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A NONZERO-SUM RISK-SENSITIVE STOCHASTIC DIFFERENTIAL GAME IN THE ORTHANT

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Abstract. We study a nonzero-sum risk-sensitive stochastic differential game for controlled reflecting diffusion processes in the nonnegative orthant. We treat two cost evaluation criteria, namely, discounted cost and ergodic cost. Under certain assumptions, we establish the existence of Nash equilibria. Also, we completely characterize a Nash equilibrium for the ergodic cost criterion in the space of stationary Markov strategies.

1. Introduction.

Risk-sensitive stochastic optimal control problem for controlled reflecting diffusion processes in the nonnegative orthant is studied in [40]. In [22] the results of [40] have been extended to zero-sum stochastic differential games in the nonnegative orthant. In this paper, we extend the results of [40] from a one-controller case to a multi-controller case in a non-cooperative setup. For notational simplicity we consider two-player case. In other words, we study two person nonzero-sum risk-sensitive stochastic differential games on infinite time horizon, where the state space is the nonnegative orthant $\overline{D} \subset \mathbb{R}^d$. Here the state of the game evolves according to a controlled reflecting diffusion process in $\overline{D}$. That is the state of the game is given by the solution of the following stochastic differential equations

$$\left.\begin{aligned} dX(t) &= \bar{b}(X(t), u_1(t), u_2(t))dt + \sigma(X(t))dW(t) - \gamma(X(t))d\xi(t), \\ d\xi(t) &= I_{\{X(t)\in\partial D\}}d\xi(t), \\ \xi(0) &= 0, \quad X(0) = x \in \overline{D}, \end{aligned}\right\} \quad (1)$$

where $\bar{b} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d$ is the drift vector of the process, $\sigma : \overline{D} \to \mathbb{R}^{d\times d}$ is the diffusion matrix, $\gamma : \overline{D} \to \overline{D}$ is a vector field which determines the direction of the reflection of the process, $W(\cdot)$ is an $\mathbb{R}^d$-valued standard Wiener process, U_i's are given compact metric spaces for $i = 1, 2$, and $u_i,\ i = 1, 2$, are U_i- valued processes which are appropriate strategies taken by the players. Inside D the process $X(\cdot)$

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behaves like diffusion process and when it hits the boundary ∂D it gets reflected inward in a direction determined by the vector field γ. The process $\xi(\cdot)$ changes only when $X(\cdot)$ hits ∂D. Such processes arise in many applications including the heavy traffic analysis of queuing networks coming from problems in manufacturing systems and communications (see [13], [15], [23], [31]). Since except for a few special cases, the direct analysis of problems in queuing network are very difficult to analyze, one tries to find a continuous approximation of it which can provide a tractable solution. It has been proved that in open queuing networks in heavy traffic, as the traffic intensity goes to unity, a suitably scaled and normalized sequence of queue length processes converges weakly in the Skorohod topology to a certain reflected diffusion process; see [30], [33], [41] for the uncontrolled case and [31], [32] for the controlled case.

It is known that (see [23], [22]) in a typical communication network there are several users, and different users may have different objectives leading to conflicts. In view of this, the analysis of such problems is often carried out using game theoretic framework. In general, in a communication network problem each of user tries to optimize a certain performance measure related to his traffic parameters, namely minimizing delays, maximizing throughput, minimizing blocking probabilities, etc. Therefore, in a communication network with heavy traffic, the basic problem is a nonzero-sum differential game with reflecting diffusion in the orthant. But establishing existence of a Nash equilibria is quite involved in both theoretically and computationally. Now, if we consider the case: where each of the player consider the the other players as single super-player and tries to find a mini-max equilibrium. Then the player can find an "optimal" strategy against the worst-case scenario, that is, the aim of each player is to guarantee the best performance under the worst-case behavior of the super-player. This kind of problem (where the state processes is a controlled reflecting diffusion process in the nonnegative orthant) has been studied in [22]. The authors [22] have established the existence of a saddle point equilibrium by studying the corresponding Isaacs equations. The analysis of zero-sum differential games considered in [22] differs substantially from its counterpart for nonzero-sum differential games, which is considered in this paper.

Corresponding to the above state dynamics we now briefly formulate the game problems. For $i = 1, 2$, let $\bar{r}_i : \overline{D} \times U_1 \times U_2 \to \mathbb{R}_+$ be the running cost functions. We are interested in two cost evaluation criteria, viz., risk-sensitive discounted cost and risk-sensitive average (ergodic) cost. For any discounted factor $\alpha > 0$ and risk-sensitive parameter $\theta > 0$, the α-discounted cost of the ith player is given by The risk-sensitive α-discounted cost of the ith player is given by

$$\mathcal{J}^{u_1,u_2}_{\alpha,i}(\theta, x) \;:=\; \frac{1}{\theta} \log E^{u_1,u_2}_x \left[e^{\theta \int_0^\infty e^{-\alpha t} \bar{r}_i(X(t),u_1(t),u_2(t))dt} \right], x \in \overline{D}, \;\; i = 1, 2,$$

where $X(t)$ is the solution of (1) corresponding to a pair of strategies (u_1, u_2) (in the next section these strategies are defined in details) chosen by the players, E^{u_1,u_2}_x denotes the expectation with respect to the law of the process $X(\cdot)$ with initial condition $X(0) = x$. The risk-sensitive ergodic cost of the ith player is given by

$$\rho^{u_1,u_2}_i(\theta, x) \;= \limsup_{T\to\infty} \frac{1}{\theta T} \log E^{u_1,u_2}_x \left[e^{\theta \int_0^T \bar{r}_i(X(t),u_1(t),u_2(t))dt} \right], x \in \overline{D}.$$

In a two person game, a Nash equilibrium is a pair of strategies such that unilateral deviation from this pair by any player is disadvantageous to him. In other words, a pair of strategies (u_1^*, u_2^*) which satisfies the following:

$$\begin{aligned}\mathcal{J}_{\alpha,1}^{u_1^*,u_2^*}(\theta,x) &\leq \mathcal{J}_{\alpha,1}^{u_1,u_2^*}(\theta,x),\ \forall\ x\in\overline{D},\\ \mathcal{J}_{\alpha,2}^{u_1^*,u_2^*}(\theta,x) &\leq \mathcal{J}_{\alpha,2}^{u_1^*,u_2}(\theta,x),\ \forall\ x\in\overline{D},\end{aligned}$$

for all strategies u_1, u_2, is called a Nash equilibrium for the α-discounted cost criterion. One can define Nash equilibrium for the ergodic cost criterion in a similar fashion. We refer to [39] for more details about risk-sensitive Nash-equilibrium.

Under certain assumptions, we establish the existence of Nash equilibria for both criteria. We obtain our results by analyzing the corresponding coupled Hamilton–Jacobi–Bellman (HJB) equations. For the average cost criterion, we prove the existence of Nash equilibria using principal eigenvalue approach.

Similar eigenvalue approach has been used in [4], [10] to study risk-sensitive stochastic optimal control problem in $\mathbb{R}^d$. Using a stochastic representation of the principal eigenfunction, the uniqueness of the same has been established in [4] in a certain class of functions. This has been achieved under a certain Lyapunov type stability condition. Without using any kind of blanket stability assumptions risk-sensitive stochastic optimal control problem in $\mathbb{R}^d$ is studied in [2]. In [8], [12] the corresponding risk-sensitive zero-sum game problems have been studied. Using principal eigenvalue approach a complete characterization of saddle point equilibria in the space of stationary Markov strategies is carried out in [12]. The risk-neutral part of our problem for bounded domain has been studied in [25] and the same for the orthant has been studied in [24]. Similar game problems are studied in [21] when the state space is a smooth bounded domain in $\mathbb{R}^d$. This type of problem for controlled diffusion process in $\mathbb{R}^d$ has been studied in [26]. Using nonlinear version of Krein-Rutman theorem the existence of principal eigenpair for the ergodic HJB equation in smooth bounded domain when the direction of reflection is co-normal is establised [3], [6].

The rest of the paper is structured as follows. Section 2 deals with detailed description of the problem. In Section 3, we study nonzero-sum games for discounted cost criterion. The nonzero-sum games for ergodic cost criterion are analyzed in Section 4. Section 5 concludes the paper.

2. **Problem description.**

We study a system which is controlled by two players; the analysis of N-players case for $N \geq 3$ is analogous. Let $D = \{x \in \mathbb{R}^d : x_i > 0,\ \forall\ i = 1, 2, \dots, d\}$ be the positive orthant in $\mathbb{R}^d$. Let $U_i,\ i = 1, 2$, be given compact metric spaces and let $V_i = P(U_i),\ i = 1, 2$, denote the space of probability measures on U_i with Prokhorov topology. Let

$$\bar{b} = (\bar{b}_1, \cdots, \bar{b}_d) : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d,$$

$$\sigma : \overline{D} \to \mathbb{R}^{d\times d},\ \sigma = [\sigma_{ij}(\cdot)]_{1\leq i,j\leq d},$$

and $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ defined in a neighborhood of ∂D, be given functions, where $\overline{D}$ is the closure of D. We extend the function $\bar{b} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d$ to

$$b = (b_1, \cdots, b_d) : \overline{D} \times V_1 \times V_2 \to \mathbb{R}^d$$

by

$$b_i(x, v_1, v_2) := \int_{U_2}\int_{U_1} \bar{b}_i(x, u_1, u_2)v_1(\,du_1)v_2(\,du_2),\ \ i = 1, \dots, d.$$

We assume that the dynamics of the controlled system is a process $\{X(t)\}_{t\geq 0}$ which is given by the solution of the following controlled reflecting stochastic differential equations (RSDE):

$$\left.\begin{array}{rcl} dX(t) & = & b(X(t), v_1(t), v_2(t))dt + \sigma(X(t))dW(t) - \gamma(X(t))d\xi(t), \\ d\xi(t) & = & I_{\{X(t)\in\partial D\}} d\xi(t), \\ \xi(0) & = & 0, \quad X(0) = x \in \overline{D}, \end{array}\right\} \quad (1)$$

where $W = (W_1, \cdots, W_d)$ is an $\mathbb{R}^d$-valued standard Wiener process, $v_i(\cdot)$ is a V_i-valued process satisfying $v_i(t) = f_i(t, X([0,t]))$ where $f_i : [0,\infty)\times C([0,\infty); \overline{D}) \to V_i$ is a measurable map and $X([0,t])(s) = X(t \wedge s), \forall\ s \in [0,\infty)$. The process v_i, satisfying the above conditions is called an admissible strategy of player i, $i = 1, 2$. Let $\mathcal{A}_i$ denotes the set of all admissible strategies of player i, $i = 1, 2$. We refer to [14] for a physical interpretation of this class of strategies. If $v_i(t) = \overline{v}_i(t, X(t))$ for some measurable $\overline{v}_i : [0,\infty) \times \mathbb{R}^d \to V_i$, then $v_i(\cdot)$ (or, by abuse of notation, the map $\overline{v}_i$ itself) is called a Markov strategy of the ith player. If the map $\overline{v}_i$ has no explicit time dependence, then the Markov strategy $\overline{v}_i$ is called a stationary Markov strategy. Let $\mathcal{M}_i, \mathcal{S}_i$, denote respectively the set of Markov, stationary Markov strategies, of the ith player, $i = 1, 2$.

In order to prove the existence of a solution of (1) we approximate $\overline{D}$ by appropriate smooth domains. Define

$$D_l^{'} = D \cap B(0, l),$$

where $B(0, l) = \{x \in \mathbb{R}^d : ||x|| < l\}$, for $l = 1, 2, \ldots$. By Theorem A2(ii) and the remark in p.28 of [16] we have that there exists domains $D_{l,m} \subset \mathbb{R}^d$ with C^∞ boundary satisfying the following conditions:
(i) The distance $d(\partial D_{l,m}, \partial D_l^{'}) < \frac{1}{m}$, $l \geq 1$,
(ii) $D_{l,n} \subset D_{l,m}$, $m \geq n, l \geq 1$, $D_{l_1,n} \subset D_{l_2,n}$, $n \geq 1, l_2 \geq l_1$. Define

$$D_m = \cup_{l=1}^{\infty} D_{l,m}, m \geq 1.$$

From the above construction one can conclude the following
(i) $D_m \uparrow \overline{D}$ where for each $m \geq 1$, D_m is a domain with C^∞ smooth boundary.
(ii) For any smooth compact subset $C \subset \overline{D}$, we have $C \subset D_{l,m}$ for sufficiently large l, m.
Now to ensure the existence of a solution of (1) we make the following assumptions.

(A0) (i) The function $\bar{b}$ is bounded, jointly continuous, Lipschitz continuous in its first argument uniformly with respect to the rest.
(ii) The functions $\sigma_{ij}, i, j = 1, 2, \ldots, d$ are Lipschitz continuous and bounded.
(iii) The function $a := \sigma\sigma^\perp$ (where $\sigma^\perp$ is the transpose of σ) is uniformly elliptic, i.e., for some positive constant δ_0

$$za(x)z^\perp \geq \delta_0||z||^2,\ x \in \overline{D},\ z \in \mathbb{R}^d,$$

where δ_0 is the ellipticity constant.

(A1) The function $\gamma = (\gamma_1, \ldots, \gamma_d)$ is such that each of the component $\gamma_i \in C_b(\mathbb{R}^d)$, and there exist $\delta_1 > 0$ such that:

- For all $x \in \Sigma_i$, $i = 1, \ldots, d$,

$$\gamma(x) \cdot \eta^i(x) \geq \delta_1 > 0$$

where $\Sigma_i = \{x \in \mathbb{R}^d : x_i = 0\}$, and $\eta^i(\cdot)$ denotes the outward normal to Σ_i.

Remark 1. Since $\gamma_i \in C_b(\mathbb{R}^d)$, for $i = 1, \ldots, d$, in view of $(A1)$, one can always choose D_m in such a way that for all m sufficiently large

$$\gamma(x) \cdot \eta^m(x) \geq \delta_1 > 0 \text{ for all } x \in \partial D_m \cap G_j, \;\; j = 1, \ldots, d,$$

where G_j is a fixed neighborhood of Σ_j and $\eta^m(\cdot)$ denotes the outward normal to ∂D_m.

Set

$$\Sigma = \{x \in \cup_{i=1}^d \Sigma_i : x \notin \cap_{j=1}^k \Sigma_{l_j},\, k \geq 2,\, l_j \in \{1, 2, \ldots, d\}\},$$

which is the smooth part of ∂D.

Under the assumptions (A0) and (A1), for a given $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, it has been proved in [7] that (1) has a unique weak solution. Adapting the approach by Zvonkin and Veretenikov (see for example [42]) one can prove that, under a pair of Markov strategies, (1) admits a unique strong solution. For more details see [Theorem 3.2, [7]].

Throughout this paper we make the following assumption.

(A2) Let $\Sigma^{'}$ denote the non-smooth part of ∂D (clearly, the surface measure of $\Sigma^{'}$ is zero). We assume that $P_x^{v_1,v_2}(X(t) \in \Sigma^{'}$ for some $t \geq 0) = 0$, for each $x \in D$, where $P_x^{v_1,v_2}$ is the probability measure on the space over which $X(\cdot)$ is a weak solution of (1) corresponding to a strategy pair (v_1, v_2) and initial state x.
We refer [23], [24], [37] for sufficient conditions ensuring **(A2)**.

For $i = 1, 2$, let $\bar{r}_i : \overline{D} \times U_1 \times U_2 \to \mathbb{R}_+$ be the running cost function. We extend $\bar{r}_i$ to $r_i : \overline{D} \times V_1 \times V_2 \to \mathbb{R}_+$ by

$$r_i(x, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{r}_i(x, u_1, u_2) v_1(\,du_1) v_2(\,du_2),$$

for $i = 1, 2$. We assume that

(A3) $\bar{r}_i$ is bounded, jointly continuous and Lipschitz in its first argument uniformly with respect to the rest, for $i = 1, 2$.

Now we describe the cost evaluation criteria.

2.1. **Discounted cost criterion.** Let $\alpha > 0$ be the discounted factor and $\theta \in (0, \Theta)$ the risk sensitive parameter. The risk-sensitive discounted cost of the ith player, $i = 1, 2$, is given by

$$\mathcal{J}_{\alpha,i}^{v_1,v_2}(\theta, x) \;:=\; \frac{1}{\theta} \ln E_x^{v_1,v_2}\Big[e^{\theta \int_0^\infty e^{-\alpha t} r_i(X(t), v_1(t), v_2(t))dt}\Big], x \in \overline{D}, \qquad (2)$$

where $X(t)$ is the solution of the RSDE (1) corresponding to $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ and $E_x^{v_1,v_2}$ denotes the expectation with respect to the law of the process (1) corresponding to the admissible strategy pair (v_1, v_2) with the initial condition $X(0) = x$. For $i = 1, 2$, Player i wishes to minimize (2) by choosing appropriate strategies v_i from $\mathcal{A}_i$, .

A pair of strategies $(v_1^*, v_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ which satisfies the following:

$$\begin{aligned} \mathcal{J}_{\alpha,1}^{v_1^*,v_2^*}(\theta, x) \;&\leq\; \mathcal{J}_{\alpha,1}^{v_1,v_2^*}(\theta, x),\; \forall\, v_1 \in \mathcal{A}_1,\, x \in \overline{D}, \\ \mathcal{J}_{\alpha,2}^{v_1^*,v_2^*}(\theta, x) \;&\leq\; \mathcal{J}_{\alpha,2}^{v_1^*,v_2}(\theta, x),\; \forall\, v_2 \in \mathcal{A}_2,\, x \in \overline{D}, \end{aligned} \qquad (3)$$

is said to be an α- discounted Nash equilibrium.

2.2. **Ergodic cost criterion.** For $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, the risk-sensitive ergodic cost of the ith player, for $i = 1, 2$, is given by

$$\rho_i^{v_1,v_2}(\theta, x) = \limsup_{T\to\infty} \frac{1}{\theta T} \log E_x^{v_1,v_2}\left[e^{\theta \int_0^T r_i(X(t),v_1(t),v_2(t))dt}\right], x \in \overline{D}. \tag{4}$$

For this cost evolution criterion the definition of Nash equilibrium is analogous.

Since the existence of a Nash equilibrium is usually established using a standard fixed point theorem. For this reason, we first endow $\mathcal{S}_i$ with a topology that makes it a compact metric space. It is known that $L^\infty(\overline{D}; \mathcal{M}_s(U_i))$ is the dual of $L^1(\overline{D}; C(U_i))$, where $\mathcal{M}_s(U_i)$ denote the space of all signed measures on U_i with the weak*-topology. Thus, in view of the Banach-Alaoglu theorem it is clear that the unit ball in $L^\infty(\overline{D}; \mathcal{M}_s(U_i))$ is weak*-compact. Again, as we know that $L^1(\overline{D}; C(U_i))$ is separable, therefore it follows that the weak*-topology of its dual is metrizable. Thus, we topologize $\mathcal{S}_i,\ i = 1, 2$, using the metrizable weak*-topology on $L^\infty(\overline{D}; \mathcal{M}_s(U_i))$. Since $\mathcal{S}_i$ is a subset of the unit ball of $L^\infty(\overline{D}; \mathcal{M}_s(U_i))$ and $\mathcal{P}(U_i)$ is closed in $\mathcal{M}_s(U_i)$, it follows that $\mathcal{S}_i$ is compact under the above weak*-topology.

Also, one can characterized the topology of $\mathcal{S}_i$ by the following convergence criterion:

For $i = 1, 2$, $v_i^n \to v_i$ in $\mathcal{S}_i$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \int_{\overline{D}} f(x) \int_{U_i} g(x, u_i) v_i^n(x)(du_i)dx \ = \ \int_{\overline{D}} f(x) \int_{U_i} g(x, u_i) v_i(x)(du_i)dx \tag{5}$$

for all $f \in L^1(\overline{D}) \cap L^2(\overline{D}),\ g \in C_b(\overline{D} \times U_i)$; see [[5], p.57] for details. Following [27] we introduce the following class of strategies: Let

$$\hat{\mathcal{S}}_i \ = \ \{\hat{v}_i : (0, \Theta) \times \overline{D} \to V_i \ \mid \hat{v}_i \text{ is measurable}\},\ i = 1, 2,$$

be a class of strategies, which will be referred to as the class of eventually stationary strategies. We endowed the space $\hat{\mathcal{S}}_i$ with the weak*-topology $L^\infty((0, \Theta) \times \mathcal{M}_s(U_i))$ (this was introduced by Warga [43] for the topology of relaxed controls). Under this topology, $\hat{\mathcal{S}}_i$ becomes a compact metrizable space. In $\hat{\mathcal{S}}_i$ we have the following convergence criterion:

For $i = 1, 2$, $\hat{v}_i^n \to \hat{v}_i$ in $\hat{\mathcal{S}}_i$ as $n \to \infty$ if and only if

$$\begin{aligned} &\lim_{n\to\infty} \int_{(0,\Theta)} \int_{\overline{D}} f(\theta, x) \int_{U_i} g(\theta, x, u_i) \hat{v}_i^n(\theta, x)(du_i) d\theta dx \qquad (6) \\ &= \int_{(0,\Theta)} \int_{\overline{D}} f(\theta, x) \int_{U_i} g(\theta, x, u_i) \hat{v}_i(\theta, x)(du_i) d\theta dx \end{aligned}$$

for all $f \in L^2((0, \Theta) \times \overline{D}) \cap L^1((0, \Theta) \times \overline{D}),\ g \in C_b((0, \Theta) \times \overline{D} \times U_i)$. Note that, corresponding to each $\hat{v}_i \in \hat{\mathcal{S}}_i, i = 1, 2$, the Markov strategies associated to it is given by $\hat{v}_i(\theta e^{-\alpha t}, X(t)), t \ge 0$, for each $\theta \in (0, \Theta)$ and $\alpha > 0$, where $X(t)$ is the solution of the following RSDE

$$\begin{aligned} dX(t) &= b(X(t), v_1(\theta e^{-\alpha t}, X(t)), v_2(\theta e^{-\alpha t}, X(t)))dt + \sigma(X(t))dW(t) \\ &\quad -\gamma(X(t))d\xi(t), \\ d\xi(t) &= I_{\{X(t)\in\partial D\}} d\xi(t), \\ \xi(0) &= 0, \quad X(0) = x \in \overline{D}. \end{aligned}$$

As we know that, $e^{-\alpha t} \to 0$, as $t \to \infty$, therefore, in the long run an element of $\hat{\mathcal{S}}_i$ "eventually" becomes an element of $\mathcal{S}_i$. This justifies the terminology [27].

Due to non-cooperative nature of this game no player would deviate from a Nash equilibrium. It is easy to see that if the Player 1 announces his strategy in advance then the Player 2 would obviously minimize his own cost. Any strategy of the Player 2 that minimizes his cost is called his optimal response corresponding to the announced strategy of the Player 1. There may exist several optimal responses of the Player 2 corresponding to each announced strategy of the Player 1. In a similar manner, one can define the optimal response of the Player 1 corresponding to each announced strategy of Player 2. Thus for a given pair of strategies of the two players, there exists a set of pair of optimal responses of the players. This defines a set-valued map from the set of strategies of the players to the power set of the set of strategies. It is easy to see that, any fixed point of this set-valued map is a Nash equilibrium. Using certain compactness and convexity together with the upper semicontinuity of this set-valued map, we can show the existence of a fixed point. In general for nonzero-sum game problems showing the upper semicontinuity of this set-valued map is by far the most difficult task.

3. **Analysis of discounted cost criterion.** In this section, we analyze the discounted cost criterion. To carry out our analysis for the α-discounted cost criterion we use the following cost criterion

$$J^{v_1,v_2}_{\alpha,i}(\theta,x) \;:=\; E^{v_1,v_2}_x\Big[e^{\theta\int_0^\infty e^{-\alpha t} r_i(X(t),v_1(t),v_2(t))dt}\Big]. \tag{7}$$

The definition of a Nash-equilibrium for this cost criterion is analogous. Since logarithm is an increasing function, the cost evaluation criteria (7) and (2) are equivalent in the sense that any Nash equilibrium for (7) is also a Nash equilibrium for the criterion (2) and vice-versa. Let $\hat{v}_i \in \hat{\mathcal{S}}_i,\ i=1,2$. Now for $\theta \in (0,\ \Theta)$ and $x \in \overline{D}$ we define the value functions corresponding to the cost criterion (7) as follows:

$$\begin{aligned}
\psi^{\hat{v}_2}_{\alpha,1}(\theta,x) &= \inf_{v_1\in\mathcal{A}_1} J^{v_1,\hat{v}_2}_{\alpha,1}(\theta,x)\\
&= \inf_{v_1\in\mathcal{A}_1} E^{v_1,\hat{v}_2}_x\Big[e^{\theta\int_0^\infty e^{-\alpha t} r_1(X(t),v_1(t),\hat{v}_2(\theta e^{-\alpha t},X(t)))dt}\Big],\\
\psi^{\hat{v}_1}_{\alpha,2}(\theta,x) &= \inf_{v_2\in\mathcal{A}_2} J^{\hat{v}_1,v_2}_{\alpha,2}(\theta,x)\\
&= \inf_{v_2\in\mathcal{A}_2} E^{\hat{v}_1,v_2}_x\Big[e^{\theta\int_0^\infty e^{-\alpha t} r_2(X(t),\hat{v}_1(\theta e^{-\alpha t},X(t)),v_2(t))dt}\Big].
\end{aligned}$$

Next, we want to prove that the value functions defined above are solutions of the corresponding HJB equations. By dynamic programming heuristics, the HJB equations for the discounted cost criterion are given by, (see [19],[38])

$$\begin{aligned}
\alpha\theta\frac{\partial\psi_2}{\partial\theta} &= \inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_2\rangle + \theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_2\Big]\\
&\quad +\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_2),\\
\psi_2(0,x) &= 1 \text{ for } x\in\overline{D},\ \ \nabla\psi_2\cdot\gamma = 0 \ \text{ on } \ (0,\Theta)\times\partial D
\end{aligned}$$

and

$$\begin{aligned}
\alpha\theta\frac{\partial\psi_1}{\partial\theta} &= \inf_{v_1\in V_1}\Big[\langle b(x,v_1,\hat{v}_2(\theta,x)),\nabla\psi_1\rangle + \theta r_1(x,v_1,\hat{v}_2(\theta,x))\psi_1\Big]\\
&\quad +\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_1),
\end{aligned}$$

$$\psi_1(0,x) \;=\; 1 \text{ for } x\in\overline{D},\; \nabla\psi_1\cdot\gamma=0 \text{ on } (0,\Theta)\times\partial D.$$

The singularity in θ at 0 and the unbounded, non-smooth nature of the orthrant pose technical difficulties in solving the above p.d.e.s. We use suitable approximation arguments to overcome these difficulties. In other words, we approximate the above p.d.e.s by a family of p.d.e.s in the smooth bounded domains $D_{l,m}$.

Lemma 3.1. *Suppose that assumptions (A0) - (A3) hold. Then for $\hat{v}_1\in\hat{\mathcal{S}}_1$ and for each $\kappa\in(0,\ \Theta)$, the p.d.e.*

$$\begin{aligned}\alpha\theta\frac{\partial\psi^{l,m}_{2,\kappa}}{\partial\theta} &= \inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi^{l,m}_{2,\kappa}\rangle+\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi^{l,m}_{2,\kappa}\Big]\\ &\quad+\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\psi^{l,m}_{2,\kappa}), \qquad (8)\\ \psi^{l,m}_{2,\kappa}(\kappa,x) &= e^{\frac{\kappa\|r_2\|_\infty}{\alpha}} \text{ for } x\in\overline{D}_{l,m},\ \nabla\psi^{l,m}_{2,\kappa}\cdot\gamma=0 \text{ on } (\kappa,\ \Theta)\times\partial D_{l,m}\end{aligned}$$

has a unique solution in $W^{1,2,p}((\kappa,\Theta)\times D)$, $\infty>p\geq 2$, for each $l,m\geq 1$, which is given by

$$\psi^{l,m}_{2,\kappa}(\theta,x) = \inf_{v_2\in\mathcal{A}_2} E^{\hat{v}_1,v_2}_x\Big[e^{\frac{\kappa\|r_2\|_\infty}{\alpha}}e^{\theta\int_0^{T_\kappa}e^{-\alpha t}r_2(X(t),\hat{v}_1(\theta e^{-\alpha t},X(t)),v_2(t))dt}\Big],$$

where $T_\kappa=\dfrac{\log(\frac{\theta}{\kappa})}{\alpha}$. Similarly, for $\hat{v}_2\in\hat{\mathcal{S}}_2$, the p.d.e.

$$\begin{aligned}\alpha\theta\frac{\partial\psi^{l,m}_{1,\kappa}}{\partial\theta} &= \inf_{v_1\in V_1}\Big[\langle b(x,v_1,\hat{v}_2(\theta,x)),\nabla\psi^{l,m}_{1,\kappa}\rangle+\theta r_1(x,v_1,\hat{v}_2(\theta,x))\psi^{l,m}_{1,\kappa}\Big]\\ &\quad+\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\psi^{l,m}_{1,\kappa}), \qquad (9)\\ \psi^{l,m}_{1,\kappa}(\kappa,x) &= e^{\frac{\kappa\|r_1\|_\infty}{\alpha}} \text{ for } x\in\overline{D}_{l,m},\ \nabla\psi^{l,m}_{1,\kappa}\cdot\gamma=0 \text{ on } (\kappa,\ \Theta)\times\partial D_{l,m},\end{aligned}$$

has a unique solution in $W^{1,2,p}((\kappa,\Theta)\times D_{l,m})$, $\infty>p\geq 2$, for each $l,m\geq 1$, which is given by

$$\psi^{l,m}_{1,\kappa}(\theta,x) = \inf_{v_1\in\mathcal{A}_1} E^{v_1,\hat{v}_2}_x\Big[e^{\frac{\kappa\|r_1\|_\infty}{\alpha}}e^{\theta\int_0^{T_\kappa}e^{-\alpha t}r_1(X(t),v_1(t),\hat{v}_2(\theta e^{-\alpha t},X(t)))dt}\Big].$$

Proof. It follows from [[21], Lemma 3.1]. □

Next we want to prove the existence of a solution to the limiting p.d.e. of the above families of p.d.e.s

Theorem 3.2. *Suppose that assumptions (A0) - (A3) hold. Then we have the following:*
(i) For each $\hat{v}_1\in\hat{\mathcal{S}}_1$, the p.d.e.

$$\begin{aligned}\alpha\theta\frac{\partial\psi_2}{\partial\theta} &= \inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_2\rangle+\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_2\Big]\\ &\quad+\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\psi_2), \qquad (10)\\ \psi_2(0,x) &= 1 \text{ for } x\in\overline{D},\ \nabla\psi_2\cdot\gamma=0 \text{ on } (0,\Theta)\times\partial D\end{aligned}$$

has a unique solution in $W^{1,2,p}_{loc}((0,\ \Theta)\times D\cup\Sigma)\cap C^{0,1}((0,\ \Theta)\times D\cup\Sigma)$, $\infty>p\geq 2$.
(ii) And, for each $\hat{v}_2\in\hat{\mathcal{S}}_2$, the p.d.e.

$$\alpha\theta\frac{\partial\psi_1}{\partial\theta} = \inf_{v_1\in V_1}\Big[\langle b(x,v_1,\hat{v}_2(\theta,x)),\nabla\psi_1\rangle+\theta r_1(x,v_1,\hat{v}_2(\theta,x))\psi_1\Big]$$

$$+\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_1), \tag{11}$$
$$\psi_1(0,x) \;=\; 1 \;\; \textit{for } x\in\overline{D},\;\; \nabla\psi_1\cdot\gamma = 0 \;\; \textit{on} \;\; (0,\Theta)\times\partial D$$

has a unique solution in $W^{1,2,p}_{loc}((0,\ \Theta)\times D\cup\Sigma)\cap C^{0,1}((0,\ \Theta)\times D\cup\Sigma)$, $\infty > p\geq 2$.

Proof. We want to prove (i), and the proof of (ii) is analogous. Let $Q\subset D$ be an open bounded domain with C^2 boundary. Then from our construction it is clear that there exist two positive integers $\bar N_1, \bar M_1$ such that $\overline{Q}\subset D_{l,m}$ for all $m\geq \bar M_1, l\geq \bar N_1$. From Lemma 3.1, it is easy to see that (8) admits a unique solution in $W^{1,2,p}((\kappa,\Theta)\times D_{l,m})$, $\infty > p\geq 2$, which is given by

$$\psi^{l,m}_{2,\kappa}(\theta,x) \;=\; \inf_{v_2\in\mathcal{A}_2} E^{\hat v_1, v_2}_x\Big[e^{\frac{\kappa\|r_2\|_\infty}{\alpha}}\, e^{\theta\int_0^{T_\kappa} e^{-\alpha t} r_2(X(t),\hat v_1(\theta e^{-\alpha t},X(t)),v_2(t))dt}\Big]. \tag{12}$$

This implies

$$|\psi^{l,m}_{2,\kappa}(\theta,x)| \leq e^{\frac{\kappa\|r_2\|_\infty}{\alpha}}\, e^{\frac{\theta\|r_2\|_\infty(1-e^{-\alpha T_\kappa})}{\alpha}} = e^{\frac{\theta\|r_2\|_\infty}{\alpha}},$$

since $e^{-\alpha T_\kappa} = \frac{\kappa}{\theta}$. Therefore

$$\|\psi^{l,m}_{2,\kappa}\|_\infty \leq e^{\frac{\Theta\|r_2\|_\infty}{\alpha}}, \tag{13}$$

Following arguments as in the proof of [[19], Lemma 2.2], one can prove that

$$\|\frac{\partial\psi^{l,m}_{2,\kappa}}{\partial\theta}\|_\infty \;\leq\; 3e^{\frac{(\Theta+2)\|r_2\|_\infty}{\alpha}}\,\frac{\|r_2\|_\infty}{\alpha}. \tag{14}$$

Let $\bar v_{2,l,m}(\cdot,\cdot)$ be a minimizing selector in (8) (existence of such a minimizing selector is ensured by [9]). Thus, we can rewrite the p.d.e (8) as a parametric family of linear elliptic p.d.e.s as follows.

$$\langle b(x,\hat v_1(\theta,x),\bar v_{2,l,m}(\theta,x)),\nabla\psi^{l,m}_{2,\kappa}\rangle + \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi^{l,m}_{2,\kappa}) = f_{l,m}(\theta,x),$$

with $\psi^{l,m}_{2,\kappa}(\kappa,x) = e^{\dfrac{\kappa\|r_2\|_\infty}{\alpha}}$ on $\overline{D}_{l,m}$ and $\nabla\psi^{l,m}_{2,\kappa}(\theta,x)\cdot\gamma(x) = 0$ on $(\kappa,\Theta)\times\partial D_{l,m}$, where $f_{l,m}(\theta,x) \;=\; \alpha\theta\frac{\partial\psi^{l,m}_{2,\kappa}(\theta,x)}{\partial\theta} \;-\; \theta r_2(x,\hat v_1(\theta,x),\bar v_{2,l,m}(\theta,x))\psi^{l,m}_{2,\kappa}(\theta,x)$. Define $\tilde b_{l,m}(\theta,x) = b(x,\hat v_1(\theta,x),\bar v_{2,l,m}(\theta,x))$. In view of our assumptions and (13), (14), for each $\theta\in(0,\Theta)$ we deduce that

$$\sup_{l,m}\|\tilde b_{l,m}(\theta,\cdot)\|_{\infty;D_{l,m}} < \infty,\ \sup_{l,m}\|f_{l,m}(\theta,\cdot)\|_{\infty;D_{l,m}} < \infty.$$

Now, applying [[28], Theorem 9.11] (see also [[19] subsection (1.6)]), it follows that

$$\|\psi^{l,m}_{2,\kappa}\|_{1,2,p;(\kappa,\Theta)\times Q} < \hat K_1,\ \text{for all } m\geq \bar M_1, l\geq \bar N_1, p\geq 2, \tag{15}$$

where the constant $\hat K_1$ is independent of m,l. From D, we choose an increasing sequence of bounded domains $\{Q_n\}_n$ such that $D\cup\Sigma = \cup_{n\geq 1}\overline{Q}_n$ and $\partial Q_n\cap\partial D$ is a C^2 portion of ∂D. A standard diagonalization procedure implies that there exist $\psi^m_{2,\kappa}\in W^{1,2,p}_{loc}((\kappa,\Theta)\times D_m)$, $2\leq p<\infty$, such that along a subsequence as $l\to\infty$

$$\psi^{l,m}_{2,\kappa} \longrightarrow \psi^m_{2,\kappa}\ \text{weakly in } W^{1,2,p}((\kappa,\Theta)\times Q). \tag{16}$$

Therefore (15) yields that $\|\psi^m_{2,\kappa}\|_{1,2,p;(\kappa,\Theta)\times Q} < \hat K_1$, for all $m\geq \bar M_1, p\geq 2$. Again, by similar diagonalization argument, we deduce that there exists $\psi_{2,\kappa}\in W^{1,2,p}_{loc}((\kappa,\Theta)\times D\cup\Sigma)$, $2\leq p<\infty$ such that as $m\to\infty$

$$\psi^m_{2,\kappa} \longrightarrow \psi_{2,\kappa}\ \text{weakly in } W^{1,2,p}((\kappa,\Theta)\times Q). \tag{17}$$

In view of the parabolic version of Morrey's lemma [[44], pp. 26-27], it is clear that $W^{1,2,p}((\kappa,\Theta)\times Q),\ p>d+1$, is compactly contained in $C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q}),\ 0<\hat{\alpha}<2-\frac{d+2}{p}$. Hence, we can extract a subsequence such that

$$\lim_{m\to\infty}\lim_{l\to\infty}\psi_{2,\kappa}^{l,m}=\psi_{2,\kappa}\ \text{ in } C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q}). \tag{18}$$

Now using (A0), (A3) and (18), we deduce that

$$\begin{aligned}&\lim_{m\to\infty}\lim_{l\to\infty}\inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_{2,\kappa}^{l,m}\rangle+\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_{2,\kappa}^{l,m}\Big]=\\&\inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_{2,\kappa}\rangle+\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_{2,\kappa}\Big],\ \text{ a.e.}\end{aligned} \tag{19}$$

Moreover, from (16),(17) and (19), letting $l\to\infty$ and then $m\to\infty$ in (8), it follows that

$$\begin{aligned}\alpha\theta\frac{\partial\psi_{2,\kappa}(\theta,x)}{\partial\theta}&=\inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_{2,\kappa}\rangle+\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_{2,\kappa}\Big]\\&\quad+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2\psi_{2,\kappa}(\theta,x)),\end{aligned} \tag{20}$$

in the sense of distribution. As we have $\psi_{2,\kappa}\in W^{1,2,p}((\kappa,\Theta)\times Q)$ for any compact subset $\overline{Q}$ of $\overline{D}$ with C^2 smooth boundary, we deduce that $\psi_{2,\kappa}$ satisfies (20) strongly. Also, since we have

$$\psi_{2,\kappa}^{l,m}(\kappa,x)=e^{\dfrac{\kappa\|r_2\|_\infty}{\alpha}}\ \text{ on } \overline{D}_{l,m} \text{ for all } l\ge\bar{N}_1, m\ge\bar{M}_1$$

using (18), it follows that $\psi_{2,\kappa}(\kappa,x)=e^{\dfrac{\kappa\|r_2\|_\infty}{\alpha}}$ on $\overline{D}$. Next we want to derive the boundary condition for the limiting solution, i.e., $\nabla\psi_{2,\kappa}(\theta,x)\cdot\gamma(x)=0$ a.e. on $(\kappa,\Theta)\times\partial D$. It is clear from our construction that, for each point $\tilde{x}_0\in\Sigma$, there exist a sequence $\{x_{l,m}\}_{l,m}$ of points of $\partial D_{l,m}$ such that $x_{l,m}\longrightarrow\tilde{x}_0$ as $m,l\to\infty$. From (18), using the fact that γ is continuous, and $\psi_{2,\kappa}\in C^1(D\cup\Sigma)$, we obtain

$$\nabla\psi_{2,\kappa}(\theta,\tilde{x}_0)\cdot\gamma(\tilde{x}_0)=\lim_{m\to\infty}\lim_{l\to\infty}\nabla\psi_{2,\kappa}^{l,m}(\theta,x_{l,m})\cdot\gamma(x_{l,m})=0.$$

Since the surface measure of Σ' (non-smooth part of ∂D) is zero, we get $\nabla\psi_{2,\kappa}(\theta,x)\cdot\gamma(x)=0$ a.e. on $(\kappa,\Theta)\times\partial D$. Therefore, $\psi_{2,\kappa}\in W^{1,2,p}_{loc}((\kappa,\Theta)\times D\cup\Sigma)\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q}),\ p\ge 2$, for each bounded C^2 domain Q in D, satisfies

$$\begin{aligned}\alpha\theta\frac{\partial\psi_{2,\kappa}(\theta,x)}{\partial\theta}&=\inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_{2,\kappa}\rangle+\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_{2,\kappa}\Big]\\&\quad+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2\psi_{2,\kappa}(\theta,x)),\\\psi_{2,\kappa}(\kappa,x)&=e^{\dfrac{\kappa\|r_2\|_\infty}{\alpha}}\ \text{ on } \overline{D},\\\nabla\psi_{2,\kappa}(\theta,x)\cdot\gamma(x)&=0 \text{ on } (\kappa,\Theta)\times\partial D.\end{aligned} \tag{21}$$

Thus, we deduce that (21) admits a solution in $W^{1,2,p}_{loc}((\kappa,\Theta)\times D\cup\Sigma)\cap C^{0,1}((\kappa,\Theta)\times D\cup\Sigma)$. Now, we want to take limit $\kappa\to 0$ in (21). To this end, we extend the function $\psi_{2,\kappa}$ to the whole of $(0,\Theta)$ as follows

$$\bar{\psi}_{2,\kappa}(\theta,x)=\begin{cases}\psi_{2,\kappa}(\theta,x) & \text{if } \theta>\kappa\\ e^{\frac{\kappa\|r_2\|_\infty}{\alpha}} & \text{if } \theta\le\kappa.\end{cases}$$

It is easy to see that, $\bar{\psi}_{2,\kappa}$ is nonnegative bounded, continuous and from (13), (14) and (15), we have

$$\sup_{0<\kappa<\Theta} \|\frac{\partial \bar{\psi}_{2,\kappa}}{\partial \theta}\|_{\infty;(\kappa,\Theta)\times\overline{D}} \ < \ \infty.$$

Furthermore, for each compact $\overline{Q} \subset \overline{D}$ with C^2-smooth boundary,

$$\sup_{0<\kappa<\Theta} \|\bar{\psi}_{2,\kappa}\|_{2,p;\overline{Q}} \ < \ \infty.$$

Also, it is easy to see that for each $0 < \theta < \Theta$, the function $\bar{\psi}_{2,\kappa}$ satisfies the following p.d.e.

$$\begin{aligned} \alpha\theta \frac{\partial \bar{\psi}_{2,\kappa}(\theta,x)}{\partial \theta} &= \inf_{v_2\in V_2} \Big[\langle b(x,\hat{v}_1(\theta,x),v_2), \nabla\bar{\psi}_{2,\kappa}\rangle + \theta r_2(x,\hat{v}_1(\theta,x),v_2)\bar{\psi}_{2,\kappa}\Big] \\ &+ \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\bar{\psi}_{2,\kappa}(\theta,x)) - \theta e^{\frac{\kappa\|r_2\|_\infty}{\alpha}} \inf_{v_2\in V_2} r_2(x,\hat{v}_1(\theta,x),v_2) I_{\{\theta\le\kappa\}}, \\ \bar{\psi}_{2,\kappa}(\kappa,x) &= e^{\frac{\kappa\|r_2\|_\infty}{\alpha}} \ \text{on} \ \overline{D}, \ \nabla\bar{\psi}_{2,\kappa}(\theta,x)\cdot\gamma(x) = 0 \ \text{on} \ (\kappa,\Theta)\times\partial D, a.e. \end{aligned} \tag{22}$$

Therefore, $\bar{\psi}_{2,\kappa} \in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q})$ for each bounded C^2 domain Q in D, is a solution to (22). Let $\varphi \in C^\infty_c((0,\Theta)\times D\cup\Sigma)$ be a test function. Then multiplying (22) by φ and integrating over $(0,\Theta)\times D\cup\Sigma$, it follows that

$$\begin{aligned} - & \int_0^\Theta\int_{D\cup\Sigma} \alpha\theta\frac{\partial\bar{\psi}_{2,\kappa}}{\partial\theta}\varphi d\theta dx + \int_0^\Theta\int_{D\cup\Sigma}\inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\bar{\psi}_{2,\kappa}\rangle \\ + & \ \theta r_2(x,\hat{v}_1(\theta,x),v_2)\bar{\psi}_{2,\kappa}\Big]\varphi d\theta dx + \frac{1}{2}\int_0^\Theta\int_{D\cup\Sigma}\mathrm{trace}(a(x)\nabla^2\bar{\psi}_{2,\kappa})\varphi d\theta dx \\ = & \int_0^\kappa\int_{D\cup\Sigma}\inf_{v_2\in V_2}\theta r_2(x,\hat{v}_1(\theta,x),v_2)e^{\frac{\kappa\|r_2\|_\infty}{\alpha}}\varphi d\theta dx. \end{aligned} \tag{23}$$

From the above estimates, we deduce that there exists $\psi_2 \in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q})$ for each bounded C^2 domain Q in D, satisfying the limiting equation. Letting $\kappa\to\infty$ in the above equation, it follows that

$$\begin{aligned} & -\int_0^\Theta\int_{D\cup\Sigma}\alpha\theta\frac{\partial\psi_2}{\partial\theta}\varphi d\theta dx + \int_0^\Theta\int_{D\cup\Sigma}\inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_2\rangle \\ & +\theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_2\Big]\varphi d\theta dx + \frac{1}{2}\int_0^\Theta\int_{D\cup\Sigma}\mathrm{trace}(a(x)\nabla^2\psi_2)\varphi d\theta dx \\ & = 0. \end{aligned} \tag{24}$$

It is known that $\psi_2 \in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}((\kappa,\Theta)\times\overline{Q})$ for each bounded C^2 domain Q in D, thus it satisfies the following:

$$\begin{aligned} \alpha\theta\frac{\partial\psi_2(\theta,x)}{\partial\theta} &= \inf_{v_2\in V_2}\Big[\langle b(x,\hat{v}_1(\theta,x),v_2),\nabla\psi_2(\theta,x)\rangle + \theta r_2(x,\hat{v}_1(\theta,x),v_2)\psi_2\Big] \\ & \quad +\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\psi_2(\theta,x)), \\ \psi_2(0,x) &= 1 \ \text{on} \ \overline{D}. \end{aligned} \tag{25}$$

Suppose Q be a domain in D with Lipschitz boundary such that its closure in $\overline{D}$ contains some part of the boundary Σ (the smooth portion ∂D). We know that $\bar{\psi}_{2,\kappa}(\theta,\cdot), \psi_{2,\kappa}(\theta,\cdot) \in W^{2,p}(Q)$, $p\ge 2$, for fixed $\theta\in(0,\Theta)$. Also, by Morrey Lemma

[[34], pp. 335-339], we have $W^{2,p}(Q)$ is compactly embedded in $C^{1,\hat{\alpha}}(\overline{Q})$. Thus, we deduce

$$\bar{\psi}_{2,\kappa}(\theta,\cdot) \longrightarrow \psi_2(\theta,\cdot) \text{ in } C^{1,\hat{\alpha}}(\overline{Q}),$$

for fixed $\theta \in (0,\Theta)$. Using the fact that $\nabla\bar{\psi}_{2,\kappa}(\theta,x)\cdot\gamma(x) = 0$ on $(0,\Theta)\times\partial D$ for all $\kappa > 0$, we deduce that $\nabla\psi_2(\theta,x)\cdot\gamma(x) = 0$ on $(0,\Theta)\times\partial D\cap\partial Q$. Since Q is arbitrary, it follows that $\nabla\psi_2(\theta,x)\cdot\gamma(x) = 0$ on $(0,\Theta)\times\partial D$. This proves the existence of a solution $\psi_2 \in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{0,1}((0,\Theta)\times D\cup\Sigma)$, $p \geq 2$, to the equation (10). For $\infty > p \geq 2$, using the dominated convergence theorem and Itô-Krylov formula as in the proof of Lemma 3.1, it follows that

$$\psi_2(\theta,x) = \inf_{v_2\in\mathcal{A}_2} E^{\hat{v}_1,v_2}_x\Big[e^{\theta\int_0^\infty e^{-\alpha t} r_2(X(t),\hat{v}_1(\theta e^{-\alpha t},X(t)),v_2(t))dt}\Big] (:= \psi^{\hat{v}_1}_{\alpha,2}(\theta,x)).$$

Hence $\psi_2 \in W^{1,2,p}_{loc}((0,\Theta)\times D\cup\Sigma)\cap C^{0,1}((0,\Theta)\times D\cup\Sigma)$, $\infty > p \geq 2$, is the unique solution to (10). □

For $\theta \in (0,\Theta)$, $\alpha > 0$ and $\hat{v}_i \in \hat{\mathcal{S}}_i$, $i = 1,2$, in view the proof of [[11], Theorem 3.1], it follows that the following estimates hold

$$1 \leq \max\left\{\psi^{\hat{v}_2}_{\alpha,1}(\theta,x), \psi^{\hat{v}_1}_{\alpha,2}(\theta,x)\right\} \leq \max_i e^{\frac{\theta\|r_i\|_\infty}{\alpha}},$$

$$\max\left\{\|\frac{\partial\psi^{\hat{v}_2}_{\alpha,1}}{\partial\theta}\|_\infty, \|\frac{\partial\psi^{\hat{v}_1}_{\alpha,2}}{\partial\theta}\|_\infty\right\} \leq 3\max_i \frac{\|r_i\|_\infty}{\alpha} e^{\frac{(\Theta+2)\|r_i\|_\infty}{\alpha}}. \quad (26)$$

In the next lemma we prove certain estimates which will be useful in proving continuity of certain maps (see Lemma 3.4). For bounded domain case [21, Lemma 3.2] authors have proved similar kind of estimates. The main idea of the proof is to use standard elliptic P.D.E. estimates.

Lemma 3.3. *Suppose that assumptions (A0) - (A3) hold. Then for $\theta \in (0,\Theta)$, $\alpha > 0$ and for each domain Q in D with C^2 boundary portion of ∂D (if the boundary of Q intersects the boundary of D), we have*

$$\sup_{\hat{v}_1\in\hat{\mathcal{S}}_1} \|\psi^{\hat{v}_1}_{\alpha,2}\|_{1,2,p;Q} < \infty, \quad \sup_{\hat{v}_2\in\hat{\mathcal{S}}_2} \|\psi^{\hat{v}_2}_{\alpha,1}\|_{1,2,p;Q} < \infty. \quad (27)$$

Proof. For any measurable minimizing selector $\bar{v}_2(\theta,x)$ of (10), rewriting the equation (10), it follows that

$$\begin{aligned} &\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\psi^{\hat{v}_1}_{\alpha,2}) + \langle b(x,\hat{v}_1(\theta,x),\bar{v}_2(\theta,x)), \nabla\psi^{\hat{v}_1}_{\alpha,2}\rangle \\ = \ &\alpha\theta\frac{\partial\psi^{\hat{v}_1}_{\alpha,2}}{\partial\theta} - \theta r_2(x,\hat{v}_1(\theta,x),\bar{v}_2(\theta,x))\psi^{\hat{v}_1}_{\alpha,2}. \end{aligned}$$

Using the estimates as in (26), it is easy to see that the r.h.s. of the equation is bounded uniformly in $\theta \in (0,\Theta)$, $\hat{v}_1 \in \hat{\mathcal{S}}_1$. Then by standard elliptic P.D.E. estimats (see, [[19], Theorem 1.8], also see [28]), we deduce that

$$\sup_{\hat{v}_1} \|\psi^{\hat{v}_1}_{\alpha,2}\|_{1,2,p;Q} < \infty. \quad (28)$$

The other estimate follows by similar argument. □

Next lemma proves continuity of certain class of functions, which will play an important role in further analysis. In order to do so, we follow the argument as the authors have used to derive similar continuity result [21, Lemma 3.5] for the bounded domain case.

Lemma 3.4. *Suppose that assumptions (A0) - (A3) hold. Then the maps* $\hat{v}_1 \mapsto \psi_{\alpha,2}^{\hat{v}_1}$ *from* $\hat{\mathcal{S}}_1 \to W_{loc}^{1,2,p}((0,\ \Theta) \times D \cup \Sigma)$, $\infty > p \geq 2$, *and* $\hat{v}_2 \mapsto \psi_{\alpha,1}^{\hat{v}_2}$ *from* $\hat{\mathcal{S}}_2 \to W_{loc}^{1,2,p}((0,\ \Theta) \times D \cup \Sigma)$, $\infty > p \geq 2$ *are continuous.*

Proof. Let $\hat{v}_1^n \to \hat{v}_1$ in $\hat{\mathcal{S}}_1$ as $n \to \infty$. Then in view of Lemma 3.3, it is easy to see that

$$\sup_{n\geq 1} \|\psi_{\alpha,2}^{\hat{v}_1^n}\|_{1,2,p;Q} < \infty. \tag{29}$$

Now we choose a sequence $\{Q_n\}$ of bounded domains from D such that $D \cup \Sigma = \cup_{n\geq 1}\overline{Q}_n$ and $\partial D \cap \partial Q_n$ is a C^2 portion of ∂D. Then by standard diagonalization argument and using the compact embedding theorem [see [1], Chapter 6], the Banach-Alaoglu theorem it follows that there exists $\tilde{\psi}_{\alpha,2} \in W_{loc}^{1,2,p}((0,\Theta) \times D \cup \Sigma) \cap C^{0,1}((0,\ \Theta) \times D \cup \Sigma)$ such that along a suitable subsequence

$$\left.\begin{aligned} &\psi_{\alpha,2}^{\hat{v}_1^n} \to \tilde{\psi}_{\alpha,2} \text{ in } W^{1,2,p}((0,\Theta) \times Q_m) \text{ weakly,}\\ &\psi_{\alpha,2}^{\hat{v}_1^n} \to \tilde{\psi}_{\alpha,2} \text{ in } C^{0,1}((0,\ \Theta) \times \overline{Q}_m) \text{ strongly,}\end{aligned}\right\} \tag{30}$$

for each $m \geq 1$. Multiplying both sides of (10) (for fixed strategy $\hat{v}_1^n$) by test function $\varphi \in C_c^{\infty}((0,\Theta) \times D \cup \Sigma)$ and integrating over $D \cup \Sigma$, we deduce that

$$\begin{aligned}
&\int_0^{\Theta} \int_{D\cup\Sigma} \alpha\theta \frac{\partial \psi_{\alpha,2}^{\hat{v}_1^n}}{\partial\theta} \varphi d\theta dx\\
&= \int_0^{\Theta} \int_{D\cup\Sigma} \inf_{v_2\in V_2} \Big[\langle b(x, \hat{v}_1^n(\theta,x), v_2), \nabla \psi_{\alpha,2}^{\hat{v}_1^n}\rangle + \theta r_2(x, \hat{v}_1^n(\theta,x), v_2)\psi_{\alpha,2}^{\hat{v}_1^n}\Big]\varphi d\theta dx\\
&+ \frac{1}{2}\int_0^{\Theta} \int_{D\cup\Sigma} \operatorname{trace}(a(x)\nabla^2 \psi_{\alpha,2}^{\hat{v}_1^n})\varphi d\theta dx.
\end{aligned}$$

Since $\tilde{\psi}_{\alpha,2} \in W_{loc}^{1,2,p}((0,\Theta) \times D \cup \Sigma) \cap C^{0,1}((0,\ \Theta) \times D \cup \Sigma)$ in view of (30), letting $n \to \infty$ in the above equation, we obtain $\tilde{\psi}_{\alpha,2}$ is a.e. solution to (10). We have $\tilde{\psi}_{\alpha,2}(0,x) = 1$, since $\psi_{\alpha,2}^{\hat{v}_1^n}(0,x) = 1$ for all $n \geq 1$. Arguing as in Theorem 3.2 we can show that $\nabla\tilde{\psi}_{\alpha,2}(\theta,x) \cdot \gamma(x) = 0$ on $(0,\Theta) \times \partial D$. Thus, $\tilde{\psi}_{\alpha,2}$ is the unique solution in $W_{loc}^{1,2,p}((0,\Theta) \times D \cup \Sigma) \cap C^{0,1}((0,\Theta) \times D \cup \Sigma)$ of the p.d.e. (10). Therefore by Theorem 3.2, it follows that $\tilde{\psi}_{\alpha,2} = \psi_{\alpha,1}^{\hat{v}_1}$. This proves the continuity of the first map. The continuity of the second map follows by similar arguments. □

For each fixed pair of strategies $(\hat{v}_1, \hat{v}_2) \in \hat{S}_1 \times \hat{S}_2$, we define

$$\hat{N}(\hat{v}_1, \hat{v}_2) \ = \ \hat{N}_1(\hat{v}_2) \times \hat{N}_2(\hat{v}_1), \tag{31}$$

where

$$\hat{N}_1(\hat{v}_2) \ = \ \Big\{\hat{v}_1^* \in \hat{\mathcal{S}}_1 | F_1(x, \hat{v}_1^*(\theta,x), \hat{v}_2(\theta,x)) = \inf_{v_1\in V_1} F_1(x, v_1, \hat{v}_2(\theta,x)) \text{ a.e. } \theta, x\Big\},$$

$$F_1(x, v_1, \hat{v}_2(\theta,x)) \ = \ \langle b(x, v_1, \hat{v}_2(\theta,x)), \nabla\psi_{\alpha,1}^{\hat{v}_2}\rangle + \theta r_1(x, v_1, \hat{v}_2(\theta,x))\psi_{\alpha,1}^{\hat{v}_2},$$

$(\theta,x) \in (0,\Theta) \times \overline{D},\ v_1 \in V_1, \hat{v}_2 \in \hat{\mathcal{S}}_2$,

$$\hat{N}_2(\hat{v}_1) \ = \ \Big\{\hat{v}_2^* \in \hat{\mathcal{S}}_2 | F_2(x, \hat{v}_1(\theta,x), \hat{v}_2^*(\theta,x)) = \inf_{v_2\in V_2} F_2(x, \hat{v}_1(\theta,x), v_2) \text{ a.e. } \theta, x\Big\},$$

$$F_2(x, \hat{v}_1(\theta,x), v_2) \ = \ \langle b(x, \hat{v}_1(\theta,x), v_2), \nabla\psi_{\alpha,2}^{\hat{v}_1}\rangle + \theta r_2(x, \hat{v}_1(\theta,x), v_2)\psi_{\alpha,2}^{\hat{v}_1},$$

$(\theta,x) \in (0,\Theta) \times \overline{D},\ v_2 \in V_2, \hat{v}_1 \in \hat{\mathcal{S}}_1$.

By a standard measurable selection theorem [9], it is easy to see that $\hat{N}_1(\hat{v}_2)$ is nonempty. From the definition of $\hat{N}_1(\hat{v}_2)$ it is clear that $\hat{N}_1(\hat{v}_2)$ is convex. Using the topology (6) of $\hat{S}_1$, it can be shown that $\hat{N}_1(\hat{v}_2)$ is closed in $\hat{\mathcal{S}}_1$, thus we have $\hat{N}_1(\hat{v}_2)$ is compact. By analogous argument we have $\hat{N}_2(\hat{v}_1)$ is nonempty, compact, convex, subset of $\hat{\mathcal{S}}_2$. This implies that $\hat{N}(\hat{v}_1, \hat{v}_2) = \hat{N}_1(\hat{v}_2) \times \hat{N}_2(\hat{v}_1)$ is a nonempty, convex and compact subset of $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$. As we have discussed earlier, to prove the existence of a Nash equilibrium, we need to prove the upper semi-continuity (u.s.c.) of certain map, in particular we need to prove u.s.c. of the set valued map: $(\hat{v}_1, \hat{v}_2) \mapsto \hat{N}(\hat{v}_1, \hat{v}_2)$ from $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2 \to 2^{\hat{\mathcal{S}}_1} \times 2^{\hat{\mathcal{S}}_2}$. As in [18], [20], [21], [26], [29], to establish u.s.c. of this set valued map we need some additional additive assumptions on the drift of the state dynamics and the cost function (ADAC) given as follows:
(A4) We assume that $\bar{b} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d$ and $\bar{r}_i : \overline{D} \times U_1 \times U_2 \to [0,\ \infty),\ i = 1, 2$ satisfy the following additive structure:

$$\begin{aligned}
\bar{b}(x, u_1, u_2) &= \bar{b}_1(x, u_1) + \bar{b}_2(x, u_2),\\
\bar{r}_i(x, u_1, u_2) &= \bar{r}_{i1}(x, u_1) + \bar{r}_{i2}(x, u_2),\ x \in \overline{D}, u_1 \in U_1, u_2 \in U_2, i = 1, 2.
\end{aligned}$$

Also, we assume that $\bar{b}_i, \bar{r}_{i1}, \bar{r}_{i2}, i = 1, 2$, satisfy conditions in (A0) and (A3).

Next lemma proves u.s.c. of certain set valued map, which will play a crucial role in establishing existence of a Nash equilibrium. In bounded domain setup similar result has been established in [21, Lemma 3.4]

Lemma 3.5. *Suppose that assumptions (A0)-(A4) hold. Then the set valued map* $(\hat{v}_1, \hat{v}_2) \mapsto \hat{N}(\hat{v}_1, \hat{v}_2)$ *from* $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2 \to 2^{\hat{\mathcal{S}}_1} \times 2^{\hat{\mathcal{S}}_2}$ *is u.s.c.*

Proof. Let $\{(\hat{v}_1^n, \hat{v}_2^n)\}_n$ be a sequence in $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$ such that $(\hat{v}_1^n, \hat{v}_2^n) \to (\hat{v}_1, \hat{v}_2) \in \hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$. For each $n \geq 1$, we choose $\hat{\hat{v}}_1^n \in \hat{N}_1(\hat{v}_2^n), n \geq 1$. Using compactness of $\hat{\mathcal{S}}_1$, we can extract a subsequence $\{\hat{\hat{v}}_1^n\}$ (denoting by the same notation without any loss of generality) such that $\hat{\hat{v}}_1^n \to \hat{\hat{v}}_1$ for some $\hat{\hat{v}}_1 \in \hat{\mathcal{S}}_1$. Thus, we have $(\hat{\hat{v}}_1^n, \hat{v}_2^n) \to (\hat{\hat{v}}_1, \hat{v}_2)$ in $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$ as $n \to \infty$. Then in view of (A4), Lemma 3.4 and the topology of $\hat{\mathcal{S}}_i, i = 1, 2$, we deduce that

$$\langle b(x, \hat{\hat{v}}_1^n(\theta, x), \hat{v}_2^n(\theta, x)), \nabla \psi_{\alpha,1}^{\hat{v}_2^n} \rangle + \theta r_1(x, \hat{\hat{v}}_1^n(\theta, x), \hat{v}_2^n(\theta, x)) \psi_{\alpha,1}^{\hat{v}_2^n}$$

converges weakly in $L^2_{loc}((0, \Theta) \times D \cup \Sigma)$ to

$$\langle b(x, \hat{\hat{v}}_1(\theta, x), \hat{v}_2(\theta, x)), \nabla \psi_{\alpha,1}^{\hat{v}_2} \rangle + \theta r_1(x, \hat{\hat{v}}_1(\theta, x), \hat{v}_2(\theta, x)) \psi_{\alpha,1}^{\hat{v}_2}.$$

In view of the Banach-Saks theorem it follows that a sequence of convex combination of the former converges strongly in $L^2_{loc}((0, \Theta) \times D \cup \Sigma)$ to the latter. Thus, along a suitable subsequence of the convergent sequence of convex combinations, it follows that (without any loss of generality denoting by the same notation)

$$\lim_{n\to\infty} F_1(x, \hat{\hat{v}}_1^n(\theta, x), \hat{v}_2^n(\theta, x)) = F_1(x, \hat{\hat{v}}_1(\theta, x), \hat{v}_2(\theta, x)),\ \text{a.e. in } \theta, x. \tag{32}$$

By analogous argument as above, for fixed $\hat{\tilde{v}}_1 \in \hat{S}_1$, we have

$$\lim_{n\to\infty} F_1(x, \hat{\tilde{v}}_1(\theta, x), \hat{v}_2^n(\theta, x)) = F_1(x, \hat{\tilde{v}}_1(\theta, x), \hat{v}_2(\theta, x)),\ \text{a.e. in } \theta, x. \tag{33}$$

Since $\hat{\hat{v}}_1^n \in \hat{N}_1(\hat{v}_2^n)$, we get

$$F_1(x, \hat{\tilde{v}}_1(\theta, x), \hat{v}_2^n(\theta, x)) \geq F_1(x, \hat{\hat{v}}_1^n(\theta, x), \hat{v}_2^n(\theta, x)),\ n \geq 1.$$

Now, using (32) and (33), we obtain

$$F_1(x, \hat{\tilde{v}}_1(\theta, x), \hat{v}_2(\theta, x)) \geq F_1(x, \hat{\hat{v}}_1(\theta, x), \hat{v}_2(\theta, x)),\ \hat{\tilde{v}}_1 \in \hat{S}_1.$$

Since $\hat{\hat{v}}_1 \in \hat{S}_1$ is arbitrary, we deduce that $\hat{\hat{v}}_1 \in \hat{N}_1(\hat{v}_2)$. Arguing as above, it can be shown that for $\hat{\hat{v}}_2^n \in \hat{N}_2(\hat{v}_1^n)$ and any limit point $\hat{\hat{v}}_2$ of $\{\hat{\hat{v}}_2^n\}$, we obtain $\hat{\hat{v}}_2 \in \hat{N}_2(\hat{v}_1)$. This shows that the above map is u.s.c. □

Following theorem proves the existence of Nash equilibria in the space of eventually stationary strategies.

Theorem 3.6. *Suppose that assumptions (A0)-(A4) hold. Then there exists an α-discounted Nash equilibrium in $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$.*

Proof. Since the above set valued map is upper semi-continuous Lemma 3.5, applying Fan's fixed point theorem [17], it follows that the map $(\hat{v}_1, \hat{v}_2) \mapsto \hat{N}(\hat{v}_1, \hat{v}_2)$ from $\hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2 \to 2^{\hat{\mathcal{S}}_1} \times 2^{\hat{\mathcal{S}}_2}$, admits a fixed point $(\hat{v}_1^*, \hat{v}_2^*) \in \hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$, i.e.,

$$(\hat{v}_1^*, \hat{v}_2^*) \in \hat{N}(\hat{v}_1^*, \hat{v}_2^*).$$

This implies that the pair $(\psi_{\alpha,1}^{\hat{v}_2^*}, \psi_{\alpha,2}^{\hat{v}_1^*})$ satisfies the following:

$$\begin{aligned}
\alpha\theta \frac{\partial \psi_{\alpha,1}^{\hat{v}_2^*}}{\partial \theta} &= \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \hat{v}_2^*(\theta, x)), \nabla \psi_{\alpha,1}^{\hat{v}_2^*} \rangle + \theta r_1(x, v_1, \hat{v}_2^*(\theta, x)) \psi_{\alpha,1}^{\hat{v}_2^*} \Big] \\
&\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_{\alpha,1}^{\hat{v}_2^*}), \\
&= \langle b(x, \hat{v}_1^*(\theta, x), \hat{v}_2^*(\theta, x)), \nabla \psi_{\alpha,1}^{\hat{v}_2^*} \rangle + \theta r_1(x, \hat{v}_1^*(\theta, x), \hat{v}_2^*(\theta, x)) \psi_{\alpha,1}^{\hat{v}_2^*} \\
&\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_{\alpha,1}^{\hat{v}_2^*}), \\
\psi_{\alpha,1}^{\hat{v}_2^*}(0, x) &= 1, \ x \in \overline{D}, \ \nabla \psi_{\alpha,1}^{\hat{v}_2^*} \cdot \gamma = 0 \ \text{ on } \ (0, \Theta) \times \partial D. \\
\alpha\theta \frac{\partial \psi_{\alpha,2}^{\hat{v}_1^*}}{\partial \theta} &= \inf_{v_2 \in V_2} \Big[\langle b(x, \hat{v}_1^*(\theta, x), v_2), \nabla \psi_{\alpha,2}^{\hat{v}_1^*} \rangle + \theta r_2(x, \hat{v}_1^*(\theta, x), v_2) \psi_{\alpha,2}^{\hat{v}_1^*} \Big] \\
&\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_{\alpha,2}^{\hat{v}_1^*}), \\
&= \langle b(x, \hat{v}_1^*(\theta, x), \hat{v}_2^*(\theta, x)), \nabla \psi_{\alpha,2}^{\hat{v}_1^*} \rangle + \theta r_2(x, \hat{v}_1^*(\theta, x), \hat{v}_2^*(\theta, x) \psi_{\alpha,2}^{\hat{v}_1^*} \\
&\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_{\alpha,2}^{\hat{v}_1^*}), \\
\psi_{\alpha,2}^{\hat{v}_1^*}(0, x) &= 1, \ x \in \overline{D}, \ \nabla \psi_{\alpha,2}^{\hat{v}_2^*} \cdot \gamma = 0 \ \text{ on } \ (0, \Theta) \times \partial D.
\end{aligned}$$

Now, using the representation of the solutions to the p.d.e.s as obtained in the proof of Theorem 3.2, it is easy to see that

$$\begin{aligned}
\psi_{\alpha,1}^{\hat{v}_2^*}(\theta, x) &= \inf_{v_1 \in \hat{\mathcal{S}}_1} J_{\alpha,1}^{v_1, \hat{v}_2^*}(\theta, x) \, (= \inf_{v_1 \in \mathcal{A}_1} J_{\alpha,1}^{v_1, \hat{v}_2^*}(\theta, x)) \\
&= J_{\alpha,1}^{\hat{v}_1^*, \hat{v}_2^*}(\theta, x), \\
\psi_{\alpha,2}^{\hat{v}_1^*}(\theta, x) &= \inf_{v_2 \in \hat{\mathcal{S}}_2} J_{\alpha,2}^{\hat{v}_1^*, v_2}(\theta, x) \, (= \inf_{v_2 \in \mathcal{A}_2} J_{\alpha,2}^{\hat{v}_1^*, v_2}(\theta, x)) \\
&= J_{\alpha,2}^{\hat{v}_1^*, \hat{v}_2^*}(\theta, x).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
J_{\alpha,1}^{v_1, \hat{v}_2^*}(\theta, x) &\geq J_{\alpha,1}^{\hat{v}_1^*, \hat{v}_2^*}(\theta, x), \ \forall \ v_1 \in \mathcal{A}_1, \\
J_{\alpha,2}^{\hat{v}_1^*, v_2}(\theta, x) &\geq J_{\alpha,2}^{\hat{v}_1^*, \hat{v}_2^*}(\theta, x), \ \forall \ v_2 \in \mathcal{A}_2.
\end{aligned}$$

This implies that the pair $(\hat{v}_1^*, \hat{v}_2^*) \in \hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2$ is a Nash equilibrium. This completes the proof. □

4. **Analysis of ergodic cost criterion.** In this section we show that for the ergodic cost evolution criterion a Nash equilibrium exists in the space of stationary Markov strategies. Also, we completely characterize Nash equilibrium in the space of stationary Markov strategies. To carry out our analysis we make the following stability hypothesis:

(A5)(Stability assumption) There exists a stochastic Lyapunov type function $V : \overline{D} \to [1, \infty)$, with following properties

(i) $V \in C^2(\overline{D})$, $\lim_{\|x\| \to \infty} V(x) = \infty$.
(ii) $\nabla V \cdot \gamma \geq 0$ on ∂D.
(iii) $L^{u_1,u_2} V(x) < \tilde{\alpha} I_{\tilde{K}} - 2\delta V(x)$, $\forall\ (x, u_1, u_2) \in \overline{D} \times U_1 \times U_2$, for some suitable compact set $\tilde{K} \subset D$ and constants $\tilde{\alpha} \geq 0$, $\delta > 0$.

Also, we make the following technical assumption about the running cost function.

(A6)(Small Cost Condition) $\theta \|r_i\|_\infty < \delta$, $\theta \in (0, \Theta)$ and δ as in (A5), $i = 1, 2$.

Since for the ergodic cost criterion we fix the risk-sensitive parameter, without loss of generality we are assuming that $\theta = 1$. Suppose that $x_0 \in D$ is an arbitrarily fixed point. Then, from our constructions it is clear that there exit l_1, m_1 large enough such that $x_0 \in D_{l,m}$ for all $l \geq l_1, m \geq m_1$.

From [40], we have the following results about the ergodic HJB equation.

Theorem 4.1. *Suppose that assumptions (A0)-(A6) hold. Then for $\tilde{v}_2 \in \mathcal{S}_2$, there exists a unique eigenpair $(\rho_1^{\tilde{v}_2}, \psi_1^{\tilde{v}_2}) \in \mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma) \cap C^1(D \cup \Sigma) \cap O(V)$, $\psi_1^{\tilde{v}_2} > 0$, $q \geq d + 1$, satisfying*

$$
\begin{aligned}
\rho_1^{\tilde{v}_2} \psi_1^{\tilde{v}_2} &= \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla \psi_1^{\tilde{v}_2} \rangle + r_1(x, v_1, \tilde{v}_2(x)) \psi_1^{\tilde{v}_2} \Big] \\
&\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_1^{\tilde{v}_2}), \quad a.e.\ x \in D \cup \Sigma, \\
\psi_1^{\tilde{v}_2}(x_0) &= 1, \ \ \nabla \psi_1^{\tilde{v}_2} \cdot \gamma = 0, \ \ on\ \partial D.
\end{aligned}
\tag{34}
$$

Moreover, There exists a compact set $\mathcal{Q}_1 \subset D$ such that for any minimizing selector v_1^* of (34) and any compact set $\mathcal{Q}_{1,1} \supset \mathcal{Q}_1$, we have

$$
\psi_1^{\tilde{v}_2}(x) = E_x^{v_1^*, \tilde{v}_2} \Big[e^{\int_0^{\tau_{1,1}^c} (r_1(X(t), v_1^*(X(t)), \tilde{v}_2(X(t))) - \rho_1^{\tilde{v}_2}) dt} \psi_1^{\tilde{v}_2}(X(\tau_{1,1}^c)) \Big], \ \forall\ x \in \mathcal{Q}_{1,1}^c,
\tag{35}
$$

where $\tau_{1,1}^c := \tau(\mathcal{Q}_{1,1}^c) = \inf\{t \geq 0 : X(t) \in \mathcal{Q}_{1,1}\}$ and $X(t)$ is the solution of (1) corresponding to $(v_1^*, \tilde{v}_2) \in \mathcal{S}_1 \times \mathcal{S}_2$.

Similarly, for each $\tilde{v}_1 \in \mathcal{S}_1$ there exists a unique eigenpair $(\rho_2^{\tilde{v}_1}, \psi_2^{\tilde{v}_1}) \in \mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma) \cap C^1(D \cup \Sigma) \cap O(V)$, $\psi_2^{\tilde{v}_1} > 0$, $q \geq d + 1$, satisfying

$$
\begin{aligned}
\rho_2^{\tilde{v}_1} \psi_2^{\tilde{v}_1} &= \inf_{v_2 \in V_2} \Big[\langle b(x, \tilde{v}_1(x), v_2), \nabla \psi_2^{\tilde{v}_1} \rangle + r_2(x, \tilde{v}_1(x), v_2) \psi_2^{\tilde{v}_1} \Big] \\
&\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_2^{\tilde{v}_1}), \quad a.e.\ x \in D \cup \Sigma, \\
\psi_2^{\tilde{v}_1}(x_0) &= 1, \ \ \nabla \psi_2^{\tilde{v}_1} \cdot \gamma = 0, \ \ on\ \partial D.
\end{aligned}
\tag{36}
$$

Moreover, There exists a compact set $\mathcal{Q}_2 \subset D$ such that for any minimizing selector v_2^* of (36) and any compact set $\mathcal{Q}_{2,1} \supset \mathcal{Q}_2$, we have

$$\psi_2^{\tilde{v}_1}(x) = E_x^{\tilde{v}_1, v_2^*}\Big[e^{\int_0^{\tau_{2,1}^c}(r_2(X(t),\tilde{v}_1(X(t)),v_2^*(X(t)))-\rho_2^{\tilde{v}_1})dt}\psi_2^{\tilde{v}_1}(X(\tau_{2,1}^c))\Big],\ \forall\ x \in \mathcal{Q}_{2,1}^c, \tag{37}$$

where $\tau_{2,1}^c := \tau(\mathcal{Q}_{2,1}^c) = \inf\{t \geq 0 : X(t) \in \mathcal{Q}_{2,1}\}$ and $X(t)$ is the solution of (1) corresponding to $(\tilde{v}_1, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$.

Proof. From [[40], Lemma 3.3], it is clear that for $l \geq l_1, m \geq m_1$, the following p.d.e.

$$\begin{aligned}
\rho_{1,\tilde{v}_2}^{l,m}\psi_{1,\tilde{v}_2}^{l,m} &= \inf_{v_1 \in V_1}\Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla\psi_{1,\tilde{v}_2}^{l,m}\rangle + r_1(x, v_1, \tilde{v}_2(x))\psi_{1,\tilde{v}_2}^{l,m}\Big] \\
&\quad + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2\psi_{1,\tilde{v}_2}^{l,m}), \\
\psi_{1,\tilde{v}_2}^{l,m}(x_0) &= 1,\ \nabla\psi_{1,\tilde{v}_2}^{l,m}\cdot\gamma = 0 \text{ on } \partial D_{l,m},
\end{aligned} \tag{38}$$

has a unique principal eigenpair $(\rho_{1,\tilde{v}_2}^{l,m}, \psi_{1,\tilde{v}_2}^{l,m}) \in \mathbb{R} \times W^{2,q}(D_{l,m})$, $q \geq d+1$, such that $\psi_{1,\tilde{v}_2}^{l,m} > 0$ and $\rho_{1,\tilde{v}_2}^{l,m} \geq 0$. Now by repeating the limiting arguments as in [[40], Theorem 3.1], one can prove the existence of a solution of (34). Similarly one can prove the existence of a solution of (36).

The representation (35), (37) of the eigenfunctions $\psi_1^{\tilde{v}_2}$, $\psi_2^{\tilde{v}_1}$ respectively, follow by similar arguments as in [[40], Lemma 3.4]. □

Remark 2. Arguing as in [[40], Remark 3.1], it follows that there exist positive constants $\hat{C}_{1,1}, \hat{C}_{2,1} > 0$ and $\hat{\beta}_{1,1}, \hat{\beta}_{2,1} \in (0, 1)$ such that $\psi_1^{\tilde{v}_2} \leq \hat{C}_{1,1}V^{\hat{\beta}_{1,1}}$ and $\psi_2^{\tilde{v}_1} \leq \hat{C}_{2,1}V^{\hat{\beta}_{2,1}}$.

Let $\{K_n\}$ be a sequence of compact sets in D such that $\cup_{n\geq 1}K_n = D \cup \Sigma$.

As in [[12], Lemma 3.3], we approximate the running cost function in the following way: for $i = 1, 2$, let $\{\phi_{i,n}\}$ be a sequence of test functions such that $\phi_{i,n} = 1$ in K_n and $\phi_{i,n} = 0$ in K_{n+1}^c. Since $\|r_i\|_\infty < \delta$, it is possible to choose constants $\delta_{i,2} > 0$ small enough such that $\|r_i\|_\infty + \delta_{i,2} < \delta$. For $(x, u_1, u_2) \in D \times U_1 \times U_2$, $i = 1, 2$, set

$$r_{i,n}(x, u_1, u_2) = \phi_{i,n}(x)r_i(x, u_1, u_2) + (1 - \phi_{i,n}(x))(\|r_i\|_\infty + \delta_{i,2}),\ \ \forall\ \ n \in \mathbb{N}.$$

It easy to see that all the results of Theorem 4.1 hold for r_i replaced by $r_{i,n}$.

Next we want to prove the representation of the eigenvalues.

Theorem 4.2. *Suppose that assumptions (A0)-(A6) hold. Then the eigenvalues obtained in the Theorem 4.1 have the the following representations*

$$\rho_1^{\tilde{v}_2} = \inf_{v_1 \in \mathcal{A}_1}\limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big]. \tag{39}$$

and

$$\rho_2^{\tilde{v}_1} = \inf_{v_2 \in \mathcal{A}_2}\limsup_{T\to\infty}\frac{1}{T}\log E_x^{\tilde{v}_1,v_2}\Big[e^{\int_0^T r_2(X(t),\tilde{v}_1(X(t)),v_2(t))dt}\Big], \tag{40}$$

Proof. Applying Itô-Krylov formula to $e^{\int_0^t (r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}\psi_1^{\tilde{v}_2}(X(t))$ and using (34), we get

$$\begin{aligned}
&E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^{T\wedge\tau_R}(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}\psi_1^{\tilde{v}_2}(X(T\wedge\tau_R))\Big]\\
&= E_x^{v_1,\tilde{v}_2}\Big[\int_0^{T\wedge\tau_R} e^{\int_0^t (r_1(X(s),v_1(s),\tilde{v}_2(X(s)))-\rho_1^{\tilde{v}_2})ds}[L\psi_1^{\tilde{v}_2}(X(t),v_1(t),\tilde{v}_2(X(t)))\\
&\qquad +(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})\psi_1^{\tilde{v}_2}(X(t))dt\Big]+\psi_1^{\tilde{v}_2}(x)\\
&\geq \psi_1^{\tilde{v}_2}(x).
\end{aligned}$$

Arguing as in [[40], Remark 3.1] (also see Remark 2), it follows that there exist positive constants $\hat{C}_{1,1},\hat{C}_{2,1}>0$ and $\hat{\beta}_{1,1},\hat{\beta}_{2,1}\in(0,1)$ such that $\psi_1^{\tilde{v}_2}\leq \hat{C}_{1,1}V^{\hat{\beta}_{1,1}}$ and $\psi_2^{\tilde{v}_1}\leq \hat{C}_{2,1}V^{\hat{\beta}_{2,1}}$. Therefore

$$\begin{aligned}
\psi_1^{\tilde{v}_2}(x) &\leq E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^{T\wedge\tau_R}(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1)dt}\psi_1^{\tilde{v}_2}(X(T\wedge\tau_R))\Big]\\
&\leq \hat{C}_{1,1}E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^{T\wedge\tau_R}(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}V^{\hat{\beta}_{1,1}}(X(T\wedge\tau_R))\Big]\\
&\leq \hat{C}_{1,1}E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^{T\wedge\tau_R}(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}V^{\hat{\beta}_{1,1}}(X(T))I_{\{T\leq\tau_R\}}\Big]\\
&\quad +\hat{C}_{1,1}E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^{T\wedge\tau_R}(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}V^{\hat{\beta}_{1,1}}(X(\tau_R))I_{\{T>\tau_R\}}\Big].
\end{aligned}\tag{41}$$

Following the steps as in [[40], Lemma 3.4] one can show that

$$\lim_{R\to\infty}E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^{T\wedge\tau_R}(r_1(X(t),v_1(t),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}V^{\hat{\beta}_{1,1}}(X(\tau_R))I_{\{T>\tau_R\}}\Big]=0.\tag{42}$$

Again, using (A5) and applying Itô-Krylov formula, we get

$$\begin{aligned}
&E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}V(X(T))\Big]\\
&\qquad\leq (V(x)+\tilde{\alpha}T)E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big].
\end{aligned}\tag{43}$$

Now (41), (42) and (43), implies

$$\psi_1^{\tilde{v}_2}(x) \quad\leq\quad (V(x)+\tilde{\alpha}T)e^{-\rho^{\tilde{v}_2}T}E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big].\tag{44}$$

Taking logarithm in (44), divide by T and letting $T\to\infty$, we obtain

$$\rho_1^{\tilde{v}_2}\leq\limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big].$$

This implies that

$$\rho_1^{\tilde{v}_2}\leq\inf_{v_1\in\mathcal{A}_1}\limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big].\tag{45}$$

Let $\bar{\tilde{v}}_1$ be a minimizing selector of (34), i.e.,

$$\begin{aligned}
\rho_1^{\tilde{v}_2}\psi_1^{\tilde{v}_2} \quad&=\quad \Big[\langle b(x,\bar{\tilde{v}}_1(x),\tilde{v}_2(x)),\nabla\psi_1^{\tilde{v}_2}\rangle+r_1(x,\bar{\tilde{v}}_1(x),\tilde{v}_2(x))\psi_1^{\tilde{v}_2}\Big]\\
&\qquad+\frac{1}{2}\operatorname{trace}(a(x)\nabla^2\psi_1^{\tilde{v}_2}).
\end{aligned}\tag{46}$$

Now, in view of (A2), using limiting argument as in Theorem 4.1 (also see [[40], Theorem 3.1]), one can prove that for each $n\in\mathbb{N}$, there exists $(\rho_{1,n},\psi_{1,n})\in\mathbb{R}\times$

$W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$, $\psi_{1,n} > 0$, satisfying

$$\begin{aligned}\rho_{1,n}\psi_{1,n} =& \Big[\langle b(x, \bar{\tilde{v}}_1(x), \tilde{v}_2(x)), \nabla \psi_{1,n}\rangle + r_{1,n}(x, \bar{\tilde{v}}_1(x), \tilde{v}_2(x))\psi_{1,n}\Big] \\ &+ \frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_{1,n}), \\ \psi_{1,n}(x_0) =& 1,\ \ \nabla\psi_{1,n}\cdot\gamma = 0,\ \text{on } \partial D.\end{aligned} \tag{47}$$

Following the steps as we have used to derive (45), we obtain

$$\rho_{1,n} \leq \limsup_{T\to\infty} \frac{1}{T}\log E_x^{\bar{\tilde{v}}_1, \tilde{v}_2}\Big[e^{\int_0^T r_{1,n}(X(t), \bar{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t)))dt}\Big]. \tag{48}$$

It is clear from our construction that $\|r_{1,n}\|_\infty \leq \|r_1\|_\infty + \delta_{1,2}$. Thus, from (48), we have $\rho_{1,n} \leq \|r_1\|_\infty + \delta_{1,2}$. Let $\hat{K}_1 = \overline{K}_{n+1}$. Therefore, it is easy to see that $\inf_{(u_1,u_2)\in U_1\times U_2} r_{1,n}(x, u_1, u_2) - \rho_{1,n} \geq 0$ for all $x \in \hat{K}_1^c$. Let

$$\tau^c_{\hat{K}_1} = \inf\{t \geq 0 : X(t) \in \hat{K}_1\}.$$

Without loss of generality we assume that $\hat{K}_1 \supset \mathcal{Q}_1$. Thus, using the representation of the eigenfunction as in Theorem 4.1, it follows that

$$\begin{aligned}\psi_{1,n}(x) &= E_x^{\bar{\tilde{v}}_1, \tilde{v}_2}\Big[e^{\int_0^{\tau^c_{\hat{K}_1}} (r_{1,n}(X(t), \bar{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t))) - \rho_{1,n})dt}\psi_{1,n}(X(\tau^c_{\hat{K}_1}))\Big], \\ &\geq \inf_{\hat{K}_1} \psi_{1,n},\ \ \forall\ x \in \hat{K}_1^c.\end{aligned}$$

Now, using Itô-Krylov's formula and Fatou's lemma, we deduce that

$$\begin{aligned}\psi_{1,n}(x) &\geq E_x^{\bar{\tilde{v}}_1, \tilde{v}_2}\Big[e^{\int_0^T (r_{1,n}(X(t), \bar{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t))) - \rho_{1,n})dt}\psi_{1,n}(X(T))\Big], \\ &\geq \inf_{\hat{K}_1} \psi_{1,n} E_x^{\hat{v}_1, \hat{v}_2}\Big[e^{\int_0^T (r_{1,n}(X(t), \bar{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t))) - \rho_{1,n})dt}\Big].\end{aligned}$$

Taking logarithm both sides, dividing by T and then letting $T \to \infty$, we get

$$\begin{aligned}\rho_{1,n} &\geq \limsup_{T\to\infty}\frac{1}{T}\log E_x^{\hat{v}_1,\hat{v}_2}\Big[e^{\int_0^T r_{1,n}(X(t), \hat{v}_1(X(t)), \hat{v}_2(X(t)))dt}\Big], \\ &\geq \limsup_{T\to\infty}\frac{1}{T}\log E_x^{\hat{v}_1,\hat{v}_2}\Big[e^{\int_0^T r_1(X(t), \hat{v}_1(X(t)), \hat{v}_2(X(t)))dt}\Big].\end{aligned} \tag{49}$$

Using Harnack's inequality and Sobolev estimate, from (47) one can easily show that $\psi_{1,n}$ is uniformly bounded in $W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$. Thus, along a suitable subsequence $\{\psi_{1,n}\}$ converges weakly in $W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$ to some $\psi_{1,*} \in W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$, and strongly in $C^{1,\hat{\alpha}}_{loc}(D\cup\Sigma)$, $\hat{\alpha} \in (0,1)$. From (48) and (49), it is clear that $\{\rho_{1,n}\}$ is a bounded sequence. Thus, along a further subsequence it converges to a constant $\rho_{1,*}$. As in Theorem 4.1, letting $n \to \infty$ in (47), we get $(\rho_{1,*}, \psi_{1,*}) \in \mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$, satisfies

$$\begin{aligned}\rho_{1,*}\psi_{1,*} &= \Big[\langle b(x, \bar{\tilde{v}}_1(x), \tilde{v}_2(x)), \nabla\psi_{1,*}\rangle + r_1(x, \bar{\tilde{v}}_1(x), \tilde{v}_2(x))\psi_{1,*}\Big] \\ &\quad +\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_{1,*}), \\ \psi_{1,*}(x_0) &= 1,\ \ \nabla\psi_{1,*}\cdot\gamma = 0,\ \text{on } \partial D.\end{aligned} \tag{50}$$

Repeating the argument as in Theorem 4.1, one can show that $\psi_{1,n} \leq \hat{C}_1 V^{\hat{\beta}}$ outside a compact set $\mathcal{Q}_1$, uniformly in n for some constant $\hat{C}_1 > 0$ and $\hat{\beta} \in (0,1)$. Thus, the limit $\psi_{1,*} \leq \hat{C}_1 V^{\hat{\beta}}$ outside $\mathcal{Q}_1$. Hence we have the following:

$$\psi_{1,*}(x) = E_x^{\bar{\tilde{v}}_1,\tilde{v}_2}\Big[e^{\int_0^{\tau_{1,1}^c}(r_1(X(t),\bar{\tilde{v}}_1(X(t)),\tilde{v}_2(X(t)))-\rho_{1,*})dt}\psi_{1,*}(X(\tau_{1,1}^c))\Big], \ \forall \ x \in \mathcal{Q}_{1,1}^c, \tag{51}$$

for any compact set $\mathcal{Q}_{1,1} \supset \mathcal{Q}_1$ (without loss of generality we are using the same notation as in Theorem 4.1).

We now want to show that $\rho_1^{\tilde{v}_2} \geq \rho_{1,*}$. If not, let $\rho_1^{\tilde{v}_2} < \rho_{1,*}$. Using (46), as in Theorem 4.1, we have for $x \in \mathcal{Q}_{1,1}^c$

$$\begin{aligned}\psi_1^{\tilde{v}_2}(x) &= E_x^{\bar{\tilde{v}}_1,\tilde{v}_2}\Big[e^{\int_0^{\tau_{1,1}^c}(r_1(X(t),\bar{\tilde{v}}_1(X(t)),\tilde{v}_2(X(t)))-\rho_1^{\tilde{v}_2})dt}\psi_1^{\tilde{v}_2}(X(\tau_{1,1}^c))\Big],\\ &\geq E_x^{\bar{\tilde{v}}_1,\tilde{v}_2}\Big[e^{\int_0^{\tau_{1,1}^c}(r_1(X(t),\bar{\tilde{v}}_1(X(t)),\tilde{v}_2(X(t)))-\rho_{1,*})dt}\psi_1^{\tilde{v}_2}(X(\tau_{1,1}^c))\Big].\end{aligned} \tag{52}$$

Now, from (51) and (52), it follows that

$$(\psi_1^{\tilde{v}_2}-\psi_{1,*})(x) \geq E_x^{\bar{\tilde{v}}_1,\tilde{v}_2}\Big[e^{\int_0^{\tau_{1,1}^c}(r_1(X(t),\bar{\tilde{v}}_1(X(t)),\tilde{v}_2(X(t)))-\rho_{1,*})dt}(\psi_1^{\tilde{v}_2}-\psi_{1,*})(X(\tau_{1,1}^c))\Big].$$

Therefore, one clearly sees that $(\psi_1^{\tilde{v}_2}-\psi_{1,*})(x) \geq 0$ for all $x \in D\cup\Sigma$ if it holds in $\mathcal{Q}_{1,1}$. Now multiplying $\psi_{1,*}$ by a suitable positive constant (say, $\hat{k}_1 = \inf_{\mathcal{Q}_{1,1}} \dfrac{\psi_1^{\tilde{v}_2}}{\psi_{1,*}}$), we obtain that $(\psi_1^{\tilde{v}_2}-\tilde{\psi}_{1,*})(x) \geq 0$ in $\mathcal{Q}_{1,1}$ and it attains its minimum value 0 in $\mathcal{Q}_{1,1}$, where $\tilde{\psi}_{1,*} = \hat{k}_1\psi_{1,*}$. It is easy to see that $\tilde{\psi}_{1,*}$ also satisfies (50). Thus, from (46) and (50) (for $\tilde{\psi}_{1,*}$), it follows that

$$\begin{aligned}&\frac{1}{2}\mathrm{trace}(a(x)\nabla^2(\psi_1^{\tilde{v}_2}-\tilde{\psi}_{1,*})) + \Big[\langle b(x,\bar{\tilde{v}}_1(x),\tilde{v}_2(x)),\nabla_x(\psi_1^{\tilde{v}_2}-\tilde{\psi}_{1,*})\rangle -\\ &(r_1(x,\bar{\tilde{v}}_1(x),\tilde{v}_2(x))-\rho_1^{\tilde{v}_2})^-(\psi_1^{\tilde{v}_2}-\tilde{\psi}_{1,*})\Big] \leq -(r_1(x,\bar{\tilde{v}}_1(x),\tilde{v}_2(x))-\rho_1^{\tilde{v}_2})^+(\psi_1^{\tilde{v}_2}-\tilde{\psi}_{1,*})\\ &\qquad -(\rho_{1,*}-\rho_1^{\tilde{v}_2})\tilde{\psi}_{1,*} \leq 0\end{aligned}$$

Thus, by an application of strong maximum principle as in [[35], Corollary 1.21] (see also [[36], Corollary 2.4], [28]), we have $\psi_1^{\tilde{v}_2} = \tilde{\psi}_{1,*}$. Since $\psi_1^{\tilde{v}_2}(x_0) = \psi_{1,*}(x_0) = 1$, we get $\psi_1^{\tilde{v}_2} = \psi_{1,*}$. Now from (46) and (50), it follows that

$$\rho_1^{\tilde{v}_2}\psi_{1,*} = \rho_{1,*}\psi_*.$$

Since $\psi_{1,*} > 0$, we have $\rho_1^{\tilde{v}_2} = \rho_{1,*}$. This contradicts the fact that $\rho_1^{\tilde{v}_2} < \rho_{1,*}$. Therefore we obtain $\rho_1^{\tilde{v}_2} \geq \rho_{1,*}$. Now combining (45) and (49), we obtain

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\log E_x^{\bar{\tilde{v}}_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),\bar{\tilde{v}}_1(X(t)),\tilde{v}_2(X(t)))dt}\Big] \leq \rho_{1,*} \leq \rho_1^{\tilde{v}_2}\\ &\qquad\leq \inf_{v_1\in\mathcal{A}_1}\limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big].\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}\rho_1^{\tilde{v}_2} &= \limsup_{T\to\infty}\frac{1}{T}\log E_x^{\bar{\tilde{v}}_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),\bar{\tilde{v}}_1(X(t)),\tilde{v}_2(X(t)))dt}\Big]\\ &= \inf_{v_1\in\mathcal{A}_1}\limsup_{T\to\infty}\frac{1}{T}\log E_x^{v_1,\tilde{v}_2}\Big[e^{\int_0^T r_1(X(t),v_1(t),\tilde{v}_2(X(t)))dt}\Big]\\ &:= \rho_1^{\tilde{v}_2}(=\inf_{v_1\in\mathcal{A}_1}\rho_1^{v_1,\tilde{v}_2}).\end{aligned} \tag{53}$$

By similar arguments one can prove the representation of $\rho_2^{\tilde{v}_1}$. This completes the proof. □

Next lemma proves the uniqueness of the eigenpairs obtained in the Theorem 4.1 in certain class of functions.

Lemma 4.3. *Suppose that assumptions (A0)-(A6) hold. Then the eigenpairs* $(\rho_1^{\tilde{v}_2}, \psi_1^{\tilde{v}_2})$, $(\rho_2^{\tilde{v}_1}, \psi_2^{\tilde{v}_1})$ *obtained in Theorem 4.1 are unique in the spaces* $\mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma) \cap C^1(D \cup \Sigma) \cap O(V^{\hat{\beta}_1})$, $q \geq d+1$ *and* $\mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma) \cap C^1(D \cup \Sigma) \cap O(V^{\hat{\beta}_2})$, $q \geq d+1$, *respectively.*

Proof. Suppose $(\bar{\rho}_1, \bar{\psi}_1) \in \mathbb{R} \times W^{2,q}_{loc}(D \cup \Sigma) \cap O(V^{\hat{\beta}_1})$, $q \geq d+1$, is another solution of (34), i.e.,

$$\bar{\rho}_1 \bar{\psi}_1 = \inf_{v_1 \in V_1} \Big[\langle b(x, v_1, \tilde{v}_2(x)), \nabla \bar{\psi}_1 \rangle + r_1(x, v_1, \tilde{v}_2(x)) \bar{\psi}_1 \Big] + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \bar{\psi}_1),$$
$$\bar{\psi}_1(x_0) = 1, \ \nabla \bar{\psi}_1(x) \cdot \gamma(x) = 0 \ \text{ on } \partial D. \tag{54}$$

Arguing as in Theorem 4.2 using the representation of the eigenvalue one can clearly see that $\bar{\rho}_1 = \rho_1^{\tilde{v}_2}$. Therefore, $\rho_1^{\tilde{v}_2}$ is the unique eigenvalue whose corresponding eigenfunction is in the space $O(V^{\hat{\beta}_1})$. Next we prove that $\psi_1^{\tilde{v}_2}$ is the unique eigenfunction provided $\psi_1^{\tilde{v}_2}(x_0) = 1$. Let $\hat{\tilde{v}}_1 \in \mathcal{S}_1$ be a measurable selector in (54), i.e.,

$$\begin{aligned} \bar{\rho}_1 \bar{\psi}_1 &= \Big[\langle b(x, \hat{\tilde{v}}_1(x), \tilde{v}_2(x)), \nabla \bar{\psi}_1 \rangle + r_1(x, \hat{\tilde{v}}_1(x), \tilde{v}_2(x)) \bar{\psi}_1 \Big] \\ &\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \bar{\psi}_1), \\ \bar{\psi}_1(x_0) &= 1, \ \nabla \bar{\psi}_1(x) \cdot \gamma(x) = 0 \ \text{ on } \partial D. \end{aligned} \tag{55}$$

Also, from (34), we have

$$\begin{aligned} \rho_1^{\tilde{v}_2} \psi_1^{\tilde{v}_2} &\leq \Big[\langle b(x, \hat{\tilde{v}}_1(x), \tilde{v}_2(x)), \nabla \psi_1^{\tilde{v}_2} \rangle + r_1(x, \hat{\tilde{v}}_1(x), \tilde{v}_2(x)) \psi_1^{\tilde{v}_2} \Big] \\ &\quad + \frac{1}{2} \operatorname{trace}(a(x) \nabla^2 \psi_1^{\tilde{v}_2}), \\ \psi_1^{\tilde{v}_2}(x_0) &= 1, \ \nabla \psi_1^{\tilde{v}_2}(x) \cdot \gamma(x) = 0 \ \text{ on } \partial D. \end{aligned} \tag{56}$$

Applying Itô-Krylov formula and using Fatou's lemma, from (55), we obtain

$$\bar{\psi}_1(x) \geq E_x^{\hat{\tilde{v}}_1, \tilde{v}_2} \left[e^{\int_0^{\tau_1^c} (r_1(X(t), \hat{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t))) - \bar{\rho}_1) dt} \bar{\psi}_1(X(\tau_1^c)) \right]. \tag{57}$$

By the similar argument as in Theorem 4.1 (see [[40] Lemma 3.4]), using (56), it follows that

$$\psi_1^{\tilde{v}_2}(x) \leq E_x^{\hat{\tilde{v}}_1, \tilde{v}_2} \left[e^{\int_0^{\tau_1^c} (r_1(X(t), \hat{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t))) - \rho_1^{\tilde{v}_2}) dt} \psi_1^{\tilde{v}_2}(X(\tau_1^c)) \right]. \tag{58}$$

Now from (57) and (58), we have (since $\rho_1^{\tilde{v}_2} = \bar{\rho}_1$)

$$\begin{aligned} &(\bar{\psi}_1(x) - \psi_1^{\tilde{v}_2}(x)) \\ &\quad \geq E_x^{\hat{\tilde{v}}_1, \tilde{v}_2} \left[e^{\int_0^{\tau_1^c} (r_1(X(t), \hat{\tilde{v}}_1(X(t)), \tilde{v}_2(X(t))) - \bar{\rho}_1) dt} (\bar{\psi}_1(X(\tau_1^c)) - \psi_1^{\tilde{v}_2}(X(\tau_1^c))) \right]. \end{aligned} \tag{59}$$

The above equation implies that $\bar{\psi}_1(x) \geq \psi_1^{\tilde{v}_2}(x)$ for all $x \in D$ if $\bar{\psi}_1(x) \geq \psi_1^{\tilde{v}_2}(x)$ in $\mathcal{Q}_1$. Multiplying $\psi_1^{\tilde{v}_2}$ by suitable positive constant we can ensure that $(\bar{\psi}_1 - \psi_1^{\tilde{v}_2})(x) \geq 0$ in $\mathcal{Q}_1$ and attains its minimum value 0 in $\mathcal{Q}_1$. Now (55) and (56) imply

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2(\bar{\psi}_1 - \psi_1^{\tilde{v}_2})) + \langle b(x, \hat{\tilde{v}}_1(x), \tilde{v}_2(x)), \nabla(\bar{\psi}_1 - \psi_1^{\tilde{v}_2})\rangle \\
&- (r_1(x, \hat{\tilde{v}}_1(x), \tilde{v}_2(x)) - \bar{\rho}_1)^-(\bar{\psi}_1 - \psi_1^{\tilde{v}_2}) \leq -(r_1(x, v_1(x)) - \bar{\rho}_1)^+(\bar{\psi}_1 - \psi_1^{\tilde{v}_2}) \leq 0, \\
&\nabla(\bar{\psi}_1 - \psi_1^{\tilde{v}_2})(x)\cdot\gamma(x) = 0 \ \text{ on } \partial D. \qquad (60)
\end{aligned}$$

Therefore by strong maximum principle as in [[35], Corollary 1.21] (see also [[36], Corollary 2.4]), we have $\bar{\psi}_1 = \psi_1^{\tilde{v}_2}$ (since $\bar{\psi}_1(x_0) = \psi_1^{\tilde{v}_2}(x_0) = 1$). This proves the uniqueness of the solution of (34). The uniqueness of $(\rho_2^{\tilde{v}_1}, \psi_2^{\tilde{v}_1})$ follows by similar argument. □

Next lemma proves continuity of certain functions.

Lemma 4.4. *Suppose that assumptions (A0) - (A6) hold. Then the maps* $\tilde{v}_1 \mapsto \psi_2^{\tilde{v}_1}$ *from* $\mathcal{S}_1 \to W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$, $\tilde{v}_1 \mapsto \rho_2^{\tilde{v}_1}$ *from* $\mathcal{S}_1 \to \mathbb{R}$, $\tilde{v}_2 \mapsto \psi_1^{\tilde{v}_2}$ *from* $\mathcal{S}_2 \to W^{2,q}_{loc}(D \cup \Sigma)$, $q \geq d+1$ *and* $\tilde{v}_2 \mapsto \rho_1^{\tilde{v}_2}$ *from* $\mathcal{S}_2 \to \mathbb{R}$ *are continuous.*

Proof. Let $\tilde{v}_1^n \to \tilde{v}_1$ in $\mathcal{S}_1$ as $n \to \infty$. Let $\bar{v}_2^n$ be a minimizing selector in (36) corresponding to $\tilde{v}_1^n$. Rewriting the equation (36), we have

$$\begin{aligned}
\langle b(x, \tilde{v}_1^n(x), \bar{v}_2^n(x)), \nabla\psi_2^{\tilde{v}_1^n}\rangle + \frac{1}{2}\,\mathrm{trace}(a(x)\nabla^2\psi_2^{\tilde{v}_1^n}) = \\
(\rho_2^{\tilde{v}_1^n} - r_2(x, \tilde{v}_1^n(x), \bar{v}_2^n(x)))\psi_2^{\tilde{v}_1^n}. \qquad (61)
\end{aligned}$$

Since $\psi_2^{\tilde{v}_1^n}(x_0) = 1$, by Harnack's inequality [[28], Corollary 9.25], for any compact subset $\tilde{K}_1 \subset D$, we get

$$\sup_{x \in \tilde{K}_1} \psi_2^{\tilde{v}_1^n}(x) \leq \tilde{C}_{\tilde{K}_1}$$

where $\tilde{C}_{\tilde{K}_1}$ is a constant independent of n. Thus, the r.h.s. of (61) is uniformly bounded in n (since $\rho_2^{\tilde{v}_1^n} \leq \|r_2\|_\infty$). Therefore from [[28], Theorem 9.11], for any bounded domain $Q \subset \tilde{K}_1$, we obtain

$$\sup_n \|\psi_2^{\tilde{v}_1^n}\|_{2,q;Q} < \infty, \ \ q \geq d+1. \qquad (62)$$

Now by Sobolev embedding theorem, Banach-Alaoglu theorem and by standard diagonalization procedure, there exists $\psi_2 \in W^{2,q}_{loc}(D \cup \Sigma) \cap C^1(D \cup \Sigma) \cap O(V)$, $q \geq d+1$ such that along a suitable subsequence

$$\left.\begin{aligned}
\psi_2^{\tilde{v}_1^n} &\to \psi_2 \text{ in } W^{2,q}_{loc}(D \cup \Sigma) \ \text{ weakly}, \\
\psi_2^{\tilde{v}_1^n} &\to \psi_2 \text{ in } C^{1,\hat{\alpha}}_{loc}(D \cup \Sigma) \ \text{ strongly},
\end{aligned}\right\} \qquad (63)$$

for some constant $\hat{\alpha} \in (0,1)$. We know that $0 < \rho_2^{\tilde{v}_1^n} \leq \|r_2\|_\infty$; thus, along a further subsequence, $\rho_2^{\tilde{v}_1^n} \to \tilde{\rho}_2$. Therefore, for $\varphi \in C_c^\infty(D \cup \Sigma)$, from (36), we have

$$\begin{aligned}
&\int_{D\cup\Sigma} \inf_{v_2 \in V_2}\Big[\langle b(x, \tilde{v}_1^n(x), v_2), \nabla\psi_2^{\tilde{v}_1^n}\rangle + (r_2(x, \tilde{v}_1^n(x), v_2) - \rho_2^{\tilde{v}_1^n})\psi_2^{\tilde{v}_1^n}\Big]\varphi dx \\
+ \ &\frac{1}{2}\int_{D\cup\Sigma} \mathrm{trace}(a(x)\nabla^2\psi_2^{\tilde{v}_1^n})\varphi dx = 0.
\end{aligned}$$

Now since $\psi_2 \in W^{2,q}_{loc}(D\cup\Sigma)$, $q \geq d+1$, letting $n \to \infty$ and using (63), it follows that $(\tilde{\rho}_2, \psi_2)$ is a solution to (36). Since $\psi_2^{\tilde{v}_1^n}(x_0) = 1$ for all $n \geq 1$, we have $\psi_2(x_0) = 1$. Also, since $\nabla\psi_2^{\tilde{v}_1^n}\cdot\gamma = 0$, on ∂D, from (63), we obtain $\nabla\psi_2\cdot\gamma = 0$, on ∂D. Following the arguments as in [[40], Remark 3.1], it follows that there exist positive constants $\hat{C}_{2,1} > 0$ and $\hat{\beta}_{2,1} \in (0,1)$ such that $\psi_2^{\tilde{v}_1^n} \leq \hat{C}_{2,1}V^{\hat{\beta}_{2,1}}$, where the constant $\hat{C}_{2,1}$ is independent of n. This indeed implies that $\psi_2 \leq \hat{C}_{2,1}V^{\hat{\beta}_{2,1}}$. Thus, in view of Lemma 4.1 (also see [[40], Theorem 3.3]), one can see that $(\tilde{\rho}_2, \psi_2)$ is the unique solution of (36) in $W^{2,p}_{loc}(D\cup\Sigma)\cap O(V^{\hat{\beta}_{2,1}})$. Therefore, it follows that $\psi_2 = \psi_2^{\tilde{v}_1}$ and $\bar{\rho}_2 = \rho_2^{\tilde{v}_1}$. Thus, we have proved the continuity of the maps $\tilde{v}_1 \mapsto \psi_2^{\tilde{v}_1}$ and $\tilde{v}_1 \mapsto \rho_2^{\tilde{v}_1}$. By similar arguments the continuity of the other maps follow. This completes the proof. □

Let $(\tilde{v}_1, \tilde{v}_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. Define

$$\tilde{N}(\tilde{v}_1, \tilde{v}_2) \;=\; \tilde{N}_1(\tilde{v}_2) \times \tilde{N}_2(\tilde{v}_1) \tag{64}$$

where

$$\tilde{N}_1(\tilde{v}_2) \;=\; \Big\{\tilde{v}_1^* \in \mathcal{S}_1 \mid \tilde{F}_1(x, \tilde{v}_1^*(x), \tilde{v}_2(x)) = \inf_{v_1\in V_1} \tilde{F}_1(x, v_1, \tilde{v}_2(x)) \text{ a.e. } x\Big\},$$

$$\tilde{F}_1(x, v_1, \tilde{v}_2(x)) \;=\; \langle b(x, v_1, \tilde{v}_2(x)), \nabla\psi_1^{\tilde{v}_2}\rangle + r_1(x, v_1, \tilde{v}_2(x))\psi_1^{\tilde{v}_2},$$

$x \in D,\; v_1 \in V_1, \tilde{v}_2 \in \mathcal{S}_2$,

$$\tilde{N}_2(\tilde{v}_1) \;=\; \Big\{\tilde{v}_2^* \in \mathcal{S}_2 \mid \tilde{F}_2(x, \tilde{v}_1(x), \tilde{v}_2^*(x)) = \inf_{v_2\in V_2} \tilde{F}_2(x, \tilde{v}_1(x), v_2) \text{ a.e. } x\Big\},$$

$$\tilde{F}_2(x, \tilde{v}_1(x), v_2) \;=\; \langle b(x, \tilde{v}_1(x), v_2), \nabla\psi_2^{\tilde{v}_1}\rangle + r_2(x, \tilde{v}_1(x), v_2)\psi_2^{\tilde{v}_1},$$

$x \in D,\; v_2 \in V_2, \tilde{v}_1 \in \mathcal{S}_1$.

A measurable selection theorem [9], ensures that $\tilde{N}_1(\tilde{v}_2)$ is nonempty. From the definition it is easy to see that $\tilde{N}_1(\tilde{v}_2)$ is convex. Also, it is clear that under the topology of $\mathcal{S}_1$, $\tilde{N}_1(\tilde{v}_2)$ is closed in $\mathcal{S}_1$. Therefore $\tilde{N}_1(\tilde{v}_2)$ is compact. Similarly $\tilde{N}_2(\tilde{v}_1)$ is a nonempty, compact, convex subset of $\mathcal{S}_2$. Thus, $\tilde{N}(\tilde{v}_1, \tilde{v}_2)$ is a nonempty, convex and compact subset of $\mathcal{S}_1 \times \mathcal{S}_2$. Now we want to prove that the map $(\tilde{v}_1, \tilde{v}_2) \mapsto \tilde{N}(\tilde{v}_1, \tilde{v}_2)$ from $\mathcal{S}_1 \times \mathcal{S}_2 \to 2^{\mathcal{S}_1} \times 2^{\mathcal{S}_2}$ is upper semi-continuous (u.s.c.).

Lemma 4.5. *Suppose that assumptions (A0)-(A6) hold. Then the set-valued map* $(\tilde{v}_1, \tilde{v}_2) \mapsto \tilde{N}(\tilde{v}_1, \tilde{v}_2)$ *from* $\mathcal{S}_1 \times \mathcal{S}_2 \to 2^{\mathcal{S}_1} \times 2^{\mathcal{S}_2}$ *is u.s.c.*

Proof. Let $\{(\tilde{v}_1^n, \tilde{v}_2^n)\}_n$ be a sequence in $\mathcal{S}_1 \times \mathcal{S}_2$ such that $(\tilde{v}_1^n, \tilde{v}_2^n) \to (\tilde{v}_1, \tilde{v}_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. Choose $\tilde{\tilde{v}}_1^n \in \tilde{N}_1(\tilde{v}_2^n), n \geq 1$. Since $\mathcal{S}_1$ is compact, there exists a subsequence (which we denote by the same notation without any loss of generality) $\{\tilde{\tilde{v}}_1^n\}$ such that $\tilde{\tilde{v}}_1^n \to \tilde{\tilde{v}}_1$ for some $\tilde{\tilde{v}}_1 \in \mathcal{S}_1$. Then $(\tilde{\tilde{v}}_1^n, \tilde{v}_2^n) \to (\tilde{\tilde{v}}_1, \tilde{v}_2)$ in $\mathcal{S}_1 \times \mathcal{S}_2$. Now by (A4), Lemma 4.4 and the topology of $\mathcal{S}_i, i = 1, 2$, we get that

$$\langle b(x, \tilde{\tilde{v}}_1^n(x), \tilde{v}_2^n(x)), \nabla\psi_1^{\tilde{v}_2^n}\rangle + r_1(x, \tilde{\tilde{v}}_1^n(x), \tilde{v}_2^n(x))\psi_1^{\tilde{v}_2^n}$$

converges weakly in $L^2_{loc}(D \cup \Sigma)$ to

$$\langle b(x, \tilde{\tilde{v}}_1(x), \tilde{v}_2(x)), \nabla\psi_1^{\tilde{v}_2}\rangle + r_1(x, \tilde{\tilde{v}}_1(x), \tilde{v}_2(x))\psi_1^{\tilde{v}_2}.$$

As a consequence of the Banach-Saks theorem, a sequence of convex combination of the former converges strongly in $L^2_{loc}(D \cup \Sigma)$ to the latter. Therefore along a suitable subsequence of the convergent sequence of convex combinations, it follows that (without any loss of generality denoting by same notations)

$$\lim_{n\to\infty} \tilde{F}_1(x, \tilde{\tilde{v}}_1^n(x), \tilde{v}_2^n(x)) = \tilde{F}_1(x, \tilde{\tilde{v}}_1(x), \tilde{v}_2(x)), \text{ a.e. in } x. \tag{65}$$

Now for fixed $\tilde{\tilde{\tilde{v}}}_1 \in \mathcal{S}_1$, using arguments similar to those above, we conclude that

$$\lim_{n\to\infty} \tilde{F}_1(x, \tilde{\tilde{\tilde{v}}}_1(x), \tilde{v}_2^n(x)) = \tilde{F}_1(x, \tilde{\tilde{\tilde{v}}}_1(x), \tilde{v}_2(x)), \text{ a.e. in } x. \tag{66}$$

Since $\tilde{\tilde{v}}_1^n \in \tilde{N}_1(\tilde{v}_2^n)$, we have

$$\tilde{F}_1(x, \tilde{\tilde{\tilde{v}}}_1(x), \tilde{v}_2^n(x)) \geq \tilde{F}_1(x, \tilde{\tilde{v}}_1^n(x), \tilde{v}_2^n(x)), \ n \geq 1.$$

Thus, from (65) and (66), it follows that

$$\tilde{F}_1(x, \tilde{\tilde{\tilde{v}}}_1(x), \tilde{v}_2(x)) \geq \tilde{F}_1(x, \tilde{\tilde{v}}_1(x), \tilde{v}_2(x)), \ \tilde{\tilde{\tilde{v}}}_1 \in \mathcal{S}_1.$$

This implies $\tilde{\tilde{v}}_1 \in \tilde{N}_1(\tilde{v}_2)$. In a similar fashion, one can show that for $\tilde{\tilde{v}}_2^n \in \tilde{N}_2(\tilde{v}_1^n)$ and any limit point $\tilde{\tilde{v}}_2$ of $\{\tilde{\tilde{v}}_2^n\}$, we have $\tilde{\tilde{v}}_2 \in \tilde{N}_2(\tilde{v}_1)$. This proves the u.s.c. of the maps. □

Next theorem proves the existence of Nash equilibria in the space of stationary Markov strategies.

Theorem 4.6. *Suppose that assumptions (A0)-(A6) hold. Then there exists a Nash equilibrium in $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. Since the set-valued map $(\tilde{v}_1, \tilde{v}_2) \mapsto N(\tilde{v}_1, \tilde{v}_2)$ from $\mathcal{S}_1 \times \mathcal{S}_2 \to 2^{\mathcal{S}_1} \times 2^{\mathcal{S}_2}$ is u.s.c., by Fan's fixed point theorem [17], it follows that there exists $(\tilde{v}_1^*, \tilde{v}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, such that

$$(\tilde{v}_1^*, \tilde{v}_2^*) \in N(\tilde{v}_1^*, \tilde{v}_2^*).$$

Thus $(\rho_1^{\tilde{v}_2^*}, \psi_1^{\tilde{v}_2^*}), (\rho_2^{\tilde{v}_1^*}, \psi_2^{\tilde{v}_1^*})$ satisfy

$$\begin{aligned}
\rho_1^{\tilde{v}_2^*}\psi_1^{\tilde{v}_2^*} &= \inf_{v_1\in V_1}\Big[\langle b(x, v_1, \tilde{v}_2^*(x)), \nabla\psi_1^{\tilde{v}_2^*}\rangle + r_1(x, v_1, \tilde{v}_2^*(x))\psi_1^{\tilde{v}_2^*}\Big] \\
&\quad + \frac{1}{2}\,\text{trace}(a(x)\nabla^2\psi_1^{\tilde{v}_2^*}), \\
&= \langle b(x, \tilde{v}_1^*(x), \tilde{v}_2^*(x)), \nabla\psi_1^{\tilde{v}_2^*}\rangle + r_1(x, \tilde{v}_1^*(x), \hat{v}_2^*(x))\psi_1^{\tilde{v}_2^*}\Big] \\
&\quad + \frac{1}{2}\,\text{trace}(a(x)\nabla^2\psi_1^{\tilde{v}_2^*}), \\
\psi_1^{\tilde{v}_2^*}(x_0) &= 1, \ \nabla\psi_1^{\tilde{v}_2^*}\cdot\gamma = 0 \ \text{ on } \ \partial D \\
\rho_2^{\tilde{v}_1^*}\psi_2^{\tilde{v}_1^*} &= \inf_{v_2\in V_2}\Big[\langle b(x, \tilde{v}_1^*(x), v_2), \nabla\psi_2^{\tilde{v}_1^*}\rangle + r_2(x, \tilde{v}_1^*(x), v_2)\psi_2^{\tilde{v}_1^*}\Big] \\
&\quad + \frac{1}{2}\,\text{trace}(a(x)\nabla^2\psi_2^{\tilde{v}_1^*}), \\
&= \langle b(x, \tilde{v}_1^*(x), \tilde{v}_2^*(x)), \nabla\psi_2^{\tilde{v}_1^*}\rangle + r_2(x, \tilde{v}_1^*(x), \tilde{v}_2^*(x)\psi_2^{\tilde{v}_1^*}\Big] \\
&\quad + \frac{1}{2}\,\text{trace}(a(x)\nabla^2\psi_2^{\tilde{v}_1^*}), \\
\psi_2^{\tilde{v}_1^*}(x_0) &= 1, \ \nabla\psi_2^{\tilde{v}_1^*}\cdot\gamma = 0 \ \text{ on } \ \partial D.
\end{aligned}$$

As in Theorem 4.2, using the representation of the eigenvalues, it follows that

$$\rho_1^{\tilde{v}_2^*} = \inf_{v_1\in\mathcal{M}_1}\rho_1^{v_1,\tilde{v}_2^*}\,(=\inf_{v_1\in\mathcal{A}_1}\rho_1^{v_1,\tilde{v}_2^*}) = \rho_1^{\tilde{v}_1^*,\tilde{v}_2^*},$$

$$\rho_2^{\tilde{v}_1^*} = \inf_{v_2\in\mathcal{M}_2} \rho_2^{\tilde{v}_1^*,v_2} (= \inf_{v_2\in\mathcal{A}_2} \rho_2^{\tilde{v}_1^*,v_2}) = \rho_2^{\tilde{v}_1^*,\tilde{v}_2^*}.$$

Therefore

$$\begin{aligned} \rho_1^{v_1,\tilde{v}_2^*} &\geq \rho_1^{\tilde{v}_1^*,\tilde{v}_2^*}, \ \forall\ v_1 \in \mathcal{A}_1, \\ \rho_2^{\tilde{v}_1^*,v_2} &\geq \rho_2^{\tilde{v}_1^*,\tilde{v}_2^*}, \ \forall\ v_2 \in \mathcal{A}_2. \end{aligned}$$

This completes the proof. □

We now want to prove that any Nash equilibrium in the space of stationary Markov strategies is a minimizing selector of the corresponding HJB equation

Theorem 4.7. *Assume (A0)-(A6). If $(\tilde{\tilde{v}}_1^*, \tilde{\tilde{v}}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ is a Nash equilibrium, i.e.,*

$$\begin{aligned} \rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} &\leq \rho_1^{\bar{v}_1,\tilde{\tilde{v}}_2^*}, \ \forall\ \bar{v}_1 \in \mathcal{A}_1, \\ \rho_2^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} &\leq \rho_2^{\tilde{\tilde{v}}_1^*,\bar{v}_2}, \ \forall\ \bar{v}_2 \in \mathcal{A}_2. \end{aligned}$$

Then $\tilde{\tilde{v}}_1^* \in \mathcal{S}_1$ is a minimizing selector of (34) (for fixed strategy $\tilde{\tilde{v}}_2^* \in \mathcal{S}_2$ of Player 2) and $\tilde{\tilde{v}}_2^* \in \mathcal{S}_2$ is a minimizing selector of (36) (for fixed strategy $\tilde{\tilde{v}}_1^* \in \mathcal{S}_1$ of Player 1).

Proof. In view of (A4), by the limiting arguments as in [[40], Theorem 3.1], one can show that corresponding to the pair of strategies $(\tilde{\tilde{v}}_1^*, \tilde{\tilde{v}}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, there exists a principal eigenpair $(\rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}, \psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}) \in \mathbb{R}\times W^{2,q}_{loc}(D\cup\Sigma)\cap C^1(D\cup\Sigma)\cap O(V)$, $q \geq d+1$, $\psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} > 0$ and $\rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} \geq 0$, satisfying the following

$$\begin{aligned} \rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}\psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} &= \langle b(x,\tilde{\tilde{v}}_1^*(x),\tilde{\tilde{v}}_2^*(x)), \nabla\psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}\rangle + r_1(x,\tilde{\tilde{v}}_1^*(x),\tilde{\tilde{v}}_2^*(x))\psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} \\ &\quad +\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}), \\ \psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}(x_0) &= 1,\ \nabla\psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}\cdot\gamma = 0,\ \text{on } \partial D. \end{aligned} \tag{67}$$

From Theorem 4.1, we obtain that for given $\tilde{\tilde{v}}_2^* \in \mathcal{S}_2$, there exists a principal eigenpair $(\rho_1^{\tilde{\tilde{v}}_2^*}, \psi_1^{\tilde{\tilde{v}}_2^*}) \in \mathbb{R} \times W^{2,q}_{loc}(D\cup\Sigma)\cap C^1(D\cup\Sigma)\cap O(V)$, $\psi_1^{\tilde{\tilde{v}}_2^*} > 0$, $q \leq d+1$, satisfying

$$\begin{aligned} \rho_1^{\tilde{\tilde{v}}_2^*}\psi_1^{\tilde{\tilde{v}}_2^*} &= \inf_{v_1\in V_1}\Big[\langle b(x,v_1,\tilde{\tilde{v}}_2^*(x)), \nabla\psi_1^{\tilde{\tilde{v}}_2^*}\rangle + r_1(x,v_1,\tilde{\tilde{v}}_2^*(x))\psi_1^{\tilde{\tilde{v}}_2^*}\Big] \\ &\quad +\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_1^{\tilde{\tilde{v}}_2^*}),\ \text{ a.e. } x\in D, \\ \psi_1^{\tilde{\tilde{v}}_2^*}(x_0) &= 1,\ \nabla\psi_1^{\tilde{\tilde{v}}_2^*}\cdot\gamma = 0 \text{ on } \partial D. \end{aligned} \tag{68}$$

Also, from Theorem 4.2 it is clear that for any minimizing selector $\tilde{\tilde{v}}_1^* \in \mathcal{S}_1$ of (68), $\rho_1^{\tilde{\tilde{v}}_2^*} = \rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}$. From (68), it follows that

$$\begin{aligned} \rho_1^{\tilde{\tilde{v}}_2^*}\psi_1^{\tilde{\tilde{v}}_2^*} &\leq \Big[\langle b(x,\tilde{\tilde{v}}_1^*(x),\tilde{\tilde{v}}_2^*(x)), \nabla\psi_1^{\tilde{\tilde{v}}_2^*}\rangle + r_1(x,\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*(x))\psi_1^{\tilde{\tilde{v}}_2^*}\Big] \\ &\quad +\frac{1}{2}\ \mathrm{trace}(a(x)\nabla^2\psi_1^{\tilde{\tilde{v}}_2^*}),\ \text{ a.e. } x\in D. \end{aligned} \tag{69}$$

Now, repeating the arguments as in [[40], Theorem 3.2], it follows that $\rho_1^{\tilde{\tilde{v}}_2^*} \leq \rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}$. But, we know that $\rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*} \leq \rho_1^{\bar{v}_1,\tilde{\tilde{v}}_2^*}$, $\forall\ \bar{v}_1 \in \mathcal{A}_1$. Thus, $\rho_1^{\tilde{\tilde{v}}_2^*} = \rho_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}$. However, by an application of strong maximum principle as in [[40], Theorem 3.3] one can prove that $\psi_1^{\tilde{\tilde{v}}_2^*} = \psi_1^{\tilde{\tilde{v}}_1^*,\tilde{\tilde{v}}_2^*}$. Therefore, from (67), (68) and (69), it follows that $\tilde{\tilde{v}}_1^*$ is a minimizing selector of (34). By repeating the arguments similar to those above, one can prove that $\tilde{\tilde{v}}_2^*$ is a minimizing selector of (36). This completes the proof. □

5. Conclusions. We have studied a nonzero-sum stochastic differential game problems where the state is given by controlled reflecting diffusion processes in the nonnegative orthant. Here we consider two cost evaluation criteria: discounted and ergodic. Under fairly general assumptions we have established the existence of α-discounted Nash equilibia in the space of eventually stationary Markov strategies. For ergodic cost criterion, using principal eigenvalue approach, under additional Lyapunov stability assumption and smallness condition on running cost function, we have established the existence of ergodic Nash equilibria in the space of stationary Markov strategies. In our analysis for ergodic cost criterion, we have crucially used (A6). It will be interesting to study the same problem without (A6).

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