Applied Mathematics & Optimization (2022) 85:27
https://doi.org/10.1007/s00245-022-09843-6

Exact Internal Controllability for a Problem with Imperfect Interface

S. Monsurrò[1] · A. K. Nandakumaran[2] · C. Perugia[3]

Accepted: 19 October 2021

Abstract
In this paper, we study the internal exact controllability for a second order linear evolution equation defined in a two-component domain. On the interface, we prescribe a jump of the solution proportional to the conormal derivatives, meanwhile a homogeneous Dirichlet condition is imposed on the exterior boundary. Due to the geometry of the domain, we apply controls through two regions which are neighborhoods of a part of the external boundary and of the whole interface, respectively. Our approach to internal exact controllability consists in proving an observability inequality by using the Lagrange multipliers method. Eventually, we apply the Hilbert Uniqueness Method, introduced by Lions, which leads to the construction of the exact control through the solution of an adjoint problem. Finally, we find a lower bound for the control time depending not only on the geometry of our domain and on the matrix of coefficients of our problem but also on the coefficient of proportionality of the jump with respect to the conormal derivatives.

Keywords Exact controllability · Second order hyperbolic equations · Imperfect interface condition · HUM

Mathematics Subject Classification 35B27 · 35Q93 · 93B05 · 93B05 · 35B37 · 35L20

✉ C. Perugia
cperugia@unisannio.it

S. Monsurrò
smonsurro@unisa.it

A. K. Nandakumaran
nands@iisc.ac.in

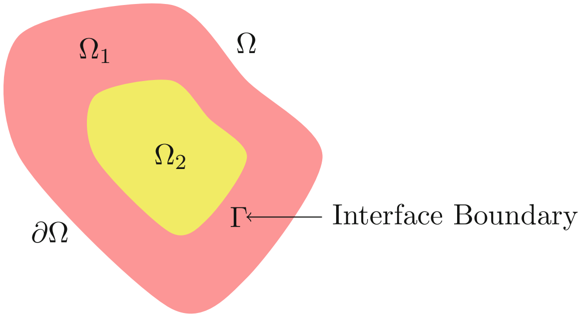
1 Department of Mathematics, Università di Salerno, 84084 Fisciano, SA, Italy

2 Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

3 Department of Science and Technology, University of Sannio, Via de Sanctis, 82100 Benevento, Italy

Published online: 10 May 2022

Fig. 1 $\Omega_2 \subset\subset \Omega$ and $\Omega_1 = \Omega \backslash \overline{\Omega}_2$



1 Introduction

In this paper, we study the internal exact controllability of an imperfect transmission hyperbolic problem. More specifically, we consider a bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, consisting of two sets Ω_1 and Ω_2, where Ω_2 is compactly contained in Ω and $\Omega_1 = \Omega \setminus \overline{\Omega}_2$. Thus our domain has an external boundary $\partial\Omega$ and an interface boundary Γ (see Fig. 1).

The hyperbolic problem is defined in the set Ω, with appropriate interface and boundary conditions on Γ and on $\partial\Omega$. Namely, on the interface separating the two components, we prescribe a jump of the solution proportional to the conormal derivatives, meanwhile a homogeneous Dirichlet condition is imposed on the exterior boundary. From the physical point of view, this problem describes the wave propagation in a composite made up of two materials with different systems in each component connected through the interface. The jump on Γ is the mathematical interpretation of imperfect interface characterized by the discontinuity of the displacement (see [1, 7, 17, 22, 33, 36, 39, 46, 47, 50, 54, 60, 61] and references therein).

The issue of exact controllability consists in acting on the trajectories of an evolution system by means of a control either through the system internally (distributed control) or through the boundary (boundary control) and asking if, given a time interval $[0, T]$, is it possible to find a control (or set of controls) leading the system to a desired state at time T, for all initial data. In a suitable functional setting, the problem of exact controllability reduces to that of observability. Roughly speaking, observability consists in deriving an estimate for the energy of an uncontrolled system, at time $t = 0$, in terms of partial measurements of its solution done on the control region. This estimate easily implies an upper bound for the norm of the initial data of the uncontrolled problem.

The observability inequality, far from being obvious, forces the control set to satisfy suitable geometric conditions. Indeed, in [6], through the microlocal approach, the authors proved that when considering a regular domain, the observability inequality holds if and only if every ray of geometric optics, propagating into the domain and reflecting on its boundary, enters the control region in time less than the control time T.

In this paper, we do not require any regularity on $\partial\Omega$ and we make use of Lagrange multipliers method to prove the above mentioned observability inequality. In general,

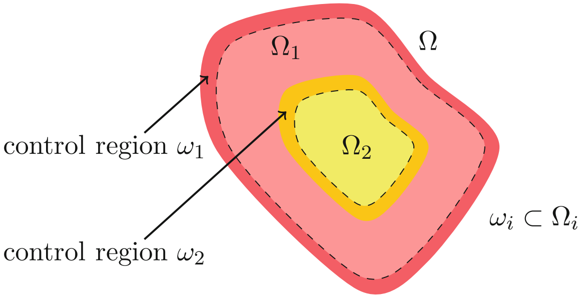


Fig. 2 ω_1-neighbourhood of $\partial\Omega$. ω_2-neighbourhood of $\Gamma = \partial\Omega_2$

when applying this technique, the assumptions on the geometry of the control region are very restrictive. They require the control set to be a neighborhood of parts of the boundary having specific structures. In our case, due to the geometry of the domain, we need to introduce a further control set which is a neighborhood of the interface Γ. More precisely, we apply controls through two regions $\omega_1 \subset \Omega_1$ and $\omega_2 \subset \Omega_2$ which are neighborhoods of a part of $\partial\Omega$ and of the whole interface Γ, respectively. In fact, ω_1 (see Fig. 2 for control regions ω_1 and ω_2) may also be a full neighborhood, for example in the case of a circle. Other novelties in our framework are the jump of the solution on the interface Γ and the resulting presence of a non-constant coefficients matrix. Due to the imperfect interface, when proving observability inequality, some difficulties arise in estimating specific surface integrals. Moreover, as usual in the hyperbolic framework, due to the finite speed of propagation of waves, the control time T in our observability inequality has to be large enough. Indeed, the control acting on ω_1 and ω_2 cannot transfer the information immediately to the whole domain Ω. However, unlike classical cases, we find a lower bound for the control time T depending not only on the geometry of our domain and on the matrix of coefficients of our problem but also on the coefficient of proportionality of the jump with respect to the conormal derivatives. The exact details are given in Sect. 3.

Once obtained the observability inequality, in order to find the exact control, we use a constructive method, introduced by Lions in [42, 43], known as Hilbert Uniqueness Method (HUM for short). The main feature is to build a control through the solution of a hyperbolic problem associated to suitable initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem by means of a functional, which turns out to be an isomorphism, thanks to the observability estimate (see Sect. 4). Let us recall that the control obtained by HUM is also an energy minimizing control.

The paper is organized as follows. In Sect. 2, we introduce the setting of the evolution problem, recall the definitions and some properties of the appropriate functional spaces required for the solutions of interface problems under consideration. For more details, we refer the reader to [22, 50] where the elliptic case is considered. Since the initial data of our problem are in a weak space, the related solution cannot be defined using the standard weak formulation. Thus, as usual when dealing with controllability problems, we need to apply the so called transposition method (see [45,Chap. 3, Sect. 9]). We also give the definition of exact controllability.

Section 3 is the core of the paper which is devoted to the proof of the observability inequality (see Lemma 3.9). To this aim, we adapt to our context some arguments introduced in [43] and [44]. By means of the Lagrange multipliers method, we derive an important identity (see Lemma 3.2). Then, we specify the required geometrical and topological assumptions on the position of the observer and on the control sets ω_1 and ω_2 (see Definitions 3.4 and 3.5) and apply the above mentioned identity in order to establish some crucial inequalities, given in Lemmas 3.3, 3.6 and 3.7. Finally, in Lemma 3.8, we find the lower bound for the control time T. Taking into account the way ω_1 is constructed, it is possible to get different regions of controllability depending on the observer point x^0. The significance of the point x^0 will be clear in Sect. 3. In Sect. 4, via HUM, we prove the exact controllability result by constructing the suitable isomorphism that allows us to identify the exact control.

The pioneer studies on exact controllability for the wave equation with transmission conditions, via HUM, go back to [43], Chapter 6. Here J. L. Lions considers a Dirichlet problem with matrix constant on each component of the domain and a control on part of the external boundary. Later on, in [49] the authors deal with the case of a Neumann boundary value problem in the same framework. In [2, 5, 12, 16, 31, 32, 56, 59] optimal control and exact controllability problems in domains with highly oscillating boundary are studied. Moreover, we refer to [9–11, 42] for exact controllability of hyperbolic problems with oscillating coefficients in fixed and perforated domains, respectively and, [34–37, 54], respectively, for optimal control and exact controllability of hyperbolic problems in composites with imperfect interface. In [55], it has been analyzed the exact boundary controllability for the same imperfect transmission problem considered in the present paper. Further, in [40] the optimal control of rigidity parameters of thin inclusions in composite materials has been investigated. In [19–21] the authors, respectively, study the correctors and approximate control for a class of parabolic equations with interfacial contact resistance, whereas in [23] the approximate controllability of linear parabolic equations in perforated domains has been studied. In [64] (see also [63]), the author studies the approximate controllability of a parabolic problem with highly oscillating coefficients in a fixed domain. The null controllability of semilinear heat equations in a fixed domain has been studied in [38]. The exact controllability and exact boundary controllability for semilinear wave equations, respectively, can be found in [41, 62]. Finally, for what concerns transmission problems in the nonlinear case, we quote [24, 25] (see also [28, 29]).

2 Statement of the Problem

Let Ω be a connected open bounded subset of $\mathbb{R}^n$, $n \geq 2$. We denote by Ω_1 and Ω_2, two non-empty open connected and disjoint subsets of Ω such that $\overline{\Omega_2} \subset \Omega$ and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$. Let us assume that the interface $\Gamma = \partial\Omega_2$ separating the two components of Ω is Lipschitz continuous and observe that by construction one has

$$\partial\Omega \cap \Gamma = \emptyset. \tag{2.1}$$

Given $T > 0$, we set $Q_1 = \Omega_1 \times (0, T)$, $Q_2 = \Omega_2 \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ and $\Gamma_T = \Gamma \times (0, T)$.

This paper aims to study the internal exact controllability of a hyperbolic imperfect transmission problem defined in the above mentioned domain. More precisely, given two open subsets ω_i of Ω_i, $i = 1, 2$, and a control $\zeta := (\zeta_1, \zeta_2)$, we consider the problem

$$\begin{cases} u_1'' - \operatorname{div}(A(x)\nabla u_1) = \zeta_1 \chi_{\omega_1} & \text{in } \; Q_1, \\ u_2'' - \operatorname{div}(A(x)\nabla u_2) = \zeta_2 \chi_{\omega_2} & \text{in } \; Q_2, \\ A(x)\nabla u_1 n_1 = -A(x)\nabla u_2 n_2 & \text{on } \; \Gamma_T, \\ A(x)\nabla u_1 n_1 = -h(x)(u_1 - u_2) & \text{on } \; \Gamma_T, \\ u_1 = 0 & \text{on } \; \Sigma, \\ u_1(0) = U_1^0, \quad u_1'(0) = U_1^1 & \text{in } \; \Omega_1, \\ u_2(0) = U_2^0, \quad u_2'(0) = U_2^1 & \text{in } \; \Omega_2, \end{cases} \tag{2.2}$$

where for any fixed $i = 1, 2$, n_i is the unitary outward normal to Ω_i and χ_{ω_i} denotes the characteristic function of the set ω_i on which acts the control ζ_i. For simplicity, we denote the wave operator by $L = \partial_{tt} - \operatorname{div}(A(x)\nabla\ )$.

Let us recall the definitions of the required function spaces to study the interface problem under consideration. They were introduced for the first time in [50] and widely studied in [22] in the homogenization framework for the analogous stationary problem. Indeed, these spaces take into account the geometry of the domain as well as the boundary and interface conditions.

As observed in [8], the space

$$V = \{v_1 \in H^1(\Omega_1) \mid v_1 = 0 \text{ on } \partial\Omega\}$$

is a Banach space endowed with the norm

$$\|v_1\|_V = \|\nabla v_1\|_{L^2(\Omega_1)}.$$

Since we do not impose any regularity on the external boundary, the condition on $\partial\Omega$ in the definition of V has to be intended in a density sense. More precisely, in view of (2.1), V can be defined as the closure of the set of the functions in $C^\infty(\Omega_1)$ with a compact support contained in Ω with respect to the $H^1(\Omega_1)$-norm. We also set

$$H_\Gamma = \Big\{ v = (v_1, v_2) \mid v_1 \in V \text{ and } v_2 \in H^1(\Omega_2) \Big\}. \tag{2.3}$$

The space H_Γ is a Hilbert space when equipped with the norm

$$\|v\|_{H_\Gamma}^2 = \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2.$$

Indeed H_Γ can be identified with $V \times H^1(\Omega_2)$, the norm above defined on H_Γ being equivalent to the standard norm in $V \times H^1(\Omega_2)$ (see [26] for details). We denote the dual of H_Γ by $(H_\Gamma)'$. It follows that (see [19]), the norms of $(H_\Gamma)'$ and $V' \times (H^1(\Omega_2))'$

are equivalent. Moreover, if $v = (v_1, v_2) \in (H_\Gamma)'$ and $u = (u_1, u_2) \in H_\Gamma$, then

$$\langle v, u\rangle_{(H_\Gamma)', H_\Gamma} = \langle v_1, u_1\rangle_{V', V} + \langle v_2, u_2\rangle_{H^1(\Omega_2)', H^1(\Omega_2)} .$$

Remark 2.1 We point out that H_Γ is a separable and reflexive Hilbert space dense in $L^2(\Omega_1) \times L^2(\Omega_2)$. Furthermore, $H_\Gamma \subseteq L^2(\Omega_1) \times L^2(\Omega_2)$ with continuous imbedding. On the other hand, one has that $L^2(\Omega_1) \times L^2(\Omega_2) \subseteq (H_\Gamma)'$, with $L^2(\Omega_1) \times L^2(\Omega_2)$ separable Hilbert space. This means that the triple $(H_\Gamma, L^2(\Omega_1) \times L^2(\Omega_2), (H_\Gamma)')$ is an evolution triple. We refer the reader to [26, 27] for a detailed analysis on this aspect. Also note that, in fact $L^2(\Omega_1) \times L^2(\Omega_2)$ can be identified with $L^2(\Omega)$ itself by observing that $v = (v_1, v_2) \in L^2(\Omega_1) \times L^2(\Omega_2)$ if and only if $v = v_1 \chi_{\Omega_1} + v_2 \chi_{\Omega_2} \in L^2(\Omega)$. By the way, due to the nature of our problem, throughout this work we prefer to adopt the notation $v = (v_1, v_2) \in L^2(\Omega_1) \times L^2(\Omega_2)$.

Let us set

$$W = \Big\{ v = (v_1, v_2) \in L^2(0, T; V \times H^1(\Omega_2)) : \\ v' = (v_1', v_2') \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2)) \Big\}, \tag{2.4}$$

which is a Hilbert space if equipped with the norm

$$\|v\|_W = \|v_1\|_{L^2(0,T;V)} + \|v_2\|_{L^2(0,T;H^1(\Omega_2))} + \left\|v_1'\right\|_{L^2(0,T;L^2(\Omega_1))} \\ + \left\|v_2'\right\|_{L^2(0,T;L^2(\Omega_2))} .$$

Note that the control ζ is such that

$$\zeta \in W'. \tag{2.5}$$

We assume that the initial data of problem (2.2) are such that

$$\begin{cases} \text{(i)} U^0 = \left(U_1^0, U_2^0\right) \in L^2(\Omega_1) \times L^2(\Omega_2), \\ \text{(ii) } U^1 = \left(U_1^1, U_2^1\right) \in (H_\Gamma)'. \end{cases} \tag{2.6}$$

Further, we suppose that A is a symmetric matrix field and there exist constants $\alpha, \beta \in \mathbb{R}$, with $0 < \alpha < \beta$ such that

$$\begin{cases} \text{(i) } a_{ij}, \dfrac{\partial a_{ij}}{\partial x_k} \in L^\infty(\Omega), \quad 1 \le i, j, k \le n, \\ \text{(ii) } (A(x)\lambda, \lambda) \ge \alpha|\lambda|^2, \ |A(x)\lambda| \le \beta|\lambda|, \end{cases} \tag{2.7}$$

for every $\lambda \in \mathbb{R}^n$ and a.e. in Ω. We put

$$M = \max_{1 \le i,j,k \le n} \max_{x \in \Omega} \left| \frac{\partial a_{ij}}{\partial x_k} \right|. \tag{2.8}$$

The function h appearing in the interface condition satisfies

$$h \in L^{\infty}(\Gamma) \text{ and there exists } h_0 \in \mathbb{R} \text{ such that } 0 < h_0 < h(x) \text{ a.e. in } \Gamma. \tag{2.9}$$

Note that the initial data (2.6) are in a weak space, hence the solution of problem (2.2) cannot be defined using the standard weak formulation. We need to apply the so called transposition method (see [45,Chap. 3, Sect. 9]), usually used in controllability problems. In some sense, it is an adjoint method where the solution is defined via an adjoint problem which provides test functions. More precisely, we define the following standard adjoint problem: for every $g = (g_1, g_2) \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2))$, consider the backward problem

$$\begin{cases} L\psi_i \equiv \psi_i'' - \operatorname{div}(A(x)\nabla\psi_i) = g_i & \text{in } Q_i,\ i = 1, 2 \\ A(x)\nabla\psi_1\, n_1 = -A(x)\nabla\psi_2\, n_2 & \text{on } \Gamma_T, \\ A(x)\nabla\psi_1\, n_1 = -h(x)(\psi_1 - \psi_2) & \text{on } \Gamma_T, \\ \psi_1 = 0 & \text{on } \Sigma, \\ \psi_i(T) = \psi_i'(T) = 0 & \text{in } \Omega_i,\ i = 1, 2. \end{cases} \tag{2.10}$$

As observed in [26], thanks to Remark 2.1, by using an approach to standard evolutionary problems based on evolution triples (there are no weak data), the usual weak formulation of problem (2.10) is valid. Hence an abstract Galerkin's method provides the existence and uniqueness result for the weak solution in W of problem (2.10), together with the a priori estimate for the solution in W. For the sake of clarity, throughout the paper, we denote by $\psi(g) = (\psi_1(g), \psi_2(g))$, the solution of problem (2.10) and when there is no ambiguity, we omit the explicit dependence on the right hand member.

Now we give the definition of solution of (2.2) in the sense of transposition.

Definition 2.2 For any fixed $\left(U^0, U^1\right) \in \left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \times (H_\Gamma)'$, we say that a function $u = (u_1, u_2) \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2))$ is a solution of problem (2.2), in the sense of transposition, if it satisfies the identity

$$\begin{aligned} &\int_{Q_1} u_1 g_1 dxdt + \int_{Q_2} u_2 g_2 dxdt \\ &= -\int_{\Omega_1} U_1^0 \psi_1'(0)dx + \left\langle U_1^1, \psi_1(0)\right\rangle_{V',V} - \int_{\Omega_2} U_2^0 \psi_2'(0)dx \\ &\quad + \left\langle U_2^1, \psi_2(0)\right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} + \langle \zeta\chi_\omega, \psi\rangle_{W',W}, \end{aligned} \tag{2.11}$$

for all $g = (g_1, g_2) \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2))$, where ψ is the solution of problem (2.10). Here we have used the notation $\zeta\chi_\omega = (\zeta_1\chi_{\omega_1}, \zeta_2\chi_{\omega_2})$.

By classical results (see [45,Chap. 3, Sect. 9, Theorems 9.3 and 9.4]), problem (2.2) admits a unique solution $u \in C\left([0, T]; L^2(\Omega_1) \times L^2(\Omega_2)\right) \cap C^1\left([0, T]; (H_\Gamma)'\right)$

satisfying the estimate

$$\begin{aligned}\|u\|_{L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))} + \|u'\|_{L^\infty(0,T;(H_\Gamma)')} \leq C(\|U^0\|_{L^2(\Omega_1)\times L^2(\Omega_2)} \\ + \|U^1\|_{(H_\Gamma)'} + \|\zeta\chi_\omega\|_{W'}),\end{aligned} \tag{2.12}$$

with C positive constant.

We denote by $u(\zeta) = (u_1(\zeta), u_2(\zeta))$ the solution of problem (2.2) in the sense above defined and, when there is no ambiguity, we omit the explicit dependence on the control.

Now, let us give the definition of exact controllability.

Definition 2.3 System (2.2) is said to be exactly controllable at time $T > 0$, if for every $\left(U^0, U^1\right)$, $\left(Z^0, Z^1\right)$ in $\left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \times (H_\Gamma)'$, there exists a control ζ belonging to W' such that the corresponding solution u of problem (2.2) satisfies

$$u(T) = Z^0, \quad u'(T) = Z^1.$$

If the controllability is achieved for the zero (null) data $Z^0 = 0$, $Z^1 = 0$, then it is known as null controllability. Since our problem is linear and reversible in time (see [65]), it is sufficient to look for controls driving the system to rest. Hence, in the sequel we prove the existence of a control $\zeta \in W'$ of (2.2) such that $u(T) = u'(T) = 0$.

In this paper, we wish to prove a controllability result, but as remarked in the introduction, it is not possible to achieve controllability without additional assumptions. Namely, if Ω_2 is star-shaped with respect to a point $x^0 \in \Omega_2$ and under suitable geometrical assumptions on the sets ω_1 and ω_2, we are able to prove that system (2.2) is exact controllable for a time $T > 0$ sufficiently large (see Theorem 4.1). To this aim, we will use a constructive method known as the Hilbert Uniqueness Method introduced by Lions (see [42, 43]). We point out that the control obtained by HUM is also the energy minimizing control. HUM is fully a PDE based method and eventually it reduces to deriving the so-called observability estimate, which is the crucial point, corresponding to an uncontrolled problem (see (3.1)). To get the observability estimate, which is a delicate estimate from below, we need to establish some fundamental results based on the Lagrange multipliers method. These results are proved in the following section.

3 The Observability Inequality

For $T > 0$, we consider the following homogeneous imperfect transmission problem

$$\begin{cases} Lz_i \equiv z_i'' - \operatorname{div}(A(x)\nabla z_i) = 0 \text{ in } Q_i,\ i = 1,2 \\ A(x)\nabla z_1 n_1 = -A(x)\nabla z_2 n_2 \quad \text{on } \Gamma_T, \\ A(x)\nabla z_1 n_1 = -h(x)(z_1 - z_2) \text{ on } \Gamma_T, \\ z_1 = 0 \quad \text{on } \Sigma, \\ z_i(0) = z_i^0, \quad z_i'(0) = z_i^1 \quad \text{in } \Omega_i,\ i = 1,2 \end{cases} \tag{3.1}$$

with the initial data

$$z^0 = \left(z_1^0, z_2^0\right) \in H_\Gamma, \quad z^1 = \left(z_1^1, z_2^1\right) \in L^2\left(\Omega_1\right) \times L^2\left(\Omega_2\right), \tag{3.2}$$

where n_i is the unitary outward normal to Ω_i, $i = 1, 2$. The weak formulation of problem (3.1) is given by

$$\begin{cases} \text{Find} z = (z_1, z_2) \text{ in } W \text{ such that} \\ \langle z_1'', v_1\rangle_{V',V} + \langle z_2'', v_2\rangle_{(H^1(\Omega_2))',H^1(\Omega_2)} + \int_{\Omega_1} A(x)\nabla z_1 \nabla v_1 dx \\ \quad + \int_{\Omega_2} A(x)\nabla z_2 \nabla v_2 \, dx + \int_\Gamma h(x)(z_1 - z_2)(v_1 - v_2)\, d\sigma_x = 0 \\ \text{for all } (v_1, v_2) \in V \times H^1(\Omega_2) \text{in } \mathcal{D}'(0, T),\ z_i(0) = z_i^0, \quad z_i'(0) = z_i^1 \text{ in } \Omega_i,\ i = 1, 2. \end{cases} \tag{3.3}$$

As already observed, in [26] the authors prove the existence and uniqueness result for the weak solution in W of problem (3.1) together with some a priori estimates.

Theorem 3.1 ([26]) *Let $T > 0$, H_Γ and W be defined as in (2.3) and (2.4). Under hypotheses (2.7), (2.9) and* (3.2)*, problem (3.1) admits a unique weak solution $z \in W$. Moreover, there exists a positive constant C, such that*

$$\|z\|_{L^\infty(0,T;H_\Gamma)} + \left\|z'\right\|_{L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))} \leq C\left(\left\|z^0\right\|_{H_\Gamma} + \left\|z^1\right\|_{L^2(\Omega_1)\times L^2(\Omega_2)}\right). \tag{3.4}$$

Let us remark that the solution of problem (3.1) has some further properties (see [45], Chapter 3, Theorem 8.2). In fact, under the same hypotheses of Theorem 3.1, the unique solution z of problem (3.1) satisfies

$$z \in C\left([0, T]; H_\Gamma\right), \ z' \in C\left([0, T]; L^2\left(\Omega_1\right) \times L^2\left(\Omega_2\right)\right).$$

Hence the initial values $z(0)$ and $z'(0)$ are meaningful in the appropriate spaces.

Now, we derive an important identity using suitable multipliers. It is essential for establishing the inverse inequalities involved in the exact controllability problem. For convenience, we use the repeated index summation convention. Moreover, when there is no ambiguity, we omit the explicit dependence on the space variable x in the matrix A and in the function h.

Lemma 3.2 *Let $q = (q_1, \ldots q_n)$ be a vector field in $(W^{1,\infty}(\Omega))^n$ and let $z = (z_1, z_2)$ be the solution of problem (3.1)–(3.2). Then, the following identity holds*

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2 q_k n_{1k}\, d\sigma_x\, dt + \frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T} An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 q_k n_{ik}\, d\sigma_x\, dt \\
&\quad - \int_{\Gamma_T} h\,(z_1 - z_2)\, q_k\, (\nabla_\sigma (z_1 - z_2))_k\; d\sigma_x\, dt \\
&\quad + \frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T}\left(|z_i'|^2 - A\nabla_\sigma z_i \nabla_\sigma z_i\right) q_k n_{ik}\, d\sigma_x\, dt \\
&= \sum_{i=1}^{2}\left(z_i', q_k\frac{\partial z_i}{\partial x_k}\right)_{\Omega_i}\Bigg|_0^T + \frac{1}{2}\sum_{i=1}^{2}\int_{Q_i}\left(|z_i'|^2 - A\nabla z_i\nabla z_i\right)\frac{\partial q_k}{\partial x_k}\, dx\, dt \\
&\quad + \sum_{i=1}^{2}\int_{Q_i} A\nabla z_i \nabla q_k \frac{\partial z_i}{\partial x_k}\, dx\, dt - \frac{1}{2}\sum_{i=1}^{2}\int_{Q_i} q_k \sum_{l,j=1}^{n}\frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_i}{\partial x_l}\frac{\partial z_i}{\partial x_j}\, dx\, dt,
\end{aligned}
\tag{3.5}
$$

where

$$
\left(z_i', q_k\frac{\partial z_i}{\partial x_k}\right)_{\Omega_i} = \int_{\Omega_i} z_i'(t) q_k \frac{\partial z_i(t)}{\partial x_k}\, dx
$$

and $\nabla_\sigma z_i = (\sigma_j z_i)_{j=1}^n$ denotes the tangential gradient of z_i on Γ for $i = 1, 2$ (see, for instance, [43,p. 137]).

Proof We prove the result for a strong solution of problem (3.1), that is under the following more regular initial data

$$
\begin{cases}
z^0 = \left(z_1^0, z_2^0\right) \in (H^2(\Omega_1)\cap V) \times H^2(\Omega_2), \\
z^1 = \left(z_1^1, z_2^1\right) \in V \times H^1(\Omega_2)\,.
\end{cases}
\tag{3.6}
$$

Indeed one can easily prove that it holds also considering the weaker hypotheses (3.2) (see for instance [43]).

Let us multiply the first equation in (3.1) by $q_k \dfrac{\partial z_1}{\partial x_k}$ and then integrate on $[0, T]$ to get

$$
\begin{aligned}
&\int_0^T \langle z_1'', q_k\frac{\partial z_1}{\partial x_k}\rangle_{V',V}\, dt + \int_{Q_1} A\nabla z_1 \nabla\left(q_k\frac{\partial z_1}{\partial x_k}\right)\, dx\, dt \\
&\quad - \int_{\Sigma} A\nabla z_1 q_k\frac{\partial z_1}{\partial x_k} n_1\, d\sigma_x\, dt - \int_{\Gamma_T} A\nabla z_1 q_k \frac{\partial z_1}{\partial x_k} n_1\, d\sigma_x\, dt = 0.
\end{aligned}
\tag{3.7}
$$

For clearness sake, let us rewrite the above identity as

$$
I_1 + I_2 + I_3 + I_4 = 0.
$$

Applying integration by parts and Gauss-Green theorem repeatedly, we get

$$
\begin{aligned}
I_1 &= \int_{\Omega_1} z_1' q_k \frac{\partial z_1}{\partial x_k}\, dx \Big|_0^T - \int_{Q_1} z_1' q_k \frac{\partial z_1'}{\partial x_k}\, dx\, dt \\
&= \int_{\Omega_1} z_1' q_k \frac{\partial z_1}{\partial x_k}\, dx \Big|_0^T - \frac{1}{2}\int_{Q_1} \frac{\partial}{\partial x_k} \left|z_1'\right|^2 q_k\, dx\, dt \\
&= \int_{\Omega_1} z_1' q_k \frac{\partial z_1}{\partial x_k}\, dx \Big|_0^T + \frac{1}{2}\int_{Q_1} \left|z_1'\right|^2 \frac{\partial q_k}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{\Sigma} \left|z_1'\right|^2 q_k n_{1k}\, d\sigma_x\, dt \\
&\quad -\frac{1}{2}\int_{\Gamma_T} \left|z_1'\right|^2 q_k n_{1k}\, d\sigma_x\, dt.
\end{aligned}
$$

Since $z_1 = 0$ on Σ implies $z_1' = 0$ on Σ by stronger regularity assumptions, the third term vanishes. Hence we get

$$
I_1 = \int_{\Omega_1} z_1' q_k \frac{\partial z_1}{\partial x_k}\, dx \Big|_0^T + \frac{1}{2}\int_{Q_1} \left|z_1'\right|^2 \frac{\partial q_k}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{\Gamma_T} \left|z_1'\right|^2 q_k n_{1k}\, d\sigma_x\, dt.
$$

Now, we compute I_2

$$
\begin{aligned}
I_2 &= \int_{Q_1} A\nabla z_1 \nabla q_k \frac{\partial z_1}{\partial x_k}\, dx\, dt + \int_{Q_1} A\nabla z_1 q_k \nabla \frac{\partial z_1}{\partial x_k}\, dx\, dt \\
&= \int_{Q_1} A\nabla z_1 \nabla q_k \frac{\partial z_1}{\partial x_k}\, dx\, dt + \int_{Q_1} A\nabla z_1 q_k \frac{\partial}{\partial x_k} \nabla z_1\, dx\, dt \\
&= \int_{Q_1} A\nabla z_1 \nabla q_k \frac{\partial z_1}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{Q_1} A\nabla z_1 \nabla z_1 \frac{\partial q_k}{\partial x_k}\, dx\, dt \\
&\quad -\frac{1}{2}\int_{Q_1} q_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k} \frac{\partial z_1}{\partial x_l} \frac{\partial z_1}{\partial x_j}\, dx\, dt \\
&\quad +\frac{1}{2}\int_{\Sigma} A\nabla z_1 \nabla z_1 q_k n_{1k}\, d\sigma_x\, dt + \frac{1}{2}\int_{\Gamma_T} A\nabla z_1 \nabla z_1 q_k n_{1k}\, d\sigma_x\, dt.
\end{aligned}
$$

Moreover, since $z_1 = 0$ on Σ, one has $\nabla z_1 = \dfrac{\partial z_1}{\partial n_1} n_1$ on Σ, that is $\dfrac{\partial z_1}{\partial x_k} = \dfrac{\partial z_1}{\partial n_1} n_{1k}$, hence I_3 becomes

$$
I_3 = -\int_{\Sigma} A n_1 n_1 q_k n_{1k} \left(\frac{\partial z_1}{\partial n_1}\right)^2\, d\sigma_x\, dt.
$$

Also note that the fourth term in the last expression for I_2 is $-\frac{1}{2}I_3$. Combining the computations for I_1, I_2, I_3 in (3.7), we can get the following identity for z_1

$$
\begin{aligned}
&\int_{\Omega_1} z_1' q_k \frac{\partial z_1}{\partial x_k}\, dx \Big|_0^T + \frac{1}{2}\int_{Q_1} \left|z_1'\right|^2 \frac{\partial q_k}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{\Gamma_T} \left|z_1'\right|^2 q_k n_{1k}\, d\sigma_x\, dt \\
&+ \int_{Q_1} A\nabla z_1 \nabla q_k \frac{\partial z_1}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{Q_1} A\nabla z_1 \nabla z_1 \frac{\partial q_k}{\partial x_k}\, dx\, dt \\
&- \frac{1}{2}\int_{Q_1} q_k \sum_{l,j=1}^{n} \frac{\partial a_{l,j}}{\partial x_k}\frac{\partial z_1}{\partial x_l}\frac{\partial z_1}{\partial x_j}\, dx\, dt + \frac{1}{2}\int_{\Sigma} A\nabla z_1 \nabla z_1 q_k n_{1k}\, d\sigma_x\, dt \\
&+ \frac{1}{2}\int_{\Gamma_T} A\nabla z_1 \nabla z_1 q_k n_{1k}\, d\sigma_x\, dt \\
&- \int_{\Sigma} A n_1 n_1 q_k n_{1k} \left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt - \int_{\Gamma_T} A\nabla z_1 n_1 q_k \frac{\partial z_1}{\partial x_k}\, d\sigma_x\, dt = 0.
\end{aligned}
\tag{3.8}
$$

Analogously, multiplying the second equation in (3.1) by $q_k \dfrac{\partial z_2}{\partial x_k}$ and then integrating on $[0, T]$, we get

$$
\begin{aligned}
&\int_{\Omega_2} z_2' q_k \frac{\partial z_2}{\partial x_k}\, dx \Big|_0^T + \frac{1}{2}\int_{Q_2} \left|z_2'\right|^2 \frac{\partial q_k}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{\Gamma_T} \left|z_2'\right|^2 q_k n_{2k}\, d\sigma_x\, dt \\
&+ \int_{Q_2} A\nabla z_2 \nabla q_k \frac{\partial z_2}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{Q_2} A\nabla z_2 \nabla z_2 \frac{\partial q_k}{\partial x_k}\, dx\, dt \\
&- \frac{1}{2}\int_{Q_2} q_k \sum_{l,j=1}^{n} \frac{\partial a_{l,j}}{\partial x_k}\frac{\partial z_2}{\partial x_l}\frac{\partial z_2}{\partial x_j}\, dx\, dt + \frac{1}{2}\int_{\Gamma_T} A\nabla z_2 \nabla z_2 q_k n_{2k}\, d\sigma_x\, dt \\
&- \int_{\Gamma_T} A\nabla z_2 n_2 q_k \frac{\partial z_2}{\partial x_k}\, d\sigma_x\, dt = 0.
\end{aligned}
\tag{3.9}
$$

Let us note that this last identity is similar to (3.8) except for the integral terms defined on the boundary Σ. This is due to the fact that z_2 is defined on Ω_2 whose boundary is only Γ.

By summing up the identities (3.8) and (3.9), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(z_i', q_k\frac{\partial z_i}{\partial x_k}\right)_{\Omega_i}\Bigg|_0^T+\frac{1}{2}\sum_{i=1}^{2}\int_{Q_i}\left(|z_i'|^2-A\nabla z_i\nabla z_i\right)\frac{\partial q_k}{\partial x_k}\,dx\,dt\\
&+\sum_{i=1}^{2}\int_{Q_i}A\nabla z_i\nabla q_k\frac{\partial z_i}{\partial x_k}\,dx\,dt\\
&-\sum_{i=1}^{2}\frac{1}{2}\int_{Q_i}q_k\sum_{l,j=1}^{n}\frac{\partial a_{l,j}}{\partial x_k}\frac{\partial z_i}{\partial x_l}\frac{\partial z_i}{\partial x_j}\,dx\,dt-\frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T}\left|z_i'\right|^2q_kn_{ik}\,d\sigma_x\,dt\\
&+\frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T}A\nabla z_i\nabla z_iq_kn_{ik}\,d\sigma_x\,dt-\sum_{i=1}^{2}\int_{\Gamma_T}A\nabla z_in_iq_k\frac{\partial z_i}{\partial x_k}\,d\sigma_x\,dt\\
&-\frac{1}{2}\int_{\Sigma}An_1n_1q_kn_{1k}\left(\frac{\partial z_1}{\partial n_1}\right)^2\,d\sigma_x\,dt=0.
\end{aligned}
\tag{3.10}
$$

The above identity is nearly close to the claimed one except for the third line. Nevertheless, if we observe that, for any fixed $i=1,2$ we have

$$
\nabla z_i=\frac{\partial z_i}{\partial n_i}n_i+\nabla_\sigma z_i \tag{3.11}
$$

on the interface Γ, by the symmetry of A, the third line of (3.10) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_{\Gamma_T}A\nabla z_i\nabla z_iq_kn_{ik}\,d\sigma_x\,dt-\int_{\Gamma_T}A\nabla z_in_iq_k\frac{\partial z_i}{\partial x_k}\,d\sigma_x\,dt=\\
&\quad=\frac{1}{2}\int_{\Gamma_T}A\left(\frac{\partial z_i}{\partial n_i}n_i+\nabla_\sigma z_i\right)\left(\frac{\partial z_i}{\partial n_i}n_i+\nabla_\sigma z_i\right)q_kn_{ik}\,d\sigma_x\,dt\\
&\qquad-\int_{\Gamma_T}A\left(\frac{\partial z_i}{\partial n_i}n_i+\nabla_\sigma z_i\right)n_iq_k\left(\frac{\partial z_i}{\partial n_i}n_{ik}+(\nabla_\sigma z_i)_k\right)\,d\sigma_x\,dt\\
&=\frac{1}{2}\int_{\Gamma_T}An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2q_kn_{ik}\,d\sigma_x\,dt+\int_{\Gamma_T}An_i\nabla_\sigma z_i\frac{\partial z_i}{\partial n_i}q_kn_{ik}\,d\sigma_x\,dt\\
&\quad+\frac{1}{2}\int_{\Gamma_T}A\nabla_\sigma z_i\nabla_\sigma z_iq_kn_{ik}\,d\sigma_x\,dt\\
&\quad-\int_{\Gamma_T}An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2q_kn_{ik}\,d\sigma_x\,dt-\int_{\Gamma_T}A\nabla_\sigma z_in_iq_k\frac{\partial z_i}{\partial n_i}n_{ik}\,d\sigma_x\,dt\\
&\quad-\int_{\Gamma_T}A\nabla z_in_iq_k(\nabla_\sigma z_i)_k\,d\sigma_x\,dt
\end{aligned}
\tag{3.12}
$$

$$= -\frac{1}{2}\int_{\Gamma_T} An_i n_i \left(\frac{\partial z_i}{\partial n_i}\right)^2 q_k n_{ik}\, d\sigma_x\, dt$$
$$+\frac{1}{2}\int_{\Gamma_T} A\nabla_\sigma z_i \nabla_\sigma z_i q_k n_{ik}\, d\sigma_x\, dt - \int_{\Gamma_T} A\nabla z_i n_i q_k (\nabla_\sigma z_i)_k\, d\sigma_x\, dt.$$

By putting (3.12) into (3.10), taking into account the interface condition in problem (3.1) and since $n_2 = -n_1$, we finally obtain the required identity

$$\sum_{i=1}^{2}\left(z_i', q_k \frac{\partial z_i}{\partial x_k}\right)_{\Omega_i}\Bigg|_0^T + \frac{1}{2}\sum_{i=1}^{2}\int_{Q_i}\left(|z_i'|^2 - A\nabla z_i \nabla z_i\right)\frac{\partial q_k}{\partial x_k}\, dx\, dt$$
$$+\sum_{i=1}^{2}\int_{Q_i} A\nabla z_i \nabla q_k \frac{\partial z_i}{\partial x_k}\, dx\, dt$$
$$-\sum_{i=1}^{2}\frac{1}{2}\int_{Q_i} q_k \sum_{l,j=1}^{n}\frac{\partial a_{l,j}}{\partial x_k}\frac{\partial z_i}{\partial x_l}\frac{\partial z_i}{\partial x_j}\, dx\, dt$$
$$-\frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T}\left|z_i'\right|^2 q_k n_{ik}\, d\sigma_x\, dt$$
$$-\sum_{i=1}^{2}\frac{1}{2}\int_{\Gamma_T} An_i n_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 q_k n_{ik}\, d\sigma_x\, dt + \sum_{i=1}^{2}\frac{1}{2}\int_{\Gamma_T} A\nabla_\sigma z_i \nabla_\sigma z_i q_k n_{ik}\, d\sigma_x\, dt$$
$$+\int_{\Gamma_T} h\,(z_1 - z_2)\, q_k\,(\nabla_\sigma(z_1 - z_2))_k\, d\sigma_x\, dt$$
$$-\int_{\Sigma} An_1 n_1 q_k n_{1k}\left(\frac{\partial z_1}{\partial n_1}\right)^2\, d\sigma_x\, dt = 0.$$

This completes the proof of the lemma. □

At this point we want to apply the above identity for a particular choice of the vector field q in order to derive the observability estimate. To this aim we adapt to our context some arguments introduced in [43, 44].

Let $x^0 \in \mathbb{R}^n$ and set

$$m(x) = x - x^0 = (x_k - x_k^0)_{k=1}^n. \tag{3.13}$$

We divide the boundary $\partial\Omega$ into two parts, i.e.

$$\partial\Omega(x^0) = \{x \in \partial\Omega : m(x)n_1(x) = m_k(x)n_{1k}(x) > 0\} \tag{3.14}$$

and

$$\partial\Omega_*(x^0) = \partial\Omega \setminus \partial\Omega(x^0).$$

We denote

$$\Sigma(x^0) = \partial\Omega(x^0) \times (0, T) \text{ and } \Sigma_*(x^0) = \partial\Omega_*(x^0) \times (0, T).$$

Further, let us define

$$R_i(x^0) = \max_{x\in\overline{\Omega}_i} |m(x)| \text{for } i = 1, 2 \text{ and } R(x^0) = \max_{x\in\overline{\Omega}} |m(x)|. \tag{3.15}$$

Some remarks are in order. Usually, in the context of controllability problems, the point x^0 can be viewed as an observer and $\partial\Omega(x^0)$ is strictly related to the action region, where the control is acting. The choice of x^0 gives various control regions according to the position of the observer and has advantages and disadvantages. For example, if Ω is a circle, geometrically, $\partial\Omega(x^0)$ is concave to the observer. More in particular, if x^0 is a point inside Ω, then $\partial\Omega(x^0) = \partial\Omega$, since the entire boundary is concave to any point inside. On the other hand, if x^0 is outside Ω, then drawing the tangents from x^0, the boundary is divided into two parts, where $\partial\Omega(x^0)$ is concave to x^0 (related to the control region) and $\partial\Omega_*(x^0)$ is convex to x^0 (not related to the control region). When dealing with internal controllability, the control region is a neighbourhood of $\partial\Omega(x^0)$. In our case, due to the geometry of the domain, we need to introduce a further control set which is a neighbourhood of the whole interface. As we will see later on, the choice of x^0 will play a fundamental role also on the control time (see Lemmas 3.8 and 3.9). In the following, we introduce the energy $E(t)$ of problem (3.1)–(3.2)

$$E(t) = \frac{1}{2}\Bigg[\int_{\Omega_1} |z_1'(t)|^2 dx + \int_{\Omega_2} |z_2'(t)|^2 dx + \int_{\Omega_1} A\nabla z_1(t)\,\nabla z_1(t) dx \\ + \int_{\Omega_2} A\nabla z_2(t)\,\nabla z_2(t) dx + \int_\Gamma h\,|z_1(t) - z_2(t)|^2\, d\sigma_x\Bigg]. \tag{3.16}$$

Let us note that $E(t)$ is conserved (see [27], Lemma 4.1), that is

$$E(t) = E(0), \quad \text{for all } t \in [0, T]. \tag{3.17}$$

We set

$$S = \frac{1}{2}\int_\Sigma An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2 m_k n_{1k}\, d\sigma_x\, dt + \frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T} An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt \\ - \int_{\Gamma_T} h\,(z_1 - z_2)\, m_k\,(\nabla_\sigma(z_1 - z_2))_k\, d\sigma_x\, dt \\ + \frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T}\left(|z_i'|^2 - A\nabla_\sigma z_i\nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt. \tag{3.18}$$

Let us observe that S is nothing else that the left-hand side of (3.5). We want to find a lower bound for S. To this aim we introduce a technical geometrical assumption concerning not only the position of the observer x^0 but also the geometry of the domain Ω_2. This geometrical property will characterize the choice of the control region related to the interface (see Definition 3.5 and Lemma 3.7).

Lemma 3.3 *Let us suppose that Ω_2 is star-shaped with respect to a point $x^0 \in \Omega_2$. Let $z = (z_1, z_2)$ the solution of problem (3.1)–(3.2). Then, for any $T > 0$, it holds*

$$S \geq \left[T\left(1 - \frac{nR(x^0)M}{\alpha}\right) - 2\max\left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0}\right)\right] E(0), \tag{3.19}$$

with M defined as in (2.8) and S as in (3.18).

Proof We take $q_k = m_k = x_k - x_k^0$, for $k = 1, \dots, n$, in the identity (3.5). Then, $\nabla q_k = \nabla m_k = e_k$, where e_k is the canonical basis element. In particular $\dfrac{\partial m_k}{\partial x_k} = 1$ and thus $\displaystyle\sum_{k=1}^{n} \frac{\partial m_k}{\partial x_k} = n$. Hence, we have

$$\begin{aligned} S =& \sum_{i=1}^{2} \left(z_i', m_k \frac{\partial z_i}{\partial x_k}\right)_{\Omega_i}\Bigg|_0^T + \frac{n}{2}\sum_{i=1}^{2}\int_{Q_i}\left(|z_i'|^2 - A\nabla z_i \nabla z_i\right)\, dx\, dt \\ &+ \sum_{i=1}^{2}\int_{Q_i} A\nabla z_i \nabla z_i\, dx\, dt - \frac{1}{2}\sum_{i=1}^{2}\int_{Q_i} m_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_i}{\partial x_l}\frac{\partial z_i}{\partial x_j}\, dx\, dt \\ =& S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{3.20}$$

We want to estimate $S_1 + S_2 + S_3 + S_4$. Let us pose

$$X_i = \left(z_i'(t), m_k \frac{\partial z_i(t)}{\partial x_k}\right)_{\Omega_i}\Bigg|_0^T \tag{3.21}$$

and

$$Y_i = \int_{Q_i}\left(|z_i'|^2 - A\nabla z_i \nabla z_i\right)\, dx\, dt \tag{3.22}$$

for $i = 1, 2$. Hence, $S_1 = X_1 + X_2$, $S_2 = \dfrac{n}{2}(Y_1 + Y_2)$ and therefore (3.20) can be rewritten as

$$\begin{aligned} S = S_1 + S_2 + S_3 + S_4 = (X_1 + X_2) + \frac{n-1}{2}(Y_1 + Y_2) \\ + \frac{1}{2}\sum_{i=1}^{2}\int_{Q_i}\left(|z_i'|^2 + A\nabla z_i \nabla z_i\right)\, dx\, dt \\ - \frac{1}{2}\sum_{i=1}^{2}\int_{Q_i} m_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_i}{\partial x_l}\frac{\partial z_i}{\partial x_j}\, dx\, dt. \end{aligned} \tag{3.23}$$

Taking into account (3.16) and the conservation law (3.17), we get

$$\begin{aligned} S_1 + S_2 + S_3 + S_4 = {} & (X_1 + X_2) + \frac{n-1}{2} (Y_1 + Y_2) + E(0)T \\ & - \frac{1}{2} \int_{\Gamma_T} h \left| z_1 - z_2 \right|^2 d\sigma_x \, dt \\ & - \frac{1}{2} \sum_{i=1}^{2} \int_{Q_i} m_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k} \frac{\partial z_i}{\partial x_l} \frac{\partial z_i}{\partial x_j} \, dx \, dt. \end{aligned} \tag{3.24}$$

By multiplying the PDEs in (3.1) by z_1 and z_2 respectively, and taking into account interface and boundary conditions, we get

$$Y_1 + Y_2 = \sum_{i=1}^{2} \left(z_i'(t), z_i(t) \right)_{\Omega_i} \Big|_0^T + \int_{\Gamma_T} h \left(z_1 - z_2 \right)^2 d\sigma_x \, dt.$$

Hence (3.24) becomes

$$\begin{aligned} S_1 + S_2 + S_3 + S_4 = {} & Z_1 + Z_2 + E(0)T + \frac{n-2}{2} \int_{\Gamma_T} h \left| z_1(t) - z_2(t) \right|^2 d\sigma_x \, dt \\ & - \frac{1}{2} \sum_{i=1}^{2} \int_{Q_i} m_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k} \frac{\partial z_i}{\partial x_l} \frac{\partial z_i}{\partial x_j} \, dx \, dt, \end{aligned} \tag{3.25}$$

where

$$Z_i = \left(z_i'(t), m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t) \right)_{\Omega_i} \Bigg|_0^T, \quad \text{for } i = 1, 2. \tag{3.26}$$

Thus, we have an $E(0)T$ term. We need to see that it is a leading term. Thus, we need to estimate the other terms in (3.25). To this aim, let us fix $i \in \{1, 2\}$. By Young inequality we get

$$\begin{aligned} \left| \left(z_i'(t), m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t) \right)_{\Omega_i} \right| & \leq \int_{\Omega_i} \left| z_i'(t) \right| \left| m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t) \right| dx \\ & \leq \frac{\mu}{2} \int_{\Omega_i} \left| z_i'(t) \right|^2 dx + \frac{1}{2\mu} \int_{\Omega_i} \\ & \quad \times \left| m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t) \right|^2 dx, \end{aligned} \tag{3.27}$$

where μ is an arbitrary positive constant. By applying Gauss–Green, it holds

$$\begin{aligned}\int_{\Omega_i} m_k \frac{\partial z_i(t)}{\partial x_k} z_i(t)\,dx &= \frac{1}{2}\int_{\Omega_i} m_k \frac{\partial}{\partial x_k} |z_i(t)|^2 \\ &= -\frac{n}{2}\int_{\Omega_i} |z_i(t)|^2\, dx + \frac{1}{2}\int_\Gamma m_k n_{ik} |z_i(t)|^2\, d\sigma_x.\end{aligned}$$

Hence, by (3.15), the second term in the right hand side of (3.27) can be estimated as

$$\begin{aligned}\int_{\Omega_i} \left| m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t)\right|^2 dx &= \int_{\Omega_i} \left| m_k \frac{\partial z_i(t)}{\partial x_k}\right|^2 dx \\ &\quad + \left[\frac{(n-1)^2}{4} - \frac{n(n-1)}{2}\right]\int_{\Omega_i} |z_i(t)|^2\, dx \\ &\quad + \frac{n-1}{2}\int_\Gamma m_k n_{ik} |z_i(t)|^2\, d\sigma_x \\ &\le (R_i(x^0))^2 \int_{\Omega_i} |\nabla z_i(t)|^2\, dx \\ &\quad + \frac{n-1}{2}\int_\Gamma m_k n_{ik} |z_i(t)|^2\, d\sigma_x,\end{aligned} \tag{3.28}$$

where, we have used the fact that $\dfrac{(n-1)^2}{4} - \dfrac{n(n-1)}{2} < 0$. Let us note that by (3.15)

$$R(x^0) = R_1(x^0) > R_2(x^0), \tag{3.29}$$

since $x^0 \in \Omega_2$, thus, by putting (3.28) into (3.27) and taking into account (2.7), we obtain

$$\begin{aligned}\left| \left(z_i'(t), m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t)\right)_{\Omega_i}\right| &\le \frac{\mu}{2}\int_{\Omega_i} \left|z_i'(t)\right|^2 dx \\ &\quad + \frac{(R(x^0))^2}{2\alpha\mu} \int_{\Omega_i} A\nabla z_i(t)\nabla z_i(t)\, dx \\ &\quad + \frac{n-1}{4\mu}\int_\Gamma m_k n_{ik} |z_i(t)|^2\, d\sigma_x.\end{aligned} \tag{3.30}$$

Let us consider the last term in (3.30), for $i = 1, 2$. We observe that

$$\left| \frac{1}{2}|z_1(t)|^2 - |z_2(t)|^2 \right| \le |z_1(t) - z_2(t)|^2, \quad \forall\, t \in [0, T].$$

Moreover, by our assumption on x^0 and since $n_1 = -n_2$ on Γ, it holds that $m_k n_{1k} \leq 0$ on Γ. Hence, we get

$$\begin{aligned}
\sum_{i=1}^{2} \int_{\Gamma} m_k n_{ik} \left|z_i(t)\right|^2 \, d\sigma_x &= \int_{\Gamma} m_k n_{1k} \left(|z_1(t)|^2 - |z_2(t)|^2 \right) d\sigma_x \\
&\leq \int_{\Gamma} m_k n_{1k} \left(\frac{1}{2} |z_1(t)|^2 - |z_2(t)|^2 \right) d\sigma_x \\
&\leq \int_{\Gamma} |m_k n_{1k}| \left| \frac{1}{2} |z_1(t)|^2 - |z_2(t)|^2 \right| d\sigma_x \\
&\leq \|m\|_{L^\infty(\Gamma)} \int_{\Gamma} \left| \frac{1}{2} |z_1(t)|^2 - |z_2(t)|^2 \right| d\sigma_x \\
&\leq \frac{R(x^0)}{h_0} \int_{\Gamma} h \left(z_1(t) - z_2(t) \right)^2 \, d\sigma_x .
\end{aligned} \tag{3.31}$$

Taking into account (3.30) and (3.31), we get the estimate

$$\begin{aligned}
\sum_{i=1}^{2} \left| \left(z_i'(t), m_k \frac{\partial z_i(t)}{\partial x_k} + \frac{n-1}{2} z_i(t) \right)_{\Omega_i} \right| &\leq \frac{\mu}{2} \sum_{i=1}^{2} \int_{\Omega_i} \left| z_i'(t) \right|^2 dx \\
&\quad + \frac{(R(x^0))^2}{2\alpha\mu} \sum_{i=1}^{2} \int_{\Omega_i} A \nabla z_i(t) \nabla z_i(t) \, dx \\
&\quad + \frac{(n-1) R(x^0)}{4\mu h_0} \int_{\Gamma} h \left(z_1(t) - z_2(t) \right)^2 \, d\sigma_x \\
&\leq \max \left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0} \right) E(t).
\end{aligned} \tag{3.32}$$

The last inequality follows by choosing $\mu = \dfrac{R(x^0)}{\sqrt{\alpha}}$ and by the definition of energy as in (3.16). Hence by (3.17) and taking into account (3.26), we readily see that

$$|Z_1 + Z_2| \leq 2 \max \left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0} \right) E(0). \tag{3.33}$$

The estimate of the last term in (3.25) is straight forward using the ellipticity and boundedness of the matrix A (see also [48]). More precisely, taking into account the energy definition in (3.16), we have

$$\left|\frac{1}{2}\sum_{i=1}^{2}\int_{Q_i} m_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_i}{\partial x_l}\frac{\partial z_i}{\partial x_j}\,dx\,dt\right| \leq \sum_{i=1}^{n}\frac{nR_i(x^0)M}{2\alpha}\int_{Q_i} A\nabla z_i \nabla z_i\,dx\,dt \leq \frac{nR(x^0)M}{\alpha}TE(0). \tag{3.34}$$

By putting (3.33) and (3.34) into (3.25), we finally arrive at the lower bound

$$S \geq -2\max\left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0}\right)E(0) + E(0)T - \frac{nR(x^0)M}{\alpha}TE(0). \tag{3.35}$$

The proof is now complete. □

We now specify the required topological assumptions on the control regions ω_1 and ω_2 in order to obtain our exact controllability result. See Fig. 1 and Fig. 2 for sample domains.

Definition 3.4 Let x^0 be as in the hypotheses of Lemma 3.3 and let $\partial\Omega(x^0)$ be defined as in (3.14). We say that $\omega_1 \subset \Omega_1$ is a neighbourhood of $\partial\Omega(x^0)$ if there exists some neighbourhood $\mathcal{O} \subset \mathbb{R}^n$ of $\partial\Omega(x^0)$ such that

$$\omega_1 = \Omega_1 \cap \mathcal{O}.$$

Definition 3.5 We say that $\omega_2 \subset \Omega_2$ is a neighborhood of Γ in Ω_2, if there exists some neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of Γ such that

$$\omega_2 = \Omega_2 \cap \mathcal{O}.$$

We will now establish a couple of important results which are crucial to get the observability inequality given in Lemma 3.9 below. In this direction, we consider the function $\tau = (\tau_1, \dots \tau_k) \in (C^1(\mathbb{R}^n))^n$ satisfying the following properties:

$$\begin{cases} \text{(i)} & \tau \cdot n_1 = 1 \text{ on} \partial\Omega, \\ \text{(ii)} & \text{supp}\,\tau \subset \omega_1, \\ \text{(iii)} & \|\tau\|_{(L^\infty(\omega_1))^n} \leq 1. \end{cases} \tag{3.36}$$

The existence of such a vectorial field is proved in [43].

Lemma 3.6 *Let ω_1 be a neighborhood of $\partial\Omega(x^0)$ and let $z = (z_1, z_2)$ the solution of problem (3.1)–(3.2). Then, for any $T > 0$, it holds*

$$\frac{1}{2}\left|\int_{\Sigma(x^0)} An_1 n_1 \left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\,dt\right| \leq 2\max\left(1, \frac{1}{\alpha}\right)E(0) + C\int_0^T\int_{\omega_1}\left(\left|z_1'\right|^2 + |\nabla z_1|^2\right)dx\,dt \tag{3.37}$$

Proof Taking $q_k = \tau_k, k = 1, \ldots, n$, in (3.5), by (3.36)i) and (3.36)ii) we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma(x^0)} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2\, d\sigma_x\, dt \\
&= \left(z_1', \tau_k\frac{\partial z_1}{\partial x_k}\right)_{\omega_1}\Bigg|_0^T \\
&\quad + \frac{1}{2}\int_0^T\int_{\omega_1}\left(|z_1'|^2 - A\nabla z_1\nabla z_1\right)\frac{\partial \tau_k}{\partial x_k}\, dx\, dt \\
&\quad + \int_0^T\int_{\omega_1} A\nabla z_1\nabla\tau_k\frac{\partial z_1}{\partial x_k}\, dx\, dt \\
&\quad - \frac{1}{2}\int_0^T\int_{\omega_1}\tau_k\sum_{l,j=1}^n\frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_1}{\partial x_l}\frac{\partial z_1}{\partial x_j}\, dx\, dt.
\end{aligned} \tag{3.38}$$

Passing to the absolute value, by (3.16), (3.36)iii), Young inequality, the conservation law and since $\tau \in (C^1(\mathbb{R}^n))^n$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left|\int_{\Sigma(x^0)} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2\, d\sigma_x\, dt\right| \leq= \frac{1}{2}\int_{\omega_1}\left|z_1'(0)\right|^2\, dx \\
&+\frac{1}{2}\int_{\omega_1}\left|z_1'(T)\right|^2\, dx + \frac{1}{2}\int_{\omega_1}|\nabla z_1(0)|^2\, dx \\
&+\frac{1}{2}\int_{\omega_1}|\nabla z_1(T)|^2\, dx + C_1\int_0^T\int_{\omega_1}\left(|z_1'|^2 + A\nabla z_1\nabla z_1\right)dx\, dt \\
&+C_2\int_0^T\int_{\omega_1} A\nabla z_1\nabla z_1\, dx\, dt + C_3\int_0^T\int_{\omega_1}|\nabla z_1|^2\, dx\, dt \leq \\
&\leq 2\max\left(1, \frac{1}{\alpha}\right)E(0) + C\int_0^T\int_{\omega_1}\left(|z_1'|^2 + |\nabla z_1|^2\right)\, dx\, dt.
\end{aligned}$$

□

We will get a similar result for the neighborhood ω_2. To this aim, let $w \in C^1(\mathbb{R}^n)$ be such that

$$\begin{cases}
\text{(i)}\quad \operatorname{supp} w \subset \omega_2, \\
\text{(ii)}\ \ 0 \leq w \leq 1 \text{ in } \ \omega_2, \\
\text{(iii)}\ w = 1 \text{ on } \ \Gamma, \\
\text{(iv)}\ \ \|\nabla w\|_{L^\infty(\mathbb{R}^n)} \leq C.
\end{cases} \tag{3.39}$$

The existence of w is quite standard, see for example, [43]. Let us denote

$$\begin{aligned} S_{\Gamma_T} = \frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T} A n_i n_i \left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt \\ -\int_{\Gamma_T} h\,(z_1 - z_2)\, m_k\, (\nabla_\sigma (z_1 - z_2))_k\, d\sigma_x\, dt \\ +\frac{1}{2}\sum_{i=1}^{2}\int_{\Gamma_T} \left(|z_i'|^2 - A\nabla_\sigma z_i \nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt. \end{aligned}$$

we want to find an upper bound for S_{Γ_T}.

Lemma 3.7 *Let ω_2 be a neighborhood of Γ and let $z = (z_1, z_2)$ the solution of problem (3.1)–(3.2). Then, for any $T > 0$, it holds*

$$\left|S_{\Gamma_T}\right| \le 2\max\left(1, \frac{R^2(x^0)}{\alpha}\right) E(0) + C\int_0^T\int_{\omega_2}\left(\left|z_2'\right|^2 + |\nabla z_2|^2\right) dx\, dt. \tag{3.40}$$

Proof Let us choose in (3.5) $q_k = m_k\, w$, $k = 1, \dots, n$. By Young inequality, (3.16) and the conservation law, we obtain

$$\begin{aligned} \left|S_{\Gamma_T}\right| &\le \frac{1}{2}\int_{\omega_2}\left|z_2'(0)\right|^2 dx + \frac{1}{2}\int_{\omega_2}\left|z_2'(T)\right|^2 dx + \frac{R^2(x^0)}{2}\int_{\omega_2}|\nabla z_2(0)|^2\, dx \\ &+\frac{R^2(x^0)}{2}\int_{\omega_2}|\nabla z_2(T)|^2\, dx + \frac{n}{2}\int_0^T\int_{\omega_2}\left(|z_2'|^2 + A\nabla z_2 \nabla z_2\right) dx\, dt \\ &+C_0\int_0^T\int_{\omega_2}\left(|z_2'|^2 + A\nabla z_2\nabla z_2\right) dx\, dt \\ &+C_1\int_0^T\int_{\omega_2} A\nabla z_2\nabla z_2\, dx\, dt + C_2\int_0^T\int_{\omega_2}|\nabla z_2|^2\, dx\, dt \\ &\le 2\max\left(1, \frac{R^2(x^0)}{\alpha}\right) E(0) + C\int_0^T\int_{\omega_2}\left(|z_2'|^2 + |\nabla z_2|^2\right) dx\, dt. \end{aligned}$$

□

Collecting together the results of Lemmas 3.3, 3.6 and 3.7, we obtain the following lower estimate.

Lemma 3.8 *Let us suppose that Ω_2 is star-shaped with respect to a point $x^0 \in \Omega_2$ satisfying*

$$R(x^0) < \frac{\alpha}{nM}. \tag{3.41}$$

Assume ω_1 and ω_2 are neighbourhoods of $\partial\Omega(x^0)$ and Γ respectively, and let $z = (z_1, z_2)$ the solution of problem (3.1)–(3.2). Then, there exists $T_0 > 0$ such that

$$\begin{aligned} E(0) \leq C_1(T) \int_0^T \int_{\omega_1} \left(\left|z_1'\right|^2 + |\nabla z_1|^2\right) dx\, dt \\ + C_2(T) \int_0^T \int_{\omega_2} \left(\left|z_2'\right|^2 + |\nabla z_2|^2\right) dx\, dt, \end{aligned} \tag{3.42}$$

for T large enough so that

$$\frac{T - T_0}{T} > \frac{nR(x^0)M}{\alpha}. \tag{3.43}$$

Proof By putting (3.37) and (3.40) into (3.19), we get

$$\begin{aligned} &\left[T\left(1 - \frac{nR(x^0)M}{\alpha}\right) - 2\max\left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0}\right)\right] E(0) \\ &\leq 2\max\left(1, R(x^0), \frac{R(x^0)}{\alpha}, \frac{R^2(x^0)}{\alpha}\right) E(0) \\ &\quad + C_1 \int_0^T \int_{\omega_1} \left(\left|z_1'\right|^2 + |\nabla z_1|^2\right) dx\, dt + C_2 \int_0^T \int_{\omega_2} \left(\left|z_2'\right|^2 + |\nabla z_2|^2\right) dx\, dt. \end{aligned}$$

Denoting

$$T_0 = 2\max\left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0}\right) + 2\max\left(1, R(x^0), \frac{R(x^0)}{\alpha}, \frac{R^2(x^0)}{\alpha}\right), \tag{3.44}$$

we obtain

$$\begin{aligned} \left[T\left(1 - \frac{nR(x^0)M}{\alpha}\right) - T_0\right] E(0) \leq C_1 \int_0^T \int_{\omega_1} \left(\left|z_1'\right|^2 + |\nabla z_1|^2\right) dx\, dt \\ + C_2 \int_0^T \int_{\omega_2} \left(\left|z_2'\right|^2 + |\nabla z_2|^2\right) dx\, dt. \end{aligned}$$

Thus, if (3.41) is satisfied and if T is large enough so that (3.43) holds, $T\left(1 - \frac{nR(x^0)M}{\alpha}\right) - T_0$ is positive and we get the result. □

Some comments are in order. For sake of simplicity, all integrals in previous lemmas are written between 0 and T. Actually they could, as well, have been written between ε and $T - \varepsilon$ with $\varepsilon > 0$ and sufficiently small. More precisely, by using (3.16) and the conservation law, the inequalities (3.19), (3.37) and (3.40) can be written as

$$
\begin{aligned}
&\frac{1}{2}\int_{\varepsilon}^{T-\varepsilon}\int_{\partial\Omega} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2 m_k n_{1k}\, d\sigma_x\, dt \\
&\quad+\frac{1}{2}\sum_{i=1}^{2}\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma} An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt \\
&\quad-\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma} h\,(z_1-z_2)\, m_k\,(\nabla_\sigma(z_1-z_2))_k\; d\sigma_x\, dt \\
&\quad+\frac{1}{2}\sum_{i=1}^{2}\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma}\left(|z_i'|^2 - A\nabla_\sigma z_i\nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt \\
&\geq \left[(T-2\varepsilon)\left(1-\frac{nR(x^0)M}{2\alpha}\right) - 2\max\left(\frac{R(x^0)}{\sqrt{\alpha}},\, \frac{(n-1)\sqrt{\alpha}}{2h_0}\right)\right]E(0), \\
&\quad\times\frac{1}{2}\left|\int_{\varepsilon}^{T-\varepsilon}\int_{\partial\Omega(x^0)} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2 m_k n_{1k}\, d\sigma_x\, dt\right| \\
&\leq 2\max\left(1,\frac{1}{\alpha}\right)E(0) \\
&\quad+C_1\int_{\varepsilon}^{T-\varepsilon}\int_{\omega_1}\left(\left|z_1'\right|^2+|\nabla z_1|^2\right) dx\, dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\left|\sum_{i=1}^{2}\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma} An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt\right. \\
&\quad-\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma} h\,(z_1-z_2)\, m_k\,(\nabla_\sigma(z_1-z_2))_k\; d\sigma_x\, dt \\
&\quad\left.+\frac{1}{2}\sum_{i=1}^{2}\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma}\left(|z_i'|^2 - A\nabla_\sigma z_i\nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt\right| \\
&\leq 2\max\left(1,\frac{R^2(x^0)}{\alpha}\right)E(0) + C_2\int_{\varepsilon}^{T-\varepsilon}\int_{\omega_2}\left(\left|z_2'\right|^2+|\nabla z_2|^2\right) dx\, dt,
\end{aligned}
$$

respectively. By arguing as in Lemma 3.8, if ε is chosen to have $T-2\varepsilon$ large enough so that

$$
\frac{(T-2\varepsilon)-T_0}{(T-2\varepsilon)} > \frac{nR(x^0)M}{\alpha}
$$

and if (3.41) is satisfied, we get

$$E(0) \leq C_1(T) \int_{\varepsilon}^{T-\varepsilon} \int_{\omega_1} \left(|z_1'|^2 + |\nabla z_1|^2 \right) dx\, dt$$
$$+C_2(T) \int_{\varepsilon}^{T-\varepsilon} \int_{\omega_2} \left(|z_2'|^2 + |\nabla z_2|^2 \right) dx\, dt. \tag{3.45}$$

Now, we can prove the observability inequality which is crucial to establish the exact controllability result.

Lemma 3.9 *(observability inequality) Let us suppose that Ω_2 is star-shaped with respect to a point $x^0 \in \Omega_2$ satisfying condition (3.41). Assume that ω_1 and ω_2 are neighbourhoods of $\partial\Omega(x^0)$ and Γ respectively and let $z = (z_1, z_2)$ be the solution of problem (3.1)–(3.2). Then there exists $T_0 > 0$ such that*

$$\begin{aligned} E(0) \leq C_1(T) \int_0^T \int_{\omega_1} \left(|z_1'|^2 + |z_1|^2 \right) dx\, dt \\ + C_2(T) \int_0^T \int_{\omega_2} \left(|z_2'|^2 + |z_2|^2 \right) dx\, dt, \end{aligned} \tag{3.46}$$

for T large enough so that

$$\frac{T - T_0}{T} > \frac{R(x^0)M}{\alpha}. \tag{3.47}$$

Proof In view of (3.42), we need to estimate ∇z_i in terms of z_i and z_i', $i = 1, 2$. Let $\omega_{01} \subset \Omega_1$ be a neighborhood of $\partial\Omega(x^0)$ and $\omega_{02} \subset \Omega_2$ be a neighborhood of Γ such that

$$\Omega \cap \omega_{0i} \subset \omega_i,\ i = 1, 2.$$

Note that (3.45) is true for any neighborhood of $\partial\Omega(x^0)$ and Γ, then it is also true for ω_{0i}, $i = 1, 2$ and we obtain

$$\begin{aligned} E(0) \leq C_1'(T) \int_{\varepsilon}^{T-\varepsilon} \int_{\omega_{01}} \left(|z_1'|^2 + |\nabla z_1|^2 \right) dx\, dt \\ + C_2'(T) \int_{\varepsilon}^{T-\varepsilon} \int_{\omega_{02}} \left(|z_2'|^2 + |\nabla z_2|^2 \right) dx\, dt. \end{aligned} \tag{3.48}$$

Let us consider $\rho \in W^{1,\infty}(\Omega)$, $\rho \geq 0$ such that

$$\begin{cases} \text{(i)}\ \ \rho(x) = 1 \text{ in } (\omega_{01} \cup \omega_{02}), \\ \text{(ii)}\ \rho(x) = 0 \text{ in } \Omega \setminus (\omega_1 \cup \omega_2). \end{cases}$$

Define the function $p(x,t) = \eta(t)\rho(x)$ in $\Omega \times (0,T)$, where $\eta(t) \in C^1([0,T])$ is such that $\eta(0) = \eta(T) = 0$ and $\eta(t) = 1$ in $(\varepsilon, T-\varepsilon)$. Thus, p satisfies

$$\begin{cases} \text{(i)} \quad p(x,t) = 1 \text{ in } (\omega_{01} \cup \omega_{02}) \times (\varepsilon, T-\varepsilon), \\ \text{(ii)} \quad p(x,t) = 0 \text{ in } (\Omega \setminus (\omega_1 \cup \omega_2)) \times (\varepsilon, T-\varepsilon), \\ \text{(iii)} \; p(x,0) = p(x,T) = 0 \text{ in } \Omega, \\ \text{(iv)} \; \dfrac{|\nabla p|^2}{p} \in L^\infty(\Omega \times (0,T)). \end{cases} \tag{3.49}$$

Multiplying the equation for z_1 in (3.1) by pz_1 and integrating by parts in Q_1, we obtain

$$\begin{aligned} &\int_0^T \langle z_1'', pz_1 \rangle_{V',V} \, dt + \int_{\omega_1 \times (0,T)} A\nabla z_1 \nabla (pz_1) \, dx \, dt \\ &- \int_{\Gamma_T} A\nabla z_1 p z_1 n_1 \, d\sigma_x \, dt = 0 \end{aligned} \tag{3.50}$$

using (3.49)ii) and the fact that $z_1 = 0$ on Σ. One more integration by parts of the first term leads to

$$\begin{aligned} &- \int_0^T \int_{\omega_1} z_1' p' z_1 \, dx \, dt - \int_0^T \int_{\omega_1} |z_1'|^2 p \, dx \, dt \\ &+ \int_0^T \int_{\omega_1} A\nabla z_1 \nabla p z_1 \, dx \, dt \\ &+ \int_0^T \int_{\omega_1} A\nabla z_1 \nabla z_1 p \, dx \, dt - \int_{\Gamma_T} A\nabla z_1 p z_1 n_1 \, d\sigma_x \, dt = 0. \end{aligned}$$

Arguing as above, we get a similar identity for z_2. Now, summing up and using the imperfect interface condition, we get

$$\begin{aligned} &\int_0^T \int_{\omega_1} A\nabla z_1 \nabla z_1 p \, dx \, dt + \int_0^T \int_{\omega_2} A\nabla z_2 \nabla z_2 p \, dx \, dt \\ &= \int_0^T \int_{\omega_1} |z_1'|^2 p \, dx \, dt + \int_0^T \int_{\omega_2} |z_2'|^2 p \, dx \, dt \\ &\quad + \int_0^T \int_{\omega_1} z_1' p' z_1 \, dx \, dt + \int_0^T \int_{\omega_2} z_2' p' z_2 \, dx \, dt \\ &\quad - \int_0^T \int_{\omega_1} A\nabla z_1 \nabla p z_1 \, dx \, dt - \int_0^T \int_{\omega_2} A\nabla z_2 \nabla p z_2 \, dx \, dt \\ &\quad - \int_{\Gamma_T} hp(z_1 - z_2)^2 \, d\sigma_x \, dt. \end{aligned} \tag{3.51}$$

We now estimate the terms on the right hand side of the above expression. To this aim let us fix $i \in \{1, 2\}$. We have

$$\int_0^T \int_{\omega_i} |z_i'|^2 p \, dx \, dt \leq \|p\|_{L^\infty(0,T;\Omega)} \int_0^T \int_{\omega_i} |z_i'|^2 \, dx \, dt \tag{3.52}$$

and by Young inequality

$$\int_0^T \int_{\omega_i} z_i' p' z_i \, dx \, dt \leq \frac{1}{2} \|p'\|_{L^\infty(0,T;\Omega)} \left(\int_0^T \int_{\omega_i} |z_i|^2 \, dx \, dt + \int_0^T \int_{\omega_i} |z_i'|^2 \, dx \, dt \right). \tag{3.53}$$

To estimate the other two terms, we apply again Young inequality and hypothesis (2.7)ii) to get

$$\begin{aligned} \left| \int_0^T \int_{\omega_i} A \nabla z_i \nabla p z_i \, dx \, dt \right| &\leq \left| \int_0^T \int_{\omega_i} \beta |\nabla z_i| |\nabla p| |z_i| \, dx \, dt \right| \\ &\leq \beta^2 \gamma \int_0^T \int_{\omega_i} p |\nabla z_i|^2 \, dx \, dt \\ &+ \frac{1}{4\gamma} \int_0^T \int_{\omega_i} \frac{|\nabla p|^2}{p} |z_i|^2 \, dx \, dt, \end{aligned} \tag{3.54}$$

for any $\gamma > 0$. By putting the above estimates into (3.51) and taking into account (3.49), we obtain

$$\begin{aligned} &\alpha \left(\int_0^T \int_{\omega_1} |\nabla z_1|^2 p \, dx \, dt + \int_0^T \int_{\omega_2} |\nabla z_2|^2 p \, dx \, dt \right) \\ &\leq C \left(\int_0^T \int_{\omega_1} |z_1'|^2 \, dx \, dt + \int_0^T \int_{\omega_2} |z_2'|^2 \, dx \, dt \right) \\ &\quad + \beta^2 \gamma \left(\int_0^T \int_{\omega_1} |\nabla z_1|^2 p \, dx \, dt + \int_0^T \int_{\omega_2} |\nabla z_2|^2 p \, dx \, dt \right), \end{aligned} \tag{3.55}$$

for some constant $C > 0$ and for any $\gamma > 0$. Thus, choosing $\gamma < \dfrac{\alpha}{\beta^2}$ and by (3.48) and (3.49), we get the desired result. □

Corollary 3.10 (equivalence of norms) *Let us suppose that Ω_2 is star-shaped with respect to a point $x^0 \in \Omega_2$ satisfying condition (3.41). Assume that ω_1 and ω_2 are neighbourhoods of $\partial\Omega(x^0)$ and Γ respectively and let $z = (z_1, z_2)$ the solution of problem (3.1)–(3.2). Then, there exists $T_0 > 0$ such that*

$$\begin{aligned} E(0) &\leq C(T) \left(\int_0^T \int_{\omega_1} \left(|z_1'|^2 + |z_1|^2 \right) dx \, dt + \int_0^T \int_{\omega_2} \left(|z_2'|^2 + |z_2|^2 \right) dx \, dt \right) \\ &\leq C_3(T) E(0), \end{aligned} \tag{3.56}$$

for T large enough so that (3.47) is satisfied.

Proof The proof is an immediate consequence of (3.4), (3.16), (3.17) and (3.46). $\square$

The above lemma essentially shows the equivalence of the standard norm in H_Γ with the norm

$$\left(\int_0^T \int_{\omega_1} \left(\left|z_1'\right|^2 + |z_1|^2\right) dx\, dt + \int_0^T \int_{\omega_2} \left(\left|z_2'\right|^2 + |z_2|^2\right) dx\, dt\right)^{1/2}.$$

It also proves the following uniqueness result: if $z_i = 0$ in $\omega_i \times (0, T)$, then $z_i = 0$ in $\Omega_i \times (0, T)$, for $i = 1, 2$.

These are the main points to develop the HUM method described in the next section.

4 HUM and the Internal Exact Controllability Result

In this section, by using the Hilbert Uniqueness Method introduced by Lions (see [42, 43]), we prove the internal exact controllability of system (2.2) stated in the following theorem.

Theorem 4.1 *Assume that (2.7) and (2.9) hold. Suppose that Ω_2 is star-shaped with respect to a point $x^0 \in \Omega_2$ satisfying $R(x^0) < \alpha/(nM)$. Let ω_1 and ω_2 be neighbourhoods of $\partial\Omega(x^0)$ and Γ, respectively. Then, for any given $\left(U^0, U^1\right)$ in $\left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \times (H_\Gamma)'$, there exist a control $\zeta \in W'$ and a time $T_0 > 0$ such that the corresponding solution of problem (2.2) satisfies*

$$u(T) = u'(T) = 0, \tag{4.1}$$

for T large enough so that

$$\frac{T - T_0}{T} > \frac{nR(x^0)M}{\alpha}. \tag{4.2}$$

Proof We point out that the exact controllability is achieved in the space $\left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \times (H_\Gamma)'$ with control in W'. In fact, we represent the control ζ in terms of the solution z of problem (3.1)–(3.2) with appropriate chosen initial data. The method is constructive, indeed one could develop it as a numerical algorithm. We briefly describe the HUM which essentially relies on the observability estimate, given in Lemma 3.9.

Given any $\left(z^0, z^1\right) \in H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$, let z the solution of problem (3.1)–(3.2). Then consider the following adjoint problem

$$\begin{cases} L\theta_i \equiv \theta_i'' - \operatorname{div}(A(x)\nabla\theta_i) = (-z_i'' + z_i)\chi_{\omega_i} & \text{in } Q_i, \quad \text{for } \; i = 1, 2 \\ A(x)\nabla\theta_1 n_1 = -A(x)\nabla\theta_2 n_2 & \text{on } \Gamma_T, \\ A(x)\nabla\theta_1 n_1 = -h(x)(\theta_1 - \theta_2) & \text{on } \Gamma_T, \\ \theta_1 = 0 & \text{on } \Sigma, \\ \theta_i(T) = \theta_i'(T) = 0 & \text{in } \Omega_i \quad \text{for } \; i = 1, 2, \end{cases} \tag{4.3}$$

where the solution $\theta = (\theta_1, \theta_2)$ is intended in the sense of transposition. Here $-z_i''\chi_{\omega_i}$, for $i = 1, 2$ is to be interpreted in a duality sense, namely

$$\langle -z_i''\chi_{\omega_i}, v_i\rangle_{W',W} = \int_0^T \int_{\omega_i} z_i' v_i' \, dx \, dt, \tag{4.4}$$

for all $v = (v_1, v_2) \in W$.

Now, if (U^0, U^1) are the initial conditions of problem (2.2) with $\zeta_i = -z_i'' + z_i$, then, by uniqueness, the null controllability problem is solved if θ satisfies

$$\theta_i(0) = U_i^0, \quad \theta_i'(0) = U_i^1 \tag{4.5}$$

for $i = 1, 2$. Thus, the key point is to choose the initial data $\left(z^0, z^1\right) \in H_\Gamma \times L^2(\Omega))$ so that the above initial conditions for θ are satisfied. This motivates us to define the linear operator

$$\Lambda : H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \to (H_\Gamma)' \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \tag{4.6}$$

as follows

$$\Lambda\left(z^0, z^1\right) = \left(\theta'(0), -\theta(0)\right). \tag{4.7}$$

Hence the null controllability problem reduces to prove that Λ is onto, since then one can solve

$$\Lambda\left(z^0, z^1\right) = (U^1, -U^0)$$

to obtain suitable initial values $\left(z^0, z^1\right)$ leading to (4.5). In fact, we prove that Λ is an isomorphism and then the solution of the above equation is unique. In this direction, we compute

$$\begin{aligned}
\left\langle \Lambda\left(z^0, z^1\right), \left(z^0, z^1\right)\right\rangle &= \left\langle \left(\theta'(0), -\theta(0)\right), \left(z^0, z^1\right)\right\rangle \\
&= \left\langle \theta_1'(0), z_1^0\right\rangle_{V',V} - \int_{\Omega_1} z_1^1 \theta_1(0) dx \\
&\quad + \left\langle \theta_2'(0), z_2^0\right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} - \int_{\Omega_2} z_2^2 \theta_2(0) dx,
\end{aligned} \tag{4.8}$$

for every $\left(z^0, z^1\right) \in H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$.

By definition of transposition solution, it is easy to see that the right hand side of the above equation satisfies

$$\left\langle \theta_1'(0), z_1^0 \right\rangle_{V',V} - \int_{\Omega_1} z_1^1 \theta_1(0) dx + \left\langle \theta_2'(0), z_2^0 \right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} - \int_{\Omega_2} z_2^2 \theta_2(0) dx$$
$$= \int_0^T \int_{\omega_1} \left(\left| z_1' \right|^2 + |z_1|^2 \right) dx\, dt + \int_0^T \int_{\omega_2} \left(\left| z_2' \right|^2 + |z_2|^2 \right) dx\, dt. \tag{4.9}$$

Thus, we have

$$\left\langle \Lambda \left(z^0, z^1 \right), \left(z^0, z^1 \right) \right\rangle = \int_{\omega_1 \times (0,T)} \left(\left| z_1' \right|^2 + |z_1|^2 \right) dx\, dt + \int_0^T \int_{\omega_2} \left(\left| z_2' \right|^2 + |z_2|^2 \right) dx\, dt. \tag{4.10}$$

In view of the equivalence of the norms stated in Corollary 3.10, the above identity shows that Λ is an isomorphism between $H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2) \right)$ and $(H_\Gamma)' \times \left(L^2(\Omega_1) \times L^2(\Omega_2) \right)$, for T large enough so that (4.2) is satisfied. Hence Theorem 4.1 holds true with exact control

$$\zeta \chi_\omega = (\zeta_1 \chi_{\omega_1}, \zeta_2 \chi_{\omega_2}) = ((-z_1'' + z_1)\chi_{\omega_1}, (-z_2'' + z_2)\chi_{\omega_2}),$$

which is an element of W'. □

We point out that, unlike classical cases, the lower bound for the control time T depends not only on the geometry of our domain and on the matrix of coefficients of our problem but also on the coefficient of proportionality of the jump of the solution of problem (2.2) with respect to the conormal derivatives via the constant h_0.

Acknowledgements This paper was completed during the visit of the second author at the University of Sannio, Department of Science and Techology, whose warm hospitality and support are gratefully acknowledged. The work was supported by the grant FFABR of MIUR. S.M. and C.P. are members of GNAMPA of INDAM.

Author Contributions The authors conceived and wrote this article in collaboration and with the same responsibility. All of them read and approved the final manuscript.

Data Availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

References

1. Auriault, J.L., Ene, H.: Macroscopic modelling of heat transfer in composites with interfacial thermal barrier. Int. J. Heat Mass Transf. **37**, 2885–2892 (1994)
2. Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Generalization of unfolding operator for highly oscillating smooth boundary domains and homogenization. Calc. Var. Partial Differ. Equ. **57**(3), 86 (2018)
3. Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Semi-linear optimal control problem on a smooth oscillating domain. Commun. Contemp. Math. **22**, 1–26 (2019). https://doi.org/10.1142/S0219199719500299
4. Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Locally periodic unfolding operator for highly oscillating rough domains. Ann. Mat. Pura Appl. **198**(6), 1931–1954 (2019). https://doi.org/10.1007/s10231-019-00848-7
5. Aiyappan, A., Nandakumaran, A.K., Sufian, Abu: Asymptotic analysis of a boundary optimal control problem on a general branched structure. Math. Methods Appl. Sci. **42**(18), 6407–6434 (2019)
6. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. SIAM J. Contin. Optim. **30**, 1024–1065 (1992)
7. Canon, E., Pernin, J.N.: Homogenization of diffusion in composite media with interfacial barrier. Rev. Roumaine Math. Pures Appl. **44**, 23–36 (1999)
8. Cioranescu, D., Saint Jean Paulin, J.: Homogenization in open sets with holes. J. Math. Anal. Appl. **71**, 590–607 (1979)
9. Cioranescu, D., Donato, P.: Some remarks on the exact controllability in a neighbourhood of the boundary of a perforated domain. In: Control of Boundaries and Stabilization (Clermont-Ferrand, 1988), pp. 75–94. Lecture Notes in Control and Information Sciences, vol. 125. Springer, Berlin (1989)
10. Cioranescu, D., Donato, P.: Exact internal controllability in perforated domains. J. Math. Pures Appl. **68**(2), 185–213 (1989)
11. Cioranescu, D., Donato, P., Zuazua, E.: Exact boundary controllability for the wave equation in domains with small holes. J. Math. Pures Appl. **71**(4), 343–377 (1992)
12. De Maio, U., Gaudiello, A., Lefter, C.: Optimal control for a parabolic problem in a domain with highly oscillating boundary. Appl. Anal. **83**(12), 1245–1264 (2004)
13. De Maio, U., Nandakumaran, A.K.: Exact internal controllability for a hyperbolic problem in a domain with highly oscillating boundary. Asymptot. Anal. **83**(3), 189–206 (2013)
14. De Maio, U., Faella, L., Perugia, C.: Optimal control problem for an anisotropic parabolic problem in a domain with very rough boundary. Ric. Mat. **63**(2), 307–328 (2014)
15. De Maio, U., Faella, L., Perugia, C.: Optimal control for a second-order linear evolution problem in a domain with oscillating boundary. Complex Var. Elliptic Equ. **6**(10), 1392–1410 (2015)
16. De Maio, U., Nandakumaran, A.K., Perugia, C.: Exact internal controllability for the wave equation in a domain with oscillating boundary with Neumann boundary condition. Evol. Equ. Control Theory **4**(3), 325–346 (2015)
17. Donato, P.: Some corrector results for composites with imperfect interface. Rend. Mat. Ser. VII **26**, 189–209 (2006)
18. Donato, P.: Homogenization of a class of imperfect transmission problems. In: Damlamian, A., Miara, B., Li, T. (eds.) Multiscale Problems: Theory, Numerical Approximation and Applications. Series in Contemporary Applied Mathematics CAM 16, pp. 109–147. Higher Education Press, Beijing (2011)
19. Donato, P., Jose, E.: Corrector results for a parabolic problem with a memory effect. ESAIM Math. Model. Numer. Anal. **44**, 421–454 (2010)
20. Donato, P., Jose, E.: Asymptotic behavior of the approximate controls for parabolic equations with interfacial contact resistance. ESAIM Control Optim. Calc. Var. **21**, 138–164 (2015). https://doi.org/10.1051/cocv/2014029
21. Donato, P., Jose, E.: Approximate controllability of a parabolic system with imperfect interfaces. Philipp. J. Sci. **144**(2), 187–196 (2015)
22. Donato, P., Monsurrò, S.: Homogenization of two heat conductors with interfacial contact resistance. Anal. Appl. **2**, 247–273 (2004)
23. Donato, P., Nabil, A.: Approximate controllability of linear parabolic equations in perforated domains. ESAIM Control Optim. Calc. Var. **6**, 21–38 (2001)

24. Donato, P., Raimondi, F.: Uniqueness result for a class of singular elliptic problems in two-component domains. J. Elliptic Parabol. Equ. **5**(2), 349–358 (2019)
25. Donato, P., Raimondi, F.: Existence and uniqueness results for a class of singular elliptic problems in two-component domains. In: Integral Methods in Science and Engineering, vol. 1. Theoretical Techniques, pp. 83–93. Birkhäuser/Springer, Cham (2017)
26. Donato, P., Faella, L., Monsurrò, S.: Homogenization of the wave equation in composites with imperfect interface: a memory effect. J. Math. Pures Appl. **87**, 119–143 (2007)
27. Donato, P., Faella, L., Monsurrò, S.: Correctors for the homogenization of a class of hyperbolic equations with imperfect interfaces. SIAM J. Math. Anal. **40**, 1952–1978 (2009)
28. Donato, P., Monsurrò, S., Raimondi, F.: Existence and uniqueness results for a class of singular elliptic problems in perforated domains. Ric. Mat. **66**(2), 333–360 (2017)
29. Donato, P., Monsurrò, S., Raimondi, F.: Homogenization of a class of singular elliptic problems in perforated domains. Nonlinear Anal. **173**, 180–208 (2018)
30. Durante, T., Mel'nyk, T.A.: Asymptotic analysis of an optimal control problem involving a thick two-level junction with alternate type of controls. J. Optim. Theory Appl. **144**(2), 205–225 (2010)
31. Durante, T., Mel'nyk, T.A.: Homogenization of quasilinear optimal control problems involving a thick multilevel junction of type 3:2:1. ESAIM Control Optim. Calc. Var. **18**(2), 583–610 (2012)
32. Durante, T., Faella, L., Perugia, C.: Homogenization and behaviour of optimal controls for the wave equation in domains with oscillating boundary. NoDEA Nonlinear Differ. Equ. Appl. **14**(5–6), 455–489 (2007)
33. Faella, L., Monsurrò, S.: Memory effects arising in the homogenization of composites with inclusions. In: Topics on Mathematics for Smart System, pp. 107–121. World Scientific Publications, Hackensack (2007)
34. Faella, L., Perugia, C.: Optimal control for evolutionary imperfect transmission problems. Bound. Value Probl. **2015**, 50 (2015). https://doi.org/10.1186/s13661-015-0310-z
35. Faella, L., Perugia, C.: Optimal control for a hyperbolic problem in composites with imperfect interface: a memory effect. Evol. Equ. Control Theory **6**(2), 187–217 (2017). https://doi.org/10.3934/eect.2017011
36. Faella, L., Monsurrò, S., Perugia, C.: Homogenization of imperfect transmission problems: the case of weakly converging data. Differ. Integral Equ. **31**, 595–620 (2018)
37. Faella, L., Monsurró, S., Perugia, C.: Exact controllability for an imperfect transmission problem. J. Math. Pures Appl. **122**, 235–271 (2019)
38. Fernandez-Cara, E.: Null controllability of the semilinear heat equation. ESAIM Control Optim. Calc. Var. **2**, 87–103 (1997)
39. Hummel, H.C.: Homogenization for heat transfer in polycrystals with interfacial resistances. Appl. Anal. **75**, 403–424 (2000)
40. Khludnev, A.M., Faella, L., Perugia, C.: Optimal control of rigidity parameters of thin inclusions in composite materials. Z. Angew. Math. Phys. **68**(2), 47 (2017). https://doi.org/10.1007/s00033-017-0792-x
41. Li, L., Zhang, X.: Exact controllability for semilinear wave equations. J. Math. Anal. Appl. **250**(2), 589–597 (2000)
42. Lions, J.L.: Contrôlabilité Exacte et Homogénéisation. I. Asymptot. Anal. **1**(1), 3–11 (1988)
43. Lions, J. L.: Contrôlabilité exacte, stabilization at perturbations de systéms distributé, Tomes 1,2 Massonn, RMA, 829, (1988)
44. Lions, J.L.: Exact controllability, stabilization and perturbations for distributed systems. SIAM Rev. **30**, 1–68 (1988)
45. Lions, J.L., Magenes, E.: Non-homogeneous boundary value problems and applications, vol. I. Springer-Verlag, Berlin Heidelberg, New York (1972)
46. Lipton, R.: Heat conduction in fine scale mixtures with interfacial contact resistance. SIAM J. Appl. Math. **58**, 55–72 (1998)
47. Lipton, R., Vernescu, B.: Composite with imperfect interface. Proc. R. Soc. Lond. Ser. A **452**, 329–358 (1996)
48. Liu, W., Williams, G.H.: Exact Neumann boundary controllability for second order hyperbolic equations. Coll. Math. **76**(1), 117–141 (1998)
49. Liu, W., Williams, G.H.: Exact Neumann boundary controllability for problems of transmission of the wave equation. Glasgow Math. J. **41**, 125–139 (1999)

50. Monsurrò, S.: Homogenization of a two-component composite with interfacial thermal barrier. Adv. Math. Sci. Appl. **13**, 43–63 (2003)
51. Monsurrò, S.: Erratum for the paper "Homogenization of a two-component composite with interfacial thermal barrier". Adv. Math. Sci. Appl. **14**, 375–377 (2004)
52. Monsurrò, S.: Homogenization of a composite with imperfect interface. Ricerch. Mat. **54**(2), 623–629 (2005)
53. Monsurrò, S.: Homogenization of a composite with very small inclusions and imperfect interface, Multi scale problems and asymptotic analysis. GAKUTO Int. Ser. Math. Sci. Appl., vol. 24, pp. 217–232. Gakkotosho, Tokyo (2006)
54. Monsurrò, S., Perugia, C.: Homogenization and exact controllability for problems with imperfect interface. Netw. Heterog. Media **14**(2), 411–444 (2019)
55. Monsurrò, S., Nandakumaran, A. K., Perugia, C.: A note on the exact boundary controllability for an imperfect transmission problem. Ric. Mat. (2021). https://doi.org/10.1007/s11587-021-00625-w
56. Nandakumaran, A.K., Sufian, A.: Oscillating PDE in a rough domain with a curved interface: homogenization of an optimal control problem. ESAIM Control Optim. Calc. Var. (2021). https://doi.org/10.1051/cocv/2020045
57. Nandakumaran, A.K., Sili, A.: Homogenization of a hyperbolic equation with highly contrasting diffusivity coefficients. Differ. Integral Equ. **29**(1/2), 37–54 (2016)
58. Nandakumaran, A.K., Prakash, R., Sardar, B.C.: Periodic controls in an oscillating domain: controls via unfolding and homogenization. SIAM J. Control Optim. **53**, 3245–3269 (2015)
59. Nandakumaran, A.K., Rajesh, M., Prakash, R.: Homogenization of an elliptic equation in a domain with oscillating boundary with non-homogeneous non-linear boundary conditions. Appl. Math. Optim. **82**, 1–34 (2018)
60. Yang, Z.: Homogenization and correctors for the hyperbolic problems with imperfect interfaces via the periodic unfolding method. Commun. Pure Appl. Anal. **13**(1), 249–272 (2014)
61. Yang, Z.: The periodic unfolding method for a class of parabolic problems with imperfect interfaces. ESAIM Math. Model. Numer. Anal. **48**(5), 1279–1302 (2014)
62. Zuazua, E.: Exact boundary controllability for the semilinear wave equation. In: Nonlinear Partial Differential Equations and their Applications, pp. 357–391. Pitman Publishing, London (1991)
63. Zuazua, E.: Approximate controllability for linear parabolic equations with rapidly oscillating coefficients. Control Cybernet. **4**, 793–801 (1994)
64. Zuazua, E.: Controllability of partial differential equations and its semi-discrete approximations. Discret. Contin. Dyn. Syst. **8**, 469–513 (2002)
65. Zuazua, E.: Controllability and observability of partial differential equations: some results and open problems (English summary). In: Handbook of Differential Equations, vol. III, pp. 527–621. Elsevier/North-Holland, Amsterdam (2007)