Appl. Comput. Harmon. Anal. 60 (2022) 396–425

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Hirschman–Widder densities

Alexander Belton [a], Dominique Guillot [b], Apoorva Khare [c,d], Mihai Putinar [e,f,*]

[a] Department of Mathematics and Statistics, Lancaster University, Lancaster, UK
[b] University of Delaware, Newark, DE, USA
[c] Department of Mathematics, Indian Institute of Science, Bangalore, India
[d] Analysis and Probability Research Group, Bangalore, India
[e] University of California at Santa Barbara, CA, USA
[f] Newcastle University, Newcastle upon Tyne, UK

A R T I C L E   I N F O

Article history:
Received 10 January 2021
Received in revised form 2 January 2022
Accepted 8 April 2022
Available online 12 April 2022
Communicated by Karlheinz Grochenig

MSC:
primary 15B48
secondary 05E05, 15A15, 30C40, 44A10, 47B34

Keywords:
Pólya frequency function
Totally positive function
Hypoexponential distribution

A B S T R A C T

Hirschman and Widder introduced a class of Pólya frequency functions given by linear combinations of one-sided exponential functions. The members of this class are probability densities, and the class is closed under convolution but not under pointwise multiplication. We show that, generically, a polynomial function of such a density is a Pólya frequency function only if the polynomial is a homothety, and also identify a subclass for which each positive-integer power is a Pólya frequency function. We further demonstrate connections between the Maclaurin coefficients, the moments of these densities, and the recovery of the density from finitely many moments, via Schur polynomials.

1. Introduction and main results

The class of Pólya frequency functions is central to the theory of total positivity. Its basic properties were announced by Schoenberg in 1947–48 [34,35] and further details were provided in subsequent work [36,37]. These functions have been actively studied ever since.

Definition 1.1 *(Schoenberg).* A function $\Lambda : \mathbb{R} \to [0, \infty)$ is a *Pólya frequency function* if it is Lebesgue integrable and non-zero at two or more points, and the Toeplitz kernel

* Corresponding author.
E-mail addresses: a.belton@lancaster.ac.uk (A. Belton), dguillot@udel.edu (D. Guillot), khare@iisc.ac.in (A. Khare), mputinar@math.ucsb.edu, mihai.putinar@ncl.ac.uk (M. Putinar).

https://doi.org/10.1016/j.acha.2022.04.002
1063-5203/

$$T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ (x, y) \mapsto \Lambda(x - y)$$

is totally non-negative. This last statement means that, for any integer $p \geq 1$ and real numbers

$$x_1 < \cdots < x_p \qquad \text{and} \qquad y_1 < \cdots < y_p,$$

the matrix $\big(\Lambda(x_j - y_k)\big)_{j,k=1}^{p}$ has non-negative determinant.

Schoenberg showed in [37] that the bilateral Laplace transform

$$\mathcal{B}\{\Lambda\}(s) := \int_{\mathbb{R}} e^{-xs}\Lambda(x)\,\mathrm{d}x$$

of a Pólya frequency function Λ converges in a open strip containing the imaginary axis, and equals on this strip the reciprocal of an entire function Ψ in the Laguerre–Pólya class [23,32], with $\Psi(0) = 1$. Conversely, any function Ψ of this form agrees with the reciprocal of the bilateral Laplace transform of some Pólya frequency function on its strip of convergence. Schoenberg also proved that a Pólya frequency function necessarily has unbounded support, and either vanishes nowhere or vanishes on a semi-axis. Members of the latter class of functions are said to be *one sided*, and Schoenberg [37] also characterized this subclass via the bilateral Laplace transform. This characterization also allows the non-smooth members of this subclass to be identified.

Theorem 1.2 *(Schoenberg). If the one-sided Pólya frequency function Λ vanishes on $(-\infty, 0)$ then $1/\mathcal{B}\{\Lambda\}$ is the restriction of an entire function Ψ in the first Laguerre–Pólya class, so has the form*

$$\Psi(s) = Ce^{\delta s}\prod_{j=1}^{\infty}(1 + \alpha_j s), \quad \textit{where } C > 0,\ \delta \geq 0,\ \alpha_j \geq 0 \textit{ and } 0 < \sum_{j=1}^{\infty}\alpha_j < \infty. \tag{1.1}$$

Conversely, if the entire function Ψ has the form (1.1) then there exists a Pólya frequency function Λ that vanishes on $(-\infty, 0)$ such that $\Psi(s) = 1/\mathcal{B}\{\Lambda\}(s)$ on an open strip containing the origin.

Such a Pólya frequency function Λ is continuous and positive on (δ, ∞) and vanishes on $(-\infty, \delta)$. Furthermore, the function Λ is smooth if and only if α_j is non-zero for infinitely many j, and is continuous unless α_j is non-zero for exactly one j.

In this note, we study the one-sided Pólya frequency functions which are continuous but non-smooth, that is, those with at least two but only finitely many non-zero terms $\alpha_1, \ldots, \alpha_m$ in (1.1), and their powers. We may normalize so that Λ is a probability density function, whence $C = 1$, and we may also assume $\delta = 0$, by replacing Λ with $x \mapsto \Lambda(x + \delta)$. Thus, we study the collection of *finitely determined* Pólya frequency functions of the form $\Lambda_{\boldsymbol{\alpha}}$, such that

$$\mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(s) = \prod_{j=1}^{m}(1 + \alpha_j s)^{-1}, \qquad \text{where } \boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_m) \text{ and } m \geq 2.$$

Some historical comments are appropriate, and here we recount this subject's early developments in chronological order. In 1947, Schoenberg [34] announced the notion of a Pólya frequency function. In their 1949 work, Hirschman and Widder [14] studied $\Lambda_{\boldsymbol{\alpha}}$ for distinct positive $\alpha_1, \ldots, \alpha_m$ and its degree of smoothness, via the Laplace transform. This was followed by Schoenberg's first full paper on Pólya frequency functions [37] in 1951. In this work, Schoenberg placed the analysis of Hirschman and Widder in a wider context, with the last part of Theorem 1.2 showing that the collection of Hirschman–Widder functions is dense, in a suitable sense, in the set of non-smooth one-sided Pólya frequency functions. Finally, Hirschman and Widder's 1955 monograph [15] contains a detailed analysis of these functions and their Laplace transforms, and provides ample evidence for the relevance of such functions to operational calculus and approximation theory.

For this reason, we adopt the following terminology for this family of functions, now allowing for non-distinct and negative α_j. For brevity, we let $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ denote the set of non-zero real numbers.

Definition 1.3. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{R}^\times)^m$, where $m \geq 2$, the corresponding *Hirschman–Widder density* is the unique continuous function $\Lambda_{\boldsymbol{\alpha}} : \mathbb{R} \to [0, \infty)$ with bilateral Laplace transform

$$\int_{\mathbb{R}} e^{-xs} \Lambda_{\boldsymbol{\alpha}}(x) \,\mathrm{d}x = \prod_{j=1}^{m} (1 + \alpha_j s)^{-1} \tag{1.2}$$

on the open half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\alpha_j^{-1} \text{ for } j = 1, \ldots, m\}$.

The next section focuses on some basic properties of these functions.

(1) Each such function $\Lambda_{\boldsymbol{\alpha}}$ exists and is unique.
(2) The function $\Lambda_{\boldsymbol{\alpha}}$ is both a Pólya frequency function and a probability density function. It is one sided if and only if all the entries of $\boldsymbol{\alpha}$ have the same sign.
(3) The function $\Lambda_{\boldsymbol{\alpha}}$ has a multiplicative representation via convolution, as well as an additive one involving one-sided exponentials. The class of Pólya frequency functions and its subclass of Hirschman–Widder densities are both semigroups for the convolution product.

However, this collection of densities is not closed under pointwise multiplication. The principal contribution of the present work is to identify when some simple algebraic operations preserve the class of Hirschman–Widder densities and when they do not. More specifically, we focus on polynomial functions and study the generic behavior of these operations and their departure from mapping the class of densities to itself.

We consider only those transforms which preserve the Pólya frequency property of infinite order, that is, with the natural number p arbitrarily large in Definition 1.1. It was known to Karlin in the 1960s [18] that there exist Pólya frequency functions whose αth powers are totally non-negative to some fixed finite order for sufficiently large α; see also [20, p. 115]. We note that requiring total non-negativity only up to some finite order no longer guarantees that the reciprocal of the Laplace transform is entire.

Our main result is as follows.

Theorem 1.4.

(1) *Suppose $m \geq 3$. There exists a subset $\mathcal{N}$ of $(0, \infty)^m$ with Lebesgue measure zero such that*

$$p \circ \Lambda_{\boldsymbol{\alpha}} : x \mapsto p\big(\Lambda_{\boldsymbol{\alpha}}(x)\big)$$

is not a Pólya frequency function for any $\boldsymbol{\alpha} \in (0, \infty)^m \setminus \mathcal{N}$ and any real polynomial p that is not a homothety, that is, $p(x) \not\equiv cx$ for any $c > 0$.

(2) *Suppose $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0, \infty)^m$, where $m \geq 2$, is such that the reciprocals $a_1 := \alpha_1^{-1}, \ldots, a_m := \alpha_m^{-1}$ form an arithmetic progression. Then $c\Lambda_{\boldsymbol{\alpha}}^n$ is a Pólya frequency function for every $c > 0$ and every integer $n \geq 1$. If, moreover, $\alpha_1 = \alpha_m$ or α_1/α_2 is irrational, then $p \circ \Lambda_{\boldsymbol{\alpha}}$ is not a Pólya frequency function for any other real polynomial p.*

We remark that the first assertion with $m = 3$ and $p(x) = x^n$ was shown in our recent work [3], and played a key role in characterizing those post-composition transforms that preserve the class of one-sided Pólya frequency functions. Theorem 1.4 shows that these $m = 3$ examples are merely the first in a large, multi-parameter family of Pólya frequency functions with the same property.

In the case where $p(x) = x^n$, note that the assumption $m \geq 3$ in the first statement is necessary. Indeed, the $m = 2$ case is covered by the second assertion, since every pair of numbers is trivially in arithmetic progression.

The exceptional null set $\mathcal{N} \subset (0, \infty)^m$ appearing in Theorem 1.4(1) has the following structure. If a tuple $\boldsymbol{\alpha}$ lies in the complement of $\mathcal{N}$ in $(0, \infty)^m$ then $p \circ \Lambda_{\boldsymbol{\alpha}}$ is not a Pólya frequency function for any non-homothetic polynomial p. The tuples in $\mathcal{N}$ are precisely those of the form $(\alpha_1, \dots, \alpha_m)$ such that the coordinates of the reciprocal tuple $(\alpha_1^{-1}, \dots, \alpha_m^{-1})$ are either $\mathbb{Q}$-linearly independent or are roots of a countable family of non-zero polynomials given in (4.1). The $\mathbb{Q}$-linearly dependent tuples lie on a countable union of hyperplanes, and the zero loci of the polynomials in (4.1) also lie on null sets. The reciprocals of these tuples form the null set $\mathcal{N}$.

As discussed in Section 2.4, the Hirschman–Widder densities are connected in multiple ways to classical probability theory. In the present work, we link them to another area: the theory of symmetric functions, and specifically Schur polynomials. In fact, this also has a possible connection to probability theory: we show below that these densities can be reconstructed from finitely many moments, via symmetric function identities.

Definition 1.5. Given a field $\mathbb{F}$ of size at least $m \geq 2$, and a tuple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ of non-negative integers $\lambda_1 \leq \dots \leq \lambda_m$, we define the corresponding *Schur polynomial* to be the polynomial extension of the function

$$s_{\boldsymbol{\lambda}}(a_1, \dots, a_m) := \frac{\det(a_j^{\lambda_k})_{j,k=1}^m}{V(a_1, \dots, a_m)}$$

for distinct $a_1, \dots, a_m \in \mathbb{F}$, where $V(a_1, \dots, a_m) := \det(a_j^{k-1}) = \prod_{1 \leq j < k \leq m}(a_k - a_j)$ is the usual Vandermonde determinant. If consecutive exponents are equal, the Schur polynomial is identically zero.

This definition differs from the one more commonly found in the literature, such as [25], in that the entries of $\boldsymbol{\lambda}$ are non-decreasing rather than non-increasing and the determinant in the numerator has exponent λ_k instead of $\lambda_k + n - k$. To switch between these two conventions is straightforward: if $\nu_j = \lambda_{m-j+1} - m + j$ then $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m)$ is such that $\nu_1 \geq \dots \geq \nu_m$ if and only if $\lambda_1 < \dots < \lambda_m$, in which case $\widetilde{s}_{\boldsymbol{\nu}}(\mathbf{a}) = s_{\boldsymbol{\lambda}}(\mathbf{a})$ for any $\mathbf{a} \in \mathbb{F}^m$, where $\widetilde{s}_{\boldsymbol{\nu}}$ is defined as in [25, (3.1)].

Schur polynomials are a distinguished family of symmetric functions, and form a basis of homogeneous symmetric polynomials. They arise naturally as characters of finite-dimensional irreducible representations of $GL_{m+1}(\mathbb{C})$ or $\mathfrak{sl}_{m+1}(\mathbb{C})$, and specialize to other families of symmetric functions.

We can now make clear the connections between Hirschman–Widder densities and Schur polynomials: both the Maclaurin coefficients and the moments of these densities are given by Schur polynomials, and are, in a certain sense, mirror images of one another.

Theorem 1.6. *Given $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, \infty)^m$, where $m \geq 2$, the Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ is represented by its Maclaurin series on $[0, \infty)$, with nth coefficient*

$$\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+) = \begin{cases} 0 & \text{if } 0 \leq n \leq m-2, \\ (-1)^{n-m+1} \alpha_1^{-1} \cdots \alpha_m^{-1} s_{(0,1,\dots,m-2,n)}(\alpha_1^{-1}, \dots, \alpha_m^{-1}) & \text{if } n \geq m-1. \end{cases}$$

The density $\Lambda_{\boldsymbol{\alpha}}$ has pth moment

$$\mu_p := \int_{\mathbb{R}} x^p \Lambda_{\boldsymbol{\alpha}}(x)\,\mathrm{d}x = p!\, s_{(0,1,\ldots,m-2,m-1+p)}(\alpha_1,\ldots,\alpha_m) \qquad \text{if } p \geq 0.$$

The parameter $\boldsymbol{\alpha}$ can be recovered, up to permutation of its entries, from the first m moments, μ_1, …, μ_m, and also from the first $m+1$ non-trivial Maclaurin coefficients, $\Lambda_{\boldsymbol{\alpha}}^{(m-1)}(0^+)$, …, $\Lambda_{\boldsymbol{\alpha}}^{(2m-1)}(0^+)$.

It may seem incongruous to require m moments but $m+1$ Maclaurin coefficients in order to recover the m coordinates α_1, …, α_m. However, this is explained by noting that $\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+)$ equals $(\alpha_1 \cdots \alpha_m)^{-1}$ times a polynomial in the reciprocal coordinates α_1^{-1}, …, α_m^{-1}, and this polynomial is 1 if $n = m-1$. Thus, the parameter $\boldsymbol{\alpha}$ is recovered from the data $\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+)/\Lambda_{\boldsymbol{\alpha}}^{(m-1)}(0^+)$ for $n = m$, …, $2m-1$.

To the best of our knowledge, the connections in Theorem 1.6 between the moments, the Maclaurin coefficients, and the moment-recovery problem for Hirschman–Widder densities have not previously been noted in the literature. While the moments are computable from first principles using probability theory, we provide another recipe: the moment-generating function may be obtained by evaluating the generating function of the complete homogeneous symmetric polynomials. In a similar spirit, the moment-recovery problem is seen to be intimately connected with the Jacobi–Trudi identity.

Pólya frequency functions, and in particular the subclass of Hirschman–Widder densities, form a foundational chapter within the wider framework of totally positive kernels. This latter concept continues to attract generations of mathematicians, with surprising new developments. It is not the intention of the present article to touch on the many ramifications and current discoveries in this subject, with one exception. The first footnote in the note by Vershik and Kerov [41] contains the following line (our translation): *"It is worth mentioning that, in works going back to the 30s, Schoenberg, but also Krein and Gantmacher, and later Karlin, have developed the theory of totally positive kernels and matrices. However, a connection with the characters of the unitary group and Weyl's formula was not remarked at that time."* That happened later, with a series of spectacular discoveries: Fourier transforms of Pólya frequency functions were rediscovered as irreducible characters of representations of the infinite symmetric group [40,28], and independently as irreducible characters of unitary representations of the infinite unitary group $U(\infty)$, [42,31]. Moreover, the classification and explicit expression of spherical functions associated to classical groups [11,13] also pointed to the same class of totally positive kernels. The string of coincidental findings of new facets of the same object does not stop here: for example, challenging computations of the characteristic functions of non-central Wishart distributions in multivariate statistics led to Pólya frequency functions and Schur polynomials [17,8,39]. Nowadays, these advances are part of group representation theory [29,41,7] or random matrix theory [20–22,10]. By reversing the arrow of discovery, we reproduce without proof in Section 3 an orbital integral which encodes analytic properties of Hirschman–Widder densities. Our work remains at an independent, elementary level, and we will indicate the simplifications this view provides.

1.1. Acknowledgments

We are grateful to the referees for their careful reading of this paper and several helpful suggestions that have improved its contents and clarified our presentation.

D.G. was partially supported by a University of Delaware Research Foundation grant, by a Simons Foundation collaboration grant for mathematicians, and by a University of Delaware strategic initiative grant.

A.K. was partially supported by Ramanujan Fellowship grant SB/S2/RJN-121/2017, MATRICS grant MTR/2017/000295, SwarnaJayanti Fellowship grants SB/SJF/2019-20/14 and DST/SJF/MS/2019/3 from SERB and DST (Govt. of India), and by grant F.510/25/CAS-II/2018(SAP-I) from UGC (Govt. of India).

M.P. was partially supported by a Simons Foundation collaboration grant.

2. Hirschman–Widder densities and symmetric functions

In this section, we prove Theorem 1.6. We begin by recalling two approaches to constructing Hirschman–Widder densities, guided by the original memoir [15].

2.1. Two constructions of Hirschman–Widder densities

Following Hirschman and Widder [15], we first establish the existence of their eponymous densities via the convolution product.

Proposition 2.1. *Let* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\mathbb{R}^\times)^m$*, where* $m \geq 2$.

(1) *The corresponding Hirschman–Widder density* $\Lambda_{\boldsymbol{\alpha}}$ *exists and is unique.*
(2) *The function* $\Lambda_{\boldsymbol{\alpha}}$ *is both a Pólya frequency function and a probability density function.*
(3) *The function* $\Lambda_{\boldsymbol{\alpha}}$ *is one sided if and only if the entries of* $\boldsymbol{\alpha}$ *all have the same sign.*

Proof. If $\alpha > 0$ then

$$\varphi_\alpha : \mathbb{R} \to [0, \infty);\ x \mapsto \begin{cases} 0 & \text{if } x < 0, \\ \alpha^{-1} e^{-\alpha^{-1} x} & \text{if } x \geq 0 \end{cases}$$

is a Pólya frequency function [34, (3)], and if $\alpha < 0$ then $\varphi_\alpha : x \mapsto \varphi_{-\alpha}(-x)$ is also a Pólya frequency function. The function

$$\Lambda_{\boldsymbol{\alpha}} := \varphi_{\alpha_1} * \cdots * \varphi_{\alpha_m} \tag{2.1}$$

is continuous, being a convolution product, and its bilateral Laplace transform is as required, since $\mathcal{B}\{\varphi_\alpha\}(s) = (1 + \alpha s)^{-1}$.

The class of Pólya frequency functions is closed under convolution [37, Lemma 5], and so $\Lambda_{\boldsymbol{\alpha}}$ is a Pólya frequency function. It is a probability density because $\mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(0) = 1$.

For uniqueness, we note that any continuous function with prescribed bilateral Laplace transform F such that $t \mapsto F(\mathrm{i}t)$ is integrable can be recovered everywhere via the Fourier–Mellin integral: here,

$$\Lambda_{\boldsymbol{\alpha}}(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{e^{xs}}{(1 + \alpha_1 s) \cdots (1 + \alpha_m s)} \, \mathrm{d}s \qquad (x \in \mathbb{R}). \tag{2.2}$$

If α_j and α_k have opposite signs then a short calculation shows that $(\varphi_{\alpha_j} * \varphi_{\alpha_k})(x) > 0$ for any $x \in \mathbb{R}$. Furthermore, if f is continuous and positive on $\mathbb{R}$ then $\varphi_\alpha * f$ is also continuous and positive on $\mathbb{R}$, for any $\alpha \in \mathbb{R}^\times$. This gives one part of (3) and the converse is immediate. □

As the Laplace transform does not behave well for the product given by pointwise multiplication, it is useful to have a second construction for the function $\Lambda_{\boldsymbol{\alpha}}$.

Given one-sided exponentials

$$\mathbf{1}_{x \geq 0}\, e^{-a_1 x}, \dots, \mathbf{1}_{x \geq 0}\, e^{-a_k x}, \qquad \text{where } k \geq 2 \text{ and } 0 < a_1 < \cdots < a_k,$$

there are, up to homothety, only finitely many choices of real coefficients $c_1, \dots, c_k$ such that the linear combination $\mathbf{1}_{x \geq 0} \sum_{j=1}^k c_j e^{-a_j x}$ is a Pólya frequency function. More generally, we have the following result, where each coefficient c_j may be a polynomial.

Definition 2.2. For a tuple of positive real numbers $\mathbf{a} = (a_1, \dots, a_k)$ such that $k \geq 1$ and $a_1 < \dots < a_k$, and a tuple of real polynomials $\mathbf{c} = (c_1, \dots, c_k)$, let

$$\Lambda_{\mathbf{a},\mathbf{c}} : \mathbb{R} \to \mathbb{R};\ x \mapsto \begin{cases} \sum_{j=1}^k c_j(x) e^{-a_j x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We let $\deg p$ denote the degree of the polynomial p, with $\deg p := -\infty$ if $p = 0$.

Proposition 2.3. *Suppose $\mathbf{a} = (a_1, \dots, a_k)$ is a tuple of positive real numbers such that $k \geq 1$ and $a_1 < \dots < a_k$.*

(1) *Given any tuple of non-negative integers $\mathbf{n} = (n_1, \dots, n_k)$, not all zero, there exists a unique tuple $\mathbf{c} = \mathbf{c}^{\mathbf{a},\mathbf{n}} = (\mathbf{c}_1^{\mathbf{a},\mathbf{n}}, \dots, \mathbf{c}_k^{\mathbf{a},\mathbf{n}})$ of real polynomials such that*

$$\mathcal{B}\{\Lambda_{\mathbf{a},\mathbf{c}}\}(s) = \prod_{j=1}^k (1 + \alpha_j s)^{-n_j}, \quad \text{where } \alpha_j := a_j^{-1} \text{ for } j = 1, \dots, k. \tag{2.3}$$

The tuple $\mathbf{c}^{\mathbf{a},\mathbf{n}}$ is such that $\deg c_j^{\mathbf{a},\mathbf{n}} \leq n_j - 1$ for all j.
In particular, if $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, \infty)^m$, $\mathbf{a} = (a_1, \dots, a_k) \in (0, \infty)^k$ and $\mathbf{m} = (m_1, \dots, m_k)$ are such that $a_1 < \dots < a_k$ and a_j^{-1} appears exactly m_j times in $\boldsymbol{\alpha}$, with $m_1 + \dots + m_k = m \geq 2$, then the function $\Lambda_{\mathbf{a},\mathbf{c}}$ with $\mathbf{c} = \mathbf{c}^{\mathbf{a},\mathbf{m}}$ is the Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$.

(2) *If the tuple of real polynomials $\mathbf{c}$ is not of the form $t\mathbf{c}^{\mathbf{a},\mathbf{n}}$ for some $t > 0$ and some non-zero tuple of non-negative integers $\mathbf{n}$, then $\Lambda_{\mathbf{a},\mathbf{c}}$ is not a Pólya frequency function.*

Proof. For (1), partial-fraction decomposition gives a family of real coefficients $c_{j,l}^{\mathbf{a},\mathbf{n}}$ such that

$$\prod_{j=1}^k (1 + \alpha_j s)^{-n_j} = \sum_{j=1}^k \sum_{l=1}^{n_j} c_{j,l}^{\mathbf{a},\mathbf{n}} (1 + \alpha_j s)^{-l}.$$

Furthermore, if $a > 0$ and $\alpha := a^{-1}$ then

$$\mathcal{B}\{\mathbf{1}_{x \geq 0}\, x^{l-1} e^{-ax}\}(s) = (l-1)!\, \alpha^l (1 + \alpha s)^{-l} \qquad (l = 1, 2, \dots).$$

It follows that setting

$$c_j^{\mathbf{a},\mathbf{n}}(x) := \sum_{l=1}^{n_j} \frac{a_j^l c_{j,l}^{\mathbf{a},\mathbf{n}}}{(l-1)!} x^{l-1} \qquad (j = 1, \dots k) \tag{2.4}$$

gives the existence of $\mathbf{c}^{\mathbf{a},\mathbf{n}}$ as required. Uniqueness follows by Lerch's theorem: if $\mathcal{B}\{\Lambda_{\mathbf{a},\mathbf{c}}\} \equiv \mathcal{B}\{\Lambda_{\mathbf{a},\mathbf{c}'}\}$ on some half plane then $\Lambda_{\mathbf{a},\mathbf{c}} = \Lambda_{\mathbf{a},\mathbf{c}'}$, as both restrict to continuous functions on $[0, \infty)$ and vanish elsewhere. Hence it suffices to show that

$$\sum_{j=1}^k p_j(x) e^{-a_j x} \equiv 0 \implies p_1(x) = \dots = p_k(x) \equiv 0$$

for any real polynomials $p_1, \dots, p_k$, but this follows because

$$\sum_{j=1}^{k} p_j(x) e^{-a_j x} \equiv 0 \implies p_1(x) + \sum_{j=2}^{k} p_j(x) e^{(a_1 - a_j)x} \equiv 0 \implies \lim_{x \to \infty} p_1(x) = 0,$$

whence $p_1 = 0$, and so on. For the final claim, it suffices to show that $\Lambda_{\mathbf{a},\mathbf{c}}$ is continuous at the origin, that is,

$$0 = \sum_{j=1}^{k} c_j^{\mathbf{a},\mathbf{m}}(0) = \sum_{j:m_j>0} a_j c_{j,1}^{\mathbf{a},\mathbf{m}},$$

where the second equality comes from (2.4) and the second sum is taken over those j for which m_j is positive. This sum vanishes because

$$\sum_{j:m_j>0} a_j c_{j,1}^{\mathbf{a},\mathbf{m}} = \lim_{s \to \infty} \sum_{j=1}^{k} \sum_{l=1}^{m_j} c_{j,l}^{\mathbf{a},\mathbf{m}} s (1 + \alpha_j s)^{-l} = \lim_{s \to \infty} s \prod_{j=1}^{k} (1 + \alpha_j s)^{-m_j} = 0.$$

For (2), suppose $\mathbf{c}$ is not of the form $t\mathbf{c}^{\mathbf{a},\mathbf{n}}$ as asserted. Then $\mathcal{B}\{\Lambda_{\mathbf{a},\mathbf{c}}\}(s)$ is a rational function of the form $p(s)/\prod_{j=1}^{k}(1 + \alpha_j s)^{m_j}$, with the numerator polynomial p not a factor of the denominator. Hence the reciprocal of $\mathcal{B}\{\Lambda_{\mathbf{a},\mathbf{c}}\}$ is not the restriction of a function belonging to the Laguerre–Pólya class. By Theorem 1.2, $\Lambda_{\mathbf{a},\mathbf{c}}$ is not a Pólya frequency function. □

2.2. Vandermonde matrices and closed-form expressions

In the first part of Theorem 1.4, the generic tuple $\boldsymbol{\alpha}$ will be taken to have distinct entries. We may assume without loss of generality that these are strictly decreasing, whence the entries of the corresponding reciprocal tuple $\mathbf{a}$ are strictly increasing. The next result provides a closed-form expression for the unique tuple $\mathbf{c}$ such that $\Lambda_{\boldsymbol{\alpha}} = \Lambda_{\mathbf{a},\mathbf{c}}$ is a Hirschman–Widder density. Note first that Proposition 2.3 implies that $\mathbf{c}$ consists of polynomials of degree zero, that is, constants.

Proposition 2.4. *Let $\boldsymbol{\alpha} \in (0, \infty)^m$ be such that $m \geq 2$ and $\alpha_1 > \cdots > \alpha_m$, and let $\mathbf{a} = (a_1, \ldots, a_m)$, where $a_j := \alpha_j^{-1}$ for $j = 1, \ldots, m$. The Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}} = \Lambda_{\mathbf{a},\mathbf{c}}$, where $\mathbf{c} = (c_1, \ldots, c_m) \in (\mathbb{R}^\times)^m$ is such that*

$$c_j = a_j \prod_{k \neq j} \frac{a_k}{a_k - a_j} \qquad (j = 1, \ldots, m).$$

In particular, the constants $c_1, \ldots, c_m$ alternate in sign and sum to zero.

Here we provide a different proof to that of Hirschman and Widder [15, Section X.2.2] and so demonstrate a connection between these densities and the theory of symmetric functions. We employ an algebraic lemma involving alternating polynomials.

In the following definition and lemma, we let $\mathbb{F}$ denote an arbitrary field.

Definition 2.5. Given any $\mathbf{a} := (a_1, \ldots, a_m) \in \mathbb{F}^m$, where $m \geq 2$, let $\widehat{\mathbf{a}}_j \in \mathbb{F}^{m-1}$ be obtained by removing the jth term from $\mathbf{a}$, so that

$$\widehat{\mathbf{a}}_j := (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m) \in \mathbb{F}^{m-1} \qquad (j \in 1, \ldots, m).$$

Recall that $V(\mathbf{a}) := \prod_{1 \leq j < k \leq m} (a_k - a_j)$ is the usual Vandermonde determinant.

Lemma 2.6. *Given any* $\mathbf{a} \in \mathbb{F}^m$, *with* $m \geq 2$, *the following identity holds in the polynomial ring* $\mathbb{F}[X]$*:*

$$V(\mathbf{a})X^l = \sum_{j=1}^{m}(-1)^{j+l-1}a_j^l V(\widehat{\mathbf{a}}_j)\prod_{k\neq j}(X+a_k) \qquad (l=0,\ldots,m-1).$$

Proof 1. Suppose first that not all the entries of $\mathbf{a}$ are distinct, say $a_p = a_q$ for $p \neq q$. Then $V(\mathbf{a})$ vanishes, as do the summands on the right-hand side for j not equal to p or q, whereas the remaining two summands cancel each other, since $\widehat{\mathbf{a}}_q$ can be obtained from $\widehat{\mathbf{a}}_p$ by $|p-q|-1$ transpositions.

We now assume that $\mathbf{a}$ has distinct entries. Since both sides are polynomials in X of degree at most $m-1$, it suffices to show they agree at $-a_p$ for $p = 1, \ldots, m$. However, when evaluated at $-a_p$, the right-hand side reduces to

$$(-a_p)^l V(\widehat{\mathbf{a}}_p)\prod_{k<p}(a_p-a_k)\prod_{k>p}(a_k-a_p),$$

which is precisely $V(\mathbf{a})(-a_p)^l$. $\square$

Proof 2. An alternative argument, suggested by one of the referees, goes as follows. The right-hand side of the proposed identity equals $(-1)^l \det B$, as seen by expanding along the first column, where

$$B = \begin{pmatrix} a_1^l & X+a_1 & a_1(X+a_1) & \cdots & a_1^{m-2}(X+a_1) \\ a_2^l & X+a_2 & a_2(X+a_2) & \cdots & a_2^{m-2}(X+a_2) \\ a_3^l & X+a_3 & a_3(X+a_3) & \cdots & a_3^{m-2}(X+a_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-1}^l & X+a_{m-1} & a_{m-1}(X+a_{m-1}) & \cdots & a_{m-1}^{m-2}(X+a_{m-1}) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & a_1 & \cdots & a_1^{m-1} \\ 1 & a_2 & \cdots & a_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & \cdots & a_m^{m-1} \end{pmatrix} (\mathbf{e}_{l+1}, X\mathbf{e}_1+\mathbf{e}_2, X\mathbf{e}_2+\mathbf{e}_3, \ldots, X\mathbf{e}_{m-1}+\mathbf{e}_m)$$

and $\{\mathbf{e}_j : 1 \leq j \leq m\}$ is the standard basis of $\mathbb{F}^m$ written as column vectors. Expanding the determinant of the final matrix along the first column yields $(-1)^{(l+1)+1}$ times the determinant of a 2×2 block diagonal matrix, whose $(1,1)$ block is lower triangular with all l diagonal entries equal to X and whose $(2,2)$ block is upper triangular with all $m-l-1$ diagonal entries equal to 1. Hence, the right-hand side of the identity equals

$$(-1)^l \det B = (-1)^l V(\mathbf{a})(-1)^l X^l = V(\mathbf{a})X^l,$$

as claimed. $\square$

Remark 2.7. Lemma 2.6 can be applied to compute the inverse of the matrix

$$E(\mathbf{a}) := \begin{pmatrix} e_0(\widehat{\mathbf{a}}_1) & e_0(\widehat{\mathbf{a}}_2) & \cdots & e_0(\widehat{\mathbf{a}}_m) \\ e_1(\widehat{\mathbf{a}}_1) & e_1(\widehat{\mathbf{a}}_2) & \cdots & e_1(\widehat{\mathbf{a}}_m) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m-1}(\widehat{\mathbf{a}}_1) & e_{m-1}(\widehat{\mathbf{a}}_2) & \cdots & e_{m-1}(\widehat{\mathbf{a}}_m) \end{pmatrix}, \tag{2.5}$$

where $\mathbf{a} = (a_1, \ldots, a_m)$ contains distinct elements of the field $\mathbb{F}$ and e_l is the elementary symmetric polynomial

$$e_0 \equiv 1 \qquad \text{and} \qquad e_l(b_1, \ldots, b_n) := \sum_{1 \le j_1 < j_2 < \cdots < j_l \le n} b_{j_1} b_{j_2} \cdots b_{j_l}. \tag{2.6}$$

These are Schur polynomials in the sense of Definition 1.5, with e_{m-l} equal to $s_{\boldsymbol{\lambda}}$ if $\boldsymbol{\lambda} := (0, 1, \ldots, l-1, l+1, \ldots, m)$; see [25, (3.9)].

A direct computation gives that

$$E(\mathbf{a})^{-1} = (-1)^{m-1} V(\mathbf{a})^{-1} D_{\pm} \begin{pmatrix} a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \\ a_1^{m-2} & a_2^{m-2} & \cdots & a_m^{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} D_V D_{\pm}, \tag{2.7}$$

where

$$D_{\pm} := \operatorname{diag}(1, -1, 1, -1, \ldots) \quad \text{and} \quad D_V := \operatorname{diag}\bigl(V(\widehat{\mathbf{a}}_1), \ldots, V(\widehat{\mathbf{a}}_m)\bigr).$$

The matrix identity (2.7) provides a closed-form expression for the inverse of a standard Vandermonde matrix over any field. This formula can also be deduced from an alternative expression for the inverse; see [26].

A further consequence of (2.7) is the following expression for the determinant of $E(\mathbf{a})$:

$$\det E(\mathbf{a}) = (-1)^{m(m-1)/2} V(\mathbf{a}). \tag{2.8}$$

This formula was proved by a different method in [25, pp. 41–42].

With Lemma 2.6 at hand, we now obtain the aforementioned closed-form expression for the Hirschman–Widder density.

Proof of Proposition 2.4. Let $\boldsymbol{\alpha}$ and $\mathbf{a}$ be as in the statement of the Proposition, and define $\mathbf{c}$ by letting

$$c_j := a_j \prod_{k \ne j} \frac{a_k}{a_k - a_j} = \frac{(-1)^{j-1} V(\widehat{\mathbf{a}}_j)}{V(\mathbf{a})} \prod_{k=1}^m a_k \qquad (j = 1, \ldots, m). \tag{2.9}$$

The bilateral Laplace transform

$$\mathcal{B}\{\Lambda_{\mathbf{a},\mathbf{c}}\}(s) = \sum_{j=1}^m \frac{(-1)^{j-1} V(\widehat{\mathbf{a}}_j)}{V(\mathbf{a})(s + a_j)} \prod_{k=1}^m a_k = \frac{a_1 \cdots a_m}{(s + a_1) \cdots (s + a_m)} = \mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(s),$$

by Lemma 2.6 with $l = 0$ and $X = s$. We therefore conclude that $\Lambda_{\boldsymbol{\alpha}} = \Lambda_{\mathbf{a},\mathbf{c}}$, by Proposition 2.3(1). That $c_1, \ldots, c_m$ are alternating follows because c_j has the same sign as $(-1)^{j-1}$, and that they sum to zero was shown in the proof of Proposition 2.3(1). $\square$

Remark 2.8. The explicit form of $\Lambda_{\mathbf{a},\mathbf{c}}$ obtained above provides a connection to the topic of *cardinal L-splines*; see [27], for example, for more on these. In [38, Section 5], the authors study the restriction of a certain L-spline to an interval $[0, \eta]$. With a slight change of notation to match that used here, this is

$$\widetilde{A}_m(-x; t) = 1 + \sum_{j=1}^m \frac{e^{-a_j x}(1-t)}{t - e^{a_j \eta}} \prod_{k \ne j} \frac{a_k}{a_k - a_j}.$$

It follows immediately from Proposition 2.4 that the Hirschman–Widder density $\Lambda_{\mathbf{a},\mathbf{c}}$ is the asymptote of this spline:

$$\Lambda_{\mathbf{a},\mathbf{c}}(x) = \lim_{t\to\pm\infty} \frac{\mathrm{d}}{\mathrm{d}x}\widetilde{A}_m(-x;t).$$

Curry and Schoenberg [6] conducted a study of Pólya frequency functions as limits of splines. Their approach was recently complemented by Okounkov from the perspective of group representation theory: if Ω is the orbit of $\operatorname{diag}(a_1,\ldots,a_m)$ under conjugation by the unitary group $U(m)$ and P is the projection from Ω to $\mathbb{R}$ given by taking the upper-left entry of each matrix then the fundamental spline $M_{m-1}(x;a_1,a_2,\ldots,a_m)$ can be viewed as the density of the projection by P of the normalized $U(m)$-invariant measure on Ω. See Section 8 in [29] for details, or [7].

We now give a determinantal representation for generic Hirschman–Widder densities, which follows directly from (2.9).

Proposition 2.9. *Let $\boldsymbol{\alpha} \in (0,\infty)^m$ have distinct coordinates and let $a_j := \alpha_j^{-1}$ for $j = 1, \ldots, m$, where $m \geq 2$. The Hirschman–Widder density*

$$\Lambda_{\boldsymbol{\alpha}}(x) = \frac{a_1\cdots a_m}{V(\mathbf{a})} \det\begin{pmatrix} e^{-a_1x} & e^{-a_2x} & e^{-a_3x} & \cdots & e^{-a_mx} \\ 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & a_3^{m-2} & \cdots & a_m^{m-2} \end{pmatrix} \qquad (x \geq 0).$$

2.3. Non-smoothness at the origin

We now take a closer look at Schoenberg's result [37, Corollary 2] that Hirschman–Widder densities are the only non-smooth, continuous one-sided Pólya frequency functions, up to homothety. In the process, we obtain what is, to the best of our knowledge, a novel connection between Pólya frequency functions and symmetric functions.

The theme in this subsection is the smoothness of the map $\boldsymbol{\alpha} \mapsto \Lambda_{\boldsymbol{\alpha}}(x)$, where x is fixed; for simplicity, we consider only $\boldsymbol{\alpha}$ with positive entries. The simplest example is such that

$$\Lambda_{(\alpha_1,\alpha_2)}(x) = \begin{cases} (\alpha_1-\alpha_2)^{-1}(e^{-\alpha_1^{-1}x} - e^{-\alpha_2^{-1}x}) & \text{if } \alpha_1 \neq \alpha_2 \text{ and } x > 0, \\ \alpha_1^{-2}xe^{-\alpha_1^{-1}x} & \text{if } \alpha_1 = \alpha_2 \text{ and } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is readily verified that the map $(\alpha_1,\alpha_2) \mapsto \Lambda_{(\alpha_1,\alpha_2)}(x)$ is continuous on $(0,\infty)^2$ for any $x \in \mathbb{R}$. Indeed, more is true.

(1) For any non-negative integer l and any $x \in \mathbb{R}^\times$, the map

$$(0,\infty)^2 \to \mathbb{R};\ (\alpha_1,\alpha_2) \mapsto \Lambda^{(l)}_{(\alpha_1,\alpha_2)}(x)$$

is real analytic.

(2) The left and right limits at 0 of the first derivative of $\Lambda_{(\alpha_1,\alpha_2)}$ are distinct:

$$\Lambda'_{(\alpha_1,\alpha_2)}(0^-) := \lim_{x\to 0^-} \Lambda'_{(\alpha_1,\alpha_2)}(x) = 0$$

$$\text{and} \quad \Lambda'_{(\alpha_1,\alpha_2)}(0^+) := \lim_{x\to 0^+} \Lambda'_{(\alpha_1,\alpha_2)}(x) = \alpha_1^{-1}\alpha_2^{-1}.$$

In particular, the Hirschman–Widder density $\Lambda_{(\alpha_1,\alpha_2)}$ is not differentiable at the origin.

The following result is a generalization of the claim (1).

Proposition 2.10. *For any non-negative integer l and any $x \in \mathbb{R}^\times$, the map $\boldsymbol{\alpha} \mapsto \Lambda_{\boldsymbol{\alpha}}^{(l)}(x)$ is real analytic on $(0,\infty)^m$, for any $m \geq 2$.*

Proof 1. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,\infty)^m$, where $m \geq 2$, and recall the identity (2.2):

$$\Lambda_{\boldsymbol{\alpha}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}tx}\prod_{j=1}^{m}(1+\mathrm{i}\alpha_j t)^{-1}\,\mathrm{d}t \qquad (x \in \mathbb{R}).$$

There is no obstruction to extending this integral to the case where $\alpha_1, \ldots, \alpha_m$ lie in the open right half-plane $\mathbb{H}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Indeed, for such $\alpha_1, \ldots, \alpha_m$, we have that

$$|e^{\mathrm{i}tx}\prod_{j=1}^{m}(1+\mathrm{i}\alpha_j t)^{-1}| \leq |t|^{-m}\prod_{j=1}^{m}|\operatorname{Re}\alpha_j|^{-1},$$

which is Lebesgue integrable with respect to t. The analyticity of the integrand in the complex variables $\alpha_1, \ldots, \alpha_m$ is then inherited by the integral.

We fix $x \in \mathbb{R}^\times$. Integration by parts yields the identity

$$(-\mathrm{i}x)^n\Lambda_{\boldsymbol{\alpha}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}tx}G_n(t)\,\mathrm{d}t$$

for any non-negative integer n, where

$$G_n(t) := \frac{\partial^n}{\partial t^n}\Big(\prod_{j=1}^{m}(1+\mathrm{i}\alpha_j t)^{-1}\Big) = O(|t|^{-n-m}) \text{ as } |t| \to \infty,$$

since $G_n(t)$ is a homogeneous polynomial in $(1+\mathrm{i}\alpha_1 t)^{-1}, \ldots, (1+\mathrm{i}\alpha_m t)^{-1}$ of degree $n+m$.

Now suppose $n \geq l - m + 2$, and note that the derivative

$$F_{l,n,\boldsymbol{\alpha}}(x) := \frac{\partial^l}{\partial x^l}\Big((-\mathrm{i}x)^n\Lambda_{\boldsymbol{\alpha}}(x)\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{i}t)^l e^{\mathrm{i}tx}G_n(t)\,\mathrm{d}t.$$

Differentiation under the integral sign is valid here as long as the function $t \mapsto |t|^l|G_n(t)|$ is integrable, and this holds because $l - n - m \leq -2$. The integrand is an analytic function in the variables $\alpha_1, \ldots, \alpha_m$ as long as these all lie in $\mathbb{H}^+$, and an inductive argument now gives the result, as $\Lambda_{\boldsymbol{\alpha}}^{(l)}(x)$ can be expressed as a linear combination of $F_{l,n,\boldsymbol{\alpha}}(x)$ and derivatives of lower order. □

Proof 2. If one uses the orbital-integral machinery and the Harish-Chandra–Itzykson–Zuber integral, as explained in Section 3, then the representation (3.2) immediately gives the stronger assertion that, for fixed $l \geq 0$ and $x \in \mathbb{R}^\times$, the map $\boldsymbol{\alpha} \mapsto \Lambda_{\boldsymbol{\alpha}}^{(l)}(x)$ is complex analytic in the half-space $\{\boldsymbol{\alpha} \in \mathbb{C}^m : \operatorname{Re}\alpha_j > 0 \text{ for } j = 1, \ldots, m\}$. □

We now examine the right-hand derivatives at the origin of our Hirschman–Widder densities; all of the left-hand derivatives here are clearly zero. Our approach will involve another instance of Schur polynomials.

We have already seen that the elementary symmetric polynomials of Remark 2.7 are Schur polynomials. Another well studied family of symmetric functions is that of complete homogeneous symmetric polynomials: if l is a non-negative integer and $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ for some $m \geq 1$ then

$$h_0 \equiv 1 \quad \text{and} \quad h_l(\mathbf{a}) := \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_l \leq m} a_{j_1} a_{j_2} \cdots a_{j_l}. \tag{2.10}$$

These are also Schur polynomials: by [25, (3.9)], we have that

$$h_l = s_{\boldsymbol{\lambda}}, \qquad \text{where} \qquad \boldsymbol{\lambda} = (0, 1, \dots, m-2, m-1+l). \tag{2.11}$$

Having defined these polynomials, we may now state and prove the following result.

Proposition 2.11. *Suppose that $\mathbf{a} = (a_1, \dots, a_m) \in (0, \infty)^m$, where $m \geq 2$, and let $\boldsymbol{\alpha} := (\alpha_1 = a_1^{-1}, \dots, \alpha_m = a_m^{-1})$. The Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ has the Maclaurin-series expansion*

$$\Lambda_{\boldsymbol{\alpha}}(x) = a_1 \cdots a_m \sum_{n=m-1}^{\infty} \frac{(-1)^{n-m+1} h_{n-m+1}(a_1, \dots, a_m)}{n!} x^n \tag{2.12}$$

valid for all $x \in [0, \infty)$. Consequently, the function $\Lambda_{\boldsymbol{\alpha}}$ is $m-2$ times continuously differentiable, but $\Lambda_{\boldsymbol{\alpha}}^{(m-1)}(0)$ does not exist.

Proof. We assume first that $a_1, \dots, a_m$ are distinct and, without loss of generality, that $a_1 < \cdots < a_m$. If n is a non-negative integer then Proposition 2.9 gives that

$$\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+) = (-1)^n \frac{a_1 \cdots a_m}{V(\mathbf{a})} \det \begin{pmatrix} a_1^n & a_2^n & \cdots & a_m^n \\ 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & \cdots & a_m^{m-2} \end{pmatrix}.$$

If $n = 0, 1, \dots, m-2$, then the matrix above has two identical rows, so its determinant vanishes. For $n \geq m-1$, moving the top row to the bottom takes $m-1$ transpositions, and now the result follows from Definition 1.5 and (2.11), together with the fact that $\Lambda_{\boldsymbol{\alpha}}$ is the restriction to $[0, \infty)$ of an entire function.

Since $\boldsymbol{\alpha} \mapsto \Lambda_{\boldsymbol{\alpha}}(x)$ is real analytic on $(0, \infty)^m$ for any $x > 0$, by Proposition 2.10, it is continuous there. As the right-hand side of (2.12) is also a continuous function of such $\boldsymbol{\alpha}$ for fixed $x > 0$, the identity holds for all $(x, \boldsymbol{\alpha}) \in (0, \infty)^{m+1}$. It also holds trivially when $x = 0$.

An obstruction to $\Lambda_{\boldsymbol{\alpha}}$ being continuously differentiable can only appear at the origin, where $\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^-)$ is always zero. The working above shows that $\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+)$ is also zero if $n = 0, \dots, m-2$, whereas $\Lambda_{\boldsymbol{\alpha}}^{(m-1)}(0^+) = h_0(a_1, \dots, a_m) = 1$. □

As a brief digression, we use Proposition 2.11 to obtain a classical identity in algebraic combinatorics, Corollary 2.13. Although the identity is well known (see [25, Chapter I.2] or [4], for example), its connection to Pólya frequency functions is not.

We begin with the following elementary observation.

Lemma 2.12. *Let* $\mathbf{a} = (a_1, \ldots, a_m) \in [0, \infty)^m$, *where* $m \geq 2$. *Then*

$$\lim_{l \to \infty} h_l(\mathbf{a})^{1/l} = \max\{a_1, \ldots, a_m\}.$$

Proof. If $a_k \geq a_j$ for $j = 1, \ldots, m$ and $l \geq m - 1$ then

$$a_k^l \leq h_l(a_1, \ldots, a_m) \leq \binom{l+m-1}{l} a_k^l \leq \frac{(2l)^{m-1} a_k^l}{(m-1)!}.$$

The result follows. □

Corollary 2.13. *Let* $\mathbf{a} = (a_1, \ldots, a_m) \in (0, \infty)^m$, *where* $m \geq 2$. *The generating function of the family of complete homogeneous symmetric polynomials in* $a_1, \ldots, a_m$ *is*

$$\sum_{l=0}^{\infty} h_l(a_1, \ldots, a_m) z^l = \prod_{j=1}^{m} \frac{1}{1 - a_j z} \qquad \textit{whenever } |z| < \min\{a_j^{-1} : j = 1, \ldots, m\}.$$

As we explain below, this corollary provides a way to obtain the moments of $\Lambda_{\boldsymbol{\alpha}}$ from its Maclaurin coefficients. The reverse inference can be drawn if one knows the moments, via arguments of Pólya [33] and Schoenberg [37]. This is discussed further in Remark 2.20 below.

Proof. This power series has radius of convergence $\min\{a_j^{-1} : j = 1, \ldots, m\}$, by Lemma 2.12. Now suppose $0 > z > \max\{-a_j^{-1} : j = 1, \ldots, m\}$ and let $s := -z^{-1}$. Setting $\alpha_j := a_j^{-1}$ for $j = 1, \ldots, m$, we see that

$$\begin{aligned}
\prod_{j=1}^{m} \frac{1}{1 - a_j z} &= \alpha_1 \cdots \alpha_m s^m \mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(s) \\
&= \alpha_1 \cdots \alpha_m s^m \int_0^\infty e^{-sx} \Lambda_{\boldsymbol{\alpha}}(x) \, \mathrm{d}x \\
&= s^m \int_0^\infty \sum_{n=m-1}^{\infty} \frac{(-1)^{n-m+1} h_{n-m+1}(a_1, \ldots, a_m)}{n!} x^n e^{-sx} \, \mathrm{d}x,
\end{aligned}$$

by Proposition 2.11. As the series

$$s^m \sum_{n=m-1}^{\infty} \frac{h_{n-m+1}(a_1, \ldots, a_m)}{n!} \int_0^\infty x^n e^{-sx} \, \mathrm{d}x = \sum_{l=0}^{\infty} h_l(a_1, \ldots, a_m) s^{-l}$$

is absolutely convergent, we may exchange the order of integration and summation in the previous formula to see that the product $\prod_{j=1}^{m} (1 - a_j z)^{-1}$ equals

$$\sum_{l=0}^{\infty} h_l(a_1, \ldots, a_m) z^l,$$

as claimed. We conclude by the identity theorem. □

2.4. Hirschman–Widder densities and probability theory

We now explore some connections to probability theory. A natural first question is to identify the random variables distributed with Hirschman–Widder density functions.

Proposition 2.14. *Let $\alpha \in (0, \infty)^m$, where $m \geq 2$. Then the Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ is the probability density function for the random variable*

$$\alpha_1 X_1 + \cdots + \alpha_m X_m,$$

where X_1, ..., X_m are independent and identically distributed exponential random variables with mean 1.

Proof. A exponential random variable X with mean 1 has density function $\mathbf{1}_{x\geq 0}\, e^{-x}$, so if $\alpha > 0$ then αX has density function $\mathbf{1}_{x\geq 0}\, \alpha^{-1} e^{-\alpha^{-1}x} = \varphi_\alpha(x)$. The result now follows by (2.1). □

Remark 2.15. Hirschman–Widder density functions are studied in the probability and statistics literature under the name of *hypoexponential densities*. They are intimately connected to the time to absorption for a finite-state Markov chain. When the entries of $\boldsymbol{\alpha}$ are equal, then $\Lambda_{\boldsymbol{\alpha}}$ is an *Erlang density*, named after the father of queueing theory; this is a special case of the gamma distribution, occurring when the shape parameter is an integer. These densities have found use in diverse applied fields, including queueing theory, population genetics, reliability analysis and cell biology. The connection to Hirschman and Widder's memoir seems to be generally unnoticed in the probability literature.

The probabilistic interpretation also leads to closed-form expressions. The explicit formula for $\Lambda_{\boldsymbol{\alpha}}(x)$ when α_1, ..., α_m are positive and distinct, which appears in Hirschman and Widder's memoir in analysis [15], also appears in probability textbooks; see, for example, Exercise 12 in Chapter I of Feller's 1966 book [9]. An explicit formula in the case where repeats may occur was obtained by Amari and Misra [2].

Remark 2.16. A second probabilistic interpretation of the density $\Lambda_{\boldsymbol{\alpha}}$ can be derived from random matrix theory. Consider the diagonal matrix $D = \operatorname{diag}(a_1, \dots, a_m)$ and its orbit Ω under unitary conjugation in the space of $m \times m$ positive semi-definite matrices. If μ is the normalized $U(m)$-invariant measure on Ω then $\Lambda_{\boldsymbol{\alpha}}(x)\, \mathrm{d}x$ is the distribution of any diagonal entry of a random positive semi-definite matrix of arbitrary size distributed according to μ. See Section 3 and [29, Section 8], or [7] for details.

Remark 2.17. Schoenberg's characterization of Pólya frequency functions, as those for which the reciprocal of the bilateral Laplace transform is the restriction of an entire function in the Laguerre–Pólya class, admits the same probabilistic interpretation as in Proposition 2.14. The Hadamard–Weierstrass factorization implies that such a function is the density function of a possibly infinite linear combination of independent and identically distributed exponential random variables, together with at most one Gaussian random variable.

We now obtain the promised closed form for the moments of the random variables distributed according to Hirschman–Widder densities.

Proof of Theorem 1.6. The first part of this result has been established in the proof of Proposition 2.11. For the second, suppose first that the entries of $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ are distinct, and let $a_j := \alpha_j^{-1}$ for $j = 1$, ..., m. Suppose further without loss of generality that $a_1 < \cdots < a_m$. If p is a non-negative integer then (2.9) gives that

$$\begin{aligned}\frac{1}{p!}\int_{-\infty}^{\infty} x^p \Lambda_{\boldsymbol{\alpha}}(x)\,\mathrm{d}x &= \frac{a_1\cdots a_m}{p!\,V(\mathbf{a})}\sum_{j=1}^{m}(-1)^{j-1}V(\widehat{\mathbf{a}}_j)\int_0^{\infty} x^p e^{-a_j x}\,\mathrm{d}x \\ &= \frac{a_1\cdots a_m}{V(\mathbf{a})}\sum_{j=1}^{m}(-1)^{j-1}V(\widehat{\mathbf{a}}_j)a_j^{-p-1} \\ &= \frac{a_1\cdots a_m}{V(\mathbf{a})}\det\begin{pmatrix} a_1^{-p-1} & a_2^{-p-1} & \cdots & a_m^{-p-1} \\ 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & \cdots & a_m^{m-2}\end{pmatrix}.\end{aligned}$$

This quantity is unchanged if, for $j = 1, \ldots, m$, we multiply the jth column by α_j^{m-2}, and multiply the whole expression by $(a_1\cdots a_m)^{m-2}$, to obtain

$$\begin{aligned}&\frac{(a_1\cdots a_m)^{m-1}}{V(\mathbf{a})}\det\begin{pmatrix} \alpha_1^{m-1+p} & \alpha_2^{m-1+p} & \cdots & \alpha_m^{m-1+p} \\ \alpha_1^{m-2} & \alpha_2^{m-2} & \cdots & \alpha_m^{m-2} \\ \alpha_1^{m-3} & \alpha_2^{m-3} & \cdots & \alpha_m^{m-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1\end{pmatrix} \\ &\qquad = \frac{(a_1\cdots a_m)^{m-1}}{V(\mathbf{a})}(-1)^{m(m-1)/2}V(\boldsymbol{\alpha})s_{(0,1,\ldots,m-2,m-1+p)}(\alpha_1,\ldots,\alpha_m).\end{aligned}$$

As

$$V(\mathbf{a}) = (a_1\cdots a_m)^{m-1}(-1)^{m(m-1)/2}V(\boldsymbol{\alpha}),$$

this gives the result when $\boldsymbol{\alpha}$ has distinct entries. The general case now follows by a continuity argument.

Finally, we explain how to recover the parameter $\boldsymbol{\alpha}$ from moments or Maclaurin coefficients. The pth moment of $\Lambda_{\boldsymbol{\alpha}}$ is

$$\mu_p := \int_{\mathbb{R}} x^p \Lambda_{\boldsymbol{\alpha}}(x)\,\mathrm{d}x = p!\,h_p(\boldsymbol{\alpha}) \qquad (p = 1, 2, \ldots). \tag{2.13}$$

The Jacobi–Trudi identity [25, (3.4)] asserts, for any increasing tuple $(\lambda_1, \ldots, \lambda_m)$, that

$$s_{(\lambda_1,\ldots,\lambda_m)}(\boldsymbol{\alpha}) = \det\big(h_{\lambda_{m-j+1}-m+k}(\boldsymbol{\alpha})\big)_{j,k=1}^{m} = \det\big(h_{\lambda_j-k+1}(\boldsymbol{\alpha})\big)_{j,k=1}^{m}$$

with the conventions of Definition 1.5 and $h_l := 0$ whenever $l < 0$. In particular,

$$e_l(\boldsymbol{\alpha}) = s_{(0,1,\ldots,m-l-1,m-l+1,\ldots,m)}(\boldsymbol{\alpha}) = \det\begin{pmatrix} A_l & 0 \\ C_l & D_l\end{pmatrix} = \det D_l \qquad (l = 1, \ldots, m),$$

where A_l is a lower-triangular matrix with $h_0(\boldsymbol{\alpha}) = 1$ as each entry of the leading diagonal and D_l is the $l \times l$ Toeplitz matrix

$$\begin{pmatrix} h_1(\boldsymbol{\alpha}) & 1 & 0 & \cdots & 0 & 0 \\ h_2(\boldsymbol{\alpha}) & h_1(\boldsymbol{\alpha}) & 1 & \cdots & 0 & 0 \\ h_3(\boldsymbol{\alpha}) & h_2(\boldsymbol{\alpha}) & h_1(\boldsymbol{\alpha}) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{l-1}(\boldsymbol{\alpha}) & h_{l-2}(\boldsymbol{\alpha}) & h_{l-3}(\boldsymbol{\alpha}) & \cdots & h_1(\boldsymbol{\alpha}) & 1 \\ h_l(\boldsymbol{\alpha}) & h_{l-1}(\boldsymbol{\alpha}) & h_{l-2}(\boldsymbol{\alpha}) & \cdots & h_2(\boldsymbol{\alpha}) & h_1(\boldsymbol{\alpha})\end{pmatrix}.$$

It follows that $e_l(\boldsymbol{\alpha}) = \det D_l = f_l(\mu_1, \ldots, \mu_l)$ for some polynomial function f_l. Hence

$$F(t) := 1 + \sum_{l=1}^{m} f_l(\mu_1, \ldots, \mu_l) t^l = \sum_{l=0}^{m} e_l(\boldsymbol{\alpha}) t^l = \prod_{j=1}^{m} (1 + \alpha_j t) \tag{2.14}$$

is determined by the moments $\mu_1, \ldots, \mu_m$ and the roots of F yield precisely the entries of $\boldsymbol{\alpha}$.

Similarly, Proposition 2.11 gives that

$$\Lambda_{\boldsymbol{\alpha}}^{(n)}(0^+)/\Lambda_{\boldsymbol{\alpha}}^{(m-1)}(0^+) = (-1)^{n-m+1} h_{n-m+1}(\alpha_1^{-1}, \ldots, \alpha_m^{-1}) \qquad (n = m, \ldots, 2m-1)$$

and the previous working shows that we may recover, up to permutation of its entries, the tuple $(\alpha_1^{-1}, \ldots, \alpha_m^{-1})$ from these ratios of Maclaurin coefficients. $\square$

Remark 2.18. The computation of the moments of $\Lambda_{\boldsymbol{\alpha}}$ in the previous proof was obtained with the assistance of the theory of symmetric functions. A more direct approach, using the probabilistic interpretation of $\Lambda_{\boldsymbol{\alpha}}$, is also available. If $X_1, \ldots, X_m$ are independent exponential random variables, each with mean one, and $\alpha_1, \ldots, \alpha_m$ are positive constants, then the random variable $X := \sum_{j=1}^{m} \alpha_j X_j$ has density $\Lambda_{\boldsymbol{\alpha}}$, by Proposition 2.14, and moment-generating function

$$\sum_{p=0}^{\infty} \frac{\mu_p}{p!} z^p = \mathbb{E}[e^{zX}] = \mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(-z) = \prod_{j=1}^{m} \frac{1}{1 - \alpha_j z}.$$

Corollary 2.13, with $\mathbf{a}$ replaced by $\boldsymbol{\alpha}$, now shows that the moments are as claimed. Alternatively, we may proceed via an explicit computation:

$$\mathbb{E}\Big[\Big(\sum_{j=1}^{m} \alpha_j X_j\Big)^p\Big] = p!\, h_p(\alpha_1, \ldots, \alpha_m)$$

for any non-negative integer p, since

$$\Big(\sum_{j=1}^{m} \alpha_j X_j\Big)^p = \sum_{p_1 + \cdots + p_m = p} \binom{p}{p_1 \cdots p_m} \alpha_1^{p_1} \cdots \alpha_m^{p_m} X_1^{p_1} \cdots X_m^{p_m},$$

where the sum is taken over non-negative integers, and $\mathbb{E}[X_1^{p_1}] = p_1!$.

We now provide a connection between Hirschman–Widder densities and certain Pólya frequency sequences. Karlin, Proschan, and Barlow proved that a probability density is a Pólya frequency function if and only if the sequence of normalized moments is a Pólya frequency sequence, in that the corresponding Toeplitz matrix is totally non-negative; see [19, Theorem 3]. This Toeplitz matrix is formed from a bi-infinite extension of the normalized-moments sequence, which here is

$$\ldots, 0, 0, 0, 1, h_1(\boldsymbol{\alpha}), h_2(\boldsymbol{\alpha}), h_3(\boldsymbol{\alpha}), \ldots$$

and the total non-negativity of the corresponding Toeplitz matrix is again the numerical shadow of the Jacobi–Trudi identity.

The observations above lead to the solution of the following moment problem.

Corollary 2.19. *Suppose* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$, *where* $m \geq 2$, *let* $\mu_0 := 1$ *and let the* $l \times l$ *Toeplitz matrix*

$$D_l := \begin{pmatrix} \mu_1 & 1 & 0 & \cdots & 0 & 0 \\ \mu_2/2! & \mu_1 & 1 & \cdots & 0 & 0 \\ \mu_3/3! & \mu_2/2! & \mu_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{l-1}/(l-1)! & \mu_{l-2}/(l-2)! & \mu_{l-3}/(l-3)! & \cdots & \mu_1 & 1 \\ \mu_l/l! & \mu_{l-1}/(l-1)! & \mu_{l-2}/(l-2)! & \cdots & \mu_2/2! & \mu_1 \end{pmatrix}$$

for $l = 1, \dots, m$. *The following are equivalent.*

(1) *There exists* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, \infty)^m$ *such that* $\boldsymbol{\mu}$ *is the truncated moment sequence of the Hirschman–Widder density* $\Lambda_{\boldsymbol{\alpha}}$.
(2) *The generating polynomial*

$$F(t) := 1 + \sum_{l=1}^{m} t^l \det D_l$$

of the determinant sequence $(\det D_1, \dots, \det D_m)$ *has all of its roots in* $(-\infty, 0)$, *so is in the first Laguerre–Pólya class.*
(3) *The sequence* $(b_n)_{n=-\infty}^{\infty}$ *of the form*

$$\dots, 0, 0, 0, b_0 = 1, \det D_1, \dots, \det D_m, 0, 0, 0, \dots$$

is a Pólya frequency sequence, in that $\det(b_{p_j - q_k})_{j,k=1}^{l}$ *is non-negative for any choice of integers* $p_1 < \dots < p_l$ *and* $q_1 < \dots < q_l$, *where* $l \geq 1$.

Proof. That (1) $\implies$ (2) follows from the proof of Theorem 1.6. For the reverse implication, it follows from (2) that there exists $\boldsymbol{\alpha} \in (0, \infty)^m$ with $e_l(\boldsymbol{\alpha}) = \det D_l$ for $l = 1, \dots, m$. Expanding the determinant along its bottom row shows that μ_l can be recovered from $\det D_l$ and $\mu_1, \dots, \mu_{l-1}$, so $\Lambda_{\boldsymbol{\alpha}}$ has moments as required.

The equivalence (2) $\iff$ (3) follows from immediately from [1, Theorem 6]. □

Remark 2.20. We conclude this section with a line of enquiry motivated by historical considerations. Pólya showed in his 1915 paper [33] that a function Ψ in the Laguerre–Pólya class with the expansion

$$\frac{1}{\Psi(s)} = \mu_0 - \frac{\mu_1}{1!} s + \frac{\mu_2}{2!} s^2 - \frac{\mu_3}{3!} s^3 + \cdots \tag{2.15}$$

and such that $\Psi(0) > 0$ has Hankel determinant

$$\det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} > 0 \qquad \text{for } n = 0, 1, 2, \dots. \tag{2.16}$$

In 1920, Hamburger went the other way [12] and deduced the existence of an underlying density function Λ from the positivity of these determinants. This is precisely what led Schoenberg to study these maps and to develop the theory of Pólya frequency functions. As observed by Schoenberg in [37], the coefficient μ_p in (2.15) is precisely the pth moment of the Pólya frequency function Λ. When $\Lambda = \Lambda_{\boldsymbol{\alpha}}$, each normalized moment is a complete homogeneous symmetric polynomial in the entries of $\boldsymbol{\alpha}$, and so this Hankel determinant is also a symmetric polynomial. The Jacobi–Trudi identity shows that the Toeplitz moment determinant in the proof of Theorem 1.6 is *monomial positive* (a positive linear combination of monomials) and even *Schur positive* (a positive linear combination of Schur polynomials). It is natural to ask if something similar applies to the Hankel moment determinant, but it turns out that neither property holds in general. For example, if $m = 2$ then

$$\mu_p = p! \sum_{l=0}^{p} \alpha_1^l \alpha_2^{p-l} \qquad \text{for } p = 0, 1, 2, \dots,$$

so

$$\det \begin{pmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} = \alpha_1^2 + \alpha_2^2,$$

which is monomial positive but not Schur positive, and

$$\det \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = 4\alpha_1^6 + 12\alpha_1^4\alpha_2^2 - 8\alpha_1^3\alpha_2^3 + 12\alpha_1^2\alpha_2^4 + 4\alpha_2^6,$$

which is not even monomial positive.

We have that $\Psi(s) = \prod_{j=1}^m (1 + \alpha_j s)$, so the formula (2.13) for the moments and the expansion (2.15) give another proof of Corollary 2.13:

$$\prod_{j=1}^{m} (1 - \alpha_j s)^{-1} = \frac{1}{\Psi(-s)} = \sum_{l=0}^{\infty} \frac{\mu_l}{l!} s^l = \sum_{l=0}^{\infty} h_l(\alpha_1, \dots, \alpha_m) s^l.$$

This line of thinking, together with (2.14), also provides the identity

$$\sum_{l=0}^{\infty} \frac{\mu_l}{l!} s^l = \prod_{j=1}^{m} (1 - \alpha_j s)^{-1} = \frac{1}{F(-s)} = \frac{1}{1 + \sum_{k=1}^m f_k(\mu_1, \dots, \mu_k)(-s)^k}.$$

Assuming, as is common in some applied areas, that moments are the only quantities available via measurements, we elaborate an alternative reconstruction scheme which parallels well known algorithms used in inverse problems. More specifically, we note that $(\mu_j)_{j=0}^{\infty}$, as a Stieltjes moment sequence, is characterized by the positivity of the Hankel determinants of the form (2.16) as well as those with a shifted index,

$$\det(\mu_{j+k+1})_{j,k=0}^{n} > 0 \qquad \text{for } n = 0, 1, 2, \dots.$$

For a proof of this, see, for instance [30, §67]. Next, we note that checking whether the polynomial $1 + \sum_{k=1}^m f_k(\mu_1, \dots, \mu_k)(-s)^k$ has only real and negative roots can be achieved by using its Sturm sequence.

A classical result attributed to Kronecker asserts that the formal series

$$\sum_{j=0}^{\infty} \gamma_j s^j$$

represents a rational function if and only if there exist positive integers n and r_0 such that

$$\det(\gamma_{r+j+k})_{j,k=0}^{n} = 0 \qquad \text{for all } r \geq r_0,$$

and the minimum value of r_0 is the degree of the denominator of the rational function. In principle, to verify this criterion involves an infinite sequence of vanishing Hankel determinants. Here, the hidden positivity in the moments of a Hirschman–Widder distribution allows a drastic reduction of Kronecker's criterion, to a single vanishing determinant.

To this aim, and in order to identify the denominator $\Psi(s) = F(s)$ in the moment generating series without splitting it into factors, we appeal to the theory of cumulants. The cumulant series for $\Lambda_{\boldsymbol{\alpha}}$ is

$$K(s) = \sum_{k=1}^{\infty} \nu_k s^k := \log \sum_{l=0}^{\infty} \frac{\mu_l}{l!} s^l = -\sum_{j=1}^{m} \log(1 - \alpha_j s)$$

and this is convergent whenever $|s| < \min\{\alpha_1^{-1}, \ldots, \alpha_m^{-1}\}$. Its derivative has the series representation

$$K'(s) = \sum_{k=1}^{\infty} k\nu_k s^{k-1} = \sum_{j=1}^{m} \frac{\alpha_j}{1 - \alpha_j s} = \sum_{j=1}^{m} (\alpha_j + \alpha_j^2 s + \alpha_j^3 s^2 + \cdots),$$

so the kth cumulant

$$\nu_k = \frac{1}{k} \sum_{j=1}^{m} \alpha_j^k \qquad (k = 1, 2, 3, \ldots). \tag{2.17}$$

As K' is the Stieltjes transform of finitely many point masses, the cumulant-generating function admits the representation

$$K(s) = \int_0^{\infty} \log\Big(\frac{t}{t-s}\Big)\, \mathrm{d}\sigma(t) \qquad (|s| < \min\{\alpha_1^{-1}, \ldots, \alpha_m^{-1}\}),$$

where σ is the sum of unit point masses at α_1^{-1}, α_2^{-1}, $\ldots$, α_m^{-1}. In other words, the cumulant-generating function K coincides, up to a constant and for small values of its argument, with the logarithmic potential of equally distributed masses at the reciprocals of the entries of $\boldsymbol{\alpha}$.

As a final step in our reconstruction process, we apply the Padé approximation scheme to the series representing $K'(s)$. We know from Kronecker's criterion that the minimal choice of n such that $\det\big((k + j + 1)\nu_{j+k+1}\big)_{j,k=0}^{n} = 0$ is when $n = m$. Elementary matrix-algebra operations single out a unique pair of polynomials, a monic polynomial $P(z)$ with degree m and $Q(z)$ with degree $m - 1$, such that

$$P(z) \sum_{k=1}^{m+1} \frac{k\nu_k}{z^k} = Q(z) + O\Big(\frac{1}{z^2}\Big).$$

Since $K'(s)$ is the Stieltjes transform of a positive measure, we infer the equality of formal series

$$\frac{1}{z} K'\Big(\frac{1}{z}\Big) = \frac{Q(z)}{P(z)}.$$

Details of the algorithmic aspects of this derivation are due to Stieltjes. They are masterfully exposed in Chapter 9 of [30]. It follows that

$$\frac{F'(-s)}{F(-s)} = K'(s) = \frac{s^{-1} Q(s^{-1})}{P(s^{-1})},$$

and so we obtain the identity

$$1+\sum_{k=1}^{m} f_k(\mu_1,\ldots,\mu_k)(-s)^k = s^m P(s^{-1}).$$

We stress that this Padé approximation procedure identifies the denominator in the moment generating series without computing its zeros.

Above, we touched on the old and eternally dominant theme of inversion of the Laplace transform. An effective method of inversion for rational functions without relying on simple-fraction decompositions appears in [24]. For a general overview of Laplace transform-inversion, we refer to the monograph [5].

3. Orbital integrals

The Fourier transforms of Pólya frequency functions can be viewed as characteristic functions of certain unitarily invariant measures defined on the space of infinite Hermitian matrices. In this section, we sketch the basic framework and some key formulas, following the ample and self-contained article of Olshanski and Vershik [29]. An alternate reference, with complete proofs and a lucid global perspective on the same topics is Faraut's article [7].

Let $U(n)$ denote the compact group of unitary transformations of $\mathbb{C}^n$, let Ω denote the orbit under unitary conjugation $S \mapsto USU^*$ of an $n\times n$ Hermitian matrix $S \in H(n)$, and let μ denote the normalized $U(n)$-invariant measure carried by Ω. The Fourier transform or characteristic function of μ is

$$f_S : H(n) \to \mathbb{C};\ B \mapsto \int_{\Omega} e^{\mathrm{i}\,\mathrm{tr}(BM)} \mu(\mathrm{d}M). \tag{3.1}$$

This function is invariant under unitary conjugation, so that

$$f_S(UBU^*) = f_S(B) \qquad \text{for all } U \in U(n) \text{ and } B \in H(n).$$

Hence $f_S(B)$ depends only on the eigenvalues of B and is a symmetric function of these eigenvalues. Furthermore, as a Fourier transform, the function f_S is positive definite.

We now consider the inductive limit of such measures and functions defined on the space $H(\infty) = \bigcup_n H(n)$ of Hermitian matrices of arbitrary size. The functions, normalized by the condition $f(0) = 1$, form a convex set and the extremal points of this set are multiplicative, in the sense that

$$f\big(\mathrm{diag}(b_1,b_2,\ldots,b_m)\big) = F(b_1)F(b_2)\cdots F(b_m)$$

for some function of a real variable F. This situation occurs precisely when the corresponding unitarily invariant measure μ on the union $H(\infty)$ is ergodic. The main classification theorem of [29], Theorem 2.9, asserts the existence of a bijective correspondence between ergodic, unitarily invariant probability measures on $H(\infty)$ and Pólya frequency functions. To be more precise, F is the Fourier transform of a Pólya frequency function attached to the ergodic measure μ. Moreover, specific invariant measures provide the building blocks of the class of Pólya frequency functions [29, Corollaries 2.5 to 2.7].

Now suppose $S = \mathrm{diag}(a_1,\ldots,a_m)$, where $a_1, \ldots, a_m$ are positive, and let $B = E_{11} = \mathrm{diag}(1,0,0,\ldots,0)$. Passing lightly over the technicalities required to extend f_S, and using the symmetry $f_S(\mathrm{i}xB) = f_B(\mathrm{i}xS)$, which follows from the tracial property, we have that

$$f_S(\mathrm{i}xE_{11}) = \int_{\Omega'} \exp\Big(-x\sum_{j=1}^{m} a_j|z_j|^2\Big)\sigma(\mathrm{d}z) \qquad (x>0) \tag{3.2}$$

where $\Omega' = S^{2m-1} \cong U(m)/U(m-1)$ is the unit sphere in $\mathbb{C}^m$ and σ is the normalized rotationally invariant (uniform) measure on the sphere.

We claim that, up to proper normalization, the above spherical average is equal, to the Hirschman–Widder distribution $\Lambda_{\boldsymbol{\alpha}}$ at the point x, where $\boldsymbol{\alpha} = (a_1^{-1}, \ldots, a_m^{-1})$. The alert reader will recognize the expression (3.2) as a Harish-Chandra–Itzykson–Zuber integral [13,16]. A variety of similar integral representations have recently been proposed as central technical ingredients in random matrix theory [20–22,10].

3.1. Explicit formulas

The aim of this brief subsection is to describe an explicit link between the spherical integral $f_S(B)$ above and Hirschman–Widder densities. Further details, including complete proofs, are contained in [29, Section 5]. Let $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{C}^m$ have corresponding diagonal matrices $S = \operatorname{diag} \mathbf{a} \in H(m)$ and $B = \operatorname{diag} \mathbf{b}$ respectively. As in (3.1) we let f_S denote the characteristic function of the $U(m)$-invariant probability measure μ with support Ω, where Ω is the $U(m)$-orbit of S under conjugation:

$$f_S(B) := \int\limits_{U(m)} e^{\mathrm{i}\,\mathrm{tr}(BUSU^*)}\ \mathrm{d}U = \int\limits_{\Omega} e^{\mathrm{i}\,\mathrm{tr}(BM)} \mu(\mathrm{d}M).$$

Since f_S is entire and symmetric as a function of the coordinates of $\mathbf{b}$, it admits a Taylor-series expansion that is convergent everywhere, so also a convergent expansion in terms of Schur polynomials:

$$f_S(\operatorname{diag} \mathbf{b}) = \sum_{\nu} c_\nu s_\nu(\mathbf{b}),$$

where the sum runs over Young diagrams with at most m rows. A computation by Olshanski and Vershik using characters of $U(m)$ and change-of-bases formulas between symmetric power-sum polynomials and Schur polynomials provides closed-form expressions for the coefficients c_ν: see [29, Theorem 5.1]. Indeed, this strategy appeared in explicit computations of Gel'fand and Naimark [11], and quite remarkably in multivariate statistics: see James [17] and the comments in [8]. From here, one derives the following expansions: see [29, Corollaries 5.2 and 5.4].

Proposition 3.1. *If the tuples $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{C}^m$ each have distinct coordinates and $S = \operatorname{diag} \mathbf{a}$ then the orbital integral f_S is given by the Harish-Chandra–Itzykson–Zuber formula:*

$$f_S(-\mathrm{i} \operatorname{diag} \mathbf{b}) = \frac{\prod_{j=0}^{m-1} j!}{V(\mathbf{a})V(\mathbf{b})} \det \begin{pmatrix} e^{b_1 a_1} & e^{b_1 a_2} & \cdots & e^{b_1 a_m} \\ e^{b_2 a_1} & e^{b_2 a_2} & \cdots & e^{b_2 a_m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{b_m a_1} & e^{b_m a_2} & \cdots & e^{b_m a_m} \end{pmatrix}.$$

If instead $B = \operatorname{diag}(1, 0, \ldots, 0) = E_{11}$ then

$$f_S(-\mathrm{i}xB) = (m-1)! \sum_{j=0}^{\infty} \frac{h_j(a_1, \ldots, a_m)}{(j+m-1)!} x^j,$$

where h_j is the jth complete homogeneous symmetric polynomial.

In particular, if $a_1, \ldots, a_m$ are positive and distinct, and $x > 0$ then, by the second part of Proposition 3.1 and (2.12),

$$f_S(\mathrm{i}xE_{11}) = (m-1)!(-x)^{1-m} \sum_{n=m-1}^{\infty} \frac{h_{n-m+1}(a_1,\dots,a_m)}{n!}(-x)^n = \frac{(m-1)!x^{1-m}}{a_1\cdots a_m}\Lambda_{\boldsymbol{\alpha}}(x),$$

where $\boldsymbol{\alpha} = (a_1^{-1},\dots,a_m^{-1})$.

In conclusion, the Hirschman–Widder density possesses the following integral and determinantal representations: if $x>0$ and $\sigma(\mathrm{d}z)$ denotes the normalized uniform measure on the sphere S^{2m-1}, then

$$\Lambda_{\boldsymbol{\alpha}}(x) = \frac{a_1\cdots a_m}{(m-1)!}x^{m-1} \int\limits_{S^{2m-1}} \exp\Bigl(-x\sum_{j=1}^{m} a_j|z_j|^2\Bigr)\sigma(\mathrm{d}z) \tag{3.3}$$

$$= \frac{a_1\cdots a_m}{V(\mathbf{a})}\det\begin{pmatrix} e^{-a_1x} & e^{-a_2x} & e^{-a_3x} & \cdots & e^{-a_mx} \\ 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & a_3^{m-2} & \cdots & a_m^{m-2} \end{pmatrix}. \tag{3.4}$$

The second representation can be obtained from the first identity in Proposition 3.1 by taking $\mathbf{b} = (-x,0,y_1,\dots,y_{m-2})$ and taking successively the jth partial derivative at zero with respect to y_j for $j=1$ to $m-2$. Thus one has an alternative, shorter route for proving many of the results contained in the previous section, if one accepts from the beginning these two complementary formulas for the Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$. We leave the choice between this and our self-contained approach to the reader.

4. Proof of the main result

With the results of the preceding section at hand, we now prove the main result of this paper. We first introduce some notation which will be used in the proof and in a later result.

Definition 4.1. Given an integer $m \geq 2$ and a finite set K of positive integers, we let

$$\Delta_m(K) := \bigcup_{k\in K} \Delta_m(k),$$

where

$$\Delta_m(k) := \{\mathbf{j} = (j_1,\dots,j_m) \in \mathbb{Z}^m : j_1,\dots,j_m \geq 0,\ j_1+\cdots+j_m = k\}$$

is the set of integer-lattice points in the scaled standard m-simplex. For any $k\geq 1$, any $\mathbf{j} = (j_1,\dots,j_m) \in \Delta_m(k)$ and any $\mathbf{a} = (a_1,\dots,a_m) \in \mathbb{R}^m$, we let

$$\binom{k}{\mathbf{j}} = \frac{k!}{\prod_{l=1}^m j_l!}, \qquad \mathbf{a}^{\mathbf{j}} := \prod_{l=1}^m a_l^{j_l}, \quad \text{and} \quad \mathbf{j}\cdot\mathbf{a} := \sum_{l=1}^m j_l a_l.$$

Proof of Theorem 1.4. *Part* (1). We note first that if the polynomial p is such that $p(0)\neq 0$ and $\Lambda_{\boldsymbol{\alpha}}$ is a Hirschman–Widder density for some $\boldsymbol{\alpha} \in (0,\infty)^m$, then $p\circ\Lambda_{\boldsymbol{\alpha}}$ equals $p(0)$ on $(-\infty,0)$, so is not integrable. Also, we have that $c\Lambda_{\boldsymbol{\alpha}}$ is negative on $(0,\infty)$ if $c<0$. Thus we need only to consider polynomials of degree at least two with zero constant term for the remainder of this proof.

Suppose first that $m\geq 4$. To construct the null set $\mathcal{N} \subset (0,\infty)^m$, we begin as follows. For any integer $n\geq 2$, we let

$$\mathcal{K}_n := \{K \subset \{1,\dots,n\} : n \in K\}$$

denote the set of subsets of $\{1, \ldots, n\}$ containing n and, for any $K \in \mathcal{K}_n$, we define the non-zero multivariate polynomial P_K by setting $P_K = f_1 - f_2$, where

$$f_i(\mathbf{a}) := (-1)^{(i-1)n} V(\widehat{\mathbf{a}}_i)^n \prod_{\mathbf{j} \in \Delta_m(K) \setminus \{n\mathbf{e}_i\}} (\mathbf{j} - n\mathbf{e}_i) \cdot \mathbf{a} \qquad (\mathbf{a} \in \mathbb{R}^m,\ i = 1, 2), \tag{4.1}$$

with $\widehat{\mathbf{a}}_1 = (a_2, a_3, \ldots, a_m)$ and $\widehat{\mathbf{a}}_2 = (a_1, a_3, \ldots, a_m)$, in accordance with Definition 2.5, and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ being the standard basis of $\mathbb{R}^m$.

To see that $P_K \neq 0$, we observe that the a_1-degree of f_1 exceeds that of f_2, which gives the claim. To see this, we note first that every factor in the product in f_1 has a linear a_1-term, as n is the maximum element of K, so

$$\deg_{a_1} f_1 = |\Delta_m(K)| - 1.$$

However, the Vandermonde determinant $V(\widehat{\mathbf{a}}_2)$ contributes $m-2$ linear a_1-terms to f_2 and therefore

$$\begin{aligned}\deg_{a_1} f_2 &= n(m-2) + |\Delta_m(K)| - |\{\mathbf{j} \in \Delta_m(K) : j_1 = 0\}| \\ &= n(m-2) + |\Delta_m(K)| - |\Delta_{m-1}(K)|.\end{aligned}$$

Using the fact that $m \geq 4$ and $n \geq 2$, it follows that

$$\begin{aligned}\deg_{a_1} f_1 - \deg_{a_1} f_2 &= |\Delta_{m-1}(K)| - n(m-2) - 1 \\ &\geq |\Delta_{m-1}(n)| - n(m-2) - 1 \\ &= \binom{n+m-2}{m-2} - n(m-2) - 1 \\ &> \frac{(n+m-2)\cdots(n+2)(n+1)}{(m-2)!} - (n+1)(m-2) \\ &\geq (n+1)\Big(\frac{m(m-1)\cdots 4}{(m-2)!} - (m-2)\Big) \\ &= (n+1)\Big(\frac{m(m-1)}{6} - (m-2)\Big) \\ &= \frac{(n+1)(m-3)(m-4)}{6} \geq 0.\end{aligned}$$

Let $\mathcal{Z}_K$ denote the zero locus of P_K in $\mathbb{R}^m$, which is a null set because P_K is non-zero. With S_m denoting the group of permutations of $\{1, \ldots, m\}$, we let $\sigma \in S_m$ act on subsets of $\mathbb{R}^m$ by permuting coordinates, so that

$$\sigma(A) := \{(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) : \mathbf{a} = (a_1, \ldots, a_m) \in A\} \qquad \text{for any } A \subset \mathbb{R}^m,$$

and note that this action is measure preserving. Finally, we let

$$H_{\mathbf{q}} := \{\mathbf{x} \in \mathbb{R}^m : \mathbf{q} \cdot \mathbf{x} = 0\}, \tag{4.2}$$

$$\widetilde{\mathcal{N}} := (0, \infty)^m \cap \Big(\bigcup_{\mathbf{q} \in \mathbb{Q}^m \setminus \{\mathbf{0}\}} H_{\mathbf{q}} \cup \bigcup_{\sigma \in S_m} \bigcup_{n=2}^{\infty} \bigcup_{K \in \mathcal{K}_n} \sigma(\mathcal{Z}_K) \Big) \tag{4.3}$$

$$\text{and} \quad \mathcal{N} := \{(a_1^{-1}, \ldots, a_m^{-1}) : \mathbf{a} \in \widetilde{\mathcal{N}}\}. \tag{4.4}$$

As a countable union of null sets, the set $\widetilde{\mathcal{N}}$ is null. Furthermore, the set $\mathcal{N}$ is also null. To see this, we note that the self-inverse map

$$f : (0,\infty)^m \to (0,\infty)^m;\ (x_1,\dots,x_m) \mapsto (x_1^{-1},\dots,x_m^{-1})$$

is Lipschitz when restricted to $[l^{-1}, l]^m$ for any positive integer l, so preserves null sets there, and

$$\mathcal{N} = \bigcup_{l=1}^{\infty} f\big(\widetilde{\mathcal{N}} \cap [l^{-1}, l]^m\big).$$

Let $\boldsymbol{\alpha} \in (0,\infty)^m \setminus \mathcal{N}$. Then the reciprocals of the entries of $\boldsymbol{\alpha}$ are linearly independent over $\mathbb{Q}$, since they are contained in no hyperplane of the form $H_{\mathbf{q}}$, so they are distinct. Thus, we may find $\mathbf{a} \in (0,\infty)^m \setminus \widetilde{\mathcal{N}}$ such that $a_1 < \cdots < a_m$, the sets $\{a_1^{-1},\dots,a_m^{-1}\}$ and $\{\alpha_1,\dots,\alpha_m\}$ are equal and $P_K(\mathbf{a}) \neq 0$ for any $n \geq 2$ and any $K \in \mathcal{K}_n$.

Now let $\mathbf{c}$ be as in Proposition 2.4, so that $\Lambda_{\mathbf{a},\mathbf{c}} = \Lambda_{\boldsymbol{\alpha}}$. If

$$c := \frac{a_1 \cdots a_m}{V(\mathbf{a})}$$

then

$$\mathbf{d} = (d_1,\dots,d_m) := c^{-1}\mathbf{c} = \big(V(\widehat{\mathbf{a}}_1), -V(\widehat{\mathbf{a}}_2), \dots, \pm V(\widehat{\mathbf{a}}_m)\big). \tag{4.5}$$

Let $\Lambda := \Lambda_{\mathbf{a},\mathbf{d}} = c^{-1}\Lambda_{\boldsymbol{\alpha}}$. If p is a polynomial of degree at least two such that $p(0) = 0$ then $p \circ \Lambda_{\boldsymbol{\alpha}} = q \circ \Lambda$, where the polynomial q is such that $q(x) = p(cx)$, so q also has degree at least two and no constant term. Thus it suffices to show that $p \circ \Lambda$ is not a Pólya frequency function, where $p(x) = \sum_{k \in K} r_k x^k$, with $K \in \mathcal{K}_n$ for some $n \geq 2$ and $r_k \neq 0$ for all $k \in K$.

For any non-negative integer k, an explicit computation reveals that the bilateral Laplace transform

$$\mathcal{B}\{\Lambda^k\}(s) = \sum_{\mathbf{j} \in \Delta_m(k)} \binom{k}{\mathbf{j}} \mathcal{B}\{\mathbf{d}^{\mathbf{j}} e^{-(\mathbf{j}\cdot\mathbf{a})x}\}(s) = \sum_{\mathbf{j} \in \Delta_m(k)} \binom{k}{\mathbf{j}} \frac{\mathbf{d}^{\mathbf{j}}}{s + \mathbf{j}\cdot\mathbf{a}} = \frac{p_k(s)}{q_k(s)} \tag{4.6}$$

for polynomials p_k and q_k, where the notation is as in Definition 4.1 and

$$q_k(s) := \prod_{\mathbf{j} \in \Delta_m(k)} (s + \mathbf{j}\cdot\mathbf{a}).$$

Thus

$$\mathcal{B}\{p \circ \Lambda\}(s) = \sum_{k \in K} r_k \mathcal{B}\{\Lambda^k\}(s) = \sum_{k \in K} r_k \frac{p_k(s)}{q_k(s)} = \frac{P(s)}{Q(s)},$$

where

$$Q(s) := \prod_{k \in K} q_k(s), \qquad \widehat{q}_k(s) := \prod_{l \in K \setminus \{k\}} q_l(s) \quad \text{and} \quad P(s) := \sum_{k \in K} r_k p_k(s) \widehat{q}_k(s).$$

For any $k \in K$, the polynomial q_k has the set of roots $\{-\mathbf{k}\cdot\mathbf{a} : \mathbf{k} \in \Delta_m(k)\}$. As the entries of $\mathbf{a}$ are linearly independent over $\mathbb{Q}$, the roots of Q are simple. Furthermore, if $\mathbf{k} \in \Delta_m(k)$ then $\widehat{q}_j(s) = 0$ whenever $j \in K \setminus \{k\}$, so

$$P(-\mathbf{k}\cdot\mathbf{a}) = r_k p_k(-\mathbf{k}\cdot\mathbf{a})\widehat{q}_k(-\mathbf{k}\cdot\mathbf{a}). \tag{4.7}$$

It follows from the definitions that

$$\widehat{q}_k(-\mathbf{k}\cdot\mathbf{a}) = \prod_{\mathbf{j}\in\Delta_m(K)\setminus\Delta_m(k)} (\mathbf{j}-\mathbf{k})\cdot\mathbf{a},$$

and to compute $p_k(-\mathbf{k}\cdot\mathbf{a})$, we use the final equality in (4.6) and the definition of q_k to see that

$$p_k(s) = \sum_{\mathbf{j}\in\Delta_m(k)} \binom{k}{\mathbf{j}} \mathbf{d}^{\mathbf{j}} \prod_{\mathbf{j}'\in\Delta_m(k)\setminus\{\mathbf{j}\}} (s+\mathbf{j}'\cdot\mathbf{a}).$$

Taking $s=-\mathbf{k}\cdot\mathbf{a}$ here and combining this with (4.7) shows that

$$P(-\mathbf{k}\cdot\mathbf{a}) = r_k \binom{k}{\mathbf{k}} \mathbf{d}^{\mathbf{k}} \prod_{\mathbf{j}\in\Delta_m(K)\setminus\{\mathbf{k}\}} (\mathbf{j}-\mathbf{k})\cdot\mathbf{a} \neq 0, \tag{4.8}$$

again using $\mathbb{Q}$-linear independence. Thus P does not vanish at any root of Q, and so Theorem 1.2 implies that $p\circ\Lambda$ is not a Pólya frequency function as long as P is not constant. However,

$$\begin{aligned}
&P(-n\mathbf{e}_1\cdot\mathbf{a}) - P(-n\mathbf{e}_2\cdot\mathbf{a})\\
&= r_n\Bigg(V(\widehat{\mathbf{a}}_1)^n \prod_{\mathbf{j}\in\Delta_m(K)\setminus\{n\mathbf{e}_1\}} (\mathbf{j}-n\mathbf{e}_1)\cdot\mathbf{a} + (-1)^{n+1}V(\widehat{\mathbf{a}}_2)^n \prod_{\mathbf{j}\in\Delta_m(K)\setminus\{n\mathbf{e}_2\}} (\mathbf{j}-n\mathbf{e}_2)\cdot\mathbf{a}\Bigg)\\
&= r_n P_K(\mathbf{a}) \neq 0,
\end{aligned}$$

and this completes the proof whenever $m\geq 4$.

We now consider the case $m=3$. When p is a monomial, this was resolved in previous work [3, Lemma 11.2] whenever the reciprocals of the entries of $\boldsymbol{\alpha}$ are linearly independent over $\mathbb{Q}$, so for Λ as above. It remains to verify that $p\circ\Lambda$ is not a Pólya frequency function when $p(x)=\sum_{k\in K} r_k x^k$, with $r_k\neq 0$ for all $k\in K$ and $K\in\mathcal{K}_n$ containing at least two elements. In this case, the polynomial P_K is non-zero, since we have that

$$\deg_{a_1} f_1 - \deg_{a_1} f_2 = |\Delta_2(K)| - n - 1 > |\Delta_2(n)| - n - 1 = 0.$$

Thus we may proceed as above, as long as we take only K containing at least two elements in the definition of $\widetilde{\mathcal{N}}$.

Part (2). We consider first the case where $\alpha_1=\cdots=\alpha_m=\alpha$. The corresponding Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ has bilateral Laplace transform

$$\mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}\}(s) = (1+\alpha s)^{-m},$$

and inverting this transform gives that

$$\Lambda_{\boldsymbol{\alpha}}(x) = \mathbf{1}_{x\geq 0}\frac{\alpha^{-m}}{(m-1)!}x^{m-1}e^{-\alpha^{-1}x}.$$

It follows immediately that, for any natural number n, the function $\Lambda_{\boldsymbol{\alpha}}^n$ is a positive multiple of $\Lambda_{\boldsymbol{\beta}}$, where $\boldsymbol{\beta}=(\alpha n^{-1},\alpha n^{-1},\ldots,\alpha n^{-1})\in(0,\infty)^{n(m-1)+1}$ has all its entries equal, and so $\Lambda_{\boldsymbol{\alpha}}^n$ is a Pólya frequency function.

Now suppose that the polynomial p is not a positive multiple of a monomial. If p has a constant term then $p \circ \Lambda_{\boldsymbol{\alpha}}$ is not integrable, so we may assume that $p(x) = \sum_{k \in K} r_k x^k$, where K is a finite set of natural numbers with at least two elements and $r_k \neq 0$ for all $k \in K$. Then

$$\mathcal{B}\{p \circ \Lambda_{\boldsymbol{\alpha}}\}(s) = \sum_{k \in K} r_k c_k (1 + \alpha k^{-1} s)^{-k(m-1)-1} = \frac{P(s)}{Q(s)},$$

where $c_k > 0$ for all $k \in K$,

$$Q(s) := \prod_{k \in K} (1 + \alpha k^{-1} s)^{k(m-1)+1} \quad \text{and} \quad P(s) := \sum_{k \in K} r_k c_k \prod_{j \in K \setminus \{k\}} (1 + \alpha j^{-1} s)^{j(m-1)+1}.$$

The polynomial P is non-constant, since each of the terms in the sum are polynomials with distinct positive degrees. Furthermore, the roots of Q are of the form $-k\alpha^{-1}$ for $k \in K$, and

$$P(-k\alpha^{-1}) = r_k c_k \prod_{j \in K \setminus \{k\}} (1 - j^{-1} k)^{j(m-1)+1} \neq 0.$$

Hence $Q(s)/P(s)$ is not the restriction of an entire function and so $p \circ \Lambda_{\boldsymbol{\alpha}}$ is not a Pólya frequency function, by Theorem 1.2.

It remains to consider the case where $a_j = a_1 + (j-1)\delta$ for $j = 1, \ldots, m$, where δ is positive and independent of j. We consider the case $m = 2$ first, so that

$$\Lambda^n_{\boldsymbol{\alpha}}(x) = \mathbf{1}_{x \geq 0} \Big(\frac{a_1 a_2}{\delta}\Big)^n \sum_{j=0}^n \binom{n}{j} (-1)^j e^{-(na_1 + j\delta)x} \tag{4.9}$$

for any natural number n. If $\mathbf{b} := (na_1, na_1 + \delta, \ldots, na_1 + n\delta)$ and $\mathbf{d} = (d_0, \ldots, d_n)$ is such that $\Lambda_{\mathbf{b},\mathbf{d}}$ is a Hirschman–Widder density, then Proposition 2.4 gives that

$$d_j = b_j \prod_{k \neq j} \frac{b_k}{b_k - b_j} = \frac{1}{n! \delta^n} \prod_{k=0}^n (na_1 + k\delta)(-1)^j \binom{n}{j}.$$

Thus $\Lambda^n_{\boldsymbol{\alpha}}$ is a positive multiple of the Hirschman–Widder density $\Lambda_{\mathbf{b},\mathbf{d}}$, so is itself a Pólya frequency function. An alternative argument for the $m = 2$ case, pointed out to us by one of the referees, goes as follows. Since

$$\Lambda^n_{\boldsymbol{\alpha}}(s) = \mathbf{1}_{x \geq 0} \Big(\frac{a_1 a_2}{\delta}\Big)^n e^{-na_1 x} (1 - e^{-\delta x})^n,$$

an explicit computation shows that

$$\begin{aligned}
\Big(\frac{\delta}{a_1 a_2}\Big)^n \mathcal{B}\{\Lambda^n_{\boldsymbol{\alpha}}\}(s) &= \frac{1}{\delta} \int_0^\infty e^{-(s+na_1)x} (1 - e^{-\delta x})^n \delta \, \mathrm{d}x \\
&= \frac{1}{\delta} \int_0^1 y^{(s+na_1)/\delta} (1-y)^n y^{-1} \, \mathrm{d}y \\
&= \frac{1}{\delta} \beta\big((s + na_1)/\delta, n+1\big),
\end{aligned}$$

where β denotes the beta function. From this it follows that

$$\frac{1}{\mathcal{B}\{\Lambda_{\boldsymbol{\alpha}}^n\}(s)} = \delta \left(\frac{\delta}{a_1 a_2}\right)^n \frac{\Gamma((s+na_1)/\delta + (n+1))}{n!\,\Gamma((s+na_1)/\delta)},$$

which is clearly a polynomial in s. Hence $\Lambda_{\boldsymbol{\alpha}}^n$ is a Pólya frequency function for all $n \geq 1$.

For $m \geq 3$, let $\boldsymbol{\beta} := (b_1^{-1}, b_2^{-1})$, where $b_1 := a_1/(m-1)$ and $b_2 := b_1 + \delta = a_m/(m-1)$. Then, by (4.9) and the previous working, the function $\Lambda_{\boldsymbol{\beta}}^{m-1}$ is a positive multiple of the Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$. Hence $\Lambda_{\boldsymbol{\alpha}}^n$ is a positive multiple of the Pólya frequency function $\Lambda_{\boldsymbol{\beta}}^{n(m-1)}$, and so is a Pólya frequency function itself.

Finally, suppose that α_1/α_2 is irrational and let $p(x) = \sum_{k\in K} r_k x^k$, where K is a finite set of natural numbers with at least two elements and $r_k \neq 0$ for all $k \in K$; as noted above, we need only consider p of this form. With the previous notation, we have that $\Lambda_{\boldsymbol{\alpha}}^k = c^k \Lambda_{\boldsymbol{\beta}}^{(m-1)k}$ for some $c > 0$, and therefore

$$\mathcal{B}\{p \circ \Lambda_{\boldsymbol{\alpha}}\}(s) = \sum_{k\in K} r_k c_k \prod_{j=0}^{(m-1)k} (s + ka_1 + j\delta)^{-1} = \frac{P(s)}{Q(s)},$$

where $c_k \neq 0$ for any $k \in K$,

$$Q(s) := \prod_{k\in K} \prod_{j=0}^{(m-1)k} (s + ka_1 + j\delta)$$

and

$$P(s) := \sum_{k\in K} r_k c_k \prod_{v\in K\setminus\{k\}} \prod_{u=0}^{(m-1)v} (s + va_1 + u\delta).$$

The roots of Q are simple, since if (j,k) and (j',k') are distinct then

$$ka_1 + j\delta = k'a_1 + j'\delta \iff \big((k-j) - (k'-j')\big)a_1 + ja_2 = (j'-j)a_2 \implies \frac{\alpha_1}{\alpha_2} = \frac{a_2}{a_1} \in \mathbb{Q}.$$

It follows that if $k \in K$ and $j \in \{0, 1, \dots, (m-1)k\}$ then

$$P(-ka_1 - j\delta) = r_k c_k \prod_{v\in K\setminus\{k\}} \prod_{u=0}^{(m-1)v} \big((v-k)a_1 + (u-j)\delta\big) \neq 0.$$

Furthermore, the polynomial P is the sum of polynomials with distinct positive degrees, so is non-constant. We conclude from Theorem 1.2 that $p \circ \Lambda_{\boldsymbol{\alpha}}$ is not a Pólya frequency function. □

We conclude with a discussion of the structure of the class of polynomials mapping a fixed Hirschman–Widder density into the class of Pólya frequency functions.

Proposition 4.2. *Suppose the entries of $\mathbf{a} = (a_1, \dots, a_m) \in (0,\infty)^m$ are linearly independent over $\mathbb{Q}$ and strictly increasing, and let $\mathbf{c}$ be as in Proposition 2.4. For the polynomial $p(x) = \sum_{k\in K} r_k x^k$, where K is a finite set of non-negative integers and $r_k \neq 0$ for all $k \in K$, the following are equivalent.*

(1) *$p \circ \Lambda_{\mathbf{a},\mathbf{c}}$ is a Pólya frequency function.*
(2) *The function*

$$\mathrm{eval}_{\widetilde{P}} : \Delta_m(K) \to \mathbb{R};\ \mathbf{k} \mapsto r_{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{c}^{\mathbf{k}} \prod_{\mathbf{j}\in\Delta_m(K)\setminus\{\mathbf{k}\}} (\mathbf{j} - \mathbf{k}) \cdot \mathbf{a}$$

is constant, where $|\mathbf{k}| := k_1 + \cdots + k_m$.

Proof. Following the proof of Theorem 1.4(1), we have that $\mathcal{B}\{p \circ \Lambda_{\mathbf{a},\mathbf{c}}\}(s) = \widetilde{P}(s)/Q(s)$, where

$$Q(s) = \prod_{\mathbf{k}\in\Delta_m(K)} (s + \mathbf{k}\cdot\mathbf{a}) \quad \text{and} \quad \widetilde{P}(s) = \sum_{\mathbf{k}\in\Delta_m(K)} r_{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{c}^{\mathbf{k}} \prod_{\mathbf{j}\in\Delta_m(K)\setminus\{\mathbf{k}\}} (s + \mathbf{j}\cdot\mathbf{a}).$$

As for P above, the polynomial $\widetilde{P}$ does not vanish at any root of Q. Thus $p \circ \Lambda_{\mathbf{a},\mathbf{c}}$ is a Pólya frequency function if and only if $\widetilde{P}$ is a constant. Since $\deg \widetilde{P} < |\Delta_m(K)|$, it suffices to check that evaluating $\widetilde{P}$ at the distinct points $\{-\mathbf{k}\cdot\mathbf{a} : \mathbf{k} \in \Delta_m(K)\}$ always yields the same answer, but this is precisely (2) above. □

Remark 4.3. We may further characterize when a polynomial $p(x) = \sum_{k\in K} r_k x^k$ maps the Hirschman–Widder density $\Lambda_{\mathbf{a},\mathbf{c}}$ into the class of all such densities. This happens if and only if Proposition 4.2(2) holds and the function $p \circ \Lambda_{\mathbf{a},\mathbf{c}}$ has unit integral. As

$$(p \circ \Lambda_{\mathbf{a},\mathbf{c}})(x) = \sum_{\mathbf{k}\in\Delta_m(K)} r_{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{c}^{\mathbf{k}} e^{-(\mathbf{k}\cdot\mathbf{a})x},$$

we obtain the additional condition

$$\sum_{\mathbf{k}\in\Delta_m(K)} r_{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} \frac{\mathbf{c}^{\mathbf{k}}}{\mathbf{k}\cdot\mathbf{a}} = 1.$$

References

[1] Michael Aissen, Albert Edrei, I.J. Schoenberg, Anne Whitney, On the generating functions of totally positive sequences, Proc. Natl. Acad. Sci. USA 37 (5) (1951) 303–307.
[2] S.V. Amari, R.B. Misra, Closed-form expressions for distribution of sum of exponential random variables, IEEE Trans. Reliab. 46 (4) (1997) 519–522.
[3] Alexander Belton, Dominique Guillot, Apoorva Khare, Mihai Putinar, Totally positive kernels, Pólya frequency functions, and their transforms, J. Anal. Math. (2022), in press, arXiv:2006.16213 [math.FA].
[4] Prasad Chaugule, Mrinal Kumar, Nutan Limaye, Chandra Kanta Mohapatra, Adrian She, Srikanth Srinivasan, Schur polynomials do not have small formulas if the determinant doesn't, in: S. Saraf (Ed.), 35th Computational Complexity Conference (CCC, 2020), Proc., in: LIPIcs Series, vol. 169, 2020, 14, 27 pp.
[5] Alan M. Cohen, Numerical Methods for Laplace Transform Inversion, Numerical Methods and Algorithms, vol. 5, Springer, New York, 2007.
[6] H.B. Curry, I.J. Schoenberg, On Pólya frequency functions IV: the fundamental spline functions and their limits, J. Anal. Math. 17 (1966) 71–107.
[7] Jacques Faraut, Rayleigh theorem, projection of orbital measures and spline functions, Adv. Pure Appl. Math. 6 (4) (2015) 261–283.
[8] Roger H. Farrell, Multivariate Calculation: Use of the Continuous Groups, Springer Series in Statistics, vol. 376, 1985.
[9] William Feller, An Introduction to Probability Theory and Its Applications, Vol. 2, John Wiley & Sons, New York, 1966.
[10] Yanik-Pascal Förster, Mario Kieburg, Holger Kösters, Polynomial ensembles and Pólya frequency functions, J. Theor. Probab. 34 (4) (2021) 1917–1950.
[11] I.M. Gel'fand, M.A. Naimark, Unitary representations of the classical groups, Tr. Mat. Inst. Steklova 36 (1950) 3–288.
[12] Hans Hamburger, Bemerkungen zu einer Fragestellung des Herrn Pólya, Math. Z. 7 (1920) 302–322.
[13] Harish-Chandra, Differential operators on a semisimple Lie algebra, Am. J. Math. 79 (1) (1957) 87–120.
[14] I.I. Hirschman Jr., D.V. Widder, The inversion of a general class of convolution transforms, Trans. Am. Math. Soc. 66 (1) (1949) 135–201.
[15] I.I. Hirschman, D.V. Widder, The Convolution Transform, Princeton Legacy Library, Princeton University Press, Princeton, 1955.
[16] C. Itzykson, J.-B. Zuber, The planar approximation. II, J. Math. Phys. 21 (3) (1980) 411–421.
[17] Alan T. James, Distributions of matrix variates and latent roots derived from normal samples, Ann. Math. Stat. 35 (2) (1964) 475–501.
[18] Samuel Karlin, Total positivity, absorption probabilities and applications, Trans. Am. Math. Soc. 111 (1964) 33–107.
[19] Samuel Karlin, Frank Proschan, Richard Eugene Barlow, Moment inequalities of Pólya frequency functions, Pac. J. Math. 11 (3) (1961) 1023–1033.

[20] Mario Kieburg, Holger Kösters, Products of random matrices from polynomial ensembles, Ann. Inst. Henri Poincaré Probab. Stat. 55 (1) (2019) 98–126.
[21] Mario Kieburg, Arno B.J. Kuijlaars, Dries Stivigny, Singular value statistics of matrix products with truncated unitary matrices, Int. Math. Res. Not. 2016 (11) (2016) 3392–3424.
[22] Arno B.J. Kuijlaars, Pablo Román, Spherical functions approach to sums of random Hermitian matrices, Int. Math. Res. Not. 2019 (4) (2019) 1005–1029.
[23] Edmond Laguerre, Sur les fonctions du genre zéro et du genre un, C. R. Math. Acad. Sci. Paris 95 (1882) 828–831.
[24] I.M. Longman, M. Sharir, Laplace transform inversion of rational functions, Geophys. J. R. Astron. Soc. 25 (1–3) (1971) 299–305.
[25] Ian G. Macdonald, Symmetric Functions and Hall Polynomials, second edition, Oxford Mathematical Monographs, Oxford Science Publications. Oxford University Press, New York, 1995. With contributions by A. Zelevinsky.
[26] N. Macon, A. Spitzbart, Inverses of Vandermonde matrices, Am. Math. Mon. 65 (2) (1958) 95–100.
[27] Charles A. Micchelli, Cardinal L-splines, in: Samuel Karlin, Charles A. Micchelli, Allan Pinkus, I.J. Schoenberg (Eds.), Studies in Spline Functions and Approximation Theory, Academic Press, 1976, pp. 203–250.
[28] A.Yu. Okounkov, Thoma's theorem and representations of the infinite bisymmetric group, Funct. Anal. Appl. 28 (1994) 100–107.
[29] Grigori Olshanski, Anatoly Vershik, Ergodic unitarily invariant measures on the space of infinite Hermitian matrices, in: R.L. Dobrushin, R.A. Minlos, M.A. Shubin, A.M. Vershik (Eds.), F.A. Berezin Memorial Volume, Contemporary Mathematical Physics, in: Amer. Math. Soc. Transl. Ser. 2, vol. 175, Amer. Math. Soc., Providence, RI, 1996, pp. 137–175.
[30] Oskar Perron, Die Lehre von den Kettenbrüchen, second edition, Chelsea Publ. Company, New York, 1950.
[31] Douglas Murray Pickrell, Mackey analysis of infinite classical motion groups, Pac. J. Math. 150 (1) (1991) 139–166.
[32] Georg Pólya, Über Annäherung durch Polynome mit lauter reellen Wurzeln, Rend. Circ. Mat. Palermo 36 (1913) 279–295.
[33] Georg Pólya, Algebraische Untersuchungen über ganze Funktionen vom geschlechte Null und Eins, J. Reine Angew. Math. 145 (1915) 224–249.
[34] I.J. Schoenberg, On totally positive functions, Laplace integrals and entire functions of the Laguerre–Pólya–Schur type, Proc. Natl. Acad. Sci. USA 33 (1) (1947) 11–17.
[35] I.J. Schoenberg, On variation-diminishing integral operators of the convolution type, Proc. Natl. Acad. Sci. USA 34 (4) (1948) 164–169.
[36] I.J. Schoenberg, On Pólya frequency functions. II. Variation-diminishing integral operators of the convolution type, Acta Sci. Math. (Szeged) 12 (B) (1950) 97–106.
[37] I.J. Schoenberg, On Pólya frequency functions. I. The totally positive functions and their Laplace transforms, J. Anal. Math. 1 (1951) 331–374.
[38] A. Sharma, J. Tzimbalario, A class of cardinal trigonometric splines, SIAM J. Math. Anal. 7 (6) (1976) 809–819.
[39] Akemichi Takemura, Zonal Polynomials, IMS Lecture Notes Monograph Series, vol. 4, 1984.
[40] Elmar Thoma, Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe, Math. Z. 85 (1964) 40–61.
[41] A.M. Vershik, S.V. Kerov, Characters and factor representations of the infinite unitary group, Sov. Math. Dokl. 26 (1982) 570–574.
[42] Dan Voiculescu, Représentations factorielles de type II_1 de $U(\infty)$, J. Math. Pures Appl. 55 (1976) 1–20.