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Semi-equivelar and vertex-transitive maps on the torus

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Abstract A vertex-transitive map X is a map on a closed surface on which the automorphism group $\mathrm{Aut}(X)$ acts transitively on the set of vertices. If the face-cycles at all the vertices in a map are of same type then the map is said to be a semi-equivelar map. Clearly, a vertex-transitive map is semi-equivelar. Converse of this is not true in general. We show that there are eleven types of semi-equivelar maps on the torus. Three of these are equivelar maps. It is known that two of these three types are always vertex-transitive. We show that this is true for the remaining one type of equivelar maps and one other type of semi-equivelar maps, namely, if X is a semi-equivelar map of type $[6^3]$ or $[3^3, 4^2]$ then X is vertex-transitive. We also show, by presenting examples, that this result is not true for the remaining seven types of semi-equivelar maps. There are ten types of semi-equivelar maps on the Klein bottle. We present examples in each of the ten types which are not vertex-transitive.

Keywords Polyhedral map on torus · Vertex-transitive map · Equivelar maps · Archimedean tiling

Mathematics Subject Classification 52C20 · 52B70 · 51M20 · 57M60

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1 Introduction

By a map we mean a polyhedral map on a surface. So, a face of a map is a p-gon for some integer $p \geq 3$. A map X is said to be *weakly regular* or *vertex-transitive* if the automorphism group $\mathrm{Aut}(X)$ acts transitively on the set $V(X)$ of vertices of X.

If v is a vertex in a map X then the faces containing v form a cycle (called the *face-cycle*) C_v in the dual graph $\Lambda(X)$ of X. Clearly, C_v is of the form $P_1 - P_2 - \cdots - P_k - P_1$, where P_i is a path consisting of n_i p_i-gons and $p_i \neq p_{i+1}$ for $1 \leq i \leq k$ (addition in the suffix is modulo k). A map X is called *semi-equivelar* (or *semi-regular*) if C_u and C_v are of same type for any two vertices u and v of X. More precisely, there exist natural numbers $p_1, \ldots, p_k \geq 3$ and $n_1, \ldots, n_k \geq 1$, $p_i \neq p_{i+1}$ such that both C_u and C_v are of the form $P_1 - P_2 - \cdots - P_k - P_1$ as above, where P_i is a path consisting of n_i p_i-gons. In this case, we say that X is *semi-equivelar of type* $[p_1^{n_1}, \ldots, p_k^{n_k}]$. (We identify $[p_1^{n_1}, \ldots, p_k^{n_k}]$ with $[p_2^{n_2}, \ldots, p_k^{n_k}, p_1^{n_1}]$ and with $[p_k^{n_k}, \ldots, p_1^{n_1}]$.) An *equivelar* map (of type $[p^q]$, (p, q) or $\{p, q\}$) is a semi-equivelar map of type $[p^q]$ for some $p, q \geq 3$. Clearly, a vertex-transitive map is semi-equivelar.

A *semi-regular* tiling of the plane $\mathbb{R}^2$ is a tiling of $\mathbb{R}^2$ by regular polygons such that all the vertices of the tiling are of same type. A *semi-regular* tiling of $\mathbb{R}^2$ is also known as *Archimedean*, or *homogeneous*, or *uniform* tiling. In Grünbaum and Shephard (1977), Grünbaum and Shephard showed that there are exactly eleven types of Archimedean tilings on the plane. These types are $[3^6]$, $[3^4, 6^1]$, $[3^3, 4^2]$, $[3^2, 4^1, 3^1, 4^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^4]$, $[4^1, 6^1, 12^1]$, $[4^1, 8^2]$, $[6^3]$. Clearly, a *semi-regular* tiling on $\mathbb{R}^2$ gives a semi-equivelar map on $\mathbb{R}^2$. But, there are semi-equivelar maps on the plane which are not (not isomorphic to) an Archimedean tiling. In fact, there exists $[p^q]$ equivelar maps on $\mathbb{R}^2$ whenever $1/p + 1/q < 1/2$ (e.g., Coxeter and Moser 1980; Fejes Tóth 1965). Thus, we have

Proposition 1.1 *There are infinitely many types of equivelar maps on the plane* $\mathbb{R}^2$.

All vertex-transitive maps on the 2-sphere are known. These are the boundaries of Platonic and Archimedean solids and two infinite families of types (namely, of types $[4^2, n^1]$ and $[3^3, m^1]$ for $4 \neq n \geq 3$, $m \geq 4$) (Grünbaum and Shephard 1977). Similarly, there are infinitely many types of vertex-transitive maps on the real projective plane (Babai 1991). Thus, there are infinitely many types of semi-equivelar maps on the 2-sphere and the real projective plane. But, for a surface of negative Euler characteristic the picture is different. In Babai (1991), Babai has shown the following.

Proposition 1.2 *A semi-equivelar map on a surface of Euler characteristic* $\chi < 0$ *has at most* -84χ *vertices.*

As a consequence of this we get

Corollary 1.3 *If the Euler characteristic* $\chi(M)$ *of a surface* M *is negative then the number of semi-equivelar maps on* M *is finite.*

We know from Datta and Nilakantan (2001) and Datta and Upadhyay (2005) that infinitely many equivelar maps exist on both the torus and the Klein bottle. Thus, infinitely many semi-equivelar maps exist on both the torus and the Klein bottle.

But, only eleven types of semi-equivelar maps on the torus and ten types of semi-equivelar maps on the Klein bottle are known in the literature. All these are quotients of Archimedean tilings of the plane (Babai 1991; Sŭch 2011a, b). Since there are infinitely many equivelar maps on the plane, it is natural to ask whether there are more types of semi-equivelar maps on the torus or the Klein bottle. Here we prove

Theorem 1.4 *Let X be a semi-equivelar map on a surface M.* (a) *If M is the torus then the type of X is* $[3^6]$, $[6^3]$, $[4^4]$, $[3^4, 6^1]$, $[3^3, 4^2]$, $[3^2, 4^1, 3^1, 4^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^1, 8^2]$ *or* $[4^1, 6^1, 12^1]$. (b) *If M is the Klein bottle then the type of X is* $[3^6]$, $[6^3]$, $[4^4]$, $[3^3, 4^2]$, $[3^2, 4^1, 3^1, 4^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^1, 8^2]$ *or* $[4^1, 6^1, 12^1]$.

Theorem 1.4 and the known examples (also the examples in Sect. 4) imply that there are exactly eleven types of semi-equivelar maps on the torus and ten types of semi-equivelar maps on the Klein bottle.

In Brehm and Kühnel (2008), Brehm and Kühnel presented a formula to determine the number of distinct vertex-transitive equivelar maps of types $[3^6]$ and $[4^4]$ on the torus. It was shown in Datta and Upadhyay (2005) that every equivelar map of type $[3^6]$ on the torus is vertex-transitive. By the similar arguments, one can easily show that a equivelar map of type $[4^4]$ on the torus is vertex-transitive [also see (Brehm and Kühnel 2008, Proposition 6)]. Thus, we have

Proposition 1.5 *Let X be an equivelar map on the torus. If the type of X is* $[3^6]$ *or* $[4^4]$ *then X is vertex-transitive.*

Here we prove

Theorem 1.6 *Let X be a semi-equivelar map on the torus. If the type of X is* $[6^3]$ *or* $[3^3, 4^2]$ *then X is vertex-transitive.*

In Sect. 4, we present examples of the other seven types of semi-equivelar maps which are not vertex-transitive. This proves

Theorem 1.7 *If* $[p_1^{n_1}, \ldots, p_k^{n_k}] = [3^2, 4^1, 3^1, 4^1]$, $[3^4, 6^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^1, 8^2]$ *or* $[4^1, 6^1, 12^1]$ *then there exists a semi-equivelar map of type* $[p_1^{n_1}, \ldots, p_k^{n_k}]$ *on the torus which is not vertex-transitive.*

In Datta and Upadhyay (2005), the first author and Upadhyay have presented examples of $[3^6]$ equivelar maps on the Klein bottle which are not vertex-transitive. In Sect. 4, we present examples of the other nine types of semi-equivelar maps on the Klein bottle which are not vertex-transitive. Thus, we have

Theorem 1.8 *If* $[p_1^{n_1}, \ldots, p_k^{n_k}]$ *is one in the list of* 10 *types in Theorem* 1.4(b) *then there exists a semi-equivelar map of type* $[p_1^{n_1}, \ldots, p_k^{n_k}]$ *on the Klein bottle which is not vertex-transitive.*

If the type of a semi-equivelar map X on the torus is different from $[3^3, 4^2]$ then, by Theorem 1.7, the vertices of X may form more than one $\mathrm{Aut}(X)$-orbits. Here we prove

Theorem 1.9 *Let X be a semi-equivelar map on the torus. Let the vertices of X form m $\mathrm{Aut}(X)$-orbits.* (a) *If the type of X is $[3^2, 4^1, 3^1, 4^1]$ then $m \leq 2$.* (b) *If the type of X is $[3^1, 6^1, 3^1, 6^1]$ then $m \leq 3$.*

Several examples of $[3^6]$ and $[4^4]$ equivelar maps on the torus are in Datta and Nilakantan (2001). From this, one can construct equivelar maps of type $[6^3]$ on the torus. In Example 4.1, we also present a semi-equivelar map of type $[3^3, 4^2]$ on the torus for the sake of completeness.

2 Proofs of Theorem 1.4 and Proposition 1.1

For $n \geq 3$, the n-gon whose edges are $u_1u_2, \dots, u_{n-1}u_n, u_nu_1$ is denoted by $u_1 - u_2 - \cdots - u_n - u_1$ or by $C_n(u_1, \dots, u_n)$. We call 3-gons and 4-gons by *triangles* and *quadrangles* respectively. A triangle $u - v - w - u$ is also denoted by uvw. If X is a map on a surface M then we identify a face of X in M with the boundary cycle of the face.

Proof of Proposition 1.1 In Datta and Nilakantan (2001), it was shown that there exists equivelar map of type $[3^8]$ on the orientable surface of genus g for each $g \geq 4$. For a fixed $g \geq 4$, let X be one such equivelar map of type $[3^8]$ on the surface M_g of genus g. Since the 2-disk $\mathbb{D}^2$ is the universal cover of M_g, by pulling back X, we get an equivelar map $\widetilde{X}$ of type $[3^8]$ on $\mathbb{D}^2$ and a polyhedral map $\eta : \widetilde{X} \to X$. From the constructions in Datta (2005) and Datta and Nilakantan (2001), we know that an equivelar map of type $[p^q]$ exists on some surface (orientable or non-orientable) of appropriate genus for each $[p^q]$ in $\{[3^7], [4^5], [4^6], [3^{3\ell-1}], [3^{3\ell}], [k^k] : \ell \geq 3, k \geq 5\}$. So, by the same arguments, equivelar maps of types $[p^q]$ exist on $\mathbb{D}^2$ for $[p^q]$ in $\{[3^7], [4^5], [4^6], [3^{3\ell-1}], [3^{3\ell}], [k^k] : \ell \geq 3, k \geq 5\}$. More generally, there exist equivelar maps of type $[p^q]$ on $\mathbb{D}^2$ whenever $1/p + 1/q < 1/2$ (cf., Coxeter and Moser 1980; Fejes Tóth 1965; Grünbaum and Shephard 1977). Since $\mathbb{R}^2$ is homeomorphic to $\mathbb{D}^2$, an equivelar map of type $[p^q]$ on $\mathbb{D}^2$ determines an equivelar map of type $[p^q]$ on $\mathbb{R}^2$. Thus, there exist equivelar maps of types $[p^q]$ on $\mathbb{R}^2$ whenever $1/p + 1/q < 1/2$. The result now follows. □

Lemma 2.1 *Let X be a semi-equivelar map on a surface M. If $\chi(M) = 0$ then the type of X is* $[3^6]$, $[3^4, 6^1]$, $[3^3, 4^2]$, $[3^2, 4^1, 3^1, 4^1]$, $[4^4]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^2, 6^2]$, $[3^2, 4^1, 12^1]$, $[3^1, 4^1, 3^1, 12^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 4^2, 6^1]$, $[6^3]$, $[3^1, 12^2]$, $[4^1, 8^2]$, $[5^2, 10^1]$, $[3^1, 7^1, 42^1]$, $[3^1, 8^1, 24^1]$, $[3^1, 9^1, 18^1]$, $[3^1, 10^1, 15^1]$, $[4^1, 5^1, 20^1]$ *or* $[4^1, 6^1, 12^1]$.

Proof Let the type of X be $[p_1^{n_1}, \dots, p_k^{n_k}]$. Consider the ℓ-tuple $(q_1^{m_1}, \dots, q_\ell^{m_\ell})$, where $q_i \neq q_j$ for $i \neq j$, $q_i = p_j$ for some j, $m_i = \sum_{p_i = q_j} n_j$ and $(m_1, q_1) > (m_2, q_2) > \cdots > (m_\ell, q_\ell)$. (Here, $(m, p) > (n, q)$ means either (i) $m > n$ or (ii) $m = n$ and $p < q$.)

Claim. $(q_1^{m_1}, \dots, q_\ell^{m_\ell}) = (3^6)$, $(3^4, 6^1)$, $(3^3, 4^2)$, (4^4), $(3^2, 6^2)$, $(3^2, 4^1, 12^1)$, $(4^2, 3^1, 6^1)$, (6^3), $(12^2, 3^1)$, $(8^2, 4^1)$, $(5^2, 10^1)$, $(3^1, 7^1, 42^1)$, $(3^1, 8^1, 24^1)$, $(3^1, 9^1, 18^1)$, $(3^1, 10^1, 15^1)$, $(4^1, 5^1, 20^1)$ or $(4^1, 6^1, 12^1)$.

Let f_0, f_1 and f_2 denote the number of vertices, edges and faces of X respectively. Let d be the degree of each vertex. Then, $d = n_1 + \cdots + n_k = m_1 + \cdots + m_\ell$ and $f_1 = f_0 \times d/2$. Clearly, the number of q_i-gons is $f_0 \times m_i/q_i$. This implies that $f_2 = f_0(m_1/q_1 + \cdots + m_\ell/q_\ell)$. Since $\chi(M) = 0$, it follows that $f_0 - f_0(m_1 + \cdots + m_\ell)/2 + f_0(m_1/q_1 + \cdots + m_\ell/q_\ell) = 0$ or

$$\left(\frac{1}{2} - \frac{1}{q_1}\right) m_1 + \cdots + \left(\frac{1}{2} - \frac{1}{q_\ell}\right) m_\ell = 1. \tag{1}$$

Since $q_i \geq 3$, it follows that $d \leq 6$. Moreover, if $d = 6$ then $\ell = 1$ and $q_1 = 3$. In this case, $(q_1^{m_1}, \ldots, q_\ell^{m_\ell}) = (3^6)$.

Now, assume $d = 5$. Then $(m_1, \ldots, m_\ell) = (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1)$ or $(1, 1, 1, 1, 1)$. It is easy to see that for $(m_1, \ldots, m_\ell) = (5), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1)$ or $(1, 1, 1, 1, 1)$, Eq. (1) has no solution. So, $(m_1, \ldots, m_\ell) = (4, 1)$ or $(3, 2)$. In the first case, $(q_1, q_2) = (3, 6)$ and in the second case, $(q_1, q_2) = (3, 4)$. Thus, $(q_1^{m_1}, \ldots, q_\ell^{m_\ell}) = (3^4, 6^1)$ or $(3^3, 4^2)$.

Let $d = 4$. Then $(m_1, \ldots, m_\ell) = (4), (3, 1), (2, 2), (2, 1, 1)$ or $(1, 1, 1, 1)$. Again, for $(m_1, \ldots, m_\ell) = (3, 1)$ or $(1, 1, 1, 1)$, Eq. (1) has no solution. So, $(m_1, \ldots, m_\ell) = (4), (2, 2)$ or $(2, 1, 1)$. In the first case, $q_1 = 4$, in the second case, $(q_1, q_2) = (3, 6)$ and in the third case, $(q_1, \{q_2, q_3\}) = (3, \{4, 12\})$ or $(4, \{3, 6\})$. Thus, $(q_1^{m_1}, \ldots, q_\ell^{m_\ell}) = (4^4), (3^2, 6^2), (3^2, 4^1, 12^1)$ or $(4^2, 3^1, 6^1)$.

Finally, assume $d = 3$. Then $(m_1, \ldots, m_\ell) = (3), (2, 1)$ or $(1, 1, 1)$. In the first case, $q_1 = 6$, in the second case, $(q_1, q_2) = (12, 3), (8, 4)$ or $(5, 10)$ and in the third case, $\{q_1, q_2, q_3\} = \{3, 7, 42\}, \{3, 8, 24\}, \{3, 9, 18\}, \{3, 10, 15\}, \{4, 5, 20\}$ or $\{4, 6, 12\}$. Thus, $(q_1^{m_1}, \ldots, q_\ell^{m_\ell}) = (6^3), (12^2, 3^1), (8^2, 4^1), (5^2, 10^1), (3^1, 7^1, 42^1), (3^1, 8^1, 24^1), (3^1, 9^1, 18^1), (3^1, 10^1, 15^1), (4^1, 5^1, 20^1)$ or $(4^1, 6^1, 12^1)$. This proves the claim.

The lemma follows from the claim. □

We need the following technical lemma for the proof of Theorem 1.4.

Lemma 2.2 *If* $[p_1^{n_1}, \ldots, p_k^{n_k}]$ *satisfies any of the following three properties then* $[p_1^{n_1}, \ldots, p_k^{n_k}]$ *can not be the type of any semi-equivelar map on a surface.*

(i) *There exists* i *such that* $n_i = 2$, p_i *is odd and* $p_j \neq p_i$ *for all* $j \neq i$.
(ii) *There exists* i *such that* $n_i = 1$, p_i *is odd,* $p_j \neq p_i$ *for all* $j \neq i$ *and* $p_{i-1} \neq p_{i+1}$.
(iii) *There exists* i *such that* $n_i = 1$, p_i *is odd,* $p_{i-1} \neq p_j$ *for all* $j \neq i - 1$ *and* $p_{i+1} \neq p_\ell$ *for all* $\ell \neq i + 1$.

(*Here, addition in the subscripts are modulo* k.)

Proof If possible let there exist a semi-equivelar map X of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ which satisfies (i). Let $A = u_1 - u_2 - u_3 - \cdots - u_{p_i} - u_1$ be a p_i-gon. Let the other face containing $u_r u_{r+1}$ be A_r for $1 \leq r \leq p_i$. (Addition in the subscripts are modulo p_i.) Consider the face-cycle of the vertex u_1. Since $p_j \neq p_i$ for all $j \neq i$ and $n_i = 2$, it follows that exactly one of A_1 and A_{p_i} is a p_i-gon. Assume, without loss, that A_1 is a p_i-gon. Since u_2 is in two p_i-gons, it follows that A_2 is not a p_i-gon. Therefore (by considering the vertex u_3, as in the case for the vertex u_1), A_3 is a p_i-gon. Continuing this way, we get $A_1, A_3, A_5, \ldots$ are p_i-gons. Since p_i is odd, it follows that A_{p_i} is a p_i-gon. Then we get three p_i-gons, namely, A, A_1 and A_{p_i}, through u_1. This is a contradiction.

Now, suppose there exists a semi-equivelar map Y of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ which satisfies (ii). Let $B = u_1 - u_2 - u_3 - \cdots - u_{p_i} - u_1$ be a p_i-gon. Let the other face containing $u_r u_{r+1}$ be B_r for $1 \leq r \leq p_i$. Consider the face-cycle of the vertex u_2. Since $p_j \neq p_i$ and $n_i = 1$, A is the only p_i-gon containing u_2. Since $p_{i-1} \neq p_{i+1}$, it follows that one of B_1 and B_2 is a p_{i-1}-gon and the other is a p_{i+1}-gon. Assume, without loss, that B_1 is a p_{i-1}-gon and B_2 is a p_{i+1}-gon. Then, by the same argument as above, $B_1, B_3, B_5, \ldots$ are p_{i-1}-gons and $B_2, B_4, \ldots$ are p_{i+1}-gons. Since p_i is odd, it follows that B_{p_i} is a p_{i-1}-gon. Then, from the face-cycle of u_1, it follows that $p_{i+1} = p_{i-1}$. This contradicts the assumption.

Finally, assume that there exists a semi-equivelar map Z of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ which satisfies (iii). Let P and Q be two adjacent faces through a vertex u_1, where P is a p_i-gon and Q is a p_{i-1}-gon. Assume that $P = u_1 - u_2 - u_3 - \cdots - u_{p_i} - u_1$ and $Q = u_1 - v_2 - v_3 - \cdots - v_{p_{i-1}-1} - u_{p_i} - u_1$. Let the other face containing $u_r u_{r+1}$ be P_r for $1 \leq r \leq p_i$. (Addition in the subscripts are modulo p_i.) Since $p_{i-1} \neq p_j$ for all $j \neq i-1$ and $p_{i+1} \neq p_\ell$ for all $\ell \neq i+1$, considering the face-cycle of u_1, it follows that P_1 is a p_{i+1}-gon. Considering the face-cycle of u_2, by the similar argument (interchanging p_{i-1} and p_{i+1}), it follows that P_2 is a p_{i-1}-gon. Continuing this way, we get $P_1, P_3, \ldots$ are p_{i+1}-gons and $P_2, P_4, \ldots$ are p_{i-1}-gons. Since p_i is odd, it follows that P_{p_i} is a p_{i+1}-gon. This is a contradiction since $P_{p_i} = Q$ is a p_{i-1}-gon and $p_{i-1} \neq p_{i+1}$. This completes the proof. □

In Maity and Upadhyay (2015, Theorem 2.1), the second author and Upadhyay have proved the following.

Proposition 2.3 *There is no semi-equivelar map of type* $[3^4, 6^1]$ *on the Klein bottle.*

Proof of Theorem 1.4 Let X be a semi-equivelar map of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ on the torus. By Lemma 2.2 (i), $[p_1^{n_1}, \ldots, p_k^{n_k}] \neq [3^2, 6^2], [3^2, 4^1, 12^1], [5^2, 10^1]$. By Lemma 2.2 (ii), $[p_1^{n_1}, \ldots, p_k^{n_k}] \neq [3^1, 4^2, 6^1], [3^1, 7^1, 42^1], [3^1, 8^1, 24^1], [3^1, 9^1, 18^1], [3^1, 10^1, 15^1], [4^1, 5^1, 20^1]$. Also, by Lemma 2.2 (iii), $[p_1^{n_1}, \ldots, p_k^{n_k}] \neq [3^1, 4^1, 3^1, 12^1]$. The result now follows by Lemma 2.1.

Let X be a semi-equivelar map of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ on the Klein bottle. As above, by Lemma 2.2, $[p_1^{n_1}, \ldots, p_k^{n_k}] \neq [3^2, 6^2], [3^2, 4^1, 12^1], [5^2, 10^1], [3^1, 4^2, 6^1], [3^1, 7^1, 42^1], [3^1, 8^1, 24^1], [3^1, 9^1, 18^1], [3^1, 10^1, 15^1], [4^1, 5^1, 20^1], [3^1, 4^1, 3^1, 12^1]$. By Proposition 2.3, $[p_1^{n_1}, \ldots, p_k^{n_k}] \neq [3^4, 6^1]$. The result now follows by Lemma 2.1. □

3 Proof of Theorem 1.6

A triangulation of a 2-manifold is called *degree-regular* if each of its vertices have the same degree. In other word, a degree-regular triangulation is an equivelar map of type $[3^k]$ for some $k \geq 3$. The triangulation E given in Fig. 1 is a degree-regular triangulation of $\mathbb{R}^2$.

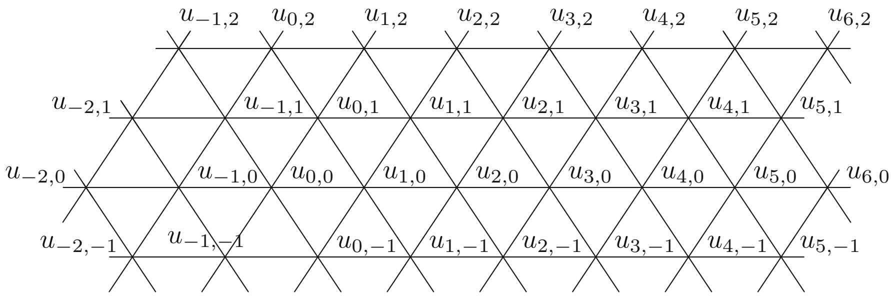


Fig. 1 Regular $[3^6]$-tiling E of $\mathbb{R}^2$

From Datta and Upadhyay (2005) we know

Proposition 3.1 *Let M be a triangulation of the plane $\mathbb{R}^2$. If the degree of each vertex of M is 6 then M is isomorphic to E.*

Using Proposition 3.1, it was shown in Datta and Upadhyay (2005) that ‘any degree-regular triangulation of the torus is vertex-transitive’. Here we prove

Lemma 3.2 *Let X be a triangulation of the torus. If X is degree-regular then the automorphism group $\mathrm{Aut}(X)$ acts face-transitively on X.*

Proof Since X is degree-regular and Euler characteristic of X is 0, it follows that the degree of each vertex in X is 6.

Since $\mathbb{R}^2$ is the universal cover of the torus, there exists a triangulation Y of $\mathbb{R}^2$ and a simplicial covering map $\eta\colon Y \to X$ [cf. (Spanier 1966, Page 144)]. Since the degree of each vertex in X is 6, the degree of each vertex in Y is 6. Because of Proposition 3.1, we may assume that $Y = E$. Let Γ be the group of covering transformations. Then $|X| = |E|/\Gamma$.

We take $V = \{u_{i,2j} = (i, j\sqrt{3}),\ u_{i,2j+1} = (i + 1/2, (2j + 1)\sqrt{3}/2) \ :\ i, j \in \mathbb{Z}\}$ as the vertex set of E. Then $H := \{x \mapsto x + a, a \in V\}$ is a subgroup of $\mathrm{Aut}(E)$ and is called the group of translations. Clearly, H is commutative.

For $\sigma \in \Gamma$, $\eta \circ \sigma = \eta$. So, σ maps the geometric carrier of a simplex to the geometric carrier of a simplex. This implies that σ induces an automorphism σ of E. Thus, we can identify Γ with a subgroup of $\mathrm{Aut}(E)$. So, X is a quotient of E by the subgroup Γ of $\mathrm{Aut}(E)$, where Γ has no fixed element (vertex, edge or face). Hence Γ consists of translations and glide reflections. Since $X = E/\Gamma$ is orientable, Γ does not contain any glide reflection. Thus $\Gamma \leq H$.

Consider the subgroup G of $\mathrm{Aut}(E)$ generated by H and the map $x \mapsto -x$. So,

$$G = \{\alpha : x \mapsto \varepsilon x + a \ :\ \varepsilon = \pm 1, a \in V\} \cong H \rtimes \mathbb{Z}_2.$$

Claim 1. G acts face-transitively on E.

Since H is vertex transitively on E, to prove Claim 1, it is sufficient to show that G acts transitively on the set of six faces containing $u_{0,0}$. This follows from the following: $u_{-1,0}u_{0,0}u_{-1,1} + u_{1,0} = u_{0,0}u_{1,0}u_{0,1} = u_{-1,-1}u_{0,-1}u_{0,0} + u_{0,1}$,

$u_{-1,0}u_{-1,-1}u_{0,0} + u_{1,0} = u_{0,0}u_{0,-1}u_{1,0} = u_{-1,1}u_{0,0}u_{0,1} + u_{0,-1}$ and $-1 \cdot u_{0,0}u_{-1,0}u_{-1,-1} = u_{0,0}u_{1,0}u_{0,1}$.

Claim 2. If $K \le H$ then $K \trianglelefteq G$.

Let $\alpha \in G$ and $\beta \in K$. Assume $\alpha(x) = \varepsilon x + a$ and $\beta(x) = x + b$ for some $a, b \in V(E)$ and $\varepsilon \in \{1, -1\}$. Then $(\alpha \circ \beta \circ \alpha^{-1})(x) = (\alpha \circ \beta)(\varepsilon(x - a)) = \alpha(\varepsilon(x - a) + b) = x - a + \varepsilon b + a = x + \varepsilon b = \beta^{\varepsilon}(x)$. Thus, $\alpha \circ \beta \circ \alpha^{-1} = \beta^{\varepsilon} \in K$. This proves Claim 2.

By Claim 2, $\Gamma \trianglelefteq G$ and hence we can assume that $G/\Gamma \le \mathrm{Aut}(E/\Gamma)$. Since, by Claim 1, G acts face-transitively on E, it follows that G/Γ acts face-transitively on E/Γ. This completes the proof since $X = E/\Gamma$. □

We need the following two lemmas for the Proof of Theorem 1.6.

Lemma 3.3 *Let X be a map on the 2-disk $\mathbb{D}^2$ whose faces are triangles and quadrangles. For a vertex x of X, let $n_3(x)$ and $n_4(x)$ be the number of triangles and quadrangles through x respectively. Suppose $(n_3(u), n_4(u)) = (3, 2)$ for each internal vertex u. Then X does not satisfy any of the following.*

(a) $1 \le n_4(w) \le 2$, $n_3(w) + n_4(w) \le 4$ *for one vertex w on the boundary, and* $(n_3(v), n_4(v)) = (0, 2)$ *for each boundary vertex $v \ne w$.*
(b) $1 \le n_3(w) \le 3$, $n_4(w) \le 2$ *and* $n_3(w) + n_4(w) \le 4$ *for one vertex w on the boundary, and* $(n_3(v), n_4(v)) = (3, 0)$ *for each boundary vertex $v \ne w$.*

Proof Let f_0, f_1 and f_2 denote the number of vertices, edges and faces of X respectively. Let n_3 (resp., n_4) denote the total number of triangles (resp., quadrangles) in X. Let there be n internal vertices and $m + 1$ boundary vertices. So, $f_0 = n + m + 1$ and $f_2 = n_3 + n_4$.

Suppose X satisfies (a). Then $n_4 = (2n+2m+n_4(w))/4$ and $n_3 = (3n+n_3(w))/3$. Since $1 \le n_4(w) \le 2$, it follows that $n_4(w) = 2$ and hence $n_3(w) \le 2$. These imply that $n_3(w) = 0$. Thus, the exceptional vertex is like other boundary vertices. Therefore, each boundary vertex is in three edges and hence $f_1 = (5n + 3m + 3)/2$. These imply $f_0 - f_1 + f_2 = (n + m + 1) - (5n + 3m + 3)/2 + (n + (n + m + 1)/2) = 0$. This is not possible since the Euler characteristic of the 2-disk $\mathbb{D}^2$ is 1.

If X satisfies (b) then $n_3 = (3n + 3m + n_3(w))/3$ and $n_4 = (2n + n_4(w))/4$. Since $1 \le n_3(w) \le 3$, it follows that $n_3(w) = 3$ and hence $n_4(w) \le 1$. These imply that $n_4(w) = 0$. Thus, the exceptional vertex is like other boundary vertices and each boundary vertex is in four edges. Thus, $f_1 = (5n + 4m + 4)/2$ and $f_2 = n_4 + n_3 = 3n/2+m+1$. Then $f_0-f_1+f_2 = (n+m+1)-(5n+4m+4)/2+(3n/2+m+1) = 0$, a contradiction again. This completes the proof. □

Lemma 3.4 *Let E_1 be the Archimedean tiling of the plane $\mathbb{R}^2$ given in Fig. 2. If X is a semi-equivelar map of $\mathbb{R}^2$ of type* $[3^3, 4^2]$ *then $X \cong E_1$.*

Proof Let the type of X be $[3^3, 4^2]$. Choose a vertex $v_{0,0}$. Let the two quadrangle through $v_{0,0}$ be $v_{-1,0} - v_{0,0} - v_{0,1} - v_{-1,1} - v_{-1,0}$ and $v_{0,0} - v_{1,0} - v_{1,1} - v_{0,1} - v_{0,0}$. Then the second quadrangle through $v_{1,0}$ is of the form $v_{1,0}-v_{2,0}-v_{2,1}-v_{1,1}-v_{1,0}$ and the second quadrangle through $v_{-1,0}$ is of the form $v_{-2,0}-v_{-1,0}-v_{-1,1}-v_{-2,1}-v_{-2,0}$.

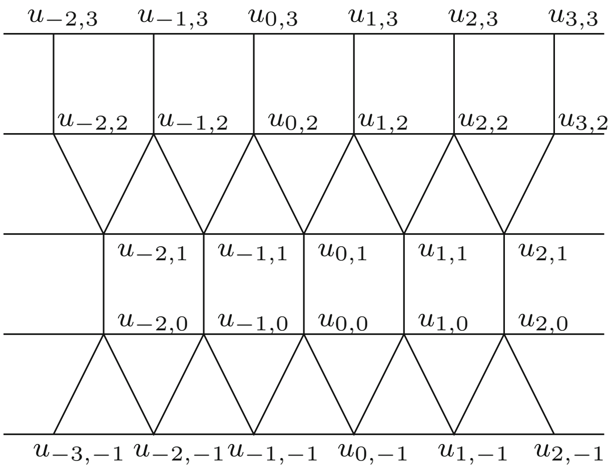


Fig. 2 Elongated triangular tiling E_1

Continuing this way, we get a path $P_0 := \cdots - v_{-2,0} - v_{-1,0} - v_{0,0} - v_{1,0} - v_{2,0} - \cdots$ in the edge graph of X such that all the quadrangles incident with a vertex of P_0 lie on one side of P_0 and all the triangles incident with the same vertex lie on the other side of P_0. If P_0 has a closed sub-path then P_0 contains a cycle W. In that case, the bounded part of X with boundary W is a map on the 2-disk $\mathbb{D}^2$ which satisfies (a) or (b) of Lemma 3.3. This is not possible by Lemma 3.3. Thus, P_0 is an infinite path. Then the faces through vertices of P_0 forms an infinite strip which is bounded by two infinite paths, say $P_{-1} = \cdots - v_{-2,-1} - v_{-1,-1} - v_{0,-1} - v_{1,-1} - v_{2,-1} - \cdots$ and $P_1 = \cdots - v_{-2,1} - v_{-1,1} - v_{0,1} - v_{1,1} - v_{2,1} - \cdots$, where the faces between P_0 and P_1 are quadrangles and the faces between P_0 and P_{-1} are triangles and the faces through $v_{i,0}$ are $v_{i-1,0} - v_{i,0} - v_{i,1} - v_{i-1,1} - v_{i-1,0}$, $v_{i,0} - v_{i+1,0} - v_{i+1,1} - v_{i,1} - v_{i,0}$, $v_{i,0}v_{i+1,0}v_{i,-1}$, $v_{i,0}v_{i,-1}v_{i-1,-1}$, $v_{i,0}v_{i-1,-1}v_{i-1,0}$.

Similarly, starting with the vertex $v_{0,1}$ in place of $v_{0,0}$ we get the paths P_0, P_1, $P_2 = \cdots - v_{-2,2} - v_{-1,2} - v_{0,2} - v_{1,2} - v_{2,2} - \cdots$, where the faces between P_1 and P_2 are triangles and the triangles through $u_{i,1}$ are $v_{i,1}v_{i+1,1}v_{i,2}$, $v_{i,1}v_{i,2}v_{i-1,2}$, $v_{i,1}v_{i-1,2}v_{i-1,1}$. Continuing this way we get paths $\cdots, P_{-2}, P_{-1}, P_0, P_1, P_2, \cdots$ such that (i) the faces between P_{2j} and P_{2j+1} are rectangles, (ii) the faces between P_{2j-1} and P_{2j} are triangles, (iii) the five faces through $v_{i,2j}$ are $v_{i-1,2j} - v_{i,2j} - v_{i,2j+1} - v_{i-1,2j+1} - v_{i-1,2j}$, $v_{i,2j} - v_{i+1,2j} - v_{i+1,2j+1} - v_{i,2j+1} - v_{i,2j}$, $v_{i,2j}v_{i+1,2j}v_{i,2j-1}$, $v_{i,2j}v_{i,2j-1}v_{i-1,2j-1}$, $v_{i,2j}v_{i-1,2j-1}v_{i-1,2j}$, and (iv) the five faces through $v_{i,2j+1}$ are $v_{i-1,2j} - v_{i,2j} - v_{i,2j+1} - v_{i-1,2j+1} - v_{i-1,2j}$, $v_{i,2j} - v_{i+1,2j} - v_{i+1,2j+1} - v_{i,2j+1} - v_{i,2j}$, $v_{i,2j+1}v_{i+1,2j+1}v_{i,2j+2}$, $v_{i,2j+1}v_{i,2j+2}v_{i-1,2j+2}$, $v_{i,2j+1}v_{i-1,2j+2}v_{i-1,2j+1}$ for all $j \in \mathbb{Z}$. Then the mapping $f : V(X) \to V(E_1)$, given by $f(v_{k,t}) = u_{k,t}$ for $k, t \in \mathbb{Z}$, is an isomorphism. This proves the lemma. $\square$

Proof of Theorem 1.6 Let X be an equivelar map of type $[6^3]$ on the torus. Let Y be the dual of X. Then Y is an equivelar map of type $[3^6]$ on the torus and $\mathrm{Aut}(Y) \equiv \mathrm{Aut}(X)$.

By Lemma 3.2, $\mathrm{Aut}(Y)$ acts face-transitively on Y. These imply, $\mathrm{Aut}(X)$ acts vertex-transitively on X. So, X is vertex-transitive.

Now, assume that X is a semi-equivelar map of type $[3^3, 4^2]$ on the torus. Since $\mathbb{R}^2$ is the universal cover of the torus, by pulling back X [using similar arguments as in the proof of Theorem 3 in Spanier (1966, Page 144)], we get a semi-equivelar map $\widetilde{X}$ of type $[3^3, 4^2]$ on $\mathbb{R}^2$ and a polyhedral covering map $\eta_1 \colon \widetilde{X} \to X$. Because of Lemma 3.4, we may assume that $\widetilde{X} = E_1$. Let Γ_1 be the group of covering transformations. Then $|X| = |E_1|/\Gamma_1$.

Let V_1 be the vertex set of E_1. We take origin $(0, 0)$ is the middle point of the line segment joining $u_{0,0}$ and $u_{1,1}$. Let $a = u_{1,0} - u_{0,0}$, $b = u_{0,2} - u_{0,0} \in \mathbb{R}^2$. Then $H_1 := \langle x \mapsto x + a, y \mapsto y + b\rangle$ is the group of all the translations of E_1. Under the action of H_1, vertices form two orbits. Consider the subgroup G_1 of $\mathrm{Aut}(E_1)$ generated by H_1 and the map $x \mapsto -x$. So,

$$G_1 = \{\alpha : x \mapsto \varepsilon x + ma + nb \,:\, \varepsilon = \pm 1, m, n \in \mathbb{Z}\} \cong H_1 \rtimes \mathbb{Z}_2.$$

Clearly, G_1 acts vertex-transitively on E_1.

Claim. If $K \leq H_1$ then $K \trianglelefteq G_1$.

Let $g \in G_1$ and $k \in K$. Assume $g(x) = \varepsilon x + ma + nb$ and $k(x) = x + pa + qb$ for some $m, n, p, q \in \mathbb{Z}$ and $\varepsilon \in \{1, -1\}$. Then $(g \circ k \circ g^{-1})(x) = (g \circ k)(\varepsilon(x - ma - nb)) = g(\varepsilon(x - ma - nb) + pa + qb) = x - ma - nb + \varepsilon(pa + qb) + ma + nb = x + \varepsilon(pa + qb) = k^{\varepsilon}(x)$. Thus, $g \circ k \circ g^{-1} = k^{\varepsilon} \in K$. This proves the claim.

For $\sigma \in \Gamma_1$, $\eta_1 \circ \sigma = \eta_1$. So, σ maps a face of the map E_1 in $\mathbb{R}^2$ to a face of E_1 (in $\mathbb{R}^2$). This implies that σ induces an automorphism σ of E_1. Thus, we can identify Γ_1 with a subgroup of $\mathrm{Aut}(E_1)$. So, X is a quotient of E_1 by the subgroup Γ_1 of $\mathrm{Aut}(E_1)$, where Γ_1 has no fixed element (vertex, edge or face). Hence Γ_1 consists of translations and glide reflections. Since $X = E_1/\Gamma_1$ is orientable, Γ_1 does not contain any glide reflection. Thus $\Gamma_1 \leq H_1$. By the claim, Γ_1 is a normal subgroup of G_1. Since G_1 acts transitively on V_1, G_1/Γ_1 acts transitively on the vertices of E_1/Γ_1. Thus, X is vertex-transitive. □

4 Examples of maps on the torus and Klein bottle

Example 4.1 Eight types of semi-equivelar maps on the torus given in Fig. 3.

It follows from Theorem 1.6 that the map T_1 is vertex-transitive.

Example 4.2 Ten types of semi-equivelar maps on the Klein bottle given in Fig. 4.

In the next two proofs, we denote the n-cycle whose edges are $u_1u_2, \ldots, u_{n-1}u_n$, u_nu_1 by $C_n(u_1, \ldots, u_n)$. This helps us to compare different sizes of cycles.

Lemma 4.3 *The semi-equivelar maps $T_2, \ldots, T_8$ in Example 4.1 are not vertex-transitive.*

Proof Let $\mathcal{G}_2$ be the graph whose vertices are the vertices of T_2 and edges are the diagonals of 4-gons of T_2. Then $\mathcal{G}_2$ is a 2-regular graph. Hence, $\mathcal{G}_2$ is a disjoint

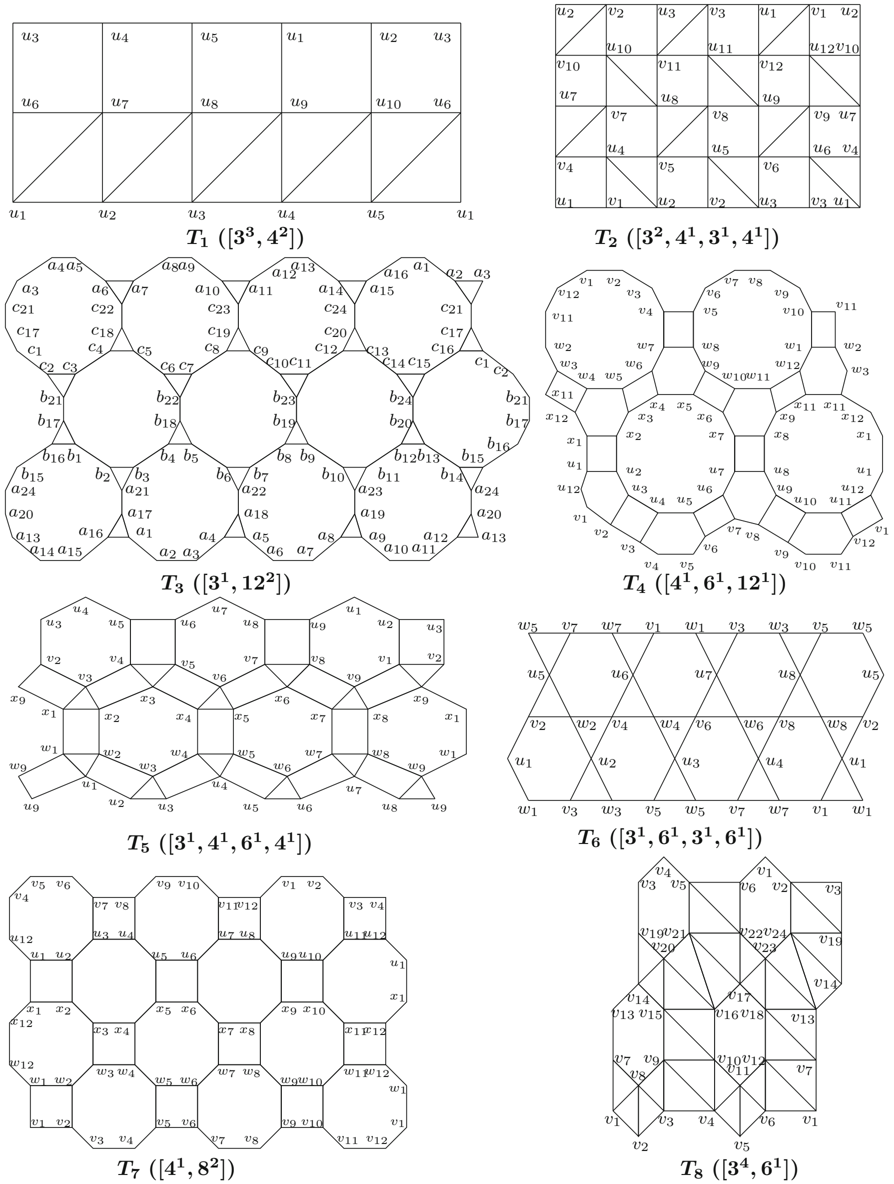


Fig. 3 Semi-equivelar maps on the torus

union of cycles. Clearly, $\mathrm{Aut}(T_2)$ acts on $\mathcal{G}_2$. If the action of $\mathrm{Aut}(T_2)$ is vertex-transitive on T_2 then it would be vertex-transitive on $\mathcal{G}_2$. But this is not possible since $C_4(u_1, u_4, u_8, u_{11})$, $C_{12}(v_1, v_4, v_9, v_{12}, v_3, v_6, v_8, v_{11}, v_2, v_5, v_7, v_{10})$ are components of $\mathcal{G}_2$ of different sizes.

Let $\mathcal{G}_3$ be the graph whose vertices are the vertices of T_3 and edges are the long diagonals of 12-gons of T_3. Then $\mathcal{G}_3$ is a 2-regular graph. Hence, $\mathcal{G}_3$ is a disjoint

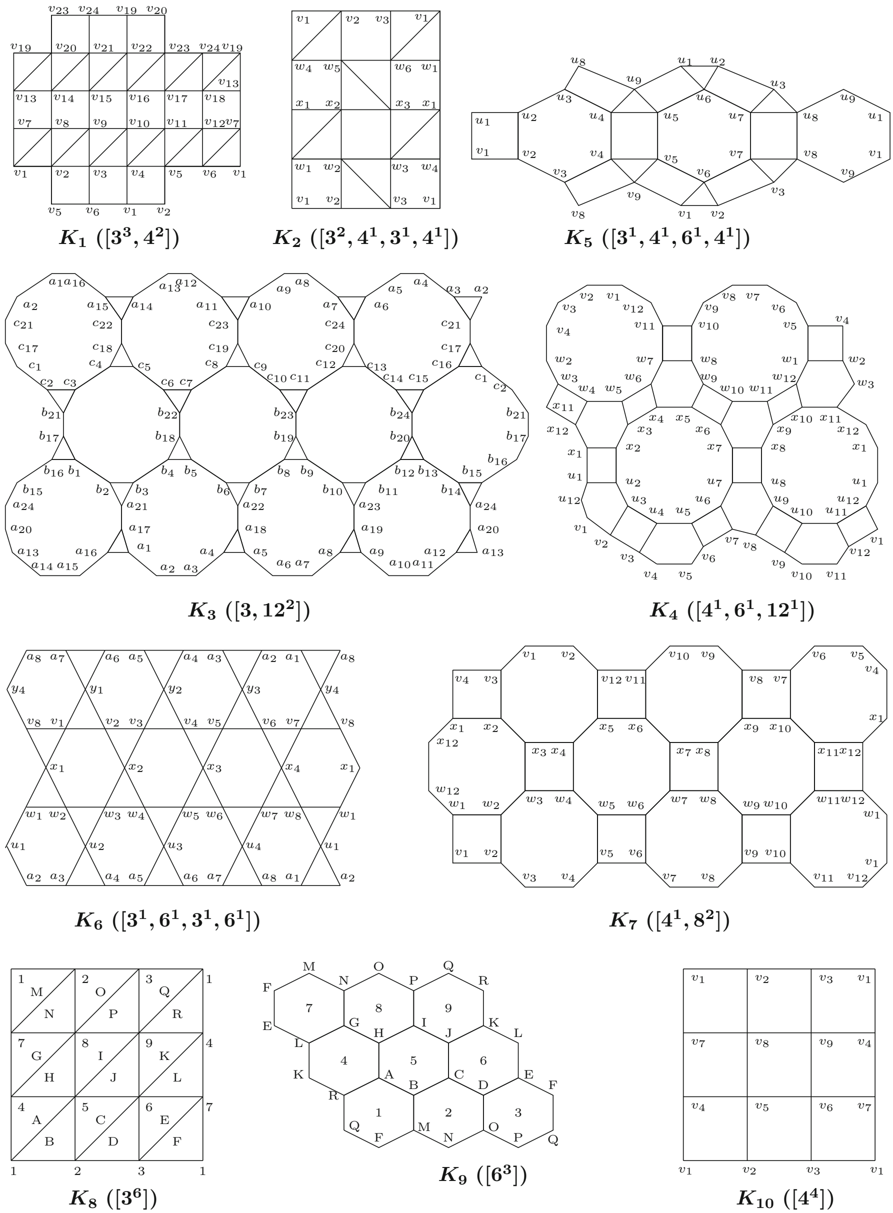


Fig. 4 Semi-eqivelar maps on the Klein bottle

union of cycles. Clearly, $\mathrm{Aut}(T_3)$ acts on $\mathcal{G}_3$. If the action of $\mathrm{Aut}(T_3)$ is vertex-transitive on T_3 then it would be vertex-transitive on $\mathcal{G}_3$. But this is not possible since $C_4(a_{17}, a_{22}, a_{19}, a_{24})$ and $C_{12}(c_1, a_6, b_9, c_{14}, a_1, b_6, c_9, a_{14}, b_1, c_6, a_9, b_{14})$ are components of $\mathcal{G}_3$ of different sizes.

Let $\mathcal{G}_4$ be the graph whose vertices are the vertices of T_4 and edges are the diagonals of 4-gons and long diagonals of 12-gons of T_4. Then $\mathcal{G}_4$ is a 2-regular graph. Clearly, $\mathrm{Aut}(T_4)$ acts on $\mathcal{G}_4$. If the action of $\mathrm{Aut}(T_4)$ is vertex-transitive on T_4 then it would be vertex-transitive on $\mathcal{G}_4$. But this is not possible since $C_8(v_2, u_4, x_5, w_{10}, v_8, u_{10}, x_{11}, w_4)$ and $C_4(x_1, u_2, x_7, u_8)$ are components of $\mathcal{G}_4$ of different sizes.

Let $\mathcal{G}_5$ be the graph whose vertices are the vertices of T_5 and edges are the diagonals of 4-gons of T_5. Then $\mathcal{G}_5$ is a 2-regular graph. Hence, $\mathcal{G}_5$ is a disjoint union of cycles. Clearly, $\mathrm{Aut}(T_5)$ acts on $\mathcal{G}_5$. If the action of $\mathrm{Aut}(T_5)$ is vertex-transitive on T_5 then it would be vertex-transitive on $\mathcal{G}_5$. But this is not possible since $C_6(x_9, v_3, x_3, v_6, x_6, v_9)$ and $C_4(u_2, v_2, x_1, w_2)$ are components of $\mathcal{G}_5$ of different sizes.

Let $\mathcal{G}_6$ be the graph whose vertices are the vertices of T_6 and edges are the long diagonals of 6-gons of T_6. Then $\mathcal{G}_6$ is a 2-regular graph. Hence, $\mathcal{G}_6$ is a disjoint union of cycles. Clearly, $\mathrm{Aut}(T_6)$ acts on $\mathcal{G}_6$. If the action of $\mathrm{Aut}(T_6)$ is vertex-transitive on T_6 then it would be vertex-transitive on $\mathcal{G}_6$. But this is not possible since $C_8(w_1, w_2, w_7, w_8, w_5, w_6, w_3, w_4)$ and $C_4(u_1, u_2, u_3, u_4)$ are components of $\mathcal{G}_6$ of different sizes.

Let $\mathcal{G}_7$ be the graph whose vertices are the vertices of T_7 and edges are the diagonals of 4-gons and common edges between any two 8-gons of T_7. Then $\mathcal{G}_7$ is a 2-regular graph. Hence, $\mathcal{G}_7$ is a disjoint union of cycles. Clearly, $\mathrm{Aut}(T_7)$ acts on $\mathcal{G}_7$. If the action of $\mathrm{Aut}(T_7)$ is vertex-transitive on T_7 then it would be vertex-transitive on $\mathcal{G}_7$. But this is not possible since $C_8(v_1, w_2, w_3, x_4, x_5, u_6, u_7, v_{12})$ and $C_{24}(v_2, w_1, w_{12}, x_{11}, x_{10}, u_9, u_8, v_{11}, v_{10}, w_9, w_8, x_7, x_6, u_5, u_4, v_7, v_6, w_5, w_4, x_3, x_2, u_1, u_{12}, v_3)$ are components of $\mathcal{G}_7$ of different sizes.

We call an edge uv of T_8 *nice* if at u (respectively, at v) three 3-gons containing u (respectively, v) lie on one side of uv and one on the other side of uv. (For example, $v_{10}v_{15}$ is nice). Observe that there is exactly one nice edge in T_8 through each vertex. Let $\mathcal{G}_8$ be the graph whose vertices are the vertices of T_8 and edges are the nice edges and the long diagonals of 6-gons. Then $\mathcal{G}_8$ is a 2-regular graph. Hence, $\mathcal{G}_8$ is a disjoint union of cycles. Clearly, $\mathrm{Aut}(T_8)$ acts on $\mathcal{G}_8$. If the action of $\mathrm{Aut}(T_8)$ is vertex-transitive on T_8 then it would be vertex-transitive on $\mathcal{G}_8$. But this is not possible since $C_4(v_7, v_{15}, v_{10}, v_{18})$ and $C_8(v_1, v_{23}, v_{17}, v_{11}, v_4, v_{20}, v_{14}\ v_8)$ are components of $\mathcal{G}_8$ of different sizes. □

Proof of Theorem 1.7 The result follows from Lemma 4.3. □

Lemma 4.4 *The maps $K_1, \ldots, K_{10}$* in Example 4.2 *are not vertex-transitive.*

Proof Let $\mathcal{H}_1$ be the graph whose vertices are the vertices of K_1 and edges are the diagonals of 4-gons of K_1. Then $\mathcal{H}_1$ is a 2-regular graph. Hence, $\mathcal{H}_1$ is a disjoint union of cycles. Clearly, $\mathrm{Aut}(K_1)$ acts on $\mathcal{H}_1$. If the action of $\mathrm{Aut}(K_1)$ is vertex-transitive on K_1 then it would be vertex-transitive on $\mathcal{H}_1$. But this is not possible since $C_6(v_7, v_{14}, v_9, v_{16}, v_{11}, v_{18})$ and $C_3(v_{20}, v_{24}, v_{22})$ are two components of $\mathcal{H}_1$ of different sizes.

There are exactly two induced 3-cycles in K_2, namely, $C_3(x_1, x_2, x_3)$ and $C_3(v_1, v_2, v_3)$. So, some vertices of K_2 are in an induced 3-cycle and some are not. Therefore, the action of $\mathrm{Aut}(K_2)$ on K_2 can not be vertex-transitive.

Like $\mathcal{G}_3$ in the proof of Lemma 4.3, let $\mathcal{H}_3$ be the graph whose vertices are the vertices of K_3 and edges are the long diagonals of 12-gons of K_3. Then, $\mathrm{Aut}(K_3)$ acts on the

2-regular graph $\mathcal{H}_3$. If the action of $\mathrm{Aut}(K_3)$ is vertex-transitive on K_3 then it would be vertex-transitive on $\mathcal{H}_3$. But this is not possible since $C_4(a_{17}, a_{22}, a_{19}, a_{24})$ and $C_{24}(a_3, b_4, c_3, a_1, b_6, c_9, a_7, b_8, c_7, a_{13}, b_2, c_5, a_{11}, b_{12}, c_{11}, a_9, b_{14}, c_1, a_{15}, b_{16}, c_{15}, a_5, b_{10}, c_{13})$ are components of $\mathcal{H}_3$ of different sizes.

Let $\mathcal{H}_4$ be the graph whose vertices are the vertices of K_4 and edges are the diagonals of 4-gons and long diagonals of 12-gons of K_4 (like $\mathcal{G}_4$ in the proof of Lemma 4.3). Then, $\mathrm{Aut}(K_4)$ acts on the 2-regular graph $\mathcal{H}_4$. If the action of $\mathrm{Aut}(K_4)$ is vertex-transitive on K_4 then it would be vertex-transitive on $\mathcal{H}_4$. But this is not possible since $C_4(v_5, w_2, v_{11}, w_8)$ and $C_8(v_2, u_4, x_5, w_{10}, v_7, u_5, x_4, w_5)$ are components of $\mathcal{H}_4$ of different sizes.

Let $\mathcal{H}_5$ be the graph whose vertices are the vertices of K_5 and edges are the diagonals of 4-gons in K_5 (like $\mathcal{G}_5$). Then, $\mathrm{Aut}(K_5)$ acts on the 2-regular graph $\mathcal{H}_5$. If the action of $\mathrm{Aut}(K_5)$ is vertex-transitive on K_5 then it would be vertex-transitive on $\mathcal{H}_5$. But this is not possible since $C_{12}(v_1, u_2, u_7, v_8, v_4, u_5, u_1, v_2, v_7, u_8, u_4, v_5)$ and $C_3(u_3, u_9, u_6)$ are components of $\mathcal{H}_5$ of different sizes.

Let $\mathcal{H}_6$ be the graph whose vertices are the vertices of K_6 and edges are the long diagonals of 6-gons of K_6 (like $\mathcal{G}_6$). Then, $\mathrm{Aut}(K_6)$ acts on the 2-regular graph $\mathcal{H}_6$. If the action of $\mathrm{Aut}(K_6)$ is vertex-transitive on K_6 then it would be vertex-transitive on $\mathcal{H}_6$. But this is not possible since $C_{24}(a_2, w_2, v_2, a_5, w_3, v_1, a_8, w_8, v_8, a_7, w_5, v_3, a_6, w_6, v_6, a_1, w_7, v_5, a_4, w_4, v_4, a_3, w_1, v_7)$ and $C_4(u_1, u_2, u_3, u_4)$ are components of $\mathcal{H}_6$ of different sizes.

Let $\mathcal{H}_7$ be the graph whose vertices are the vertices of K_7 and edges are the diagonals of 4-gons and common edges between any two 8-gons in K_7 (like $\mathcal{G}_7$). Then $\mathrm{Aut}(K_7)$ acts on the 2-regular graph $\mathcal{H}_7$. If the action of $\mathrm{Aut}(K_7)$ is vertex-transitive on K_7 then it would be vertex-transitive on $\mathcal{H}_7$. But this is not possible since $C_{24}(v_1, w_2, w_3, x_4, x_5, v_{11}, v_{10}, w_9, w_8, x_7, x_6, v_{12}, v_2, w_1, w_{12}, x_{11}, x_{10}, v_8, v_9, w_{10}, w_{11}, x_{12}, x_1, v_3)$ and $C_{12}(v_5, w_6, w_7, x_8, x_9, v_7, v_6, w_5, w_4, x_3, x_2, v_4)$ are components of $\mathcal{H}_7$ of different sizes.

Let $\mathrm{Skel}_1(K_8)$ be the edge graph of K_8 and $\mathcal{N}_8$ be the non-edge graph (i.e., the complement of $\mathrm{Skel}_1(K_8)$) of K_8. If $\mathrm{Aut}(K_8)$ acts vertex-transitively then $\mathrm{Aut}(K_8)$ acts vertex-transitively on $\mathrm{Skel}_1(K_8)$ and hence on $\mathcal{N}_8$. But, this is not possible since $\mathcal{N}_8$ is the union of two cycles of different lengths, namely, $\mathcal{N}_8 = C_6(2, 4, 3, 5, 7, 9) \sqcup C_3(1, 6, 8)$.

Consider the triangles $C = 256$ and $O = 238$ in K_8. If there exists $\alpha \in \mathrm{Aut}(K_8)$ such that $\alpha(C) = O$ then α acts on $\mathcal{N}_8 = C_6(2, 4, 3, 5, 7, 9) \sqcup C_3(1, 6, 8)$ and hence $\alpha(6) = 8$, $\alpha(\{2, 5\}) = \{2, 3\}$. This is not possible, since 25 is a long diagonal in $C_6(2, 4, 3, 5, 7, 9)$ where as 23 is a short diagonal in $C_6(2, 4, 3, 5, 7, 9)$. Thus, the action of $\mathrm{Aut}(K_8)$ on K_8 is not face-transitive. Observe that K_9 is the dual of K_8. Hence the action of $\mathrm{Aut}(K_9) = \mathrm{Aut}(K_8)$ on K_9 is not vertex-transitive.

There are exactly four induced 3-cycles in K_{10}, namely, $C_3(v_1, v_2, v_3)$, $C_3(v_1, v_4, v_7)$, $C_3(v_2, v_5, v_8)$ and $C_3(v_3, v_6, v_9)$. Let $\mathcal{H}_{10} := C_3(v_1, v_2, v_3) \cup C_3(v_1, v_4, v_7) \cup C_3(v_2, v_5, v_8) \cup C_3(v_3, v_6, v_9)$. Clearly, $\mathrm{Aut}(K_{10})$ acts on $\mathcal{H}_{10}$. If the action of $\mathrm{Aut}(K_{10})$ is vertex-transitive on K_{10} then it would be vertex-transitive on $\mathcal{H}_{10}$. But this is not possible since the degrees of all the vertices in $\mathcal{H}_{10}$ are not same. □

Proof of Theorem 1.8 The result follows from Lemma 4.4. □

5 Proof of Theorem 1.9

Lemma 5.1 *Let X be a map on the 2-disk $\mathbb{D}^2$ whose faces are triangles and quadrangles. For a vertex x of X, let $n_3(x)$ and $n_4(x)$ be the number of triangles and quadrangles through x respectively. Then X does not satisfy all the following four properties.* (i) $(n_3(u), n_4(u)) = (3, 2)$ *for each internal vertex u,* (ii) $n_3(w) \leq 3$, $n_4(w) \leq 2$, $n_3(w) + n_4(w) \leq 4$, $(n_3(w), n_4(w)) \neq (3, 0), (0, 2)$ *for one vertex w on the boundary,* (iii) $(n_3(v), n_4(v)) = (1, 1)$ *or* $(2, 1)$ *for each boundary vertex $v \neq w$, and* (iv) $n_3(v_1) + n_3(v_2) = 3$ *for each boundary edge v_1v_2 not containing w.*

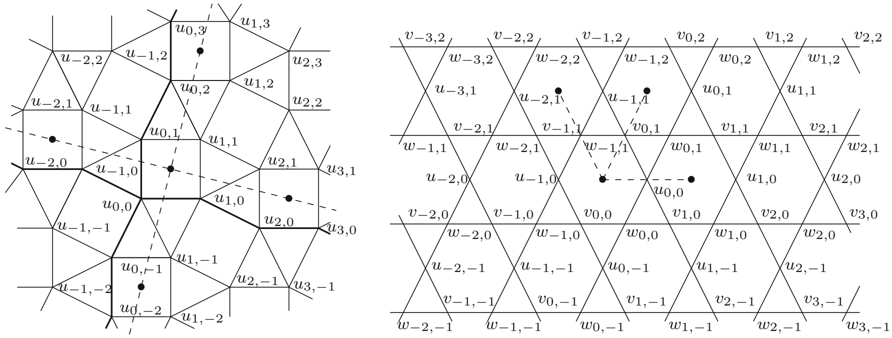
Proof Let f_0, f_1 and f_2 denote the number of vertices, edges and faces of X respectively. Let n_3 (resp., n_4) denote the total number of triangles (resp., quadrangles) in X. Let there be n internal vertices and $m + 1$ boundary vertices. So, $f_0 = n + m + 1$ and $f_2 = n_3 + n_4$.

Suppose X satisfies (i), (ii), (iii) and (iv). First assume that m is even. Let $m = 2p$. Then $n_3 = (3n + 2p + p + n_3(w))/3$ and $n_4 = (2n + 2p + n_4(w))/4$. So, $n_3(w) \in \{0, 3\}$ and $n_4(w) \in \{0, 2\}$. Since $1 \leq n_3(w) + n_4(w) \leq 4$, these imply $(n_3(w), n_4(w)) \in \{(3, 0), (0, 2)\}$, a contradiction. So, m is odd. Let $m = 2q + 1$. Then $n_4 = (2n + 2q + 1 + n_4(w))/4$. So, $n_4(w) = 1$. Now, $n_3 = (3n + 2q + q + \varepsilon + n_3(w))/3$, where $\varepsilon = 1$ or 2 depending on whether the number of boundary vertices which are in one triangle is $q + 1$ or q. So, $\varepsilon + n_3(w) = 3$. This implies that the alternate vertices on the boundary are in 1 and 2 triangles and the degrees of $q + 1$ boundary vertices are 4 and the degrees of the other $q + 1$ vertices are 3. Thus, $f_2 = (n + q + 1)/2 + (n + q + 1)$ and $f_1 = (5n + 4(q + 1) + 3(q + 1))/2$. Then $f_0 - f_1 + f_2 = (n + 2q + 2) - (5n + 7q + 7)/2 + (3n + 3q + 3)/2 = 0$. This is not possible since the Euler characteristic of the 2-disk $\mathbb{D}^2$ is 1. This completes the proof. □

Lemma 5.2 *Let X be a map on the 2-disk $\mathbb{D}^2$ whose faces are triangles and hexagons. For a vertex x of X, let $n_3(x)$ and $n_6(x)$ be the number of triangles and hexagons through x respectively. Then X does not satisfy all the following three properties.* (i) $(n_3(u), n_6(u)) = (2, 2)$ *for each internal vertex u,* (ii) $n_3(w), n_6(w) \leq 2$, $1 \leq n_3(w)+n_6(w) \leq 3$, *for one vertex w on the boundary, and* (iii) $(n_3(v), n_6(v)) = (1, 1)$ *for each boundary vertex $v \neq w$.*

Proof Let f_0, f_1 and f_2 denote the number of vertices, edges and faces of X respectively. Let n_3 (resp., n_6) denote the total number of triangles (resp., hexagons) in X. Let there be n internal vertices and $m + 1$ boundary vertices. So, $f_0 = n + m + 1$ and $f_2 = n_3 + n_6$.

Suppose X satisfies (i), (ii) and (iii). Then $n_3 = (2n + m + n_3(w))/3$ and $n_6 = (2n + m + n_6(w))/6$. So, $n_6(w) - n_3(w) = 6n_6 - 3n_3 = 3(2n_6 - n_3)$. Since $0 \leq n_3(w), n_6(w) \leq 2$, these imply $n_6(w) - n_3(w) = 0$. So, $n_6(w) = n_3(w)$. Since $1 \leq n_3(w) + n_4(w) \leq 3$, these imply that $n_6(w) = n_3(w) = 1$. Thus, the exceptional vertex is like other boundary vertices. Therefore, each boundary vertex is in three edges and hence $f_1 = (4n + 3(m + 1))/2$. So, $m + 1$ is even, say $m + 1 = 2\ell$.



a Snub square tiling E_2 **b** Trihexagonal tiling E_6

Fig. 5 Two Archimedean tiling of the plane

Thus, $f_1 = 2n + 3\ell$. Now, since $n_6(w) = n_3(w) = 1$, $f_2 = n_3 + n_6 = (2n + m + 1)/3 + (2n + m + 1)/6 = (2n + m + 1)/2 = n + \ell$. Then $f_0 - f_1 + f_2 = (n + 2\ell) - (2n + 3\ell) + (n + \ell) = 0$. This is not possible since the Euler characteristic of the 2-disk $\mathbb{D}^2$ is 1. This completes the proof. □

Lemma 5.3 *Let E_2 and E_6 be the Archimedean tilings of $\mathbb{R}^2$ given in Fig. 5. Let Y be a semi-equivelar map on the plane $\mathbb{R}^2$.* (a) *If the type of Y is* $[3^2, 4^1, 3^1, 4^1]$ *then* $Y \cong E_2$. (b) *If the type of Y is* $[3^1, 6^1, 3^1, 6^1]$ *then* $Y \cong E_6$.

Proof If the type of Y is $[3^2, 4^1, 3^1, 4^1]$ then by the similar arguments as in the proof of Lemma 3.4, we get $Y \cong E_2$. In this case, to show that the path in Y (similar to the path P_0 in the proof of Lemma 3.4) corresponding to the path $\cdots - u_{-2,0} - u_{-1,0} - u_{0,0} - u_{1,0} - u_{2,0} - u_{3,0} - \cdots$ in E_2 is an infinite path, we need to use that there is no map on the 2-disk $\mathbb{D}^2$ which satisfies (i)–(iv) of Lemma 5.1.

If the type of Y is $[3^1, 6^1, 3^1, 6^1]$ then by the similar arguments as in the proof of Lemma 3.4, we get $Y \cong E_6$. In this case, to show that the path in Y corresponding to the path $\cdots - v_{-2,0} - w_{-2,0} - v_{-1,0} - w_{-1,0} - v_{0,0} - w_{0,0} - v_{1,0} - w_{1,0} - v_{2,0} - w_{2,0} - \cdots$ in E_6 is an infinite path, we need to use that there is no map on the 2-disk $\mathbb{D}^2$ which satisfies (i)–(iii) of Lemma 5.2. □

Proof of Theorem 1.9 Let X be a semi-equivelar map of type $[3^2, 4^1, 3^1, 4^1]$ on the torus. By similar arguments as in the proof of Theorem 1.6 and using Lemma 5.3(a), we assume that there exists a polyhedral covering map $\eta_2 \colon E_2 \to X$. Let Γ_2 be the group of covering transformations. Then $|X| = |E_2|/\Gamma_2$.

Let V_2 be the vertex set of E_2. We take origin $(0, 0)$ is the middle point of the line segment joining $u_{0,0}$ and $u_{1,1}$ (see Fig. 5a). Let $a = u_{2,0} - u_{0,0}$, $b = u_{0,2} - u_{0,0} \in \mathbb{R}^2$. Consider the translations $x \mapsto x + a$, $x \mapsto x + b$. Then $H_2 := \langle x \mapsto x + a, x \mapsto x + b \rangle$ is the group of all the translations of E_2. Under the action of H_2, vertices form four orbits. Consider the subgroup G_2 of $\mathrm{Aut}(E_2)$ generated by H_2 and the map (the half rotation) $x \mapsto -x$. So,

$$G_2 = \{\alpha : x \mapsto \varepsilon x + ma + nb \;:\; \varepsilon = \pm 1, m, n \in \mathbb{Z}\} \cong H_2 \rtimes \mathbb{Z}_2.$$

Clearly, under the action of G_2, vertices of E_2 form two orbits. The two orbits are $O_1 = \{u_{i,j} : i + j \text{ is odd}\}$ and $O_2 = \{u_{i,j} : i + j \text{ is even}\}$.

Claim. If $K \leq H_2$ then $K \trianglelefteq G_2$.

Let $g \in G_2$ and $k \in K$. Assume $g(x) = \varepsilon x + ma + nb$ and $k(x) = x + pa + qb$ for some $m, n, p, q \in \mathbb{Z}$ and $\varepsilon \in \{1, -1\}$. Then $(g \circ k \circ g^{-1})(x) = (g \circ k)(\varepsilon(x - ma - nb)) = g(\varepsilon(x - ma - nb) + pa + qb) = x - ma - nb + \varepsilon(pa + qb) + ma + nb = x + \varepsilon(pa + qb) = k^{\varepsilon}(x)$. Thus, $g \circ k \circ g^{-1} = k^{\varepsilon} \in K$. This proves the claim.

For $\sigma \in \Gamma_2$, $\eta_2 \circ \sigma = \eta_2$. So, σ maps a face of the map E_2 (in $\mathbb{R}^2$) to a face of E_2 (in $\mathbb{R}^2$). This implies that σ induces an automorphism σ of E_2. Thus, we can identify Γ_2 with a subgroup of $\mathrm{Aut}(E_2)$. So, X is a quotient of E_2 by a subgroup Γ_2 of $\mathrm{Aut}(E_2)$, where Γ_2 has no fixed element (vertex, edge or face). Hence Γ_2 consists of translations and glide reflections. Since $X = E_2/\Gamma_2$ is orientable, Γ_2 does not contain any glide reflection. Thus $\Gamma_2 \leq H_2$. By the claim, Γ_2 is a normal subgroup of G_2. Thus, G_2/Γ_2 acts on $X = E_2/\Gamma_2$. Since O_1 and O_2 are the G_2-orbits, it follows that $\eta_2(O_1)$ and $\eta_2(O_2)$ are the (G_2/Γ_2)-orbits. Since the vertex set of X is $\eta_2(V_2) = \eta_2(O_1) \sqcup \eta_2(O_2)$ and $G_2/\Gamma_2 \leq \mathrm{Aut}(X)$, part (a) follows.

Let X be a semi-equivelar map of type $[3^1, 6^1, 3^1, 6^1]$ on the torus. By similar arguments as in the proof of Theorem 1.6 and using Lemma 5.3 (b), we assume that there exists a polyhedral covering map $\eta_6 \colon E_6 \to X$. Let Γ_6 be the group of covering transformations. Then $|X| = |E_6|/\Gamma_6$.

Let V_6 be the vertex set of E_6. We take origin (0, 0) is the middle point of the line segment joining $u_{-1,0}$ and $u_{0,0}$ (see Fig. 5b). Let $r = u_{1,0} - u_{0,0} = v_{1,0} - v_{0,0} = w_{1,0} - w_{0,0}$, $s = u_{0,1} - u_{0,0} = v_{0,1} - v_{0,0} = w_{0,1} - w_{0,0}$ and $t = u_{-1,1} - u_{0,0} = v_{-1,1} - v_{0,0} = w_{-1,1} - w_{0,0}$. Consider the translations $x \mapsto x + r$, $x \mapsto x + s$ and $x \mapsto x + t$. Then $H_6 := \langle x \mapsto x + r, x \mapsto x + s, x \mapsto x + t\rangle$ is the group of all the translations of E_6. Since H_6 is a group of translations it is abelian. Under the action of H_6, vertices form three orbits. The orbits are $O_u = \{u_{i,j} : i, j \in \mathbb{Z}\}$, $O_v = \{v_{i,j} : i, j \in \mathbb{Z}\}$, $O_w = \{w_{i,j} : i, j \in \mathbb{Z}\}$.

As before, we can identify Γ_6 with a subgroup of H_6. So, X is a quotient of E_6 by a group Γ_6, where $\Gamma_6 \leq H_6 \leq \mathrm{Aut}(E_6)$. Since H_6 is abelian, Γ_6 is a normal subgroup of H_6. Thus, H_6/Γ_6 acts on $X = E_6/\Gamma_6$. Since O_u, O_v and O_w are the H_6-orbits, it follows that $\eta_6(O_u)$, $\eta_6(O_v)$ and $\eta_6(O_w)$ are the (H_6/Γ_6)-orbits. Since the vertex set of X is $\eta_6(V_6) = \eta_6(O_u) \sqcup \eta_6(O_v) \sqcup \eta_6(O_w)$ and $H_6/\Gamma_6 \leq \mathrm{Aut}(X)$, part (b) follows. □

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