if $q > n/(n+2\sigma)$ the above integral reduces to

$$\frac{1}{2}B\left(n/2, (nq/2+\sigma q - n/2)\right) \int_0^1 y^{n-nq}\, dy.$$

This is finite only if $q < (1+n)/n$. Hence, for $n/(n+2\sigma) < q < 1 + \frac{1}{n}$,

$$\int_{|x|^2+y^2\leq 1} |P_y^\sigma(x)|^q\, dx\, dy < \infty.$$

On the other hand for $q > 1$, using the estimate of Jacobian in (2.1) and the asymptotic behavior of ϕ_0 given in (2.6), it follows from Theorem 3.2 that

$$\int_{|x|^2+y^2\geq 1} |P_y^\sigma(x)|^q\, dx\, dy$$

$$\leq \int_{|x^2|+y^2\geq 1} \frac{y^{2\sigma q}}{(4^\sigma \Gamma(\sigma))^q} \left(\sqrt{|x|^2+y^2}\right)^{-(l/2+|\Sigma_0^+|+\sigma+1/2)q} e^{-|\rho|q\sqrt{|x|^2+y^2}}\, |\phi_0(x)|^q\, dx\, dy$$

$$\leq C \int_{|x|^2+y^2\geq 1} y^{2\sigma q}\, e^{-\frac{|\rho|(q+1)}{2}\sqrt{|x|^2+y^2}}\, e^{-\frac{|\rho|(q-1)}{2}\sqrt{|x|^2+y^2}}\, |\phi_0(x)|^q\, dx\, dy$$

$$\leq C \int_{\left\{(H,y)\in\overline{\mathfrak{a}^+}\times(0,\infty):|H|^2+y^2\geq 1\right\}} y^{2\sigma q}\, e^{-\frac{|\rho|(q-1)|y|}{2}}\, e^{-\frac{|\rho|(q+1)|H|}{2}}$$

$$\times\, |H|^{|\Sigma_0^+|q}\, e^{-q\,\rho(H)}\; e^{2\rho(H)}\; dH\; dy$$
$$\leq \left(\int_0^\infty y^{2\sigma q}\; e^{-|\rho|(q-1)|y|/2}\; dy\right)\left(\int_{\mathfrak{a}^+} |H|^{|\Sigma_0^+|q}\; e^{-\frac{3}{2}(q-1)\rho(H)}\; dH\right) < \infty.$$

This completes the proof. $\square$

We are now in a position to prove Theorem 1.8. We follow similar ideas which are used to the proof of [35, Theorem B].

Proof of Theorem 1.8. We first prove (1). Let u be the solution of (1.5) with boundary value $f \in H^\sigma(X)$, and let

$$\mathcal{U}(\lambda, k, \eta) = \mathcal{F}\left(\widetilde{u}(\lambda, k)\right)(\eta), \quad \textit{for } \lambda \in \mathfrak{a}^*, k \in K, \eta \in \mathbb{R}_+$$

be the composition of the Helgason and the Euclidean Fourier transform on $X \times \mathbb{R}$. Multiplying y^2 on both sides of the equation (1.5) and taking the composition of Helgason and Euclidean Fourier transform on $X \times \mathbb{R}$ it follows that

$$\frac{\partial^2}{\partial \eta^2}\left(\left(|\lambda|^2 + |\rho|^2 + \eta^2\right)\mathcal{U}(\lambda, k, \eta)\right) - (1 - 2\sigma)\frac{\partial}{\partial \eta}\left(\eta\, \mathcal{U}(\lambda, k, \eta)\right) = 0$$

which is equivalent to

$$\left\{(|\lambda|^2 + |\rho|^2 + \eta^2)\frac{\partial^2}{\partial^2 \eta} + (3 + 2\sigma)\eta\frac{\partial}{\partial \eta} + (1 + 2\sigma)\right\}\mathcal{U}(\lambda, k, \eta) = 0. \tag{5.2}$$

Let $t = \frac{\eta}{\sqrt{|\lambda|^2+|\rho|^2}}$ and we define

$$v(\lambda, k, t) = \mathcal{U}(\lambda, k, \eta).$$

Then equation (5.2) reduces to

$$\mathcal{D}_{\sigma,t} v(\lambda, k, t) := \left\{(1 + t^2)\frac{d^2}{dt^2} + (2\sigma + 3)t\frac{d}{dt} + (2\sigma + 1)\right\} v(\lambda, k, t) = 0.$$

Since $f(x) = u(x, 0)$ for $x \in X$, by the Euclidean Fourier inversion formula we have

$$\widetilde{f}(\lambda, k) = u(\cdot, 0)\widetilde{\ }(\lambda, k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{U}(\lambda, k, \eta)\; d\eta = \frac{\sqrt{|\lambda|^2 + |\rho|^2}}{\sqrt{2\pi}}\int_{\mathbb{R}} v(\lambda, k, t)\; dt.$$

Therefore, the function v satisfies

$$\mathcal{D}_{\sigma,t}v(\lambda,k,t)=0,\ \text{ and } \int_{\mathbb{R}} v(\lambda,k,t)\,dt=\frac{\sqrt{2\pi}}{\sqrt{|\lambda|^2+|\rho|^2}}\widetilde{f}(\lambda,k),$$

for almost every $(\lambda,k)\in\mathfrak{a}^*\times K$. Hence, the function v is given by

$$v(\lambda,k,t)=\frac{\sqrt{2\pi}}{\sqrt{|\lambda|^2+|\rho|^2}}\widetilde{f}(\lambda,k)\ \psi(t), \tag{5.3}$$

where ψ satisfies

$$\mathcal{D}_{\sigma,t}\psi=0,\ \text{ and } \int_{\mathbb{R}}\psi(t)\ dt=1. \tag{5.4}$$

The equation $D_{\sigma,t}\psi=0$ has a fundamental system of solutions spanned by

$$\begin{aligned}\psi_1(t)&={}_2F_1\left(\frac{1}{2},\sigma+\frac{1}{2};\frac{1}{2};-t^2\right)=(1+t^2)^{-\sigma-1/2},\\ \psi_2(t)&=t\,{}_2F_1\left(1,\sigma+1;\frac{3}{2};-t^2\right).\end{aligned}$$

Using (5.3) it is now easy to check that

$$\begin{aligned}&\int_{\mathfrak{a}^*\times K\times\mathbb{R}}|\mathcal{U}(\lambda,k,\eta)|^2(|\lambda|^2+|\rho|^2+\eta^2)^{\sigma+\frac{1}{2}}\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda\ dk\ d\eta\\ &=\int_{\mathfrak{a}^*\times K\times\mathbb{R}}|v(\lambda,k,t)|^2\ (|\lambda|^2+|\rho|^2)^{\sigma+1}\ \left(1+t^2\right)^{\sigma+\frac{1}{2}}\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda\ dk\ dt\\ &=2\pi\int_{\mathfrak{a}^*\times K}|\widehat{f}(\lambda,k)|^2\ (|\lambda|^2+|\rho^2|)^{\sigma}\ |\mathbf{c}(\lambda)|^{-2}\ d\lambda\ dk\int_{\mathbb{R}}|\psi(t)|^2\ (1+t^2)^{\sigma+\frac{1}{2}}\ dt. \end{aligned}\tag{5.5}$$

Since $f\in H^\sigma$, it follows that $u\in H^{\sigma+\frac{1}{2}}$ if and only if $\psi\in L^2(\mathbb{R},(1+t^2)^{\sigma+\frac{1}{2}}\,dt)$. It is easy to check from the asymptotic properties of hypergeometric function that $\psi_2\notin L^2(\mathbb{R},(1+t^2)^{\sigma+\frac{1}{2}}\,dt)$ (see [1, Theorem 2.3.2]). Hence, we choose $\psi(t)$ to be a constant multiple of $\psi_1(t)=(1+t^2)^{-\sigma-\frac{1}{2}}$. From (5.1) we get that $\|\psi_1\|_{L^1(\mathbb{R})}=\sqrt{\pi}\,\Gamma(\sigma)/\Gamma(\sigma+\frac{1}{2})$. Hence, using (5.4) it follows that

$$\psi(t)=\frac{\Gamma(\sigma+1/2)}{\sqrt{\pi}\Gamma(\sigma)}\psi_1(t).$$

We now observe that

$$\int_{\mathbb{R}}|\psi(t)|^2\ (1+t^2)^{\sigma+\frac{1}{2}}\ dz=\frac{\Gamma(\sigma+1/2)}{\sqrt{\pi}\Gamma(\sigma)},$$

and hence from (5.5),

$$\|u\|^2_{H^{\sigma+\frac{1}{2}}(X\times\mathbb{R}_+)} = \frac{2\sqrt{\pi}\Gamma(\sigma+\frac{1}{2})}{\Gamma(\sigma)}\|f\|_{H^\sigma(X)}.$$

This completes the proof of part (1). We now prove part (2). We first observe that

$$\|T_\sigma f\|^q_{L^q(X\times\mathbb{R}_+)} = \int_0^\infty \|f * P^\sigma_y\|^q_{L^q(X)}\, dy.$$

Also, from Theorem 3.2 it follows that for each $y > 0$ the function $P^\sigma_y \in L^q(X)$, for all $q > 1$. Therefore, by Kunze-Stein phenomenon (Remark 1.9), for $1 \le p < q \le 2$

$$\|f * P^\sigma_y\|_{L^q(X)} \le C\|f\|_{L^p(X)}\|P^\sigma_y\|_{L^q(X)}.$$

Therefore, by Lemma 5.1 it follows that

$$T_\sigma : L^p(X) \to L^q(X\times\mathbb{R}_+), \tag{5.6}$$

is a bounded map, for $1 \le p < q < (n+1)/n$. We also observe that

$$T_\sigma : L^\infty(X) \to L^\infty(X\times\mathbb{R}_+), \tag{5.7}$$

is a bounded map, as the integral $\int_X P^\sigma_y(x)\,dx = 1$ for all $y > 0$. By Riesz Thorin interpolation theorem it now follows from (5.6) and (5.7) that

$$T_\sigma : L^p(X) \to L^q(X\times\mathbb{R}_+), \tag{5.8}$$

is bounded for $1 \le p < \infty$ and $p < q < (\frac{n+1}{n})p$. We now prove that

$$\|T_\sigma f\|_{L^q(X\times\mathbb{R}_+)} \le C\|f\|_{L^p(X)},$$

for $p > 1$ and $q = (\frac{n+1}{n})p$. By (5.7) and Marcinkiewicz interpolation theorem it is enough to show that

$$T_\sigma : L^1(X) \to L^{(1+n)/n,\infty}(X\times\mathbb{R}_+).$$

Using Theorem 3.2 and the boundedness of the function ϕ_0 we get that

$$|T_\sigma f(x,y)| \le \int_X |f(z)|P^\sigma_y(z^{-1}x)\ |f(z)|\ dz \le Cy^{-n}\ \|f\|_{L^1(X)}$$

$$+Cy^{2\sigma} \int_{|z^{-1}x|^2+y^2\ge 1} \sqrt{(|z^{-1}x|^2+y^2)}^{(-l/2-1/2-\sigma-|\Sigma_0^+|)}\ e^{-|\rho|\sqrt{(|z^{-1}x|^2+y^2)}}|f(z)|\ dz$$

$$\le Cy^{-n}\ \|f\|_{L^1(X)} + Cy^{-n}\ \|f\|_{L^1(X)}\ \sup y \in \mathbb{R}_+ \left(y^{2\sigma+n} e^{-|\rho| y}\right)$$
$$\le Cy^{-n}\ \|f\|_{L^1(X)}.$$

Hence, $|T_a f(x,y)| > \lambda$ implies that $y \le \left(\frac{C\|f\|_{L^1(X)}}{\lambda}\right)^{\frac{1}{n}} = b$ (say). Then Chebyshev's inequality yields

$$\begin{aligned}
&m\left(\{(x,y) \in X \times \mathbb{R}_+ : |T_a f(x,y)| > \lambda\}\right) \\
&= m\left(\{(x,y) \in X \times \mathbb{R}_+ : y < b, |T_a f(x,y)| > \lambda\}\right) \\
&\le \frac{1}{\lambda} \int\limits_{\{(x,y)\in X\times\mathbb{R}_+ : y<b\}} |T_a f(x,y)|\ dx\ dy \\
&\le \frac{C_a}{\lambda} \int\limits_X |f(z)| \int\limits_{\{(x,y)\in X\times\mathbb{R}_+ : y<b\}} P_y^\sigma(z^{-1}x)\ dx\ dy\ dz \\
&\le \frac{C_\sigma}{\lambda} \|f\|_{L^1(X)}\, b = C_\sigma \left(\frac{\|f\|_{L^1(X)}}{\lambda}\right)^{1+\frac{1}{n}}.
\end{aligned}$$

The last inequality follows because of the fact that $\int_X P_y^\sigma(x)\,dx = 1$ for all $y > 0$. This completes the proof. □

6. Poincaré-Sobolev inequality

In this section we prove Theorem 1.12. For the convenience of the reader we restate the theorem here.

Theorem 6.1. *Let* $\dim X = n \ge 3$ *and* $0 < \sigma < \min\{l + 2|\Sigma_0^+|, n\}$. *Then for* $2 < p \le \frac{2n}{n-\sigma}$ *there exists* $S = S_{n,\sigma,p} > 0$ *such that for all* $f \in H^{\frac{\sigma}{2}}(X)$

$$\|(-\Delta - |\rho|^2)^{\sigma/4} f\|^2_{L^2(X)} \ge S\|f\|^2_{L^p(X)}. \tag{6.1}$$

Proof. We first observe that it is enough to prove the result for $f \in C_c^\infty(X)$. It also suffices to show that

$$\int\limits_X f(x)\ (-\Delta - |\rho|^2)^{-\sigma/2} f(x)\ dx \le C\|f\|^2_{L^{p'}(X)}. \tag{6.2}$$

Indeed, if (6.2) holds, then by Hölder's inequality

$$\begin{aligned}
|\langle f, g\rangle| &= \left|\left\langle \left(-\Delta - |\rho|^2\right)^{\sigma/4} f, \left(-\Delta - |\rho|^2\right)^{-\sigma/4} g\right\rangle\right| \\
&\le \left\|\left(-\Delta - |\rho|^2\right)^{\sigma/4} f\right\|_{L^2(X)} \left\|\left(-\Delta - |\rho|^2\right)^{-\sigma/4} g\right\|_{L^2(X)}
\end{aligned}$$

$$= \left\langle \left(-\Delta - |\rho|^2\right)^{\sigma/2} f, f \right\rangle^{1/2} \left\langle \left(-\Delta - |\rho|^2\right)^{-\sigma/2} g, g \right\rangle^{1/2}$$
$$\leq C^{\frac{1}{2}} \left\langle \left(-\Delta - |\rho|^2\right)^{\sigma/2} f, f \right\rangle^{\frac{1}{2}} \|g\|_{L^{p'}(X)},$$

and hence

$$\|f\|_{L^p(X)} \leq C_n^{\frac{1}{2}} \left\langle \left(-\Delta - |\rho|^2\right)^{\sigma/2} f, f \right\rangle^{\frac{1}{2}}.$$

We now prove (6.2). Let k_σ be the Schwartz kernel for the operator $(-\Delta - |\rho|^2)^{-\sigma/2}$. We have the following well-known estimates due to Anker and Ji [4, Theorem 4.2.2], for $0 < \sigma < l + 2|\Sigma_0^+|$

$$\begin{aligned} k_\sigma(x) &\asymp |x|^{\sigma - l - 2|\Sigma_0^+|} \, \phi_0(x), \;\; |x| \geq 1, \\ &\asymp |x|^{\sigma - n}, \;\; |x| < 1. \end{aligned} \tag{6.3}$$

To prove (6.2), it is enough to show that

$$\|f * k_\sigma\|_{L^p(X)} \leq C\|f\|_{L^{p'}(X)}.$$

Let χ be the characteristic function of the unit ball $\mathbf{B}(o,1)$ and $k_\sigma^0(x) = \chi(x)\, k_\sigma(x)$ and $k_\sigma^\infty = k - k_0$. Now, by Young's inequality we have that

$$\|f * k_\sigma^0\|_{L^p(X)} \leq C\|f\|_{L^{p'}(X)} \, \|k_\sigma^0\|_{L^{p/2}(X)},$$

and

$$\|k_\sigma^0\|_{L^{p/2}(X)}^{\frac{p}{2}} \asymp \int_0^1 |t|^{(\sigma-n)p/2} \, |t|^{|\Sigma^+|} \, t^{l-1} \, dt.$$

The right-hand side is finite if $p < \frac{2n}{n-\sigma}$. Using the fact that for $r < 1$, the volume of the ball $B(o,r)$ in X is of order r^n, it is easy to check that $k_\sigma^0 \in L^{\frac{n}{n-\sigma},\infty}(X)$. By Young's inequality for weak type spaces [24, Theorem 1.4.24. page 63] it follows that

$$\|f * k_\sigma^0\|_{L^{\frac{2n}{n-\sigma}}(X)} \leq C\|f\|_{L^{\frac{2n}{n+\sigma}}(X)}.$$

Therefore, we have for all $p \leq \frac{2n}{n-\sigma}$,

$$\|f * k_\sigma^0\|_{L^p(X)} \leq C\|f\|_{L^{p'}(X)}. \tag{6.4}$$

Next, we shall show that for $p > 2$,

$$\|f * k_\sigma^\infty\|_{L^p(X)} \leq C_p\|f\|_{L^{p'}(X)}.$$

To prove this we shall use complex interpolation theorem and the idea of [36, Theorem 4.1]. For $\Re z \geq -\frac{1}{2}$, we define an analytic family of linear operators T_z from (X, dx) to itself as follows:

$$T_z f = f * (k_\sigma^\infty)^{1+z}.$$

For $z = -\frac{1}{2} + iy$, we have

$$\begin{aligned}\|T_z f\|_{L^\infty(X)} &= \|f * (k_\sigma^\infty)^{\frac{1}{2}+iy}\|_{L^\infty(X)} \\ &\leq C \sup_{\{x \in X : |x| \geq 1\}} \varphi_0(x)^{\frac{1}{2}} |x|^{(\sigma - l)/2 - |\Sigma_0^+|} \|f\|_{L^1(X)} \\ &\leq C\|f\|_{L^1(X)}.\end{aligned}$$

For $z = \epsilon + iy, \epsilon > 0$, we have

$$\begin{aligned}\|T_z f\|_{L^2(X)}^2 &= \int_{\mathbb{R}} \int_K \left|\widetilde{f}(\lambda, k)\right|^2 \left|\widehat{(k_\sigma^\infty)^{1+\epsilon+iy}}(\lambda)\right|^2 |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk \\ &\leq \sup \left|\widehat{(k_\sigma^\infty)^{1+\epsilon+iy}}(\lambda)\right|^2 \|f\|_{L^2(X)}^2.\end{aligned}$$

Now, by Theorem 2.1 it follows that for $\lambda \in \mathfrak{a}^*$ and $\epsilon > 0$

$$\begin{aligned}|\widehat{(k_\sigma^\infty)^{1+\epsilon+iy}}(\lambda)| &\leq \left| \int_{\{x \in X : |x| \geq 1\}} |x|^{(\sigma - l - 2|\Sigma_0^+|)(1+\epsilon+iy)} \, (\phi_0(x))^{1+\epsilon+iy} \, \phi_{-\lambda}(x) \, dx \right| \\ &\leq \int_{\{x \in X : |x| \geq 1\}} \phi_0(x)^{2+\epsilon} \, dx < \infty,\end{aligned}$$

and hence $\|T_z f\|_{L^\infty(X)} \leq \|f\|_{L^2(X)}$. Hence, by analytic interpolation for $p > 2$,

$$\|f * k_\sigma^\infty\|_{L^p(X)} = \|T_0 f\|_{L^p(X)} \leq C\|f\|_{L^{p'}(X)}. \tag{6.5}$$

Therefore, from (6.4) and from (6.5), it follows that for all $2 < p \leq \frac{2n}{n-\sigma}$,

$$\|f * k_\sigma\|_{L^p(X)} \leq C\|f\|_{L^{p'}(X)}.$$

This completes the proof. □

As a corollary of the theorem above we have the following

Corollary 6.2. *Let $2 < p \leq \frac{2n}{n-2}$ and $\dim X = n \geq 3$. Then there exists $S_{n,p} > 0$ such that for all $u \in H^1(X)$,*

$$\|\nabla u\|_{L^2(X)}^2 - |\rho|^2 \|u\|_{L^2(X)}^2 \geq S_{n,p} \|u\|_{L^p(X)}^2.$$

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