



## Topological Insulators in Amorphous Systems

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Much of the current understanding of topological insulators, which informs the experimental search for topological materials and systems, is based on crystalline band theory, where local electronic degrees of freedom at different crystal sites hybridize with each other in ways that produce nontrivial topology. Here we provide a novel theoretical demonstration of realizing topological phases in *amorphous systems*, as exemplified by a set of sites randomly located in space. We show this by constructing hopping models on such random lattices whose gapped ground states are shown to possess nontrivial topological nature (characterized by Bott indices) that manifests as quantized conductances in systems with a boundary. Our study adds a new dimension, beyond crystalline solids, to the search for topological systems by pointing to the promising possibilities in amorphous solids and other engineered random systems.

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*Introduction.*—The band insulating state of many fermions has received renewed attention in recent times [1–3]. Clues that such gapped phases may support additional nontrivial physics—attributed to topology—was provided by the discovery of the integer quantum Hall effect [4] and the theoretical work that followed [5–7]. These ideas saw a resurgence with the discovery of the two dimensional spin Hall insulator [8–13], soon followed [14–17] by the three dimensional topological insulator (see Refs. [1–3]). The topological character of such insulators is robust to a small amount of impurities and disorder that preserve certain symmetries, and their surfaces typically host gapless modes which open possible new directions for technological applications [1–3].

Soon, a complete classification of gapped phases of noninteracting fermions appeared [18–21] (see recent reviews [22,23]). This classification hinges on the ten symmetry classes of Altland and Zirnbauer [24,25]. In any given spatial dimension  $d$  only five symmetry classes of the ten host topologically nontrivial phases (see, e.g., Kitaev [21]). Two gapped systems are considered to be topologically equivalent if the ground state of one can be reached starting from the ground state of the other by a symmetry preserving adiabatic deformation of the Hamiltonian which does not close the gap during the deformation process. The ground state comprised of filled bands can be viewed as a map from the Brillouin zone of the crystal to a “symmetric space” whose character is determined by the symmetries of the system. Whether such a map allows for nontrivial “winding” decides if a symmetry class supports topological phases in that dimension. Well-known lattice models of Haldane [7], Kitaev [26], Kane-Mele [9], Bernevig-Hughes-Zhang (BHZ) [12] etc., fall into this paradigm, which has since been expanded to include crystalline symmetries [27]. These theoretical advances have strongly influenced, perhaps even constrained as we

shall see in this Letter, the search for topological systems to be focused on crystalline materials, both solid state [28–31] and synthetic [32]. For example, significant effort has been invested in the automated search of crystalline topological materials [33,34].

While robustness to disorder is the defining property of any topological state, usually it is stated with the caveat “disorder should not be large enough to close the gap.” Most studies that have attempted to address the question of disorder in topological systems have, therefore, an implicit crystalline lattice and a “small” disorder. The effect of on site Anderson disorder on a  $d = 3$  topological insulator was studied in Ref. [35], where it was shown that with increasing disorder a topological state transits to a metal. On the other hand, studies on surface disorder have shown that the protected Dirac cone can emerge beneath the surface [36]. The effect of disorder in Kitaev chains has also been studied [37]. Other pertinent studies include, how an on site disorder can induce [38] a topological phase in a trivial system which has the necessary ingredients to produce topological phases, and an interesting concept of “statistical topological insulators” which requires another statistical symmetry apart from the symmetry protecting the topological phase [39]. Studies of weak topological insulators have shown that they are surprisingly robust to disorder [40]. Physics of topological phases in noncrystalline lattices have received relatively less attention. Most studies have focused on quasicrystalline systems [41–43], for example, realizing a weak topological insulator phase in such a system [42]. An interesting unexplored question, both from theoretical and practical perspectives, is whether a completely random set of points, i.e., a random lattice, such as that realized by impurities in a material, can host topological phases. This is the question that we address in this Letter. Random lattices have previously been explored in the context of lattice field theory [44], however, the possibility of topological phases in them has not been so far addressed.

In this Letter, we theoretically establish that amorphous systems can host topologically insulating phases. We provide a demonstration of this by constructing models, using familiar ingredients, on random lattices where fermions hop between sites within a finite range. By tuning parameters such as the density of sites, we show that the system undergoes a quantum phase transition from a trivial to a topological phase. We characterize the topological nature by obtaining the topological invariant (cf. Bott index [45]), and associated quantized transport signatures. We also address interesting features of such quantum phase transitions. This is achieved through a detailed study of all nontrivial symmetry classes (A, AII, D, DIII, and C) in two dimensions. We also provide a demonstration of a  $\mathbb{Z}_2$  topological insulator in three dimensions. This work opens a new direction in the experimental search for topological quantum matter by demonstrating their possibility in as yet unexplored, amorphous systems. We discuss several examples including glassy systems and other engineered random systems.

*Model.*—We consider a region (box) of  $d$ -dimensional space whose volume is  $V$ . Our random lattice is constructed by placing  $N$  sites, labeled by ( $I = 1, \dots, N$ ), randomly in the box (see Fig. 1). The positions of the sites are sampled from an uncorrelated uniform distribution. The collection of points is characterized by a single parameter, viz. the density  $\rho = N/V$ . The sites are identical, each of which hosts  $L$  single particle states. The states at the site  $I$  are denoted by  $|I\alpha\rangle$  and the associated fermion operators are denoted by  $c_{I,\alpha}^\dagger$  and  $c_{I,\alpha}$  ( $\alpha = 1, \dots, L$ ). The “flavor” label  $\alpha$  can stand for the spin or a (spin-)orbital quantum number as appropriate.

Fermions hop from site to site, possibly changing their flavor in the process. We mimic a realistic system by

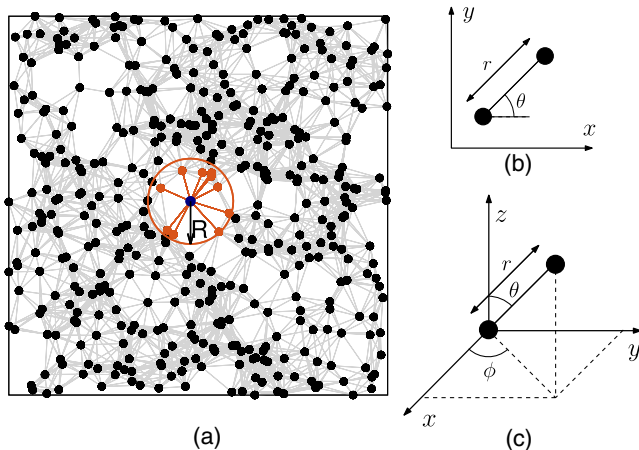


FIG. 1. Random lattice. (a) A typical realization of a random lattice. Sites are indicated by dark spots. Light lines indicate the hoppings between the sites that are within the distance  $\leq R$  from each other. (b),(c) The separation between two sites is described by a distance  $r$  and an angle  $\theta$  in two dimensions (b) and by  $(\theta, \phi)$  in three dimensions (c).

considering a finite range  $R$  of hopping (see Fig. 1). A generic Hamiltonian of such a system is

$$\mathcal{H} = \sum_{I\alpha} \sum_{J\beta} t_{\alpha\beta}(\mathbf{r}_{IJ}) c_{I,\alpha}^\dagger c_{J,\beta}. \quad (1)$$

Here,  $J$  runs over all sites that are within a distance of  $R$  of  $I$ , i.e.,  $|\mathbf{r}_{IJ}| \leq R$  (with  $\mathbf{r}_{IJ}$  being the vector from  $I$  to  $J$ ). The  $L \times L$  matrix  $t_{\alpha\beta}(\mathbf{r})$  depends on the vector  $\mathbf{r} = r\hat{\mathbf{r}}$  via

$$t_{\alpha\beta}(\mathbf{r}) = t(r)T_{\alpha\beta}(\hat{\mathbf{r}}). \quad (2)$$

The distance dependence of the hopping is captured by  $t(r)$  and  $T_{\alpha\beta}(\hat{\mathbf{r}})$  contains both the orbital and angular dependencies. This form is motivated by hoppings found in spin-orbit coupled systems that are common in the realization of topological phases. The unit vector  $\hat{\mathbf{r}}$  in two spatial dimensions is described by an angle  $\theta$  with respect to a chosen axis [see Fig. 1(b)], while in three dimensions we parametrize  $\hat{\mathbf{r}}$  by polar and azimuthal angles  $(\theta, \phi)$ . The hopping matrix for  $\mathbf{r} = \mathbf{0}$ ,

$$t_{\alpha\beta}(\mathbf{0}) = \epsilon_{\alpha\beta}, \quad (3)$$

describes the “on site energy” or the “atomic” Hamiltonian of the system. For,  $r \neq 0$ , we choose

$$t(r) = C\Theta(R - r)e^{-r/a}, \quad (4)$$

where the constant  $C$  is chosen such that  $t(r)$  is a unit energy when  $r = a$ , i.e.,  $C = e$ . The step function  $\Theta$  enforces the cutoff distance  $R$ . In the construction that follows, the forms of  $\epsilon_{\alpha\beta}$  and  $T_{\alpha\beta}(\hat{\mathbf{r}})$  will be motivated by systems that are experimentally relevant [2,12]. Finally, to investigate the physics of topological edge states we will study the system with and without periodic boundary conditions on the box. In the remaining discussion the scale  $a$  is set to unity, and all other lengths are measured in units of  $a$ .

*Two dimensional systems.*—We begin the discussion of our results with two dimensional systems. As is known, there are five symmetry classes (A, AII, D, DIII, and C) that allow topological phases in two dimensions. We construct Hamiltonians for each of these classes that respect all the relevant symmetries (See Table SI in the Supplemental Material [46]). We have performed analysis of systems in all these five classes, and discuss the results of class A in detail as a representative (see Ref. [46] for discussion of other classes).

The symmetry class A possesses no intrinsic symmetries like time reversal, charge conjugation, or sublattice. The class A system is realized with two orbitals per site ( $L = 2$ ) with  $\epsilon_{\alpha\beta} = \begin{pmatrix} 2+M & (1-i)\lambda \\ (1+i)\lambda & -(2+M) \end{pmatrix}$  and

$$T_{\alpha\beta}(\hat{\mathbf{r}}) = \begin{pmatrix} \frac{-1+t_2}{2} & \frac{-ie^{-i\theta} + \lambda[\sin^2\theta(1+i)-1]}{2} \\ \frac{-ie^{i\theta} + \lambda[\sin^2\theta(1-i)-1]}{2} & \frac{1+t_2}{2} \end{pmatrix}. \quad (5)$$

There are three dimensionless parameters  $\lambda$ ,  $t_2$ , and  $M$ . While  $\lambda$  adds to interorbital mixing,  $t_2$  signifies intraorbital hopping.  $M$  (mass term) quantifies the difference in the on site energies of the two orbitals in the atomic limit. This structure can be obtained by starting from the BHZ model [12,57] by introducing  $t_2$ , which breaks charge conjugation symmetry.  $M$  is the parameter we tune to investigate the possibility of topological phases for various values of  $\rho$ , the density of sites. We study filling of one fermion per site.

Results of a particular realization of the random lattice for a  $24 \times 24$  system with  $\rho = 1$ ,  $R = 4$ , and  $M = -0.5$  is shown in Fig. 2. Figure 2(a) shows the energy eigenvalues of the system with and without the periodic boundary conditions. In the presence of periodic boundary conditions the system clearly shows an energy gap; i.e, it is an insulator. In the absence of periodic boundary conditions in both directions we see a set of energy eigenvalues in the midgap region. A typical midgap state, we find, is an “edge” state which is localized on the “surfaces” of the box; see Fig. 2(b) (see also Ref. [46], Fig. S7 for different geometries). On the other hand, a typical bulk state has a localized character (see Ref. [46], Sec. S6). We studied the transport in this system by coupling it to leads and keeping two opposite surfaces, which host the edge states, open. Figure 2(c) shows the two terminal conductances that we calculate using the nonequilibrium Green’s function formalism [58] (see Ref. [46], Sec. S2 for details). Remarkably, when the energy of the incoming fermions is in the “bulk gap” we find that the conductance is quantized to unity even in a completely random system.

Are the edge states that we see of topological origin? The presence of an edge state that provides for a quantized

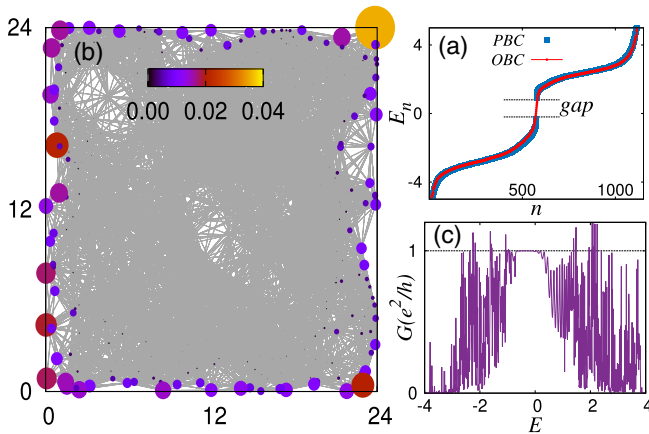


FIG. 2. Class A random lattice model ( $d = 2$ ): (Area  $24 \times 24$ ,  $R = 4$ ,  $M = -0.5$ ,  $t_2 = 0.25$ ,  $\lambda = 0.5$ ,  $\rho = 1$ ). (a) Energy eigenvalues  $E_n$  versus the state number  $n$ . The system with periodic boundary conditions (PBC) shows a gap, while that with open boundaries (OBC) shows midgap states. (b) The wave function of the midgap state is localized on the edge. The size and the color of the blob indicates the probability of finding a fermion at that site. (c) Two terminal conductance ( $G$ ) as a function of incident fermion energy showing a quantized value in the energy gap.

conductance in a completely random lattice, if surprising, is suggestive of its topological origin. We adapt the Bott index of Loring and Hastings [45] to random lattices to characterize the nontrivial topology of our system via an invariant. Briefly, a nontrivial Bott index quantifies the obstruction to the construction of localized Wannier orbitals from the occupied states [45]. The Bott index is given by  $(1/2\pi)\text{Im}\{\text{tr}[\log(WUW^\dagger U^\dagger)]\}$ , where  $U$  and  $W$  are obtained as follows. The position coordinates of all the orbitals  $|I\alpha\rangle \equiv (x_{I\alpha}, y_{I\alpha})$  are rescaled into two angles  $(\theta_{I\alpha}, \phi_{I\alpha})$  defined between  $(0, 2\pi)$ . Diagonal matrices  $\Theta$  and  $\Phi$  are defined with elements  $\theta_{I\alpha}$  and  $\phi_{I\alpha}$  as diagonal elements. The matrices  $U$  and  $W$  are obtained as  $U = P \exp(i\Theta)P$  and  $W = P \exp(i\Phi)P$ , where  $P$  is the projector to the occupied states [45]. In a crystalline system, the Bott index is the same as the Thouless-Kohmoto-Nightingale-den Nijs invariant [6] of 2D class A (details are discussed in the Ref. [46], Sec. S3). Given this correspondence, it is natural to expect robust edge states when the Bott index is nontrivial. For the parameters of Fig. 2, we find the Bott index to be  $-1$ , a nontrivial value. Taken together, the presence of the bulk gap, surface localized midgap states, quantized transport, and a nontrivial Bott index unequivocally demonstrates the topological character of this amorphous system.

Under what condition does a random lattice show topological phases? We address this question by obtaining a phase diagram in the  $M$ - $\rho$  plane, i.e., by varying the mass parameter and the density of sites. Figure 3(a) shows the Bott index for a particular configuration as a function of the mass parameter  $M$  at  $\rho = 0.6$ . For  $-2 \lesssim M \lesssim 1.2$  the system is in a topological phase with two quantum phase transitions at  $M \approx -2$  and at  $M \approx 1.2$ . The phase diagram is obtained by averaging over several hundred realizations of the random lattice for all densities. Figure 3(b) shows a contour plot of the Bott index in the  $M$ - $\rho$  plane. We see that there is a large regime of parameters in the density and  $M$  where the system is topological. An important point is that a critical density  $\rho_c$  is needed to obtain such a topological phase. The existence of such a critical density is expected to be “universal,” although its precise value will be determined by the specific microscopic parameters. For any  $\rho > \rho_c$ , note that there are two values of  $M$  at which the system has a phase transition. We investigated these phase transitions by studying the scaling properties of the gap and the Bott index as a function of system size. We find that the gap indeed vanishes in the thermodynamic limit at both the transitions albeit with different exponents (see Ref. [46], Sec. S4). There is also a concomitant sharper jump in the Bott index with increasing system size (see Ref. [46], Sec. S3).

It is interesting to contrast the physics here with traditional quantum Hall (QH) systems [59–61] which also fall in class A. The QH state which is robust for small disorder, is destroyed when the disorder energy scale matches the QH gap set by the external magnetic field and the system

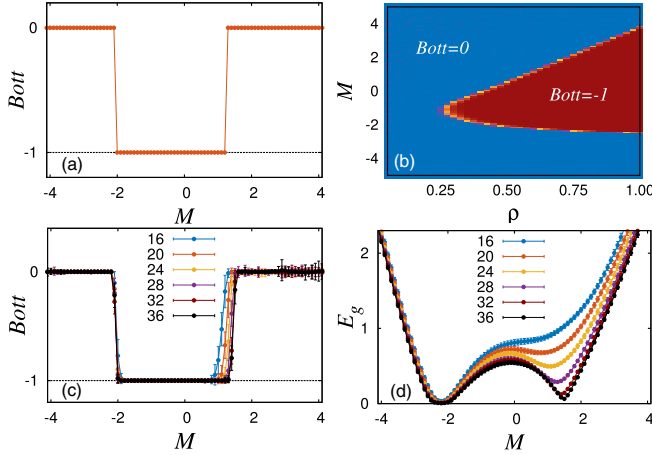


FIG. 3. Phase diagram. (a) Bott index for a particular realization of the random lattice at  $\rho = 0.6$ . (Area  $24 \times 24$ ,  $R = 4$ ,  $t_2 = 0.25$ ,  $\lambda = 0.5$ ) (b) Contour plot of configuration averaged Bott index in the  $M$ - $\rho$  plane. Red region indicates the topologically nontrivial phase. (c) Configuration averaged Bott index for various system sizes. (d) Configuration averaged energy gap  $E_g$  for various system sizes; (c) and (d) are also for  $\rho = 0.6$ . Configuration average is performed over 320 realizations of the random lattice. Other parameters are kept same as in (a).

transits to a thermodynamically gapless insulator with Fermi energy in the localized states [62,63]. On the other hand, on decreasing  $\rho$ , i.e., effectively increasing disorder, the topological gapped state found here goes to a trivial gapped state through a gap closing transition. Transport in the disorder induced trivial phase of the QH system will be

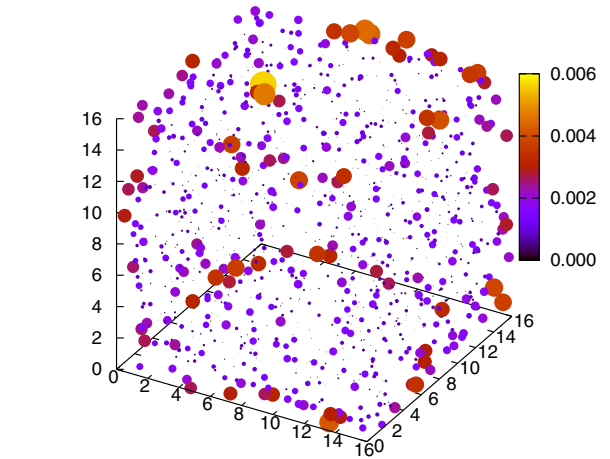


FIG. 4.  $\mathbb{Z}_2$  system in three dimensions. The midgap state localized on the surface. The size and the color of the blob indicate the probability of finding a fermion at the site. ( $\rho = 0.6$ ,  $V = 16^3$ ,  $M = 0.5$ ,  $R = 4$ ).

governed by variable range hopping, while the trivial phase in Fig. 3(b) will show activated transport akin to a typical band insulator.

*Three dimensional system.*—Given the vast interest enjoyed by the three dimensional topological insulator (a system with a  $\mathbb{Z}_2$  invariant), we also investigate the possibility of realizing this in a random lattice. To this end, we consider a system with four orbitals ( $L = 4$ ) (see Ref. [64]) with a Hamiltonian described by  $\epsilon_{\alpha\beta} = \text{Diag}(-3 + M, -3 + M, 3 - M, 3 - M)$ , where  $M$  is the mass parameter, and

$$T_{\alpha\beta}(\theta, \phi) = \frac{1}{2} \begin{pmatrix} 1 & 0 & -i \cos \theta & -ie^{-i\phi} \sin \theta \\ 0 & 1 & -ie^{i\phi} \sin \theta & i \cos \theta \\ -i \cos \theta & -ie^{-i\phi} \sin \theta & -1 & 0 \\ -ie^{i\phi} \sin \theta & i \cos \theta & 0 & -1 \end{pmatrix}. \quad (6)$$

This system has the required time reversal symmetry. For an appropriate set of parameters ( $M = 0.5$  and  $\rho = 0.6$ ), we find indeed that there are midgap states in an open system whose wave functions are localized on the boundary (see Fig. 4). This strongly indicates the realization of a topological state.

*Perspective.*—The possibility of topological phases in a completely random system opens up several new avenues for experiments.

**Bulk amorphous materials:** This work has clearly demonstrated that amorphous solids (“glasses”) with spin-orbit coupled motifs can host topologically insulating phases. Search for such materials offers to be an exciting research direction.

**Engineered systems:** Engineered random systems are also possible, of which we discuss two examples. First, two

dimensional systems can be made by choosing an insulating surface on which suitable “motifs” such as atoms, molecules, or nanoclusters with appropriate orbitals are deposited at random. Note that this process will require far less control than conventional layered crystalline materials. The electronic states of these motifs will then hybridize to produce the required topological phase. Second is the possibility of creating three dimensional systems starting from a suitable large band gap trivial insulator. The idea then is to place “impurity atoms,” again with suitable orbitals and “friendly” chemistry with the host, not unlike the process of  $\delta$  doping of phosphorus in silicon [65]. The hybridization of the impurity orbitals would again produce a topological insulating state in the impurity bands under favorable conditions. An important challenge in these cases will be to engineer the bandwidth of such systems to make

them useful for room temperature applications. We expect that our results will motivate material science experts to address these issues. Finally, we note that suggestions of this Letter are distinct from systems made out of polycrystals of topological insulators (see, for example, Ref. [66]).

Our work also provides some interesting new directions for theoretical research. There are two equivalent ways to view topological phases. The first one is “kinematic”, i.e., based on the homotopy of ground state wave functions of systems in a given symmetry class as discussed in the introduction. The second approach [20], probably better suited for the random system, is based on asking if the  $(d-1)$ -dimensional surface of a gapped  $d$ -dimensional system resists localization. This absence of localization on the  $(d-1)$  surface can be used to characterize the topology of the  $d$ -dimensional bulk. Localization is prevented by the presence of a topological term in the action (nonlinear  $\sigma$  model) which describes the low energy modes of the  $(d-1)$ -dimensional surface. While one usually writes down such  $\sigma$  models based on symmetry considerations, an interesting question in the current context would be to uncover how such topological terms can arise in the random lattice setting.

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