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Idele class groups with modulus

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A B S T R A C T

We prove Bloch's formula for the Chow group of 0-cycles with modulus on smooth projective varieties over finite fields. The proof relies on two new results in global ramification theory.

Contents

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1. Introduction

1.1. Motivation

In the classical theory of algebraic cycles, Bloch's formula describes the Chow groups of a smooth variety over a field in terms of certain Zariski or Nisnevich cohomology groups of the K-theory sheaves. This provides a direct bridge between algebraic cycles and algebraic K-theory. For codimension one cycles, such a formula was known to ancients and for codimension two cycles (on surfaces), this was shown by Bloch [6]. The most general case was shown by Quillen [37]. A further progress was made much later by Kato [20] and Kerz [24] who showed that Bloch's formula also holds if one uses Milnor K-theory instead of Quillen K-theory. From the point of view of algebraic cycles, this is a significant improvement because Milnor K-theory is much simpler to study than Quillen K-theory.

The theory of Chow groups with modulus is a recent generalization of the classical Chow groups. One of the main objectives of Chow groups with modulus is to provide a cycle theoretic description of the K-theory of singular varieties in the same way as the classical Chow groups do for the K-theory of smooth varieties. Bloch's formula for the Chow groups with modulus is one of the most important steps in reaching this goal. However, very few cases of this formula are currently known (see [4], [15] and [31]), and its general case is one of the most challenging problems in the theory of Chow groups with modulus at present.

In this paper, we shall focus only on Bloch's formula for the Chow group of 0-cycles. Even this special case appears to be so hard at the moment that one is not sure if this formula would indeed be true eventually. Recently, Kerz and Saito [28] have conjectured a weaker version of this formula for smooth varieties over perfect fields. The main contribution of this paper is to verify Bloch's formula for smooth projective varieties over finite fields. We do this by proving two fundamental results in the global ramification theory over finite fields: a reciprocity theorem for the idele class group with modulus à la Kato-Saito [22], and a comparison theorem for two known notions of abelianized étale fundamental groups with modulus. Below we describe our main results.

1.2. Bloch's formula over finite fields

Let us fix a reduced quasi-projective scheme X of pure dimension $d \geq 1$ over a field k and an effective Cartier divisor $D \subset X$. Let $\mathcal{K}^M_{d,(X,D)}$ be the Nisnevich sheaf of relative

Milnor K-theory on X, defined as the kernel of the canonical surjection $\mathcal{K}^M_{d,X} \twoheadrightarrow \iota_*(\mathcal{K}^M_{d,D})$, where $\iota\colon D \hookrightarrow X$ is the inclusion. We let $C_{KS}(X,D) = H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)})$ and call it 'the Kato-Saito idele class group'. Let $x \in X \setminus D$ be a regular closed point. One then knows by [20] (see also [18, Lemma 3.7]) that there is a canonical isomorphism $\mathbb{Z} \xrightarrow{\cong} K^M_0(k(x)) \xrightarrow{\cong} H^d_x(X, \mathcal{K}^M_{d,(X,D)})$, where the latter is the Nisnevich cohomology with support. Hence, using the 'forget support' map for x and extending it linearly to the free abelian group on all regular closed points of $X \setminus D$, we get the 'cycle class map'

$$\mathrm{cyc}_{X|D}\colon \mathcal{Z}_0(X_{\mathrm{reg}} \setminus D) \to C_{KS}(X,D). \tag{1.1}$$

It is not known if this map factors through $\mathrm{CH}_0(X|D)$ even if X is smooth. The following result settles Bloch's formula for smooth projective varieties over finite fields.

Theorem 1.1. *Let k be either a finite field or an algebraic closure of a finite field. Let X be a smooth projective variety over k and let $D \subset X$ be an effective Cartier divisor. Then the cycle class map induces an isomorphism*

$$\mathrm{cyc}_{X|D}\colon \mathrm{CH}_0(X|D) \xrightarrow{\cong} C_{KS}(X,D).$$

If X is not regular, it is perhaps not expected that Theorem 1.1 would hold. Nonetheless, we can prove the following version of Bloch's formula. Let X be an integral normal projective variety of dimension $d \geq 1$ over a finite field and D an effective Cartier divisor on X. Assume that $X \setminus D$ is regular. Let $\mathcal{I}_D$ denote the ideal sheaf of D. For any $n \geq 1$, let $nD \subset X$ be the effective Cartier divisor defined by $\mathcal{I}^n_D$.

Theorem 1.2. *The cycle class map induces an isomorphism of pro-abelian groups*

$$\mathrm{cyc}^\bullet_{X|D}\colon \{\mathrm{CH}_0(X|nD)\}_{n\in\mathbb{N}} \xrightarrow{\cong} \{C_{KS}(X,nD)\}_{n\in\mathbb{N}}.$$

1.3. Reciprocity theorem for Kato-Saito idele class group

Let X be an integral and normal quasi-projective variety over a field k and $D \subset X$ a closed subscheme of pure codimension one whose complement U is regular. In this case, there is a notion of fundamental group with modulus $\pi^{\mathrm{ab}}_1(X,D)$ due to Deligne and Laumon [33]. However, it is not known if $\pi^{\mathrm{ab}}_1(X,D)$ admits a class field theory in terms of the Kato-Saito idele class group with modulus. This is arguably the main obstacle in finding a bridge between the cycle-theoretic and K-theoretic ramified class field theories. In this paper, we completely solve this problem, which immediately implies Theorem 1.1.

Our solution to the above problem consists of two main steps. The first step is a reciprocity theorem for $C_{KS}(X,D)$ which goes as follows. In [18], we introduced a new fundamental group with modulus $\pi^{\mathrm{adiv}}_1(X,D)$ (see § 5.1) and constructed a reciprocity map $\rho_{X|D}\colon C_{KS}(X,D) \to \pi^{\mathrm{adiv}}_1(X,D)$. The key difference between $\pi^{\mathrm{adiv}}_1(X,D)$ and

$\pi_1^{\mathrm{ab}}(X, D)$ is that the former measures the ramification of a finite étale covering of U at the generic points of D while the latter at the points on the scheme theoretic inverse image of D on the normalizations of integral curves not contained in D. If X is projective over k, there are degree maps $\deg\colon C_{KS}(X, D) \to \mathbb{Z}$ and $\deg'\colon \pi_1^{\mathrm{adiv}}(X, D) \to \widehat{\mathbb{Z}}$. We let $C_{KS}(X, D)_0 = \mathrm{Ker}(\deg)$ and $\pi_1^{\mathrm{adiv}}(X, D)_0 = \mathrm{Ker}(\deg')$. We consider $\pi_1^{\mathrm{adiv}}(X, D)$ and $C_{KS}(X, D)$ as topological abelian groups with the profinite and the discrete topology, respectively. We show the following.

Theorem 1.3. *Assume that k is a finite field. Then the reciprocity map $\rho_{X|D}$ is injective with dense image. It induces an isomorphism of finite groups*

$$\rho^0_{X|D}\colon C_{KS}(X, D)_0 \xrightarrow{\cong} \pi_1^{\mathrm{adiv}}(X, D)_0$$

if X is projective over k.

1.4. Comparison between two fundamental groups with modulus

In the second step, we prove the following result which may independently act as a key tool in understanding the ramifications of a finite morphism from a normal to a regular scheme. This eliminates the difference between the two notions of ramifications discussed above.

Theorem 1.4. *Let X be a smooth projective variety of pure dimension $d \geq 1$ over a finite field and let $D \subset X$ be an effective Cartier divisor. Then the surjections $\pi_1^{\mathrm{ab}}(X \setminus D) \twoheadrightarrow \pi_1^{\mathrm{adiv}}(X, D)$ and $\pi_1^{\mathrm{ab}}(X \setminus D) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)$ have identical kernels. In particular, there is a canonical isomorphism of profinite groups*

$$\pi_1^{\mathrm{adiv}}(X, D) \cong \pi_1^{\mathrm{ab}}(X, D).$$

When D_{red} is a simple normal crossing divisor on X, the isomorphism between the tame quotients of $\pi_1^{\mathrm{adiv}}(X, D)$ and $\pi_1^{\mathrm{ab}}(X, D)$ can be deduced from the main results of [29] and [14].

1.5. Application to Nisnevich descent for Chow groups with modulus

If $D \subset X$ is an effective Cartier divisor on a smooth scheme, the Nisnevich hypercohomology of the sheafified cycle complex $z^p(X|D, *)$ with modulus [5] gives rise to the motivic cohomology with modulus $H^p_{\mathcal{M}}(X|D, \mathbb{Z}(q))$ together with a canonical map $\mathrm{CH}^p(X|D) \to H^{2p}_{\mathcal{M}}(X|D, \mathbb{Z}(p))$. The Nisnevich descent for the Chow groups with modulus asks whether this map is an isomorphism. Using [17, Lemma 3.9] and [40, Theorem 1], we get the following consequence of Theorem 1.1. This improves [40, Theorem 2].

Corollary 1.5. *Let X be a smooth projective variety of pure dimension $d \geq 1$ over a finite field and let $D \subset X$ be an effective Cartier divisor such that D_{red} is a simple normal crossing divisor. Then the canonical map of pro-abelian groups*

$$\{\mathrm{CH}_0(X|nD)\}_{n\in\mathbb{N}} \to \{H^{2d}_{\mathcal{M}}(X|nD, \mathbb{Z}(d))\}_{n\in\mathbb{N}}$$

is an isomorphism.

1.6. Application to the functoriality of Kato-Saito idele class group

A very useful application of the main results and which was not known before is the following transfer property of $C_{KS}(X, D)$.

Corollary 1.6. *Let X be a smooth projective variety of pure dimension $d \geq 1$ over a finite field k and let $D \subset X$ be an effective Cartier divisor. Let $f\colon X' \to X$ be a proper map from a smooth projective variety X' of dimension $d' \geq 1$ over k. Let $D' \subset X'$ be an effective Cartier divisor such that $f^*(D) \subset D'$. Then there is a push-forward map*

$$f_*\colon C_{KS}(X', D') \to C_{KS}(X, D).$$

This is an isomorphism if $D' = f^*(D)$ and $f\colon X' \setminus D' \to X \setminus D$ is an isomorphism.

We end the discussion of our main results and their applications with the following.

Conjecture 1.7. *Corollary 1.6 holds if k is any perfect field.*

1.7. An overview of proofs

We now give a brief overview of the contents of this paper. The proofs of our main results are based two key ingredients among several steps.

The first is to show the finiteness of the degree zero Kato-Saito idele class group $C_{KS}(X, D)_0$ when the base field is finite (see Theorem 4.9). It is known that the Kerz-Saito idele class group $\mathrm{CH}_0(X|D)_0$ in this case is finite (see [11, Theorem 8.1]). Since the latter is a quotient of the Wiesend class group $W(X \setminus D)$ and there is a continuous reciprocity homomorphism from $W(X \setminus D)$ to the limit (over n) of $C_{KS}(X, nD)$, one would expect that this would descend to a map of pro-abelian groups $\{\mathrm{CH}_0(X|nD)\} \to \{C_{KS}(X, nD)\}$. This would imply the finiteness of $C_{KS}(X, D)_0$. Unfortunately, this approach breaks down because of the nature of the topology of $W(X\setminus D)$ (see [18, § 4.5] for details). We use some results in K-theory of surfaces and some class field theory results of Bloch [7] and Kato-Saito [22] to independently prove the finiteness theorem. As an application, we prove a very general finiteness theorem for the Kato-Saito idele class group of non-proper schemes over finite fields (see Corollary 4.10).

The second key ingredient is a moving lemma for the Chow group of 0-cycles with modulus (see Theorem 7.1). Combined with some results of Kato [21] and Matsuda [35], this allows us to prove the equivalence between the filtrations of the first étale cohomology of the function field of X whose Pontryagin duals define the two fundamental groups with modulus $\pi_1^{\mathrm{adiv}}(X, D)$ and $\pi_1^{\mathrm{ab}}(X, D)$.

Besides the finiteness theorem, the proof of Theorem 1.3 also relies on some results about $\pi_1^{\mathrm{adiv}}(X, D)$ already shown in [18] and the ramification theory results proven in [18] and § 5.4. The proofs of the remaining main results are based on Theorem 1.3 and the above key steps.

We prove the finiteness theorem in § 2, § 3 and § 4. Theorem 1.3 is proven in § 5. We construct the cycle class map in pro-setting in § 6 which is used in the proof of Theorem 1.2. The moving lemma is proven in § 7 and § 8. We prove the remaining main results in § 9.

1.8. Notation

In this paper, k will be the base field of characteristic $p \geq 0$. For most parts, this will be a finite field in which case we shall let $q = p^s$ be the order of k for some integer $s \geq 1$. A *scheme* will usually mean a separated and essentially of finite type k-scheme. We shall denote the category of such schemes by $\mathbf{Sch}_k$. The product $X \times_{\mathrm{Spec}\,(k)} Y$ in $\mathbf{Sch}_k$ will be written as $X \times Y$. If $k \subset k'$ is an extension of fields and $X \in \mathbf{Sch}_k$, we shall let $X_{k'}$ denote $X \times \mathrm{Spec}\,(k')$. For a subscheme $D \subset X$, we shall let $|D|$ denote the set of points lying on D.

For a morphism $f\colon X' \to X$ of schemes and $D \subset X$ a subscheme, we shall write $D \times_X X'$ as $f^*(D)$. For a point $x \in X$, we shall let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$ and $k(x)$ the residue field of $\mathcal{O}_{X,x}$. We shall let $\mathcal{O}^h_{X,x}$ (resp. $\mathcal{O}^{sh}_{X,x}$) denote the Henselization (resp. strict Henselization) of $\mathcal{O}_{X,x}$. We shall let $\overline{\{x\}}$ denote the closure of $\{x\}$ with its integral subscheme structure. We shall let $k(X)$ denote the total ring of quotients of X. If X is reduced, we shall let X_n denote the normalization of X.

If X is a Noetherian scheme, we shall consider sheaves on X and their cohomology with respect to the Nisnevich topology unless mentioned otherwise. We shall let $X_{(q)}$ (resp. $X^{(q)}$) denote the set of points on X of dimension (resp. codimension) q. We shall let $\pi_1^{\mathrm{ab}}(X)$ denote the abelianized étale fundamental group of X. We shall consider $\pi_1^{\mathrm{ab}}(X)$ as a topological abelian group with its profinite topology.

For a field K of characteristic $p > 0$ with a discrete valuation λ, we shall let K_λ denote the fraction field of the Henselization of the ring of integers associated to λ. For a Galois extension (possibly infinite) $K \subset L$ of fields, we shall let $G(L/K)$ denote the Galois group of L over K. We shall let G_K denote the absolute Galois group of K. All Galois groups will be considered as topological abelian groups with their profinite topology. If A is an abelian group, we let $A/\infty = \varprojlim_{m \in \mathbb{N}} A/mA$.

Throughout this paper, we shall freely use the notations and terminology of [18] without recalling them. In particular, we refer the reader to [18, § 2, 3] for the notion of Parshin chains and their Milnor K-theory and the idele class group $C(X, D)$.

2. Idele class group of a surface

Our goal in the next several sections is to prove first of the two key steps for proving the main results: the finiteness of the degree zero Kato-Saito idele class group $C_{KS}(X, D)_0$. The proof of this step is by induction on the dimension of the scheme. In this section, we recall the degree maps for the Kato-Saito idele class group with modulus and prove a crucial step showing that the kernel of the degree map for a surface is a torsion group of bounded exponent. Recall that for any excellent scheme X of dimension d and closed subscheme $D \subset X$, the Kato-Saito idele class group with modulus $C_{KS}(X, D)$ is the Nisnevich cohomology group $H^d_{\mathrm{nis}}(X, \mathcal{K}^M_{d,(X,D)})$, where $\mathcal{K}^M_{i,(X,D)}$ is the sheaf of relative Milnor K-groups (see [18, § 2.4]).

2.1. The degree map for Kato-Saito idele class group

We define the degree map for $C_{KS}(X, D)$ and prove some of its properties. Let k be any field and X a projective integral scheme of dimension $d \geq 1$ over k. Let $D \subset X$ be a nowhere dense closed subscheme. Let K denote the function field of X.

For a closed subscheme $Y \subset X$, we let $C^Y_{KS}(X, D) = H^d_Y(X, \mathcal{K}^M_{d,(X,D)})$ denote the Nisnevich cohomology with support. For a closed point $x \in X$, we let $C_{KS}(X_x, D_x) = C^x_{KS}(\mathrm{Spec}\,(\mathcal{O}^h_{X,x}), \mathrm{Spec}\,(\mathcal{O}^h_{X,x}/\mathcal{I}^h_{D,x}))$.

By [20, § 1], there is, for every $n \geq 0$, a complex of Nisnevich sheaves on X given by

$$\begin{aligned} \mathcal{K}^M_{n,X} \to \bigoplus_{x \in X^{(0)}} (\iota_x)_*(K^M_n(k(x))) \to \bigoplus_{x \in X^{(1)}} (\iota_x)_*(K^M_{n-1}(k(x))) \to \cdots \\ \cdots \to \bigoplus_{x \in X^{(d-1)}} (\iota_x)_*(K^M_{n-d+1}(k(x))) \xrightarrow{\partial} \bigoplus_{x \in X^{(d)}} (\iota_x)_*(K^M_{n-d}(k(x))) \to 0. \end{aligned} \tag{2.1}$$

An elementary cohomological argument shows that by taking the cohomology of the sheaves in this complex, one gets a canonical homomorphism $\nu_x \colon H^r_x(X, \mathcal{K}^M_{d,X}) \to K^M_{d-r}(k(x))$ for $x \in X^{(r)}$, and a commutative square

$$\begin{tikzcd} H^r_y(X, \mathcal{K}^M_{d,X}) \arrow[r, "\nu_y"] \arrow[d, "\partial"'] & K^M_{d-r}(k(y)) \arrow[d, "\partial"] \\ H^{r+1}_x(X, \mathcal{K}^M_{d,X}) \arrow[r, "\nu_x"] & K^M_{d-r-1}(k(x)) \end{tikzcd} \tag{2.2}$$

for $y \in X^{(r)}$ and $x \in X^{(r+1)} \cap \overline{\{y\}}$ (e.g., see [22, (2.1.2)]).

For any $n \geq 0$, let $\mathrm{CH}_n^F(X)$ denote the Chow group of cycles of dimension n on X in the sense of [12, Chapter 1]. We prove the following.

Lemma 2.1. *Let $Y \subset X$ be a closed immersion. Then the following hold.*

(1) *There is a canonical homomorphism*

$$\nu_Y : C^Y_{KS}(X, D) \to \mathrm{CH}_0^F(Y)$$

such that for any regular closed point $x \in Y \setminus D$, the composition

$$\lambda_x : \mathbb{Z} \xrightarrow{\cong} C^x_{KS}(X, D) \to C^Y_{KS}(X, D) \to \mathrm{CH}_0^F(Y)$$

has the property that $\lambda_x(1) = [x]$, the cycle class of x.

(2) *If $\iota : Z \hookrightarrow Y$ is a closed immersion, then the diagram*

$$\begin{tikzcd} C^Z_{KS}(X, D) \arrow[r] \arrow[d, "\iota_*"'] & \mathrm{CH}_0^F(Z) \arrow[d, "\iota_*"] \\ C^Y_{KS}(X, D) \arrow[r] & \mathrm{CH}_0^F(Y) \end{tikzcd} \tag{2.3}$$

commutes, where the left vertical arrow is the canonical homomorphism between cohomology groups with support and the right vertical arrow is the push-forward homomorphism.

(3) *The map $C^Y_{KS}(X, D) \to \mathrm{CH}_0^F(Y)$ is an isomorphism if $Y \subset X_{\mathrm{reg}} \setminus D$.*

Proof. To prove parts (1) and (2) of the lemma, it suffices to consider the case when $D = \emptyset$ using the canonical map $C^Y_{KS}(X, D) \to C^Y_{KS}(X)$ which is clearly functorial for closed immersions $Z \subset Y$.

We now recall that for any Nisnevich sheaf $\mathcal{F}$ on X, the coniveau spectral sequence for $H^*_Y(X, \mathcal{F})$ is of the form

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)} \cap Y} H^{p+q}_x(X, \mathcal{F}) \Rightarrow H^{p+q}_Y(X, \mathcal{F}). \tag{2.4}$$

Using the cohomological vanishing and the exact sequence

$$H^{p+q-1}_{\mathrm{nis}}(X_x \setminus \{x\}, \mathcal{F}) \to H^{p+q}_x(X, \mathcal{F}) \to H^{p+q}_{\mathrm{nis}}(X_x, \mathcal{F})$$

for $x \in X^{(p)}$, we see that $E_1^{p,q} = 0$ for $q \geq 1$. Hence, the above spectral sequence degenerates to an exact sequence

$$\bigoplus_{y \in Y_{(1)}} H^{d-1}_y(X, \mathcal{F}) \xrightarrow{\partial} \bigoplus_{x \in Y_{(0)}} H^d_x(X, \mathcal{F}) \to H^d_Y(X, \mathcal{F}) \to 0. \tag{2.5}$$

Applying this to $\mathcal{K}^M_{d,X}$, we get an exact sequence

$$\bigoplus_{y\in Y_{(1)}} H^{d-1}_y(X,\mathcal{K}^M_{d,X}) \xrightarrow{\partial} \bigoplus_{x\in Y_{(0)}} H^d_x(X,\mathcal{K}^M_{d,X}) \to C^Y_{KS}(X) \to 0. \tag{2.6}$$

We now consider the diagram

$$\begin{tikzcd} \bigoplus\limits_{y\in Y_{(1)}} H^{d-1}_y(X,\mathcal{K}^M_{d,X}) \arrow[r, "\partial"] \arrow[d] & \bigoplus\limits_{x\in Y_{(0)}} H^d_x(X,\mathcal{K}^M_{d,X}) \arrow[r] \arrow[d] & C^Y_{KS}(X) \arrow[r] \arrow[d, dotted] & 0 \\ \bigoplus\limits_{y\in Y_{(1)}} K^M_1(k(y)) \arrow[r, "\partial"] & \bigoplus\limits_{x\in Y_{(0)}} K^M_0(k(x)) \arrow[r] & \mathrm{CH}^F_0(Y) \arrow[r] & 0, \end{tikzcd} \tag{2.7}$$

where the square on the left is the commutative square of (2.2) with $r = d-1$. It is well known that the boundary map ∂ in the bottom row is the map which takes a rational function to its divisor and the right horizontal bottom arrow is the cycle class map (e.g., see the proof of [20, Theorem 3]). In particular, the bottom row is exact. The top row is the exact sequence (2.6). The first part of the lemma now follows. Furthermore, it is clear from the above construction that the (2.3) is commutative if $Z \subset Y$, proving (2).

If $Y \subset X_{\mathrm{reg}} \setminus D$, then we can assume by excision that X is regular with $D = \emptyset$ so that $\mathcal{K}^M_{d,(X,D)} \cong \mathcal{K}^M_{d,X}$. In this case, the two vertical arrows in (2.7) on the left are isomorphisms by [18, Lemma 3.7]. It follows that the third vertical arrow (which is now an honest map) is also an isomorphism. This proves (3). This also shows that for a closed point $x \in (X_{\mathrm{reg}} \cap Y) \setminus D$, the composite $\lambda_x \colon H^d_x(X,\mathcal{K}^M_{d,X}) \to \mathrm{CH}^F_0(Y)$ has the property stated in part (1) of the lemma. We have thus finished the proof. $\square$

Since X is projective, there is a push-forward map $\deg\colon \mathrm{CH}^F_0(Y) \to \mathbb{Z}$, which takes a closed point to the degree of its residue field over k. By composition with ν_Y, we get a degree map

$$\deg\colon C^Y_{KS}(X,D) \to \mathbb{Z} \tag{2.8}$$

which clearly factors through the degree map $\deg\colon C_{KS}(X,D) \to \mathbb{Z}$. We let $C^Y_{KS}(X,D)_0$ be the kernel of the degree map. We define $C_{KS}(X,D)_0$ similarly.

Recall that [22, Lemma 1.6.3] provides a recipe for constructing a homomorphism from $C_{KS}(X,D)$ to an abelian group. This says that giving a homomorphism from $C_{KS}(X,D)$ to an abelian group A is same as defining group homomorphisms $K^M_d(k(P)) \to A$ (where P runs through all maximal Parshin chains in X) which annihilate the images (by the residue homomorphisms) of the Milnor K-groups of certain Q-chains. We shall refer to this as the Kato-Saito recipe in the sequel.

One can easily check (using (2.6) with $Y = X$) that the above degree map $C_{KS}(X,D) \to \mathbb{Z}$ is same as the one obtained by the Kato-Saito recipe, where for a maximal Parshin chain $P = (p_0, \ldots, p_d)$, we define our desired homomorphism to be the composite

$$K_d^M(k(P)) \cong H^0_{P_d}(X, \mathcal{K}^M_{d,(X,D)}) \xrightarrow{\partial} H^1_{P_{d-1}}(X, \mathcal{K}^M_{d,(X,D)}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} H^d_{P_0}(X, \mathcal{K}^M_{d,(X,D)}) \to \mathbb{Z},$$

in which ∂ denotes the boundary map and $P_i = (p_0, \ldots, p_i)$. The last arrow is induced by (2.1).

Lemma 2.2. *Let $f\colon Y \to X$ be a projective morphism with Y integral. Let $E \subset Y$ be a closed subscheme such that $f^*(D) \subset E$. Assume that the image of f is not contained in $D \cup X_{\rm sing}$ and the image of E is nowhere dense in X. Suppose that the Kato-Saito recipe defines a push-forward map $f_*\colon C_{KS}(Y,E) \to C_{KS}(X,D)$. Then the diagram*

$$\begin{tikzcd} C_{KS}(Y,E) \arrow[d, "f_*"'] \arrow[dr, "\deg_Y"] & \\ C_{KS}(X,D) \arrow[r, "\deg_X"'] & \mathbb{Z} \end{tikzcd} \tag{2.9}$$

is commutative.

Proof. By our assumption, there is a dense open $U \subset X$ away from D such that U and $f^{-1}(U)$ are both regular. Furthermore, $E \cap f^{-1}(U) = \emptyset$. It follows from [22, Theorem 2.5] that $C_{KS}(Y,E)$ is generated by the classes of closed points in $f^{-1}(U)$. Hence, it suffices to show that for every closed point $y \in f^{-1}(U)$, one has $\deg_X \circ f_*([y]) = \deg_Y([y])$. But this follows immediately from Lemma 2.1 and the fact that the proper push-forward map on the classical Chow group of 0-cycles commutes with the degree map. □

One easy consequence of Lemma 2.2 is the following.

Corollary 2.3. *Let $D \subset D'$ be two nowhere dense closed subschemes. Then the canonical map $C_{KS}(X,D')_0 \to C_{KS}(X,D)_0$ is surjective.*

Lemma 2.4. *Let $f\colon X' \to X$ be a projective birational morphism and let $D' \subset f^*(D)$. Then there is a commutative diagram*

$$\begin{tikzcd} C_{KS}(X,D) \arrow[d, "f^*"'] \arrow[dr, "\deg_X"] & \\ C_{KS}(X',D') \arrow[r, "\deg_{X'}"'] & \mathbb{Z}. \end{tikzcd}$$

Proof. By [22, Theorem 2.5], it suffices to show that $\deg_{X'} \circ f^*([x]) = \deg_X([x])$ for every closed point $x \in (X_{\mathrm{reg}} \cap f(U)) \setminus D$, where U is an open subset of X' on which f is an isomorphism. But this is obvious by Lemma 2.1. $\square$

Let $C(X, D)$ denote the idele class group with modulus due to Kerz [26] (see [18, § 3]). Recall from [26, Theorem 8.2] (or [18, Theorem 3.8]) that when $X \setminus D$ is regular, there are canonical maps

$$\mathbb{Z} \cong K_0^M(k(x)) \xrightarrow{\tau_x} C(X, D) \xrightarrow[\cong]{\psi_{X|D}} C_{KS}(X, D) \tag{2.10}$$

for every $x \in X_{(0)} \setminus D$, where τ_x is induced by the inclusion of the Parshin chain of length zero and the composite arrow is the forget support map $\tau'_x \colon K_0^M(k(x)) \cong C^x_{KS}(X, D) \to C_{KS}(X, D)$. Furthermore, $\psi_{X|D}$ is an isomorphism. There is a degree map $\deg \colon C(X, D) \to \mathbb{Z}$ by [18, Proposition 4.8], whose kernel is $C(X, D)_0$.

The following result shows the compatibility between the degree maps for $C(X, D)$ and $C_{KS}(X, D)$ via $\psi_{X|D}$.

Lemma 2.5. *Assume that $X \setminus D$ is regular. Then the diagram*

$$\begin{tikzcd} C(X,D) \arrow[d, "\psi_{X|D}"'] \arrow[dr, "\deg"] & \\ C_{KS}(X,D) \arrow[r, "\deg"'] & \mathbb{Z} \end{tikzcd} \tag{2.11}$$

is commutative.

Proof. Let $U = X \setminus D$. Since $\psi_{X|D}$ is an isomorphism, it suffices to show using [22, Theorem 2.5] that for every closed point $x \in U$, the above diagram commutes if we replace $C(X, D)$ by $K_0^M(k(x))$ via the canonical map $\tau_x \colon K_0^M(k(x)) \to C(X, D)$, where we consider $\{x\}$ as a Parshin chain on $(U \subset X)$. However, we have $\deg \circ \tau_x(1) = [k(x) : k]$ by [18, Proposition 4.8]. Since $\psi_{X|D} \circ \tau_x(1) = \tau'_x(1)$, we are done by Lemma 2.1 which says that $\deg \circ \tau'_x(1) = [k(x) : k]$. $\square$

2.2. Some K-theory results

In this subsection, we prove some K-theory results of general interest which will be used in the proof of the finiteness theorem. The following is an elementary but very useful lemma.

Lemma 2.6. *Let Z be a Noetherian scheme over a field of characteristic $p > 0$ and let $W \subset Z$ be a closed subscheme defined by a nilpotent ideal sheaf. Then* $H^i_{\mathrm{nis}}(Z, \mathcal{K}^M_{j,(Z,W)})$

is a p-primary torsion group of bounded exponent (which depends only on j) for all $i \geq 0$ and $j \geq 1$.

Proof. We show that the sheaf $\mathcal{K}^M_{j,(Z,W)}$ itself is p-primary torsion of bounded exponent for every $j \geq 1$. This will prove the lemma. By the definition of $\mathcal{K}^M_{j,(Z,W)}$ (see [18, (2.2)]), it suffices to prove the statement for $j = 1$. In this case, we can use the iterative process to reduce the problem to the case when the ideal sheaf $\mathcal{I}_W$ defining W is square-zero. But then, we must have $\mathcal{K}^M_{1,(Z,W)} \cong \mathcal{I}_W$ and the latter is a p-torsion sheaf. □

Lemma 2.7. *Let $f: X_n \to X$ be the normalization morphism for a reduced Noetherian scheme X such that f is a finite morphism (e.g., X is essentially of finite type over a field). Then we can find a conductor closed subscheme $Y \hookrightarrow X$ such that for $Y' = f^*(Y)$, the canonical restriction map*

$$f_*(\mathcal{K}_{2,X_n})/\mathcal{K}_{2,X} \to f_*(\mathcal{K}_{2,Y'})/\mathcal{K}_{2,Y}$$

is an isomorphism of Nisnevich sheaves on X.

Proof. We fix a conductor subscheme $Y \subset X$ (which always exists) for f and look at the commutative diagram of Nisnevich sheaves (on X)

$$\begin{tikzcd}
\mathcal{K}_{2,(X,Y)} \arrow[r] \arrow[d, "f^*"'] & \mathcal{K}_{2,X} \arrow[r] \arrow[d, "f^*"] & \mathcal{K}_{2,Y} \arrow[r] \arrow[d, "f^*"] & 0 \\
f_*(\mathcal{K}_{2,(X_n,Y')}) \arrow[r] & f_*(\mathcal{K}_{2,X_n}) \arrow[r] & f_*(\mathcal{K}_{2,Y'}) \arrow[r] & 0.
\end{tikzcd} \tag{2.12}$$

Note that the maps $\mathcal{K}_{2,X} \to \mathcal{K}_{2,Y}$ and $\mathcal{K}_{2,X_n} \to \mathcal{K}_{2,Y'}$ are surjective by [25, Proposition 10, Theorem 13] because the surjectivity clearly holds for the Milnor K-theory sheaf. In particular, the top row is exact. The bottom row is exact because f is finite, and a finite push-forward is an exact functor on the category of Nisnevich sheaves.

On the other hand, we also have a double relative K-theory exact sequence

$$\mathcal{K}_{2,(X,Y)} \xrightarrow{f^*} f_*(\mathcal{K}_{2,(X_n,Y')}) \to \mathcal{K}_{1,(X,X_n,Y)} \to \mathcal{K}_{1,(X,Y)} \xrightarrow{f^*} f_*(\mathcal{K}_{1,(X_n,Y')}), \tag{2.13}$$

where $\mathcal{K}_{1,(X,X_n,Y)}$ is the sheaf of double relative K-theory. Since $\mathcal{K}_{1,(X,Y)} \cong (1+\mathcal{I}_Y)^\times \hookrightarrow (1+\mathcal{I}_{Y'})^\times \cong \mathcal{K}_{1,(X_n,Y')}$, and $\mathcal{K}_{1,(X,X_n,Y)} \cong \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{Y'} \Omega^1_{Y'/Y}$ by [13, Theorem 0.2], a combination of (2.12) and (2.13) yields (via a diagram chase) an exact sequence

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{Y'} \Omega^1_{Y'/Y} \to f_*(\mathcal{K}_{2,X_n})/\mathcal{K}_{2,X} \to f_*(\mathcal{K}_{2,Y'})/\mathcal{K}_{2,Y} \to 0. \tag{2.14}$$

Comparing this exact sequence for Y and $2Y$ (where $mY \subset X$ is defined by $\mathcal{I}_Y^m$), we get the desired isomorphism if we choose our conductor subscheme to be $2Y$. □

The next result is of independent interest and plays a fundamental role in the study of 0-cycles on singular schemes. Recall that for any $X \in \mathbf{Sch}_k$, the edge map of the Thomason-Trobaugh spectral sequence yields a split surjection $K_1(X) \twoheadrightarrow H^0_{\text{nis}}(X, \mathcal{O}^\times_X)$. We let $SK_1(X)$ denote the kernel of this surjection.

Proposition 2.8. *Let X be a reduced quasi-projective surface over a field and let $f\colon X_n \to X$ be the normalization map. Then we can find a conductor closed subscheme $Y \hookrightarrow X$ such that for $Y' = f^*(Y)$, there is an exact sequence*

$$0 \to \frac{SK_1(X_n)}{SK_1(X)} \to \frac{SK_1(Y')}{SK_1(Y)} \to C_{KS}(X)_0 \to C_{KS}(X_n)_0 \to 0.$$

Proof. A version of this result is shown in [30, Proposition 2.3] for surfaces over $\mathbb{C}$. We shall modify that argument and, in particular, use Lemma 2.7 to prove the proposition. By [25, Proposition 10, Theorem 13], we can replace $C_{KS}(X)$ by $H^2_{\text{nis}}(X, \mathcal{K}_{2,X})$, where $\mathcal{K}_{*,X}$ is the Quillen K-theory sheaf. The same for X_n too.

We choose a conductor subscheme $Y \hookrightarrow X$ and consider the exact sequence of Nisnevich sheaves on X:

$$\mathcal{K}_{2,X} \to f_*(\mathcal{K}_{2,X_n}) \to f_*(\mathcal{K}_{2,X_n})/\mathcal{K}_{2,X} \to 0.$$

Since $H^i_{\text{nis}}(Z, \mathcal{K}_{2,Z}) = 0$ for $i > 0$ for a semilocal scheme Z, the Leray spectral sequence tells us that $H^i_{\text{nis}}(X_n, \mathcal{K}_{2,X_n}) \cong H^i_{\text{nis}}(X, f_*(\mathcal{K}_{2,X_n}))$ for all $i \geq 0$. Since the kernel and cokernel of the map $\mathcal{K}_{2,X} \to f_*(\mathcal{K}_{2,X_n})$ are supported on Y, it follows from the above sheaf exact sequence that there is an exact cohomology sequence

$$\begin{aligned} 0 \to \frac{H^1_{\text{nis}}(X_n, \mathcal{K}_{2,X_n})}{H^1_{\text{nis}}(X, \mathcal{K}_{2,X})} &\to H^1_{\text{nis}}(X, f_*(\mathcal{K}_{2,X_n})/\mathcal{K}_{2,X}) \\ &\to H^2_{\text{nis}}(X, \mathcal{K}_{2,X}) \to H^2_{\text{nis}}(X_n, \mathcal{K}_{2,X_n}) \to 0. \end{aligned} \tag{2.15}$$

We can replace the last two terms by their degree zero subgroups without disturbing the exactness.

Since

$$H^2_{\text{nis}}(X, \mathcal{K}_{3,X}) \to SK_1(X) \to H^1_{\text{nis}}(X, \mathcal{K}_{2,X}) \to 0 \tag{2.16}$$

is exact by the Thomason-Trobaugh spectral sequence [43, Theorem 10.8] and since $H^2_{\text{nis}}(X, \mathcal{K}_{3,X}) \twoheadrightarrow H^2_{\text{nis}}(X_n, \mathcal{K}_{3,X_n})$, the left-end term in (2.15) is same as the quotient $SK_1(X_n)/SK_1(X)$. Since the edge map of the spectral sequence induces an isomorphism $SK_1(Z) \cong H^1_{\text{nis}}(Z, \mathcal{K}_{2,Z})$ for any Noetherian scheme Z of Krull dimension at most one, what we are left to show is that the canonical map

$$H^1_{\text{nis}}(X, f_*(\mathcal{K}_{2,X_n})/\mathcal{K}_{2,X}) \to H^1_{\text{nis}}(Y, f_*(\mathcal{K}_{2,Y'})/\mathcal{K}_{2,Y})$$

is an isomorphism if we replace Y by some of its infinitesimal thickenings. But this is a direct consequence of Lemma 2.7 (and exactness of f_*). $\square$

Lemma 2.9. *Let Y be a one-dimensional Noetherian k-scheme, where k is a finite field. Then $SK_1(Y) = 0$ if Y is affine and $SK_1(Y)$ is a torsion group of bounded exponent if Y is projective.*

Proof. The affine case follows from [36]. We therefore assume that Y is projective. The Thomason-Trobaugh spectral sequence [43, Theorem 10.8] yields isomorphisms (see (2.16))

$$SK_1(Y) \cong H^1_{\mathrm{nis}}(Y, \mathcal{K}_{2,Y}) \cong H^1_{\mathrm{nis}}(Y, \mathcal{K}^M_{2,Y}) \cong H^1_{\mathrm{zar}}(Y, \mathcal{K}^M_{2,Y}). \tag{2.17}$$

By Lemma 2.6, we can therefore assume that Y is reduced. Let $f\colon Y_n \to Y$ be the normalization map.

Since the kernel of the first arrow in the exact sequence

$$\mathcal{K}_{2,Y} \xrightarrow{f^*} f_*(\mathcal{K}_{2,Y_n}) \to f_*(\mathcal{K}_{2,Y_n})/\mathcal{K}_{2,Y} \to 0$$

is supported on Y_{sing}, there is an exact cohomology sequence

$$H^0_{\mathrm{nis}}(Y, f_*(\mathcal{K}_{2,Y_n})/\mathcal{K}_{2,Y}) \to H^1_{\mathrm{nis}}(Y, \mathcal{K}_{2,Y}) \to H^1_{\mathrm{nis}}(Y_n, \mathcal{K}_{2,Y_n}) \to 0. \tag{2.18}$$

Combining this with Lemma 2.7, we can find a conductor closed subscheme $Z \subset Y$ such that one has an exact sequence

$$H^0_{\mathrm{nis}}(Z, f_*(\mathcal{K}_{2,Z'})/\mathcal{K}_{2,Z}) \to H^1_{\mathrm{nis}}(Y, \mathcal{K}_{2,Y}) \to H^1_{\mathrm{nis}}(Y_n, \mathcal{K}_{2,Y_n}) \to 0, \tag{2.19}$$

where $Z' = f^*(Z)$. Since $K_2(Z') = H^0_{\mathrm{nis}}(Z, f_*(\mathcal{K}_{2,Z'})) \twoheadrightarrow H^0_{\mathrm{nis}}(Z, f_*(\mathcal{K}_{2,Z'})/\mathcal{K}_{2,Z})$, (2.17) and (2.19) together give us an exact sequence

$$K_2(Z') \to SK_1(Y) \to SK_1(Y_n) \to 0.$$

Since $\dim(Z') = 0$, we see that Z'_{red} is the spectrum of a finite product of finite fields. Hence, $K_2(Z'_{\mathrm{red}}) = 0$. It follows from Lemma 2.6 that $K_2(Z')$ is a p-primary torsion group of bounded exponent. Since Y_n is a smooth projective curve over k, the K-theory localization sequence and the affine case of the lemma shows that the push-forward map $K_*(Y_n) \to K_*(k)$ induces an isomorphism $SK_1(Y_n) \cong (k^\times)^r$, where r is the number of irreducible components of Y. The result follows. $\square$

2.3. Torsion in the idele class group of a surface

Let X be an integral projective scheme of dimension two over a finite field k and $D \subset X$ a nowhere dense closed subscheme. The following result is the starting point of the proof of Theorem 4.9.

Proposition 2.10. *$C_{KS}(X, D)_0$ is a torsion group of bounded exponent.*

Proof. Using the exact sequence

$$SK_1(D) \to C_{KS}(X, D)_0 \to C_{KS}(X)_0 \to 0$$

and Lemma 2.9, it suffices to show that $C_{KS}(X)_0$ is a torsion group of bounded exponent. Using Proposition 2.8 and Lemma 2.9, we can assume that X is normal.

Let $f\colon \widetilde{X} \to X$ be a resolution of singularities of X with reduced exceptional divisor E. It is then shown in the proof of [31, Theorem 1.4] that there are isomorphisms

$$C_{KS}(X) \cong C_{KS}(\widetilde{X}, nE) \cong \mathrm{CH}_0(\widetilde{X}|nE) \tag{2.20}$$

for all $n \gg 0$, where the last group is the Chow group of 0-cycles with modulus (see § 7.1). Using Lemma 2.4, we get $C_{KS}(X)_0 \cong \mathrm{CH}_0(\widetilde{X}|nE)_0$. We are now done by [4, Corollary 8.7]. □

3. Generic fibration over a curve

Let X be an integral projective scheme of dimension two over a finite field k. We assume that there exists a projective dominant morphism $f\colon X \to S$, where S is an integral projective curve over k. We assume that there exists a dense open $S' \subset S$ such that the fibers of f over S' are integral. We assume also that f has a section over the generic point of S and the generic fiber is regular.

We shall deduce an important corollary of Proposition 2.10 under the above set-up. By Lemma 2.1, we have the maps $C_{KS}(X, D) \xrightarrow{\nu_X} \mathrm{CH}^F_0(X) \xrightarrow{f_*} \mathrm{CH}^F_0(S)$. We let $C_{KS}(X, D)_S$ denote the kernel of the composite map. It follows from Lemma 2.1 that there are inclusions

$$C_{KS}(X, D)_S \hookrightarrow C_{KS}(X, D)_0 \hookrightarrow C_{KS}(X, D). \tag{3.1}$$

Let $t \in S$ denote the generic point of S. For any $s \in S$, we let $X_s = f^{-1}(s)$ denote the scheme theoretic fiber. For every proper closed subset $Z \subset S$, there is an exact sequence of Nisnevich cohomology groups with support

$$H^1_{\mathrm{nis}}(f^{-1}(S \setminus Z), \mathcal{K}^M_{2,(X,D)}) \to H^2_{f^{-1}(Z)}(X, \mathcal{K}^M_{2,(X,D)}) \to H^2_{\mathrm{nis}}(X, \mathcal{K}^M_{2,(X,D)}).$$

Taking the inductive limit over all proper closed subsets Z, we get an exact sequence

$$SK_1(X_t) \xrightarrow{\partial} \bigoplus_{s \in S_{(0)}} C_{KS}^{X_s}(X, D) \to C_{KS}(X, D) \to 0, \tag{3.2}$$

where the second map is surjective by our assumption and [22, Theorem 2.5].

We now consider the diagram

$$\begin{tikzcd} SK_1(X_t) \arrow[r, "\partial"] \arrow[d, "f_*"'] & \bigoplus_{s \in S_{(0)}} C_{KS}^{X_s}(X, D) \arrow[r] \arrow[d, "f_*"] & C_{KS}(X, D) \arrow[r] \arrow[d, "f_*"] & 0 \\ K_1^M(k(t)) \arrow[r, "\mathrm{div}"] & \bigoplus_{s \in S_{(0)}} \mathbb{Z} \arrow[r] & \mathrm{CH}_0^F(S) \arrow[r] & 0, \end{tikzcd} \tag{3.3}$$

where the left vertical arrow is induced by the push-forward map $f_*\colon K_1(X_t) \to K_1(k(t)) \cong K_1^M(k(t))$ which exists because X_t is regular. The bottom exact sequence is the standard one which defines $\mathrm{CH}_0^F(S)$. The middle vertical arrow is the direct sum of the compositions of the maps $C_{KS}^{X_s}(X, D) \to \mathrm{CH}_0^F(X_s)$ from Lemma 2.1 and the push-forward maps $f_*\colon \mathrm{CH}_0^F(X_s) \to \mathrm{CH}_0^F(\mathrm{Spec}\,(k(s))) \cong \mathbb{Z}$. The right vertical arrow is the composition of the map $\nu_X\colon C_{KS}(X, D) \to \mathrm{CH}_0^F(X)$ from Lemma 2.1 and the push-forward map $f_*\colon \mathrm{CH}_0^F(X) \to \mathrm{CH}_0^F(S)$.

Lemma 3.1. *The diagram* (3.3) *is commutative.*

Proof. To show that the right square in (3.3) commutes, it suffices to show that it commutes when restricted to each direct summand $C_{KS}^{X_s}(X, D)$. Furthermore, we can replace $C_{KS}^{X_s}(X, D)$ and $C_{KS}(X, D)$ by $C_{KS}^{X_s}(X)$ and $C_{KS}(X)$, respectively because f_* factors through the latter groups.

We now fix $s \in S_{(0)}$ and consider the diagram

$$\begin{tikzcd} C_{KS}^{X_s}(X) \arrow[r] \arrow[d] & \mathrm{CH}_0^F(X_s) \arrow[r, "f_*"] \arrow[d] & \mathbb{Z} \arrow[d] \\ C_{KS}(X) \arrow[r] & \mathrm{CH}_0^F(X) \arrow[r, "f_*"] & \mathrm{CH}_0^F(S), \end{tikzcd} \tag{3.4}$$

where the vertical arrows are induced by the inclusions $X_s \hookrightarrow X$ and $\{s\} \hookrightarrow S$.

The middle (resp. right) vertical arrow in (3.3) is the composition of the top (resp. bottom) horizontal arrows in (3.4) by the definition of f_*. The left square in (3.4) commutes by Lemma 2.1 and the right square is well known to be commutative (see [12]). This shows that the right square of (3.3) commutes.

We now show that the left square in (3.3) commutes. Since X_t is a regular scheme of dimension one over the infinite field $k(t)$, it is well known (e.g., see [24]) that the Gersten sequence (2.1) gives an exact sequence

$$K_2^M(K) \xrightarrow{\partial} \bigoplus_{x\in(X_t)_{(0)}} K_1^M(k(x)) \to SK_1(X_t) \to 0, \tag{3.5}$$

and the left vertical arrow in (3.3) is induced by the norm maps $K_1^M(k(x)) \to K_1^M(k(t))$ for $x \in (X_t)_{(0)}$. Note that this norm map is same as the composition $K_1^M(k(x)) \xrightarrow{(\iota_x)_*} K_1(X_t) \xrightarrow{f_*} K_1(k(t))$. Hence, it suffices to show that the left square commutes after we replace the map ∂ by its composition with

$$H^1_x(X_t, \mathcal{K}^M_{2,X_t}) \xrightarrow{\cong} K_1^M(k(x)) \to SK_1(X_t) \cong H^1_{\mathrm{nis}}(X_t, \mathcal{K}^M_{2,X_t})$$

on the left for $x \in (X_t)_{(0)}$. Furthermore, since the map $f_*\colon C^{X_s}_{KS}(X,D) \to K_0^M(k(s))$ factors through $C^{X_s}_{KS}(X) \to K_0^M(k(s))$, we can also replace ∂ by its composition with $C^{X_s}_{KS}(X,D) \to C^{X_s}_{KS}(X)$ on the right.

We now fix a closed point $x \in X_t$ and let $Y \subset X$ be the closure of $\{x\}$ with integral subscheme structure. Then f restricts to a finite dominant map $f\colon Y \to S$. The composite map $K_1^M(k(x)) \to H^1_{\mathrm{nis}}(X_t, \mathcal{K}^M_{2,X_t}) \xrightarrow{\partial} \bigoplus_{s\in S_{(0)}} C^{X_s}_{KS}(X)$ factors through the composition

$$K_1^M(k(x)) \xrightarrow{\cong} H^1_x(X, \mathcal{K}^M_{2,X}) \xrightarrow{\partial} \bigoplus_{y\in Y_{(0)}} C^y_{KS}(X) \cong \bigoplus_{s\in S_{(0)}} C^{Y_s}_{KS}(X) \to \bigoplus_{s\in S_{(0)}} C^{X_s}_{KS}(X),$$

where the left arrow is an isomorphism by [18, Lemma 3.7] because $x \in X_{\mathrm{reg}}$. We therefore have to show that the left square in the diagram

$$\begin{tikzcd} K_1^M(k(x)) \arrow[r, "\partial"] \arrow[d, "N"'] & \bigoplus_{s\in S_{(0)}} C^{Y_s}_{KS}(X) \arrow[r] \arrow[d, "f_*"] & \bigoplus_{s\in S_{(0)}} \mathrm{CH}_0^F(Y_s) \arrow[dl, "f_*"] \\ K_1^M(k(t)) \arrow[r, "\mathrm{div}"] & \bigoplus_{s\in S_{(0)}} \mathbb{Z} & \end{tikzcd} \tag{3.6}$$

is commutative, where N is the norm map.

Since the triangle on the right is commutative by the definition of $f_*\colon C^{Y_s}_{KS}(X) \to \mathbb{Z}$ for $s \in S_{(0)}$, we need to show that big outer diagram in (3.6) commutes. But this is classical because the top composite horizontal arrow is the boundary map of the complex (2.1). One knows that this is same as the boundary map $\partial\colon K_1(k(x)) \to \bigoplus_{s\in Y_{(0)}} K_0(k(s))$ on Quillen K-groups (see [16, Lemma 11.2]) and Quillen [37] showed that this map takes an element of $K_1(k(x))$ to its divisor. The diagram therefore commutes by [12, Proposition 1.4]. $\square$

Let $V(X_t) = \mathrm{Ker}(SK_1(X_t) \to K_1(k(t)))$. Since the kernel of $f_*\colon C^{X_s}_{KS}(X,D) \to \mathrm{CH}^F_0(\mathrm{Spec}\,(k(s)))$ is same as $C^{X_s}_{KS}(X,D)_0$, the commutative diagram (3.3) induces a 3-term complex

$$V(X_t) \xrightarrow{\partial} \bigoplus_{s \in S_{(0)}} C^{X_s}_{KS}(X,D)_0 \to C_{KS}(X,D)_S \to 0. \tag{3.7}$$

Since f has a section over $\mathrm{Spec}\,(k(t))$, it follows that the left vertical arrow in (3.3) is split surjective. A diagram chase shows that the middle arrow in (3.7) is surjective. For any integer $m \geq 1$, let

$$A_m = \underset{s \in S_{(0)} \setminus S'_{\mathrm{reg}}}{\oplus} C^{X_s}_{KS}(X,D)_0/m \ \text{ and } \ C^{S'_{\mathrm{reg}}}_{KS}(X,D)_0 = \underset{s \in (S'_{\mathrm{reg}})_{(0)}}{\oplus} C^{X_s}_{KS}(X,D)_0. \tag{3.8}$$

In view of (3.1), an important consequence of Proposition 2.10 is the following.

Corollary 3.2. *For all $m \gg 0$, the exact sequence (3.2) induces a 3-term complex*

$$V(X_t) \xrightarrow{\partial} A_m \bigoplus C^{S'_{\mathrm{reg}}}_{KS}(X,D)_0 \to C_{KS}(X,D)_0 \to 0$$

in which the middle arrow is surjective.

For a morphism of schemes $T \to S$, we let $X_T = X \times_S T$. We now let $s \in S \setminus S'_{\mathrm{reg}}$ be a closed point. We let $k(t)_s$ denote the total quotient ring of $\mathcal{O}^h_{S,s}$. We let $Z = \mathrm{Spec}\,(\mathcal{O}^h_{S,s})$. Note that the normalization Z_n of Z is canonically isomorphic to $\mathcal{O}^h_{S_n,s}$, where the latter is the Henselization of $\mathcal{O}_{S_n,s}$ with respect to its Jacobson radical. In particular, if $\nu\colon S_n \to S$ denotes the normalization map, then $\mathcal{O}^h_{S_n,s}$ is the product of the Henselian discrete valuation rings $\mathcal{O}^h_{S_n,s_i}$ for $1 \leq i \leq r$, where $\Sigma = \{s_1, \dots, s_r\} = \nu^{-1}(s)$. Let $k(t)_i$ denote the quotient field of $\mathcal{O}^h_{S_n,s_i}$ so that $k(t)_s = \prod_{i=1}^{r} k(t)_i$. Note that $r = 1$ if $s \in S_{\mathrm{reg}}$.

By [22, Proposition 4.2], there exists a nowhere dense closed subscheme $D' \subset X_{S_n}$ containing $D \times_S S_n$ and a norm map of sheaves $\nu_*(\mathcal{K}^M_{2,(X_{S_n},D')}) \to \mathcal{K}^M_{2,(X,D)}$. Taking the cohomology with support, we get a push-forward map $\nu_*\colon C^{X_\Sigma}_{KS}(X_{S_n}, D') \to C^{X_s}_{KS}(X,D)$. Note that we have used here the fact that the push-forward of Nisnevich sheaves under a finite map is an exact functor. By excision, the above is same as the map $\nu_*\colon C^{X_\Sigma}_{KS}(X_{Z_n}, D'_{Z_n}) \to C^{X_s}_{KS}(X_Z, D_Z)$. We thus get a commutative diagram

$$
\begin{tikzcd}
 & \bigoplus\limits_{s' \in S_{(0)}} C^{X_{s'}}_{KS}(X, D) \arrow[r, two heads] & C^{X_s}_{KS}(X, D) \arrow[d, "\cong"] \\
SK_1(X_t) \arrow[ur, "\partial"] \arrow[r, "\alpha_s"] \arrow[dr, "\alpha_s"'] & SK_1(X_{k(t)_s}) \arrow[r, "\partial"] & C^{X_s}_{KS}(X_Z, D_Z) \\
 & \prod\limits_{i=1}^{r} SK_1(X_{k(t)_i}) \arrow[r, "\partial"] \arrow[u, "\cong"', "\nu_*"] & C^{X_\Sigma}_{KS}(X_{Z_n}, D'_{Z_n}), \arrow[u, "\nu_*"']
\end{tikzcd}
\tag{3.9}
$$

where the horizontal arrow on the top is the projection and α_s is induced by the canonical inclusion $k(t) \subset k(t)_s$. One gets a similar commutative diagram by reducing all the abelian groups modulo any integer $m \geq 1$.

For a fixed integer $m \geq 1$, we let $SK_1(X_t, m, S')$ be the kernel of the canonical map $SK_1(X_t) \xrightarrow{\alpha_s} \prod\limits_v SK_1(X_{k(t)_v})/m$, where v runs through all closed points of S_n lying over $S \setminus S'_{\mathrm{reg}}$. We let $V(X_t, m, S') = SK_1(X_t, m, S') \cap V(X_t)$. It follows from (3.9) that $SK_1(X_t, m, S')$ is annihilated by the composite map

$$
SK_1(X_t) \xrightarrow{\partial} \bigoplus_{s \in S_{(0)}} C^{X_s}_{KS}(X, D)/m \twoheadrightarrow \bigoplus_{s \in S_{(0)} \setminus S'_{\mathrm{reg}}} C^{X_s}_{KS}(X, D)/m.
$$

Since $C^{X_s}_{KS}(X, D)_0/m \hookrightarrow C^{X_s}_{KS}(X, D)/m$, it follows therefore from (3.7) that $V(X_t, m, S')$ is annihilated by the composite map

$$
V(X_t) \xrightarrow{\partial} \bigoplus_{s \in S_{(0)}} C^{X_s}_{KS}(X, D)_0/m \twoheadrightarrow \bigoplus_{s \in S_{(0)} \setminus S'_{\mathrm{reg}}} C^{X_s}_{KS}(X, D)_0/m. \tag{3.10}
$$

Using Corollary 3.2, (3.1) and Lemma 2.1, we obtain the following consequence of Proposition 2.10.

Corollary 3.3. *For all $m \gg 0$, the exact sequence (3.2) restricts to a 3-term complex*

$$
V(X_t, m, S') \xrightarrow{\partial} C^{S'_{\mathrm{reg}}}_{KS}(X, D)_0 \to C_{KS}(X, D)_0.
$$

If $f^{-1}(S'_{\mathrm{reg}})$ is regular and $D \cap f^{-1}(S'_{\mathrm{reg}}) = \emptyset$, then we have a complex

$$
V(X_t, m, S') \xrightarrow{\partial} \bigoplus_{s \in (S'_{\mathrm{reg}})_{(0)}} \mathrm{CH}^F_0(X_s)_0 \to C_{KS}(X, D)_0. \tag{3.11}
$$

3.1. Local Kato-Saito idele class group

Let X be a Noetherian scheme of Krull dimension $d \geq 1$. Let $\mathcal{P}_r(X)$ denote the set of all Parshin chains of length r on X. If $P \in \mathcal{P}_r(X)$, we let $P_i = (p_0, \ldots, p_i)$ for $0 \leq i \leq r$.

Let $x \in X$ be a closed point and $X_x = \operatorname{Spec}(\mathcal{O}^h_{X,x})$. We write $\mathcal{P}^x_r(X)$ for the set of Parshin chains of length r on X beginning with x and $\mathcal{Q}^x(X)$ for the set of Q-chains on X beginning with x. Let $\mathcal{F}$ be a Nisnevich sheaf on X. For $P = (p_0, \ldots, p_r) \in \mathcal{P}^x_r(X)$ and $q \geq 0$, we have the sequence of boundary maps

$$H^q_P(X, \mathcal{F}) \xrightarrow{\partial} H^{q+1}_{P_{r-1}}(X, \mathcal{F}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} H^{q+r-1}_{P_1}(X, \mathcal{F}) \xrightarrow{\partial} H^{q+r}_x(X, \mathcal{F}). \tag{3.12}$$

We let $\partial^x_P \colon H^q_P(X, \mathcal{F}) \to H^{q+r}_x(X, \mathcal{F})$ denote the composite map.

If $Q = (p_0, \ldots, p_{s-1}, p_{s+1}, \ldots, p_r) \in \mathcal{Q}^x(X)$ is a Q-chain with break at s, we let $B'(Q)$ be the set of all $y \in X$ such that $Q_y = (p_0, \ldots, p_{s-1}, y, p_{s+1}, \ldots, p_r) \in \mathcal{P}^x_r(X)$. For any $y \in B'(Q)$, we have the restriction (localization) map $H^q_Q(X, \mathcal{F}) \to H^q_{Q_y}(X, \mathcal{F})$. For $a \in H^q_Q(X, \mathcal{F})$, we shall let a_{Q_y} denote the image of a under this restriction. The following is a straightforward extension of [22, Lemma 1.6.3].

Lemma 3.4. *We have the following.*

(1) *The map*

$$\bigoplus_{P \in \mathcal{P}^x_d(X)} H^0_P(X, \mathcal{F}) \xrightarrow{\sum_P \partial^x_P} H^d_x(X, \mathcal{F})$$

is surjective.

(2) *If $d \geq 2$ and A is an abelian group, then the group $\operatorname{Hom}(H^d_x(X, \mathcal{F}), A)$ is canonically isomorphic to the group of all families $(g_P)_{P \in \mathcal{P}^x_d(X)}$ of homomorphisms $g_P \colon H^0_P(X, \mathcal{F}) \to A$ satisfying the following reciprocity law (R).*

(R) For any $0 < s < d$, any Q-chain $Q \in \mathcal{Q}^x(X)$ of length d with break at s, and any $a \in H^0_Q(X, \mathcal{F})$, the element $g_{Q_y}(a_{Q_y}) = 0$ for almost all $y \in B'(Q)$ and

$$\sum_{y \in B'(Q)} g_{Q_y}(a_{Q_y}) = 0.$$

(3) *For $d = 1$, there is an exact sequence*

$$0 \to H^0_{\mathrm{nis}}(X_x, \mathcal{F}) \to H^0_{\mathrm{nis}}(X_x \setminus \{x\}, \mathcal{F}) \to H^1_x(X, \mathcal{F}) \to 0.$$

Proof. We let $U = X_x \setminus \{x\}$. Then U is a Noetherian scheme of Krull dimension $d-1$ and the correspondence $\mathcal{P}_{d-1}(U) \to \mathcal{P}_d(X_x)$, which takes P to $P_x := (x, P)$, is a bijection. Let $f \colon X_x \to X$ denote the canonical map and let $f_* \colon \mathcal{P}_d(X_x) \to \mathcal{P}^x_d(X)$ denote the induced map. For $P \in \mathcal{P}_{d-1}(U)$, we write $f_*(P_x)$ also as P_x.

We now have a commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{P \in \mathcal{P}_{d-1}(U)} H^0_P(U, \mathcal{F}) \arrow[r, "\cong"] \arrow[d, two heads, "c_U"'] & \bigoplus\limits_{P \in \mathcal{P}_d(X_x)} H^0_P(X_x, \mathcal{F}) \arrow[d, "\sum_P \partial^x_P"] & \bigoplus\limits_{P \in \mathcal{P}^x_d(X)} H^0_P(X, \mathcal{F}) \arrow[l, "\cong"', "f^*"] \arrow[d, "\sum_P \partial^x_P"] \\
H^{d-1}_{\mathrm{nis}}(U, \mathcal{F}) \arrow[r, two heads, "\partial"] & H^d_x(X_x, \mathcal{F}) & H^d_x(X, \mathcal{F}). \arrow[l, "f^*", "\cong"']
\end{tikzcd} \tag{3.13}$$

The arrow c_U is surjective by [22, Lemma 1.6.3]. The first assertion of the lemma therefore follows from a diagram chase. The second assertion follows from the isomorphism $H^{d-1}_{\mathrm{nis}}(U, \mathcal{F}) \xrightarrow{\cong} H^d_x(X, \mathcal{F})$ for $d \geq 2$ and by applying [22, Lemma 1.6.3] to $H^{d-1}_{\mathrm{nis}}(U, \mathcal{F})$. The last part is obvious. $\square$

Lemma 3.5. *Let $f\colon Y \hookrightarrow X$ be a closed immersion of integral schemes. Let $D \subset X$ be a nowhere dense closed subscheme. Let $E \subset Y$ be a nowhere dense closed subscheme such that the Kato-Saito recipe gives a push-forward map $f_*\colon C_{KS}(Y, E) \to C_{KS}(X, D)$. Let $x \in Y \setminus D$ be a closed point. Let $Z \subset X$ be a closed subset such that $x \in Z$ and $Z \subset X_{\mathrm{reg}}$. Then the diagram*

$$\begin{tikzcd}
C^x_{KS}(Y, E) \arrow[r] \arrow[d, "f_*"'] & C_{KS}(Y, E) \arrow[d, "f_*"] \\
\mathrm{CH}^F_0(Z) \arrow[r] & C_{KS}(X, D)
\end{tikzcd} \tag{3.14}$$

is commutative.

Proof. Note that the composition of the bottom horizontal arrow and left vertical arrow is the given by

$$C^x_{KS}(Y, E) \to \mathrm{CH}^F_0(\mathrm{Spec}\,(k(x))) \cong K^M_0(k(x)) \cong C^x_{KS}(X, D) \to C^Z_{KS}(X, D) \\ \to C_{KS}(X, D).$$

Let $d = \dim(X)$ and $r = \dim(Y)$. Let $P = (p_0, \ldots, p_r) \in \mathcal{P}^x_r(Y)$ be a maximal Parshin chain. By Lemma 3.4 (Part 1), it suffices to show that the square on the right in the diagram

$$\begin{tikzcd}
K^M_r(k(P)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H^0_P(Y, \mathcal{K}^M_{r,(Y,E)}) \arrow[r] \arrow[d, "\partial^x_P"] & C_{KS}(Y, E) \arrow[d, "f_*"] \\
H^{d-r}_P(X, \mathcal{K}^M_{d,(X,D)}) \arrow[r, "\partial^x_P"] & K^M_0(k(x)) \arrow[r] & C_{KS}(X, D)
\end{tikzcd} \tag{3.15}$$

is commutative. Since the square on the left is clearly commutative and the left horizontal arrow on the top is an isomorphism, all we need to show is that the big outer rectangle commutes. But this follows from [22, Proposition 2.9(i)]. □

3.2. Bloch's map

Let X be a projective integral scheme of dimension $d \geq 2$ over a finite field k. Let $f\colon X \to S$ be a dominant projective morphism between integral schemes in $\mathbf{Sch}_k$ of relative dimension one. Assume that there exists a dense open subscheme $S' \subset S$ such that the induced morphism $f'\colon X' := f^{-1}(S') \to S'$ is smooth over S' and the fibers of f' are integral. Assume also that f' has a section. Since f' is the restriction of f to the open subscheme X', we shall often write it simply as f in the sequel if no confusion arises.

Let $j\colon S' \hookrightarrow S$ be the inclusion. Let $T' \subset S'$ be an integral curve and let $Y' = X' \times_{S'} T'$. Since $Y' \to T'$ is a smooth projective morphism to an integral curve of relative dimension one whose all fibers are integral, it follows that Y' is integral.

Let T (resp. Y) be the scheme theoretic closure of T' (resp. Y') in S (resp. X). Then T and Y are integral closed subschemes of S and X, respectively. Let $\gamma\colon T \hookrightarrow S$ and $\gamma'\colon Y \to X$ be the inclusion maps. Let $t \in T$ denote the generic point of T. For any point $s \in S$, we let $X_s = f^{-1}(s)$ be the scheme theoretic fiber.

The long cohomology sequence with support gives us the boundary map $\partial\colon SK_1(Y_t) \to \bigoplus_{s\in T_{(0)}} H^2_{X_s}(Y, \mathcal{K}^M_{2,Y})$. For every $s \in T_{(0)}$, Lemma 2.1 says that there is a canonical map $\gamma'_*\colon H^2_{X_s}(Y, \mathcal{K}^M_{2,Y}) \to \mathrm{CH}^F_0(X_s)$. Let $\partial_s\colon SK_1(Y_t) \to \mathrm{CH}^F_0(X_s)$ denote the composition

$$SK_1(Y_t) \xrightarrow{\partial} \bigoplus_{u\in T_{(0)}} H^2_{X_u}(Y, \mathcal{K}^M_{2,Y}) \twoheadrightarrow H^2_{X_s}(Y, \mathcal{K}^M_{2,Y}) \xrightarrow{\gamma'_*} \mathrm{CH}^F_0(X_s), \tag{3.16}$$

where the middle arrow is the projection. By (3.7), the map ∂_s restricts to $\partial_s\colon V(Y_t) \to H^2_{X_s}(Y, \mathcal{K}_{2,Y})_0 \to \mathrm{CH}^F_0(X_s)_0$. Summing these maps over $s \in T_{(0)}$ and composing with the inclusion of the direct sums of Chow groups via $T_{(0)} \hookrightarrow S_{(0)}$, we get a boundary map $\partial_{X,t}\colon V(Y_t) \to \bigoplus_{s\in S_{(0)}} \mathrm{CH}^F_0(X_s)_0$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
V(Y_t) \arrow[r, "\partial_{X,t}"] \arrow[d, "j^*"', "\cong"] & \bigoplus_{s\in S_{(0)}} \mathrm{CH}^F_0(X_s)_0 \arrow[d, two heads, "j^*"] \\
V(Y_t) \arrow[r, "\partial_{X',t}"] & \bigoplus_{s\in S'_{(0)}} \mathrm{CH}^F_0(X_s)_0.
\end{tikzcd} \tag{3.17}$$

We let $V(Y_t, m, T')$ be as in Corollary 3.3 and write it as $V(Y_t, m, S')$. Composing the lower horizontal arrow in (3.17) with the inclusion $V(Y_t, m, S') \hookrightarrow V(Y_t)$, we get the boundary map

$$\partial_{X',t} \colon V(Y_t, m, S') \to \bigoplus_{s \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0. \tag{3.18}$$

This boundary map is same as the one used in [22, Theorem 5.4] and was first considered by Bloch [7, Theorem 4.2] in a special case.

For $s \in S'_{\mathrm{reg}}$, Lemma 2.1 yields a well defined map $\mathrm{CH}_0^F(X_s) \to C_{KS}(X, D)_0$. As a consequence of the proof of Corollary 3.3, we get the following.

Theorem 3.6. *Assume that S' is regular and $D \cap X' = \emptyset$. Then the composition*

$$V(Y_t, m, S') \xrightarrow{\partial_{X',t}} \bigoplus_{s \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \to C_{KS}(X, D)_0$$

is zero for all $m \gg 0$.

Proof. By [22, Proposition 2.9, Lemma 2.10], we can find a closed subscheme $E \subset Y$ containing $\gamma'^*(D)$ such that $E \cap X' = \emptyset$ and the Kato-Saito recipe provides a push-forward map $\gamma'_* \colon C_{KS}(Y, E) \to C_{KS}(X, D)$. Lemma 2.2 says that this map is degree preserving.

Let $s \in T'_{\mathrm{reg}}$ be a closed point. Our first claim is that the diagram

$$\begin{tikzcd} C^{X_s}_{KS}(Y, E) \arrow[rr] \arrow[d, "\gamma'_*"'] & & C_{KS}(Y, E) \arrow[d, "\gamma'_*"] \\ \mathrm{CH}_0^F(X_s) \arrow[r, "\cong"] & C^{X_s}_{KS}(X, D) \arrow[r] & C_{KS}(X, D) \end{tikzcd} \tag{3.19}$$

commutes. Since $X_s \subset Y_{\mathrm{reg}}$ and $X_s \cap E = \emptyset$, it follows from Lemma 2.1 that the left vertical arrow is an isomorphism. Hence, the claim follows from Lemma 3.5.

The diagram

$$\begin{tikzcd} V(Y_t, m, T') \arrow[r, "\partial"] \arrow[d, "\cong"'] & \bigoplus_{s \in T'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \arrow[d, "\gamma'_*"] \\ V(Y_t, m, S') \arrow[r, "\partial_{X',t}"] & \bigoplus_{s \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \end{tikzcd} \tag{3.20}$$

commutes by the construction of the map $\partial_{X',t}$ above (see (3.16)).

Our second claim is that the bottom boundary map in (3.20) becomes trivial after composing with the projection $\bigoplus_{s \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \twoheadrightarrow \bigoplus_{s \in S'_{(0)} \setminus T'_{\mathrm{reg}}} \mathrm{CH}_0^F(X_s)_0$ for all $m \gg 0$.

To prove the claim, we first note that the composite map is anyway zero if $s \notin T'$. For points in T', we project the horizontal arrows in (3.20) to closed points in the finite set T'_{sing}, which gives us a commutative diagram

$$\begin{tikzcd} V(Y_t, m, T') \arrow[r, "\partial"] \arrow[d, "\cong"'] & \bigoplus_{s \in T'_{\mathrm{sing}}} \mathrm{CH}_0^F(X_s)_0/m \arrow[d, "\gamma'_*"] \\ V(Y_t, m, S') \arrow[r, "\partial_{X',t}"] & \bigoplus_{s \in T'_{\mathrm{sing}}} \mathrm{CH}_0^F(X_s)_0/m \end{tikzcd} \tag{3.21}$$

for any $m \geq 1$.

It follows from (3.10) that the top horizontal arrow in (3.21) is zero for all $m \gg 0$. Hence, the same holds for the bottom horizontal arrow. To finish the proof of the claim, it suffices therefore to show that $\mathrm{CH}_0^F(X_s)_0$ is a finite group for every $s \in T'_{\mathrm{sing}}$. But this follows because X_s is a smooth projective curve over $k(s)$ and hence $\mathrm{CH}_0^F(X_s)_0 \cong J(X_s)(k(s))$, and the latter group is finite.

It follows from the claim that the bottom horizontal arrow in (3.20) factors through $V(X_t, m, S') \xrightarrow{\partial_{X',t}} \bigoplus_{s \in (T'_{\mathrm{reg}})_{(0)}} \mathrm{CH}_0^F(X_s)_0$ for all $m \gg 0$. Hence, the theorem is equivalent to showing that the composition

$$V(Y_t, m, S') \xrightarrow{\partial} \bigoplus_{s \in (T'_{\mathrm{reg}})_{(0)}} \mathrm{CH}_0^F(X_s)_0 \to C_{KS}(X, D)_0$$

is zero for all $m \gg 0$.

For this, we consider the commutative diagram

$$\begin{tikzcd} V(Y_t, m, T') \arrow[r, "\partial"] \arrow[d, "\cong"'] & \bigoplus_{s \in (T'_{\mathrm{reg}})_{(0)}} \mathrm{CH}_0^F(X_s)_0 \arrow[r] \arrow[d, "\gamma'_*"] & C_{KS}(Y, E)_0 \arrow[d, "\cong"', "\gamma'_*"] \\ V(Y_t, m, S') \arrow[r, "\partial"] & \bigoplus_{s \in (T'_{\mathrm{reg}})_{(0)}} \mathrm{CH}_0^F(X_s)_0 \arrow[r] & C_{KS}(X, D)_0 \end{tikzcd} \tag{3.22}$$

for all $m \gg 0$, where the right square commutes by our first claim that (3.19) is commutative. But the top composite arrow is zero for all $m \gg 0$ by Corollary 3.3. We conclude that the same holds for the bottom composite arrow. This finishes the proof of the theorem. $\square$

4. The finiteness theorem

Let k be a finite field and X an integral projective scheme of dimension $d \geq 1$ over k. Let $D \subset X$ be a nowhere dense closed subscheme. Let K denote the function field of X. Our goal in this section is to prove the finiteness of $C_{KS}(X, D)_0$. This will be done in several steps beginning with the case of curves.

4.1. The case of curves

Suppose first that X is a curve. In this case, we have an exact sequence

$$H^0_{\mathrm{nis}}(D, \mathcal{O}_D^\times) \to C_{KS}(X, D)_0 \to C_{KS}(X)_0 \to 0.$$

The first term is clearly finite. So we need to see that the third term is finite. We do this by comparing with the normalization $f\colon X_n \to X$. We have an exact sequence

$$H^0_{\mathrm{nis}}(Y, \mathcal{O}_Y^\times) \to C_{KS}(X)_0 \to C_{KS}(X_n)_0 \to 0$$

by Lemma 2.4, where $Y \subset X_n$ is a finite closed subscheme. We are therefore reduced to proving that $C_{KS}(X)_0$ is finite when X is a smooth projective curve over a finite field. But this is classical as $C_{KS}(X)_0 = J(k)$, where J is the Jacobian variety of X.

4.2. Two general reduction steps

The following are two reduction steps which apply without special condition on X.

Lemma 4.1. *Assume that $C_{KS}(X, D)_0$ is finite when X is normal. Then the same holds for arbitrary X.*

Proof. Let $f\colon X_n \to X$ be the normalization map. Then, it follows from [22, Proposition 4.2] that there is a nowhere dense closed subscheme $D' \subset X_n$ such that $D'_{\mathrm{red}} \subset f^{-1}(X_{\mathrm{sing}} \cup D_{\mathrm{red}})$ and $f^*(D) \subset D'$. Moreover, the norm map between the Milnor K-theory of the residue fields of the maximal Parshin chains defines a norm map between the Nisnevich sheaves

$$N_{X_n/X}\colon f_*(\mathcal{K}^M_{d,(X_n,D')}) \to \mathcal{K}^M_{d,(X,D)}. \tag{4.1}$$

Taking the cohomology, this defines a push-forward map $f_*\colon C_{KS}(X_n, D') \to C_{KS}(X, D)$.

By construction, the above map has the property that for a regular closed point $x \in X_n \setminus D'$ such that $f(x) \in X_{\mathrm{reg}}$, the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cong"] & C^{x}_{KS}(X_n, D') \arrow[r] \arrow[d, "\cong"', "f_*"] & C_{KS}(X_n, D') \arrow[d, "f_*"] \\
& C^{f(x)}_{KS}(X, D) \arrow[r] & C_{KS}(X, D)
\end{tikzcd} \tag{4.2}$$

is commutative.

We conclude from [22, Theorem 2.5] that $f_* \colon C_{KS}(X_n, D') \to C_{KS}(X, D)$ is surjective. In particular, it is surjective on the degree zero subgroups by Lemma 2.2. □

Remark 4.2. The proof of Lemma 4.1 yields more than what its statement asserts. Namely, the map $f_* \colon C_{KS}(X_n, D') \to C_{KS}(X, D)$ is surjective even if X is not projective. This surjectivity will be used in the proof of Corollary 4.10.

Lemma 4.3. *Assume that $C_{KS}(X, D)_0$ is finite when D_{red} is an effective Weil divisor whose complement is regular. Then the same holds for arbitrary D.*

Proof. Given any irreducible component of $D_{\mathrm{red}} \cup X_{\mathrm{sing}}$, we can find an irreducible prime divisor of X containing this component. This implies that there is a reduced Weil divisor E (with reduced induced closed subscheme structure) containing $D_{\mathrm{red}} \cup X_{\mathrm{sing}}$. But this implies that $D \subset mE$ for all $m \gg 0$. Since E is nowhere dense, the canonical map $C_{KS}(X, mE) \to C_{KS}(X, D)$ is surjective for all $m \gg 0$. Hence, it is surjective on the degree zero subgroups. We are therefore done. □

4.3. The case of generic fibration

Let X and D be as above with $d \geq 2$. Let S be an integral projective scheme of dimension $d-1$ over k and $f \colon X \to S$ a dominant morphism with the following properties:

(1) $f_*(\mathcal{O}_X) \cong \mathcal{O}_S$;
(2) There is a regular dense open $S' \subset S$ such that the restriction $f \colon f^{-1}(S') \to S'$ is smooth of relative dimension one;
(3) The fibers of $f \colon f^{-1}(S') \to S'$ are integral;
(4) The map $f \colon f^{-1}(S') \to S'$ has a section $\iota : S' \hookrightarrow f^{-1}(S')$;
(5) $D \cap f^{-1}(S') = \emptyset$.

Definition 4.4. A morphism $f \colon X \to S$ which satisfies the above properties will be called a generic fibration of relative dimension one.

We shall now prove the finiteness theorem when X admits a generic fibration of relative dimension one over a $(d-1)$-dimensional integral scheme. We shall prove this using Theorem 3.6, an exact sequence of Bloch [7, Theorem 4.2] and its generalization by Kato-Saito [22, Theorem 5.4].

Lemma 4.5. *Suppose that there exists a generic fibration $f\colon X \to S$ of relative dimension one. Then $C_{KS}(X,D)_0$ is finite.*

Proof. We shall follow the notations of the definition of generic fibration of relative dimension one. We let $\eta \in S$ be the generic point. We let $\widetilde{S} \subset X$ be the scheme theoretic closure of $\iota(S')$ in X. Note that $\widetilde{S}$ is an integral scheme such that $f|_{\widetilde{S}}\colon \widetilde{S} \to S$ is projective and birational. We let $X' = f^{-1}(S')$ and write $\widetilde{S} \cap X'$ as S'. We let $\iota\colon \widetilde{S} \hookrightarrow X$ denote the inclusion. By [22, Proposition 2.9, Lemma 2.10], we can find a closed subscheme $\widetilde{D} \subset \widetilde{S}$ containing $\iota^*(D)$ such that $\widetilde{D}_{\mathrm{red}} = (\iota^*(D))_{\mathrm{red}}$ and the Kato-Saito recipe gives a push-forward map $\iota_*\colon C_{KS}(\widetilde{S},\widetilde{D}) \to C_{KS}(X,D)$.

We now consider the commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{x \in S'_{(0)}} \mathbb{Z} \arrow[r, equal] \arrow[d, hook] & \bigoplus\limits_{x \in S'_{(0)}} \mathbb{Z} \arrow[r, two heads] \arrow[d, hook, "\iota_*"] & C_{KS}(\widetilde{S},\widetilde{D}) \arrow[d, "\iota_*"] \\
\bigoplus\limits_{x \in X'_{(0)}} \mathbb{Z} \arrow[r] & \bigoplus\limits_{x \in S'_{(0)}} \mathrm{CH}_0^F(X_s) \arrow[r, two heads] \arrow[d] & C_{KS}(X,D) \arrow[d] \\
 & M \arrow[r, two heads] \arrow[d] & N \arrow[d] \\
 & 0 & 0,
\end{tikzcd} \tag{4.3}$$

where M and N are defined so that the middle and the right columns are exact. The horizontal arrows on the right side are given by Lemma 2.1 and are surjective by [22, Theorem 2.5]. Note also that since $\mathrm{CH}_0^F(X_s)_0 = \mathrm{Ker}(\mathrm{CH}_0^F(X_s)_0 \xrightarrow{f_*} \mathrm{CH}_0^F(\mathrm{Spec}\,(k(s)))$, it follows that $M = \bigoplus\limits_{x \in S'_0} M_s$ is such that the composition

$$\mathrm{CH}_0^F(X_s)_0 \hookrightarrow \mathrm{CH}_0^F(X_s) \to M_s$$

is an isomorphism for all $s \in S'_{(0)}$. Hence, we can identify M with $\bigoplus\limits_{x \in S'_0} \mathrm{CH}_0^F(X_s)_0$.

Next, it follows from Lemma 2.2 that the right column in (4.3) induces an exact sequence

$$C_{KS}(\widetilde{S},\widetilde{D})_0 \xrightarrow{\iota_*} C_{KS}(X,D)_0 \to N \tag{4.4}$$

By induction on $\dim(X)$, our task therefore remains to show that N is finite. Using the above description of M and the surjection $M \twoheadrightarrow N$, it suffices to show that the composite map

$$\bigoplus_{x \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \hookrightarrow \bigoplus_{x \in S'_{(0)}} \mathrm{CH}_0^F(X_s) \to C_{KS}(X, D) \to N \tag{4.5}$$

factors through a finite quotient.

We now choose a point $t \in S'_{(1)}$ and let $T \subset S$ be its scheme theoretic closure. Then $T \subset S$ is an integral curve whose generic point lies in S'. We let $\gamma\colon T \hookrightarrow S$ denote the inclusion map and let $T' = S' \cap T$. We let $Y' = X' \times_S T'$. Then the restriction $Y' \to T'$ of f is a smooth projective morphism with integral fibers of dimension one. This implies that Y' must be integral. We let $Y \subset X$ be the scheme theoretic closure of Y' in X. This gives us a commutative square of integral projective schemes

$$\begin{array}{ccc} Y & \xrightarrow{\gamma'} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\gamma} & S. \end{array} \tag{4.6}$$

It follows from Theorem 3.6 that the composite map

$$V(Y_t, m_t, S') \xrightarrow{\partial_{X',t}} \bigoplus_{s \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \to C_{KS}(X, D)_0 \tag{4.7}$$

is zero for all $m_t \gg 0$. We write $V(Y_t, m_t, S')$ as $V(X_t, m_t, S')$ (note that $Y_t = X_t$ for any $t \in S'_{(1)}$) and let

$$\partial_{X'}\colon \bigoplus_{t \in S'_{(1)}} V(X_t, m_t, S') \to \bigoplus_{x \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0$$

denote the sum of the boundary maps $\partial_{X',t}$ over the points $t \in S'_{(1)}$.

We now let $\pi_1^{\mathrm{ab}}(X_\eta)_0 := \mathrm{Ker}(\pi_1^{\mathrm{ab}}(X_\eta) \xrightarrow{f_*} \pi_1^{\mathrm{ab}}(\mathrm{Spec}\,(k(\eta))))$. It follows from a theorem of Katz and Lang [23] that $\pi_1^{\mathrm{ab}}(X_\eta)_0$ is finite. By [22, Theorem 5.4] (the exact sequence of Bloch and Kato-Saito), there exists an exact sequence

$$\bigoplus_{t \in S'_{(1)}} V(X_t, m_t, S') \xrightarrow{\partial_{X'}} \bigoplus_{x \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \to \pi_1^{\mathrm{ab}}(X_\eta)_0 \to 0, \tag{4.8}$$

where for every $t \in S'_{(1)}$, we can take m_t to be any nonzero integer which annihilates $\pi_1^{\mathrm{ab}}(X_\eta)_0$.

We saw above in (4.7) and in § 3 that if for $t \in S'_{(1)}$, we choose m_t large enough such that m_t annihilates the resulting $C_{KS}(Y, E)_0$, $\bigoplus_{t \in T'_{\mathrm{sing}}} \mathrm{CH}_0^F(X_s)_0$ and $\pi_1^{\mathrm{ab}}(X_\eta)_0$, then the forget support map

$$\bigoplus_{x \in S'_{(0)}} \mathrm{CH}_0^F(X_s)_0 \to C_{KS}(X, D)$$

annihilates the image of $\partial_{X'}$. It follows that the composite map in (4.5) factors through the finite group $\pi_1^{\mathrm{ab}}(X_\eta)_0$. We have therefore proven the lemma. □

4.4. The case of schemes fibered by curves

We shall say that a general fiber of a morphism between schemes has property $\mathcal{P}$ if all (scheme theoretic) fibers over a dense open subset of the base have property $\mathcal{P}$. Let $f: X \to S$ be a dominant projective morphism between integral schemes over k whose generic fiber has dimension one. Let F be the function field of S. We shall say that f is 'nice' if the following hold:

(1) X and S are normal;
(2) f is generically smooth;
(3) $f_*\mathcal{O}_X = \mathcal{O}_S$;
(4) All fibers of f are geometrically connected;
(5) General fibers of f are geometrically integral of dimension one.

For a finite field extension $F \hookrightarrow F'$, let S' denote the normalization of S in F' and let X' denote the normalization of X in the compositum KF' inside an algebraic closure of K. Then f induces a projective morphism $f': X' \to S'$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\psi} & S \end{array} \tag{4.9}$$

is commutative. The horizontal arrows are finite dominant and vertical arrows are projective dominant morphisms between integral schemes.

Lemma 4.6. *If $f: X \to S$ is nice, then so is $f': X' \to S'$.*

Proof. The property (1) follows by construction. We let $Y = X_F$. Then Y is a smooth projective geometrically integral curve over F. This implies that $Y_{F'}$ also has the same property. In particular, $Y_{F'}$ coincides with the normalization of Y in the composite field KF'. Equivalently, $Y_{F'}$ is the generic fiber of the map $f': X' \to S'$. This proves (2).

Since we showed above that the generic fiber $X'_{F'}$ of f' is geometrically reduced and connected, it follows that $H^0(X'_{F'}, \mathcal{O}_{X'_{F'}}) = F'$ (see [9, Lemma 33.9.3]). An easy application of Stein factorization tells us that $f_*\mathcal{O}_{X'} = \mathcal{O}_{S'}$ (e.g., see [9, Lemma 37.49.6]). In particular, all fibers of f' are geometrically connected. This proves (3) and (4). The first part of (5) now easily follows from [9, Lemmas 37.24.4, 37.25.5]. The second part is an easy consequence of the fact that f is dominant and hence flat over a dense open subset of S by the generic flatness. □

Lemma 4.7. *We can choose F' to be a purely inseparable extension such that the resulting map $f' \colon X' \to S'$ is nice.*

Proof. We let $S_n \to S$ be the normalization morphism and let Y denote the normalization of X. Then Y is a normal integral scheme and f induces a projective dominant map $f_1 \colon Y \to S_n$. Since the generic fiber of f_1 is a projective limit of open subschemes of Y and the latter is normal, it follows that Y_F is integral and normal. Since $\dim(Y_F) = 1$, it follows that Y_F is an integral regular curve over F.

By [9, Lemma 33.27.3], there exists a finite purely inseparable field extension $F \hookrightarrow F'$ such that the normalization Z of $(Y_F \otimes_F F')_{\mathrm{red}}$ is geometrically normal. Since $\dim(Y_F) = 1$, it follows that Z is geometrically regular. In particular, it is geometrically reduced. Since $Y_F \otimes_F F' \to Y_F$ is a base change of the radicial morphism $\mathrm{Spec}\,(F') \to \mathrm{Spec}\,(F)$, it is also radicial. It follows that $Y_F \otimes_F F'$ is irreducible. We conclude that Z is an integral and geometrically regular (equivalently, smooth) projective curve over F'.

We let S' denote the normalization of S in F' and let X' be the normalization of Y in the composite field KF'. Let $f' \colon X' \to S'$ be the induced morphism. Since Y_F is a projective limit of open subschemes of Y, it follows that the generic fiber of f' is the normalization Z' of Y_F in KF'. On the other hand, it is easily seen that the canonical map $Z \to Z'$ is a birational morphism between normal projective curves over F' (with function field KF'), and hence is an isomorphism.

Since X' is same as the normalization of X in the composite field KF', we conclude that there exists a commutative square such as (4.9) for which the generic fiber of f' is a smooth projective curve integral over the function field F' of S'. We have thus shown (1) and (2). The proof of (3), (4) and (5) is identical to the proof of the second part of Lemma 4.6. □

Lemma 4.8. *Let $f \colon X \to S$ be a dominant projective morphism between integral schemes over k whose generic fiber has dimension one. Then $C_{KS}(X, D)_0$ is finite.*

Proof. Let F denote the function field of S. By Lemma 4.7, there is a purely inseparable finite extension $F \subset F'$ such that the resulting map $f' \colon X' \to S'$ is nice. By [22, Proposition 4.2], there is a nowhere dense closed subscheme $D' \subset X'$ containing $\phi^*(D)$ (see (4.9)) and a push-forward map $\phi_* \colon C_{KS}(X', D') \to C_{KS}(X, D)$. This map is degree preserving by Lemma 2.2. It follows from [22, Corollary 4.10] that $\phi_* \colon C_{KS}(X', D') \to C_{KS}(X, D)$ is surjective. Hence, it is surjective on the degree zero subgroups. It suffices therefore to prove the lemma when $f \colon X \to S$ is nice. We shall assume this to be the case in the rest of the proof and divide the proof into various sub-cases. We let Y denote the generic fiber of f.

We first consider the case when $D \cap Y = \emptyset$ and $Y(F) \neq \emptyset$. Since f is projective, it follows that the scheme theoretic image of D in S is a nowhere dense closed subscheme.

We can therefore find an open dense subscheme $S' \subset S$ such that $D \cap f^{-1}(S') = \emptyset$. Since Y is smooth over F and $Y(F) \neq \emptyset$, we can assume (after shrinking S' if necessary) that f is smooth over S' and has a section over S'. It follows that $f\colon X \to S$ is a generic fibration of relative dimension one. We conclude the finiteness of $C_{KS}(X,D)_0$ by Lemma 4.5.

We now consider the case when $D \cap Y = \emptyset$. We can find a finite Galois extension $F \subset F'$ such that the resulting map $f'\colon X' \to S'$ has the property that $Y'(F') \neq \emptyset$, where Y' is the generic fiber of f'. It follows from Lemma 4.6 that f' is also nice. Since X is normal, there exists a push-forward map $\phi_*\colon C_{KS}(X',D') \to C_{KS}(X,D)$ where we can take $D' = \phi^*(D)$. It follows from [22, Corollary 4.10] that $\mathrm{Coker}(\phi_*)$ is finite. Since ϕ_* is degree preserving by Lemma 2.2, it follows that

$$C_{KS}(X',D')_0 \xrightarrow{\phi_*} C_{KS}(X,D)_0 \to \mathrm{Coker}(\phi_*) \tag{4.10}$$

is exact. Since we have shown above that $C_{KS}(X',D')_0$ is finite, it follows that so is $C_{KS}(X,D)_0$.

We now consider the remaining case $D \cap Y \neq \emptyset$. We let $T = \{x \in Y^{(1)} | \overline{\{x\}} \subset D\}$, where the closure of $\{x\}$ is taken in X. It is clear that T is a finite set. We can now choose a closed subscheme $D' \subset D$ such that

$$\mathcal{I}_{D'}\mathcal{O}_{X,x} = \mathcal{I}_D\mathcal{O}_{X,x} \ \text{ for } \ x \in X^{(1)} \setminus T \ \text{ and } \ D' \cap Y = \emptyset. \tag{4.11}$$

It follows from the choice of D' that $\mathcal{I}_{D'}\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ for every $x \in T$. By [22, Theorem 8.3, Proposition 8.4, Corollary 8.5], there exists an exact sequence

$$\bigoplus_{x \in T} C(x, i_x) \to C_{KS}(X,D) \to C_{KS}(X,D') \to 0, \tag{4.12}$$

where $i_x \gg 0$ is an integer and $C(x, i_x)$ is finite for each $x \in T$. We remark here that this is shown in [22, Proposition 8.4] when T is a singleton. However, an easy induction on the cardinality of T yields the general case.

Taking the degree zero parts, we get an exact sequence

$$\bigoplus_{x \in T} C(x, i_x) \to C_{KS}(X,D)_0 \to C_{KS}(X,D')_0 \to 0. \tag{4.13}$$

We have shown earlier that $C_{KS}(X,D')_0$ is finite and this concludes the proof. $\square$

4.5. The general case of finiteness theorem

Let k be a finite field. Let X be a projective integral scheme in $\mathbf{Sch}_k$ of dimension $d \geq 1$ and $D \subset X$ a nowhere dense closed subscheme. Let K be the function field of X. We shall now prove the finiteness of $C_{KS}(X,D)_0$ in general. The result is the following.

Theorem 4.9. *The degree zero idele class group $C_{KS}(X,D)_0$ is finite. In particular, $C(X,D)_0$ is finite if $X \setminus D$ is regular.*

Proof. We only need to prove the finiteness of $C_{KS}(X,D)_0$ in view of [18, Theorem 3.8] (see (2.10)) and Lemma 2.5.

First of all, we can assume that X is normal by Lemma 4.1. Next, by enlarging D if necessary, we can assume that its complement is regular by Lemma 4.3. We let $Y = D_{\mathrm{red}}$ with reduced closed subscheme structure and let $V = X \setminus Y$. Note that $X_{\mathrm{sing}} \subset Y$. We can now find a finite field extension $k \hookrightarrow k'$ such that if X' is the normalization of X in the composite field Kk' and if Y' is the inverse image of Y under the projection $\phi\colon X' \to X$, then there is a blow-up $g\colon X'' \to X'$ with center away from Y' and a dominant projective morphism $f\colon X'' \to \mathbb{P}^{d-1}_{k'}$. This is proven in [22, Lemma 9.5] if k is infinite, where one can take $k' = k$. The finite field case is easily reduced to the infinite field because we can get $g\colon X'' \to X'$ and $f\colon X'' \to \mathbb{P}^{d-1}_{E}$, where E is a pro-ℓ extension of k for some prime ℓ. Any such construction is then actually defined over a finite extension k' of k.

Note here that the push-forward $\phi_*\colon C_{KS}(X',D') \to C_{KS}(X,D)$ of (4.10) exists if we let $D' = f^*(D)$ because X is normal (see [22, Proposition 4.2]). By [22, Corollary 4.10], the cokernel of ϕ_* is finite, so it suffices to prove the theorem for X'. We can therefore assume that there is a blow-up $g\colon \widetilde{X} \to X$ with center away from Y and a dominant projective morphism $f\colon \widetilde{X} \to \mathbb{P}^{d-1}_{k}$. We let $U \subset X$ be an open dense subscheme over which g is an isomorphism. Then $Y \subset U$.

By [22, Lemma 9.6], the Kato-Saito recipe gives a push-forward map $g_*\colon C_{KS}(\widetilde{X},\widetilde{D}) \to C_{KS}(X,D)$ for any nowhere dense closed subscheme $\widetilde{D} \subset \widetilde{X}$ containing $g^*(D)$. The map g_* has the property that for any closed point $x \in g^{-1}(U_{\mathrm{reg}}) \setminus \widetilde{D}$, the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & C^x_{KS}(\widetilde{X},\widetilde{D}) \arrow[r] \arrow[d, "g_*"', "\cong"] & C_{KS}(\widetilde{X},\widetilde{D}) \arrow[d, "g_*"] \\
& C^x_{KS}(X,D) \arrow[r] & C_{KS}(X,D)
\end{tikzcd} \tag{4.14}$$

commutes where the horizontal arrows are the forget support maps.

It follows from this diagram and [22, Theorem 2.5] that g_* is surjective. Since g_* is degree preserving by Lemma 2.2, this implies that $g_*\colon C_{KS}(\widetilde{X},\widetilde{D})_0 \to C_{KS}(X,D)_0$ is also surjective. It suffices therefore to show that $C_{KS}(\widetilde{X},\widetilde{D})_0$ is finite. But this follows from Lemma 4.8 because f is a dominant morphism between integral schemes of relative dimension one and hence its generic fiber has dimension one. This concludes the proof of the theorem. □

As a consequence of Theorem 4.9, we obtain the following new finiteness theorem for the Kato-Saito idele class group of non-projective schemes.

Corollary 4.10. *Let X be an integral quasi-projective scheme of dimension $d \geq 1$ over a finite field and $D \subset X$ a nowhere dense closed subscheme. Assume that X is not projective over k. Then $C_{KS}(X, D)$ is finite.*

Proof. Thanks to Remark 4.2, we can assume that X is normal. We can now find an open immersion $j: X \hookrightarrow \overline{X}$ such that $\overline{X}$ is an integral and normal projective scheme. By blowing up the complement of X (with the reduced closed subscheme structure) in $\overline{X}$ and normalizing again, we can furthermore assume that $\dim(\overline{X} \setminus X) = d - 1$. We let $Y = \overline{X} \setminus X$ with reduced closed subscheme structure.

We let $\overline{D} \subset \overline{X}$ be the scheme theoretic closure of D in $\overline{X}$. We let $Z = \overline{D} \cup \overline{X}_{\rm sing}$. As D is a dense open subscheme of $\overline{D}$, it follows that $\dim(\overline{D} \setminus D) = \dim(Y \cap \overline{D}) \leq d - 2$. Since $\overline{X}$ is normal, we must also have $\dim(Y \cap Z) \leq d - 2$. Since $\dim(Y) = d - 1$, we find that $Y \setminus Z \neq \emptyset$. We choose a closed point $x \in Y \setminus Z$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^Y_{KS}(\overline{X}, \overline{D})_0 \arrow[r] \arrow[d] & C^Y_{KS}(\overline{X}, \overline{D}) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \arrow[d, equal] \\
0 \arrow[r] & C_{KS}(\overline{X}, \overline{D})_0 \arrow[r] \arrow[d] & C_{KS}(\overline{X}, \overline{D}) \arrow[r, "\deg"] \arrow[d, two heads] & \mathbb{Z} \\
& C_{KS}(X, D) \arrow[r, equal] & C_{KS}(X, D). &
\end{tikzcd} \tag{4.15}$$

All rows as well as the middle column of this diagram are exact. The right vertical arrow on the bottom is surjective by [22, Theorem 2.5]. It follows from Lemma 2.1 that $\deg(\lambda_x(1)) \neq 0$. In particular, the cokernel of composite map $C^Y_{KS}(\overline{X}, \overline{D}) \to C_{KS}(\overline{X}, \overline{D}) \to \mathbb{Z}$ is finite. Since $C_{KS}(\overline{X}, \overline{D})_0$ is finite by Theorem 4.9, a diagram chase shows that $C_{KS}(X, D)$ must be finite. □

5. The reciprocity theorem

In this section, we shall prove our main reciprocity theorem. We begin by recalling the fundamental groups with modulus and the reciprocity map.

5.1. Recollection of fundamental groups with modulus

Let K be a Henselian discrete valuation field. Let $G_K^{(\bullet)}$ be the Abbes-Saito filtration of G_K (see [1] or [18, § 6.1]). Let L/K be a finite separable extension and $n \geq 0$ an integer. Recall from [18, § 7.1] that the ramification of L/K is said to be bounded by n if $G_K^{(n)}$ is contained in $\mathrm{Gal}(\overline{K}/L)$ under the inclusions $G_K^{(n)} \subset G_K \supset \mathrm{Gal}(\overline{K}/L)$, where $\overline{K}$ denotes a fixed separable closure of K.

Let k be a field and $X \in \mathbf{Sch}_k$ an integral normal scheme of dimension $d \geq 1$. Let $D \subset X$ be an effective Weil divisor and C the support of D with reduced closed subscheme structure. Set $U = X \setminus C$. Let K denote the function field of X. For any generic point λ of C, let K_λ denote the Henselization of K along λ. Recall from [18, Definition 7.5] that the co-1-skeleton (or divisorial) étale fundamental group with modulus $\pi_1^{\mathrm{adiv}}(X, D)$ is a quotient of $\pi_1^{\mathrm{ab}}(U)$ which classifies finite abelian covers $f \colon U' \to U$ having the property that for every generic point λ of C and every point $\lambda' \in f^{-1}(\lambda)$, the extension of fields $K_\lambda \hookrightarrow K'_{\lambda'}$ has ramification bounded by n_λ. Here, X' is the normalization of X in $K' = k(U')$ and $f \colon X' \to X$ is the resulting map. We thus have continuous surjective homomorphisms of profinite groups

$$G_{k(X)}^{\mathrm{ab}} \twoheadrightarrow \pi_1^{\mathrm{ab}}(U) \overset{q_{X|D}}{\twoheadrightarrow} \pi_1^{\mathrm{adiv}}(X, D) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X).$$

We let $\pi_1^{\mathrm{ab}}(U)_0$ be the kernel of the composite continuous homomorphism $\deg' \colon \pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X) \to G_k$. We define $\pi_1^{\mathrm{adiv}}(X, D)_0$ similarly.

For any field L, let us write $H^1(L) = H^1_{\mathrm{et}}(L, \mathbb{Q}/\mathbb{Z})$ and let $\mathrm{fil}^{\mathrm{ms}}_\bullet H^1(L)$ denote the Matsuda filtration of $H^1(L)$ if L is a Henselian discrete valuation field (see [18, § 6.2]). Recall the following from [18, § 7.5]. We write $D = \sum_{x \in X^{(1)}} n_x \overline{\{x\}}$ as an element of the group of Weil divisors $\mathrm{Div}(X)$. Let $\mathrm{Div}_C(X)$ denote the set of closed subschemes of X of pure codimension one whose support is C. This is a directed set with respect to inclusion. Note also that every $D' \in \mathrm{Div}_C(X)$ defines a unique effective Weil divisor on X, which we shall also denote by D'.

Definition 5.1. Let $\mathrm{fil}_D H^1(K)$ denote the subgroup of characters $\chi \in H^1(K)$ such that for every $x \in X^{(1)}$, the image χ_x of χ under the canonical map $H^1(K) \to H^1(K_x)$ lies in $\mathrm{fil}^{\mathrm{ms}}_{n_x} H^1(K_x)$.

Let $\mathrm{fil}^c_D H^1(K)$ be the subgroup of characters $\chi \in H^1(U)$ such that for every integral curve $Y \subset X$ not contained in D and normalization Y_n, the finite map $\nu \colon Y_n \to X$ has the property that the image of χ under $f^* \colon H^1(U) \to H^1(\nu^{-1}(U))$ lies in $\mathrm{fil}_{\nu^*(D)} H^1(k(Y))$.

Since $n_x = 0$ for all $x \in U^{(1)}$, it follows from various properties of the Matsuda filtration (see [18, §6]) that $\mathrm{fil}_D H^1(K)$ lies inside $H^1(U)$ under the canonical inclusion $H^1(U) \hookrightarrow H^1(K)$. We shall therefore write $\mathrm{fil}_D H^1(K)$ also as $\mathrm{fil}_D H^1(U)$. We shall use the following properties of $\mathrm{fil}_D H^1(K)$. For a profinite (or discrete) group G, let $G^\vee$ denote the Pontryagin dual of G (see [18, § 7.4]).

Proposition 5.2. *The canonical map* $\varinjlim_{D' \in \mathrm{Div}_C(X)} \mathrm{fil}_{D'} H^1(K) \to H^1(U)$ *is an isomorphism. The canonical homomorphism of profinite groups* $\pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^{\mathrm{adiv}}(X, D)$ *defines an isomorphism of discrete groups* $(\pi_1^{\mathrm{adiv}}(X, D))^\vee \cong \mathrm{fil}_D H^1(K)$.

Proof. See [18, Proposition 7.13, Theorem 7.16]. □

Definition 5.3. We let $\pi_1^{\mathrm{ab}}(X, D)$ be the Pontryagin dual of the discrete group $\mathrm{fil}^{\mathrm{c}}_D H^1(K)$ and call it the 1-skeleton (or curve based) étale fundamental group with modulus of the pair (X, D). We let $\pi_1^{\mathrm{ab}}(X, D)_0$ be the kernel of the canonical continuous composite homomorphism $\deg' \colon \pi_1^{\mathrm{ab}}(X, D) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X) \to G_k$.

The fundamental group $\pi_1^{\mathrm{ab}}(X, D)$ was first considered by Deligne and Laumon [33] when k is finite to study ramifications at infinity of abelian coverings of smooth quasi-projective schemes. This group can be identified with the Pontryagin dual of the group of rank one lisse $\overline{\mathbb{Q}}_\ell$ sheaves on U with ramification bounded by D, a notion due to Deligne (see [11]).

It is clear from the definitions that there are surjective continuous homomorphisms of profinite abelian groups $\pi_1^{\mathrm{adiv}}(X, D) \overset{q_{X|D}}{\twoheadleftarrow} \pi_1^{\mathrm{ab}}(U) \overset{q'_{X|D}}{\twoheadrightarrow} \pi_1^{\mathrm{ab}}(X, D)$. Furthermore, we have the following analogue of Proposition 5.2 (see [11, Proposition 3.9] and [28, Proposition 2.10]).

Proposition 5.4. *The canonical map* $\pi_1^{\mathrm{ab}}(U) \to \varprojlim_{D' \in \mathrm{Div}_C(X)} \pi_1^{\mathrm{ab}}(X, D')$ *is an isomorphism if* k *is perfect.*

Since $\pi_1^{\mathrm{ab}}(U)_0$ is closed in $\pi_1^{\mathrm{ab}}(U)$, we get from the definition of the map $\deg'$, Propositions 5.2, 5.4 and [39, Corollary 1.1.8] the following.

Corollary 5.5. *Assume* k *is perfect. Then we have the isomorphisms of profinite groups*

$$\varprojlim_{D' \in \mathrm{Div}_C(X)} \pi_1^{\mathrm{adiv}}(X, D')_0 \overset{\cong}{\longleftarrow} \pi_1^{\mathrm{ab}}(U)_0 \overset{\cong}{\longrightarrow} \varprojlim_{D' \in \mathrm{Div}_C(X)} \pi_1^{\mathrm{ab}}(X, D')_0.$$

It is evident from the definitions that $\pi_1^{\mathrm{adiv}}(X, D) \cong \pi_1^{\mathrm{ab}}(X, D)$ if $d = 1$. On the other hand, it is not even clear a priori that there is any map in either direction between $\pi_1^{\mathrm{adiv}}(X, D)$ and $\pi_1^{\mathrm{ab}}(X, D)$ when $d \geq 2$. One of the central results of this paper is that these groups are in fact isomorphic for any $d \geq 1$ when X is regular and k is finite.

Remark 5.6. One can mimic the construction of [18, § 7] (where the ramification bound has to be redefined by restricting to curves) to show that there is a Galois category such that $\pi_1^{\mathrm{ab}}(X, D)$ is the abelianization of the automorphism group $\pi_1(X, D)$ of the associated fiber functor. But we shall not need this Tannakian interpretation of $\pi_1^{\mathrm{ab}}(X, D)$.

5.2. The reciprocity map for $C(X, D)$

We let $C_{U/X}$ denote the idele class group of $(U \subset X)$. Recall from [18, § 3.3] that this is the quotient of the direct sum $I_{U/X}$ of the Milnor K-groups of the residue fields of Parshin chains on $(U \subset X)$ by the boundaries of the Milnor K-groups of the residue fields of Q-chains on $(U \subset X)$. $C_{U/X}$ is a topological abelian group which has the quotient topology, induced by the canonical topology of the Milnor K-groups of the residue fields of Parshin chains and $C(X,D)$ has discrete topology. The main result of [18] is the following.

Theorem 5.7. *Assume that k is finite. Then there exist continuous reciprocity homomorphisms $\rho_{U/X}\colon C_{U/X} \to \pi_1^{\mathrm{ab}}(U)$ and $\rho_{X|D}\colon C(X,D) \to \pi_1^{\mathrm{adiv}}(X,D)$ such that the diagram*

$$\begin{array}{ccc} C_{U/X} & \xrightarrow{\rho_{U/X}} & \pi_1^{\mathrm{ab}}(U) \\ \downarrow & & \downarrow \\ C(X,D) & \xrightarrow{\rho_{X|D}} & \pi_1^{\mathrm{adiv}}(X,D) \end{array} \tag{5.1}$$

is commutative, where the vertical arrows are the canonical surjections.

The map $\rho_{U/X}$ has the property that it takes the canonical generator of $K_0^M(k(x))$ for a regular closed point $x \in U$ to the image of the Frobenius substitution at x.

Let $\widetilde{C}_{U/X} = \varprojlim_{|D|=X\setminus U} C(X,D)$ with the inverse limit topology. The limit of the degree maps $C(X,D) \to \mathbb{Z}$ defines a continuous homomorphism $\deg\colon \widetilde{C}_{U/X} \to \mathbb{Z}$. We let $(\widetilde{C}_{U/X})_0$ be the kernel of this map (see [18, Corollary 4.9]). It is clear that there is a canonical continuous homomorphism $C_{U/X} \to \widetilde{C}_{U/X}$. Moreover, Theorem 5.7 and [18, Corollary 7.17] together say that $\rho_{U/X}$ factors through a continuous reciprocity homomorphism $\widetilde{\rho}_{U/X}\colon \widetilde{C}_{U/X} \to \pi_1^{\mathrm{ab}}(U)$ which is the limit of the maps $\rho_{X|D}$.

5.3. Completions of idele class groups

Let k be any field and $X \in \mathbf{Sch}_k$ an integral projective scheme. Let $C \subset X$ be a nowhere dense closed reduced subscheme such that $U = X \setminus D$ is regular. We shall let $\mathbb{N}$ be the set of all positive integers. We shall say that $m_1 \le' m_2$ if m_1 divides m_2. This makes $\mathbb{N}$ a directed set. We let $I = \mathrm{Div}_C(X) \times \mathbb{N}$, where $\mathrm{Div}_C(X)$ is as in § 5.1, and say that $(D_1, m_1) \le (D_2, m_2)$ if $D_1 \subseteq D_2$ and $m_1 \le' m_2$. It is easy to see that I is a directed set with this coordinate-wise partial order. This partial order naturally makes I a diagram category. An inverse (resp. direct) system $\{G_{ij}\}_{(i,j)\in I}$ in a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from I to $\mathcal{C}$.

If $\mathcal{C}$ admits all small limits and $\{G_{ij}\}_{(i,j)\in I}$ is an inverse system in $\mathcal{C}$, then it is well known that there are canonical isomorphisms

$$\varprojlim_{i\in \mathrm{Div}_C(X)} \varprojlim_{j\in\mathbb{N}} G_{ij} \cong \varprojlim_{(i,j)\in I} G_{ij} \cong \varprojlim_{j\in\mathbb{N}} \varprojlim_{i\in \mathrm{Div}_C(X)} G_{ij}. \tag{5.2}$$

Similarly, if $\mathcal{C}$ admits all small colimits and $\{G_{ij}\}_{(i,j)\in I}$ is a direct system in $\mathcal{C}$, then there are canonical isomorphisms

$$\varinjlim_{i \in \mathrm{Div}_C(X)} \varinjlim_{j \in \mathbb{N}} G_{ij} \cong \varinjlim_{(i,j) \in I} G_{ij} \cong \varinjlim_{j \in \mathbb{N}} \varinjlim_{i \in \mathrm{Div}_C(X)} G_{ij}. \tag{5.3}$$

We shall apply the above machinery to the contravariant functor $I \to \mathbf{Ab}$ given by $(D, m) \mapsto C(X, D)/m$. We shall denote this by $\{C(X, D)/m\}$. Before we study the completions of this inverse system, we state the following crucial result.

Lemma 5.8. *For every $D \in \mathrm{Div}_C(X)$, the map $C(X, D)_0 \to C(X, D)_0/\infty$ is an isomorphism. In particular, the map $(\widetilde{C}_{U/X})_0 \to \varprojlim_{D,m} C(X, D)_0/m$ is an isomorphism.*

Proof. The first isomorphism is immediate from Theorem 4.9. The second isomorphism follows from the first isomorphism, [18, Corollary 4.9] and (5.2). $\square$

It follows from [18, Proposition 4.8] that the sequence

$$0 \to C(X, D)_0/m \to C(X, D)/m \xrightarrow{(\deg)/n} \mathbb{Z}/m \to 0 \tag{5.4}$$

is exact for every $(D, m) \in I$.

Lemma 5.9. *For every $m \in \mathbb{N}$, the canonical map $\widetilde{C}_{U/X}/m \to \varprojlim_{D} C(X, D)/m$ is an isomorphism.*

Proof. Let us denote the map of the lemma by θ_m. It follows from [18, Proposition 4.8], the Mittag-Leffler property of $\{C(X, D)_0/m\}$ and injectivity of $C(X, D)_0/m \to C(X, D)/m$ that there is a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (\widetilde{C}_{U/X})_0/m \arrow[r] \arrow[d] & \widetilde{C}_{U/X}/m \arrow[r, "(\deg)/n"] \arrow[d, "\theta_m"] & \mathbb{Z}/m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim_{D} C(X, D)_0/m \arrow[r] & \varprojlim_{D} C(X, D)/m \arrow[r, "(\deg)/n"] & \mathbb{Z}/m \arrow[r] & 0.
\end{tikzcd} \tag{5.5}$$

It suffices therefore to show that the left vertical arrow in this diagram is an isomorphism. It follows from [18, Corollary 4.9] and Theorem 4.9 that $(\widetilde{C}_{U/X})_0$ is profinite. Hence, the desired isomorphism follows from Lemma 5.12 below. $\square$

Since each $C(X, D)_0$ is finite by Theorem 4.9, it follows from [39, Lemma 1.1.5] that the pro-abelian group $\{\varprojlim_{D} C(X, D)_0/m\}_{m \in \mathbb{N}}$ is Mittag-Leffler. Using Lemma 5.8 and (5.5), we therefore conclude the following.

Corollary 5.10. *There exists a commutative diagram of short exact sequences of continuous homomorphisms of topological abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & (\widetilde{C}_{U/X})_0 \arrow[r] \arrow[d, equal] & \widetilde{C}_{U/X} \arrow[r, "(\deg)/n"] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (\widetilde{C}_{U/X})_0 \arrow[r] \arrow[d, "\theta_{U/X}"', "\cong"] & \widetilde{C}_{U/X}/\infty \arrow[r, "(\deg)/n"] \arrow[d, "\cong"', "\theta_{U/X}"] & \widehat{\mathbb{Z}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim_{D,m} C(X,D)_0/m \arrow[r] & \varprojlim_{D,m} C(X,D)/m \arrow[r, "(\deg)/n"] & \widehat{\mathbb{Z}} \arrow[r] & 0,
\end{tikzcd} \tag{5.6}$$

where the top vertical arrows are completion maps and the bottom vertical arrows are isomorphisms. In particular, $\widetilde{C}_{U/X}$ *is dense in the profinite group* $\widetilde{C}_{U/X}/\infty$ *and there is a short exact sequence*

$$0 \to \widetilde{C}_{U/X} \to \widetilde{C}_{U/X}/\infty \xrightarrow{(\deg)/n} \widehat{\mathbb{Z}}/\mathbb{Z} \to 0. \tag{5.7}$$

Proof. We only need to give an argument for density of $\widetilde{C}_{U/X}$ in $\widetilde{C}_{U/X}/\infty$ as everything else is clear from what we have shown above. But this density follows immediately from [18, Lemma 7.11] using a diagram chase in (5.6). $\square$

Corollary 5.11. *For every* $m \in \mathbb{N}$ *and* $D \in \mathrm{Div}_C(X)$*, the reciprocity map* $\rho_{U/X}$ *gives rise to a commutative diagram*

$$\begin{tikzcd}
 & & & C_{U/X} \arrow[llldd] \arrow[ldd] \arrow[ddd, "\rho_{U/X}"] \\
 & & & \\
\widetilde{C}_{U/X} \arrow[rr, hook] \arrow[rd, "\widetilde{\rho}_{U/X}"] \arrow[dddd, two heads, "p_{X|D}"'] & & \widetilde{C}_{U/X}/\infty \arrow[rd, "\rho^{\infty}_{U/X}", "\cong"'] \arrow[dddd, two heads, "p_{X|D}"'] & \\
 & \pi_1^{\mathrm{ab}}(U) \arrow[rr, equal] \arrow[dddd, two heads, "q_{X|D}"'] & & \pi_1^{\mathrm{ab}}(U) \arrow[dddd, two heads, "q_{X|D}"] \\
 & & & \\
 & & & \\
C(X,D) \arrow[rr, two heads] \arrow[rd, "\rho_{X|D}"'] & & C(X,D)/m \arrow[rd, "\rho_{X|D}/m"] & \\
 & \pi_1^{\mathrm{adiv}}(X,D) \arrow[rr, two heads] & & \pi_1^{\mathrm{adiv}}(X,D)/m.
\end{tikzcd} \tag{5.8}$$

Proof. The only thing we need to show is that $\widetilde{\rho}_{U/X}$ factors through $\rho^{\infty}_{U/X}$ and the latter is an isomorphism. Everything else is clear from the construction of $\rho_{U/X}$. Concerning the factorization, it suffices to show that the canonical map $\pi_1^{\mathrm{ab}}(U) \to \varprojlim_{D,m} \pi_1^{\mathrm{adiv}}(X,D)/m$ is an isomorphism. However, it follows from [18, Corollary 7.17] and Lemma 5.12 that the map $\pi_1^{\mathrm{ab}}(U)/m \to \varprojlim_D \pi_1^{\mathrm{adiv}}(X, D)/m$ is an isomorphism for every $m \in \mathbb{N}$. Taking the limit over $m \in \mathbb{N}$, using (5.2) and the known isomorphism $\pi_1^{\mathrm{ab}}(U) \xrightarrow{\cong} \varprojlim_m \pi_1^{\mathrm{ab}}(U)/m$, we get the desired isomorphism.

To see that $\rho_{U/X}^{\infty}$ is an isomorphism, we observe using [18, Lemmas 7.9, 8.3, Corollary 7.17] that $\rho_{U/X}^{\infty}$ coincides with the reciprocity map of [22, Theorem 9.1]. But the latter is an isomorphism. □

Lemma 5.12. *Let $G = \varprojlim_i G_i$ be the limit of an inverse system of compact Hausdorff topological abelian groups $\{G_i\}$ whose transition maps are surjective. Let $m \in \mathbb{N}$ be any integer. Then the canonical map $G/m \to \varprojlim_i G_i/m$ is an isomorphism.*

Proof. We have a short exact sequence of pro-abelian groups

$$0 \to \{mG_i\} \to \{G_i\} \to \{G_i/m\} \to 0. \tag{5.9}$$

Since $\{G_i\} \twoheadrightarrow \{mG_i\}$ is a surjective morphism of pro-abelian groups and $\{G_i\}$ is Mittag-Leffler, it follows that $\{mG_i\}$ is also Mittag-Leffler. In particular, ${\varprojlim_i}^1 mG_i = 0$. We thus get a short exact sequence of limits

$$0 \to \varprojlim_i mG_i \to G \to \varprojlim_i G_i/m \to 0. \tag{5.10}$$

We now consider the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & mG \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & G/m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim_i mG_i \arrow[r] & G \arrow[r] & \varprojlim_i G_i/m \arrow[r] & 0.
\end{tikzcd} \tag{5.11}$$

Since the transition maps of $\{G_i\}$ are surjective, it follows that $G \twoheadrightarrow G_i$ for each i. Hence, $mG \twoheadrightarrow mG_i$ for every i. This implies from [39, Corollary 1.1.8] that $\overline{mG} = \varprojlim_i mG_i$, where $\overline{mG}$ denotes the closure of mG in G. On the other hand, the multiplication map $G \xrightarrow{m} G$ is continuous as G is a topological abelian group. Since each G_i is compact Hausdorff, it follows that G is compact Hausdorff. Hence, mG must be closed in G. This implies that the left vertical arrow in (5.11) is an isomorphism. A diagram chase now shows that the right vertical arrow is also an isomorphism, as desired. □

5.4. A result from local ramification theory

We need one more ingredient to prove Theorem 1.3. This is about a property of the ramification filtrations. Let K be a Henselian discrete valuation field with ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$ and residue field $\mathfrak{f}$. We have the inclusions $K \hookrightarrow K^{sh} \hookrightarrow \overline{K}$, where $\overline{K}$ is a fixed separable closure of K and K^{sh} is the strict Henselization of K. Let $\mathrm{fil}^{\mathrm{ms}}_{\bullet} H^1(K)$ denote the Matsuda filtration of $H^1(K) := H^1_{\mathrm{et}}(K, \mathbb{Q}/\mathbb{Z})$ (see [18, § 6.2]).

Proposition 5.13. *There is a short exact sequence*

$$0 \to \mathrm{fil}^{\mathrm{ms}}_0 H^1(K) \to H^1(K) \to H^1(K^{sh}) \to 0.$$

For $m \geq 1$, the canonical square

$$\begin{array}{ccc} \mathrm{fil}^{\mathrm{ms}}_m H^1(K) & \longrightarrow & H^1(K) \\ \downarrow & & \downarrow \\ \mathrm{fil}^{\mathrm{ms}}_m H^1(K^{sh}) & \to & H^1(K^{sh}) \end{array}$$

is Cartesian.

Proof. Recall that K^{sh} is the quotient field of the strict Henselization $\mathcal{O}_K^{sh}$ of $\mathcal{O}_K$. Since $\mathcal{O}_K$ is Henselian, it is easy to see that K^{sh} is same as the maximal unramified extension of K inside $\overline{K}$. The exact sequence in the proposition now follows from a property of the Abbes-Saito filtration and [18, Theorem 6.1(1)].

We now prove that the square in the proposition is Cartesian. Since $\mathrm{fil}^{\mathrm{ms}}_{\bullet} H^1(K)$ is an exhaustive filtration of $H^1(K)$ by [18, Theorem 6.1(3)], it suffices to show that for every $m \geq 2$, the square

$$\begin{array}{ccc} \mathrm{fil}^{\mathrm{ms}}_{m-1} H^1(K) & \longrightarrow & \mathrm{fil}^{\mathrm{ms}}_m H^1(K) \\ \downarrow & & \downarrow \\ \mathrm{fil}^{\mathrm{ms}}_{m-1} H^1(K^{sh}) & \to & \mathrm{fil}^{\mathrm{ms}}_m H^1(K^{sh}) \end{array}$$

is Cartesian. Equivalently, it suffices to show that the map

$$\phi^*_m \colon \mathrm{gr}^{\mathrm{ms}}_m H^1(K) \to \mathrm{gr}^{\mathrm{ms}}_m H^1(K^{sh}),$$

induced by the inclusion $\phi \colon K \hookrightarrow K^{sh}$, is injective.

We first assume that either $p \neq 2$ or $m \neq 2$. In this case, it follows from [35, Proposition 3.2.3] that the refined Artin conductor

$$\mathrm{rar}_K \colon \mathrm{gr}^{\mathrm{ms}}_m H^1(K) \to \frac{\mathfrak{m}_K^{-m}}{\mathfrak{m}_K^{-m+1}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K} \tag{5.12}$$

induced by the map

$$F^r d\colon W_r(K) \to \Omega^1_K;\quad \underline{a} \mapsto \sum_{i=0}^{r-1} a_i^{p^i-1} da_i, \tag{5.13}$$

is injective.

Since rar_K is clearly functorial in K, it suffices to show that the map $\frac{\mathfrak{m}_K^{-m}}{\mathfrak{m}_K^{-m+1}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K} \to \frac{\mathfrak{m}_{K^{sh}}^{-m}}{\mathfrak{m}_{K^{sh}}^{-m+1}} \otimes_{\mathcal{O}_K^{sh}} \Omega^1_{\mathcal{O}_K^{sh}}$ is injective. However, the map $\mathcal{O}_K \to \mathfrak{m}_K^{-m}$ $(1 \mapsto \pi_K^{-m})$ induces an isomorphism $\mathfrak{f} \xrightarrow{\cong} \frac{\mathfrak{m}_K^{-m}}{\mathfrak{m}_K^{-m+1}}$. In particular, we have an isomorphism of $\mathfrak{f}$-vector spaces $\alpha_{\mathfrak{f}}\colon \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{f} \xrightarrow{\cong} \frac{\mathfrak{m}_K^{-m}}{\mathfrak{m}_K^{-m+1}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}$. Since $\mathcal{O}_K^{sh}$ is unramified over $\mathcal{O}_K$, we can choose π_K to be a uniformizer of K^{sh} as well. This implies that $\alpha_{\mathfrak{f}}$ is compatible with the similar isomorphism $\alpha_{\overline{\mathfrak{f}}}$, where $\overline{\mathfrak{f}}$ is a separable closure of $\mathfrak{f}$. Hence, we are reduced to showing that the map $\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{f} \to \Omega^1_{\mathcal{O}_K^{sh}} \otimes_{\mathcal{O}_K^{sh}} \overline{\mathfrak{f}}$ is injective.

Since $\mathcal{O}_K^{sh}$ is étale over $\mathcal{O}_K$, the map $\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{O}_K^{sh} \to \Omega^1_{\mathcal{O}_K^{sh}}$ is an isomorphism. Hence, we need to show that the map $\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{f} \to \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \overline{\mathfrak{f}}$ is injective. But this is obvious since $\Omega^1_{\mathcal{O}_K}$ is a free $\mathcal{O}_K$-module.

We now assume $p = m = 2$. Let $\mathfrak{f}^{1/2}$ be the field extension of $\mathfrak{f}$ obtained by adjoining square roots of elements of $\mathfrak{f}$. We define $\overline{\mathfrak{f}}^{1/2}$ similarly. Then [44, Proposition 1.17] says that there is an injective modified refined Artin conductor map

$$\mathrm{rar}'_K\colon \mathrm{gr}^{\mathrm{ms}}_2 H^1(K) \to \left(\frac{\mathfrak{m}_K^{-2}}{\mathfrak{m}_K^{-1}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}\right) \otimes_{\mathfrak{f}} \mathfrak{f}^{1/2} \cong \mathfrak{m}_K^{-2}\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{f}^{1/2}, \tag{5.14}$$

where the isomorphism on the right depends on the choice of the uniformizer π_K. Note that even if Yatagawa works with complete fields in the latter part of her paper, the proof of the above works for any Henselian discrete valuation field with no modification.

Since π_K is also a uniformizer of K^{sh}, it suffices to show that the map $\mathfrak{m}_K^{-2}\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{f}^{1/2} \to \mathfrak{m}_{K^{sh}}^{-2}\Omega^1_{\mathcal{O}_K^{sh}} \otimes_{\mathcal{O}_K^{sh}} \overline{\mathfrak{f}}^{1/2}$ is injective. As we argued in the previous case, this reduces to showing that the map $\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{f}^{1/2} \to \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \overline{\mathfrak{f}}^{1/2}$ is injective. But this follows again by the fact that $\Omega^1_{\mathcal{O}_K}$ is a free $\mathcal{O}_K$-module. $\square$

We shall use the following consequence of Proposition 5.13 in the proof of Theorem 1.3. Let k be a finite field and $X \in \mathbf{Sch}_k$ an integral normal scheme of dimension d. Let $D \subset X$ be an effective Weil divisor with $U = X \setminus |D|$. Let λ be a generic point of $|D|$. Let η denote the generic point of X and $K = k(\eta)$. Let K_λ denote the Henselization of K at λ. Let $P = (p_0, \ldots, p_{d-2}, \lambda, \eta)$ be a Parshin chain on $(U \subset X)$.

Recall the following notations from [22, § 3.3] or [18, § 2.3]. Let $V \subset K$ be a d-DV which dominates P. Let $V = V_0 \subset \cdots \subset V_{d-2} \subset V_{d-1} \subset V_d = K$ be the chain of valuation rings in K induced by V. Since X is normal, it is easy to check that for any such chain, one must have $V_{d-1} = \mathcal{O}_{X,\lambda}$. Let V' be the image of V in $k(\lambda)$. Let $\widetilde{V}_{d-1}$

be the unique Henselian discrete valuation ring having an ind-étale local homomorphism $V_{d-1} \to \widetilde{V}_{d-1}$ such that its residue field E_{d-1} is the quotient field of $(V')^h$. Then V^h is the inverse image of $(V')^h$ under the quotient map $\widetilde{V}_{d-1} \twoheadrightarrow E_{d-1}$. It follows that its function field $Q(V^h)$ is a Henselian discrete valuation field whose ring of integers is $\widetilde{V}_{d-1}$ (see [22, § 3.7.2]).

It then follows that there are canonical inclusions of discrete valuation rings

$$\mathcal{O}_{X,\lambda} \hookrightarrow \widetilde{V}_{d-1} \hookrightarrow \mathcal{O}^{sh}_{X,\lambda}. \tag{5.15}$$

Moreover, we have (see the proof of [22, Proposition 3.3])

$$\mathcal{O}^h_{X,P'} \cong \prod_{V \in \mathcal{V}(P)} \widetilde{V}_{d-1}, \tag{5.16}$$

where $\mathcal{V}(P)$ is the set of d-DV's in K which dominate P. As an immediate consequence of Proposition 5.13, we therefore get the following.

Corollary 5.14. *For every $m \geq 1$, the square*

$$\begin{tikzcd}
\mathrm{fil}^{\mathrm{ms}}_m H^1(K_\lambda) \arrow[r] \arrow[d] & H^1(K_\lambda) \arrow[d] \\
\mathrm{fil}^{\mathrm{ms}}_m H^1(Q(V^h)) \arrow[r] & H^1(Q(V^h))
\end{tikzcd}$$

is Cartesian.

5.5. Proof of Theorem 1.3

Let k be a finite field and $X \in \mathbf{Sch}_k$ an integral normal scheme. Let $D \subset X$ be a closed subscheme of pure codimension one such that $U = X \setminus D$ is regular. The density assertion of Theorem 1.3 is a direct consequence of the Chebotarev-Lang density theorem [42, Theorem 5.8.16]. By [18, Lemma 8.4], the heart of the proof is to show that

$$\rho_{X|D} \colon C(X,D)_0 \to \pi_1^{\mathrm{adiv}}(X,D)_0 \tag{5.17}$$

is an isomorphism if X is projective over k. The finiteness claim will then follow from Theorem 4.9.

We first show that (5.17) is injective. Using Theorem 4.9, this is equivalent to showing that the map $\rho^\vee_{X|D} \colon (\pi_1^{\mathrm{adiv}}(X,D)_0)^\vee \to (C(X,D)_0)^\vee$ is surjective. We fix a character $\chi \in (C(X,D)_0)^\vee$. We choose $m \in \mathbb{N}$ large enough such that $C(X,D)_0 \cong C(X,D)_0/m$ using Theorem 4.9. Using [18, Proposition 4.8, Lemma 7.10], χ lifts to a character of $C(X,D)/m$. Choose one such lift and denote its image in $C(X,D)^\vee$ also by χ. We let $\widetilde{\chi} = p^\vee_{X|D}(\chi)$ and consider the commutative diagram (see [18, § 5.6])

$$\begin{tikzcd}
\mathrm{fil}_D H^1(K) \arrow[r, "\rho^\vee_{X|D}"] \arrow[d, hook, "q^\vee_{X|D}"'] & C(X,D)^\vee \arrow[d, hook, "p^\vee_{X|D}"] & (C(X,D)/m)^\vee \arrow[l, hook'] \arrow[dd] \\
H^1(U) \arrow[r, "\widetilde{\rho}^\vee_{U/X}"] \arrow[drr, bend right, "\cong", "(\rho^\infty_{U/X})^\vee"'] & (\widetilde{C}_{U/X})^\vee & \\
& & (\widetilde{C}_{U/X}/\infty)^\vee. \arrow[ul]
\end{tikzcd} \tag{5.18}$$

Since $\chi \in (C(X,D)/m)^\vee$, it follows from Corollary 5.10 that $\widetilde{\chi} \in (\widetilde{C}_{U/X}/\infty)^\vee$. We conclude from Corollary 5.11 that there exists $\chi' \in H^1(U)$ such that $\widetilde{\rho}^\vee_{U|X}(\chi') = \widetilde{\chi}$. It suffices to show that χ' lies in the image of $q^\vee_{X|D}$.

We let $C = D_{\mathrm{red}}$. We fix a point $x \in \mathrm{Irr}_C$ and let χ'_x be the image of χ' in $H^1(K_x)$. We need to show that $\chi'_x \in \mathrm{fil}^{\mathrm{ms}}_{n_x} H^1(K_x)$, where n_x is the multiplicity of D at x. By Corollary 5.14, it suffices to show that for some maximal Parshin chain $P = (p_0, \dots, p_{d-2}, x, \eta)$ on $(U \subset X)$ and d-DV $V \subset K$ dominating P, the image of χ'_x in $H^1(Q(V^h))$ lies in the subgroup $\mathrm{fil}^{\mathrm{ms}}_{n_x} H^1(Q(V^h))$.

Choose any Parshin chain as above and call it P_0. Let $V \subset K$ be a d-DV dominating P_0 and let $\widehat{\chi}_x$ denote the image of χ'_x in $H^1(Q(V^h))$. Let $V = V_0 \subset \cdots \subset V_{d-1} \subset V_d = K$ be the chain of valuation rings associated to V. Then $Q(V^h)$ is a d-dimensional Henselian local field whose ring of integers is $\widetilde{V}_{d-1}$ (see [18, §2.3]). By [18, Theorem 6.3], it suffices to show that $\{\alpha, \widehat{\chi}_x\} = 0$ for every $\alpha \in K^M_d(\widetilde{V}_{d-1}, I_D) = K^M_d(\widetilde{V}_{d-1}, \mathfrak{m}^{n_x})$ under the pairing $K^M_d(Q(V^h)) \times H^1(Q(V^h)) \to H^{d+1}(Q(V^h))$. Here, $\mathfrak{m}$ is the maximal ideal of $\widetilde{V}_{d-1}$.

Now, we are given that $\widetilde{\chi}$ annihilates $\mathrm{Ker}(p_{X|D})$ and the latter is the sum of images of $K^M_d(\mathcal{O}^h_{X,P'}, I_D) \to C_{U/X}$, where P runs through all maximal Parshin chains on $(U \subset X)$. In particular, $\chi' \circ \rho_{U/X}$ annihilates the image of $K^M_d(\mathcal{O}^h_{X,P'_0}, I_D) \to C_{U/X}$. It follows from (5.16) that $\chi' \circ \rho_{U/X}$ annihilates the image of $K^M_d(\widetilde{V}_{d-1}, \mathfrak{m}^{n_x}) \to C_{U/X}$. Equivalently, $\{\alpha, \widehat{\chi}_x\} = 0$ for every $\alpha \in K^M_d(\widetilde{V}_{d-1}, \mathfrak{m}^{n_x})$. We have thus proven the desired claim, and hence the injectivity of (5.17).

We now show that (5.17) is surjective. Since $\pi_1^{\mathrm{adiv}}(X,D)_0$ is a profinite group and since $C(X,D)_0$ is finite by Theorem 4.9, it suffices to show that the image of $C(X,D)_0$ is dense in $\pi_1^{\mathrm{adiv}}(X,D)_0$. By [18, Lemma 7.11], we need to show that every element of $(\pi_1^{\mathrm{adiv}}(X,D)_0)^\vee$ which vanishes on $C(X,D)_0$ is zero. We fix $\chi \in (\pi_1^{\mathrm{adiv}}(X,D)_0)^\vee$ which vanishes on $C(X,D)_0$.

By [18, Corollary 8.5], there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\widehat{\mathbb{Z}})^\vee \arrow[r] \arrow[d, "\cong"'] & (\pi_1^{\mathrm{adiv}}(X,D))^\vee \arrow[r] \arrow[d, "\rho^\vee_{X|D}"] & (\pi_1^{\mathrm{adiv}}(X,D)_0)^\vee \arrow[r] \arrow[d, "\rho^\vee_{X|D}"] & 0 \\
0 \arrow[r] & \mathbb{Z}^\vee \arrow[r] & C(X,D)^\vee \arrow[r] & (C(X,D)_0)^\vee \arrow[r] & 0,
\end{tikzcd} \tag{5.19}$$

where the left vertical arrow is an isomorphism.

This diagram shows that χ lifts to an element $\chi' \in (\pi_1^{\mathrm{adiv}}(X,D))^\vee$. Since $\rho^\vee_{X|D}(\chi) = 0$, a diagram chase shows that there exists an element $\chi'' \in (\widehat{\mathbb{Z}})^\vee$ such that $\rho^\vee_{X|D}(\chi' - \chi'') = 0$. On the other hand, since the image of the map $\rho_{X|D}$ is dense, it follows from [18, Lemma 7.11] that the middle vertical arrow in (5.19) is injective. It follows that $\chi' = \chi''$. Equivalently, $\chi = 0$. The proof of Theorem 1.3 is now complete. □

Corollary 5.15. *Let X and D be as in Theorem 1.3 with X projective. Then for every $m \in \mathbb{N}$, the reciprocity map $\rho_{X|D}$ induces an isomorphism of finite groups*

$$\rho_{X|D} \colon C(X,D)/m \xrightarrow{\cong} \pi_1^{\mathrm{adiv}}(X,D)/m.$$

Proof. Use [18, Lemma 8.4] and Theorem 1.3. □

6. Reciprocity theorem for $\pi_1^{\mathrm{ab}}(X,D)$

In this section, we shall prove a reciprocity theorem for $\pi_1^{\mathrm{ab}}(X,D)$ in the pro-setting as D varies over a set of closed subschemes of pure codimension one X all of which have the same support. The main application for us will be in the proof of Theorem 1.2.

Let k be a finite field and X a projective integral normal scheme of dimension $d \geq 1$ over k. We let $C \subset X$ be a reduced closed subscheme of pure codimension one with complement U. We assume that U is regular. We shall need the following results.

Lemma 6.1. *Let $f \colon X' \to X$ be a morphism of integral normal schemes whose image is not contained in C. Let $D' \subset X'$ be an effective Weil divisor. Assume that $f^*(D)$ is an effective Weil divisor on X' such that $f^*(D) \leq D'$. Then there is a push-forward map*

$$f_* \colon \pi_1^{\mathrm{ab}}(X', D') \to \pi_1^{\mathrm{ab}}(X,D).$$

Proof. Obvious from Definition 5.3. □

Lemma 6.2. *Assume that k is finite and X is projective over k. Then the group $\pi_1^{\mathrm{ab}}(X,D)_0$ is finite. In particular, the map $\pi_1^{\mathrm{ab}}(X,D)_0 \to \pi_1^{\mathrm{ab}}(X,D)_0/\infty$ is an isomorphism.*

Proof. This is shown in [27, Corollary 1.2] when $D \subset X$ is an effective Cartier divisor. But one does not need the latter assumption. The reason is that since D is anyway a closed subscheme which defines a Weil divisor, its inverse image $f^*(D)$ becomes an effective Cartier divisor on X' where $f \colon X' \to X$ is any smooth alteration. We can replace X by any such alteration using Lemma 6.1 because the map $f_* \colon \pi^{\mathrm{ab}}(f^{-1}(U)) \to \pi_1^{\mathrm{ab}}(U)$ has finite cokernel. □

The following is the key step for proving the main result of this section.

Lemma 6.3. *For every $D \in \mathrm{Div}_C(X)$, there exists $D' \in \mathrm{Div}_C(X)$ such that the composite map $\widetilde{C}_{U/X} \xrightarrow{\widetilde{\rho}_{U/X}} \pi_1^{\mathrm{ab}}(U) \xrightarrow{q'_{X|D}} \pi_1^{\mathrm{ab}}(X, D)$ factors through*

$$\rho^{c}_{X|D} \colon C(X, D') \to \pi_1^{\mathrm{ab}}(X, D).$$

Proof. We fix a closed subscheme $D \in \mathrm{Div}_C(X)$. For any $D' \in \mathrm{Div}_C(X)$, let $F_{D'} = \mathrm{Ker}(\widetilde{C}_{U/X} \twoheadrightarrow C(X, D'))$. It follows from [18, Proposition 4.8] that $F_{D'} = \mathrm{Ker}((\widetilde{C}_{U/X})_0 \twoheadrightarrow C(X, D')_0)$. Hence, it suffices to show that the composite map $(\widetilde{C}_{U/X})_0 \xrightarrow{\widetilde{\rho}_{U/X}} \pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)$ annihilates $F_{D'}$ for some D'. Using [18, Lemma 8.4], we see that $\widetilde{\rho}_{U/X}$ induces the maps $(\widetilde{C}_{U/X})_0 \xrightarrow{\widetilde{\rho}_{U/X}} \pi_1^{\mathrm{ab}}(U)_0 \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)_0$. Hence, we need to show that this composite map annihilates $F_{D'}$ for some D'.

Now, we know from Theorem 4.9 that each $C(X, D')_0$ is finite. It follows from [18, Corollary 4.9] that $(\widetilde{C}_{U/X})_0 \xrightarrow{\cong} \varprojlim_{D' \in \mathrm{Div}_C(X)} C(X, D')_0$. In particular, $(\widetilde{C}_{U/X})_0$ is profinite. Since the maps $(\widetilde{C}_{U/X})_0 \xrightarrow{\widetilde{\rho}_{U/X}} \pi_1^{\mathrm{ab}}(U)_0 \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)_0$ are continuous, it follows that the images of $F_{D'}$ under these maps are closed subgroups. We let $E_{D'}$ be the image of $F_{D'}$ in $\pi_1^{\mathrm{ab}}(X, D)_0$ under the composite map.

We now let $\chi \in (\pi_1^{\mathrm{ab}}(X, D)_0)^{\vee}$ be a continuous character. Then the composite $\chi' := \chi \circ q'_{X|D} \circ \widetilde{\rho}_{U/X}$ is a continuous character of $(\widetilde{C}_{U/X})_0$. Since $(\widetilde{C}_{U/X})_0$ is profinite as we just saw, χ' factors through some $C(X, D_\chi)_0$. In other words, $\chi(E_{D_\chi}) = 0$. Since $\pi_1^{\mathrm{ab}}(X, D)_0$ is finite by Lemma 6.2, $(\pi_1^{\mathrm{ab}}(X, D)_0)^{\vee}$ is also finite (see [39, Example 2.9.5]). We can therefore choose $D' \in \mathrm{Div}_C(X)$ which dominates D_χ for all $\chi \in (\pi_1^{\mathrm{ab}}(X, D)_0)^{\vee}$. It is then clear that χ annihilates $E_{D'}$ for all $\chi \in (\pi_1^{\mathrm{ab}}(X, D)_0)^{\vee}$. We have thus shown that $E_{D'}$ is a closed subgroup of $\pi_1^{\mathrm{ab}}(X, D)_0$ which is annihilated by all characters of the latter group. But then the Pontryagin duality theorem says that $E_{D'}$ must be zero. Equivalently, $F_{D'}$ lies in the kernel of the composite map $(\widetilde{C}_{U/X})_0 \xrightarrow{\widetilde{\rho}_{U/X}} \pi_1^{\mathrm{ab}}(U)_0 \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)_0$. This proves the lemma. $\square$

Let $D \in \mathrm{Div}_C(X)$ and $nD \subset X$ the closed subscheme defined by $\mathcal{I}_D^n$, where $\mathcal{I}_D$ is the sheaf of ideals defining D. For every $n \in \mathbb{N}$, the set of closed subschemes $n'D \in \mathrm{Div}_C(X)$ which have the property described in Lemma 6.3 is linearly ordered by inclusion. Since X is Noetherian, there exists smallest integer $\lambda(n) \in \mathbb{N}$ such that $\lambda(n) \geq \lambda(n-1)$ and $\lambda(n)D$ satisfies the property asserted in Lemma 6.3. That is, the composite map $\widetilde{C}_{U/X} \xrightarrow{\widetilde{\rho}_{U/X}} \pi_1^{\mathrm{ab}}(U) \xrightarrow{q'_{X|D}} \pi_1^{\mathrm{ab}}(X, nD)$ factors through

$$\rho^{c}_{X|nD} \colon C(X, \lambda(n)D) \to \pi_1^{\mathrm{ab}}(X, nD).$$

We let $\lambda \colon \mathbb{N} \to \mathbb{N}$ be the above function. We write $\lambda(1)D = D_\rho$. As a consequence of Lemma 6.3, we conclude the following.

Theorem 6.4. *The reciprocity map $\rho_{U/X}\colon C_{U/X} \to \pi_1^{\mathrm{ab}}(U)$ of Theorem 5.7 induces a continuous homomorphism between the topological pro-abelian groups*

$$\rho^\bullet_{X|D}\colon \{C(X,nD)\}_{n\in\mathbb{N}} \to \{\pi_1^{\mathrm{ab}}(X,nD)\}_{n\in\mathbb{N}}$$

such that $\varprojlim_{n\in\mathbb{N}} \rho^\bullet_{X|D} = \widetilde{\rho}_{U/X}$.

From the construction of $\widetilde{\rho}_{U/X}$, we actually get the following commutative diagram of short exact sequences of topological pro-abelian groups. We ignore to write the indexing set $\mathbb{N}$.

$$\begin{tikzcd}
0 \arrow[r] & \{C(X,nD)_0\} \arrow[r] \arrow[d, "\rho^\bullet_{X|D}"'] & \{C(X,nD)\} \arrow[r, "\deg/n_0"] \arrow[d, "\rho^\bullet_{X|D}"'] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \{\pi_1^{\mathrm{ab}}(X,nD)_0\} \arrow[r] & \{\pi_1^{\mathrm{ab}}(X,nD)\} \arrow[r, "\deg'/n_0"] & \widehat{\mathbb{Z}} \arrow[r] & 0,
\end{tikzcd} \tag{6.1}$$

where the right vertical arrow is the profinite completion map and $n_0 \in \mathbb{N}$ depends only on U and not on $\mathrm{Div}_C(X)$. Moreover, taking the limits of the vertical arrows, we get the commutative diagram of short exact sequences of topological abelian groups

$$\begin{tikzcd}
0 \arrow[r] & (\widetilde{C}_{U/X})_0 \arrow[r] \arrow[d, "\widetilde{\rho}_{U/X}"'] & \widetilde{C}_{U/X} \arrow[r, "\deg/n_0"] \arrow[d, "\widetilde{\rho}_{U/X}"'] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \pi_1^{\mathrm{ab}}(U)_0 \arrow[r] & \pi_1^{\mathrm{ab}}(U) \arrow[r, "\deg'/n_0"] & \widehat{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd} \tag{6.2}$$

by Proposition 5.4 and Corollary 5.5. Note that the projective limits over $\mathrm{Div}_C(X)$ and $\{nD\}_{n\in\mathbb{N}}$ coincide.

Our reciprocity theorem for the 1-skeleton étale fundamental group with modulus is the following.

Theorem 6.5. *For each $D \in \mathrm{Div}_C(X)$, the reciprocity map of pro-abelian groups*

$$\rho^\bullet_{X|D}\colon \{C(X,nD)\}_{n\in\mathbb{N}} \to \{\pi_1^{\mathrm{ab}}(X,nD)\}_{n\in\mathbb{N}}$$

is injective and has dense image.

Proof. The proof of the density assertion is same as in Theorem 1.3. We shall prove pro-injectivity.

Using the commutative diagram (6.1), it suffices to show the injectivity at the level of the degree zero subgroups. Before we do this, we claim that $\rho^c_{X|nD}\colon C(X,\lambda(n)D)_0 \to \pi_1^{\mathrm{ab}}(X,nD)_0$ is surjective. Indeed, as the latter group is finite by Lemma 6.2, it suffices to show that this map has dense image. But the proof of this is identical to that of the surjectivity of the map (5.17) in Theorem 1.3, which only used that $\rho_{X|D}$ has dense image and not its injectivity.

Let $n \in \mathbb{N}$ and write $n' = \lambda(n)$. We let F_n denote the kernel of $C(X, n'D)_0 \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, nD)_0$. Using the Pontryagin duality for finite groups, it suffices to show that the ind-abelian group $\{F_n^\vee\}$ is zero.

Choose a character $\chi \in F_n^\vee$ and lift it to a character of $C(X, n'D)_0$. We denote this lift also by χ. We choose $m \in \mathbb{N}$ large enough such that $C(X, n'D)_0 \cong C(X, n'D)_0/m$ using Theorem 4.9. Using [18, Proposition 4.8, Lemma 7.10], χ lifts to a character of $C(X, n'D)/m$. Choose one such lift and denote its image in $C(X, n'D)^\vee$ also by χ. We let $\widetilde{\chi} = p^\vee_{X|n'D}(\chi)$ and consider the commutative diagram (see (5.18))

$$\begin{tikzcd}
\mathrm{fil}^c_{nD} H^1(K) \arrow[r, "(\rho^c_{X|nD})^\vee"] \arrow[d, hook, "{q'}^\vee_{X|nD}"'] & C(X, n'D)^\vee \arrow[d, hook, "p^\vee_{X|n'D}"] & (C(X, n'D)/m)^\vee \arrow[l, hook'] \arrow[dd] \\
H^1(U) \arrow[r, "\widetilde{\rho}^\vee_{U/X}"] \arrow[drr, bend right, "\cong", "(\rho^\infty_{U/X})^\vee"'] & (\widetilde{C}_{U/X})^\vee & \\
 & & (\widetilde{C}_{U/X}/\infty)^\vee . \arrow[ul]
\end{tikzcd} \tag{6.3}$$

Since $\chi \in (C(X, n'D)/m)^\vee$, it follows from Corollary 5.10 that $\widetilde{\chi} \in (\widetilde{C}_{U/X}/\infty)^\vee$. We conclude from Corollary 5.11 that there exists $\chi' \in H^1(U)$ such that $\widetilde{\rho}^\vee_{U/X}(\chi') = \widetilde{\chi}$. It follows from Proposition 5.4 that $\chi' \in \mathrm{fil}^c_{n_1 D} H^1(K)$ for some $n_1 \gg n$.

We let χ_ρ be the image of χ' under the map $(\pi_1^{\mathrm{ab}}(X, n_1 D))^\vee \to (\pi_1^{\mathrm{ab}}(X, n_1 D)_0)^\vee$. Then it follows that the image of χ_ρ in $(C(X, \lambda(n_1)D)_0)^\vee$ lies in the image of $(\rho^c_{X|n_1 D})^\vee$. Since $F_n^\vee = \mathrm{Coker}((\rho^c_{X|nD})^\vee)$ by [18, Lemma 7.10], it follows that χ dies under the canonical map $F_n^\vee \to F_{n_1}^\vee$. Since $F_n^\vee$ is finite, we can find $n_1 \gg n$ such that the map $F_n^\vee \to F_{n_1}^\vee$ is zero. We have thus shown that the ind-abelian group $\{F_n^\vee\}$ is zero. This finishes the proof. □

Corollary 6.6. *The morphism of pro-abelian groups*

$$\rho^\bullet_{X|D} \colon \{C(X, nD)_0\}_{n\in\mathbb{N}} \to \{\pi_1^{\mathrm{ab}}(X, nD)_0\}_{n\in\mathbb{N}}$$

is an isomorphism.

Proof. The map $\rho^\bullet_{X|D}$ is injective by Theorem 6.5. Moreover, we also showed in the proof of this theorem that $\rho^\bullet_{X|D}$ is surjective. □

Corollary 6.7. *The morphism of pro-abelian groups*

$$\rho^{\bullet}_{X|D} \colon \{C(X, nD)/m\}_{n\in\mathbb{N}} \to \{\pi_1^{\mathrm{ab}}(X, nD)/m\}_{n\in\mathbb{N}}$$

is an isomorphism for every $m \in \mathbb{N}$*.*

Proof. The proof is identical to that of Corollary 5.15 using Corollary 6.6. □

The following result will be improved in § 9.2 when X is regular. We nevertheless need this weaker version to prove Bloch's formula for normal varieties.

Theorem 6.8. *For every* $D \in \mathrm{Div}_C(D)$*, the identity map of* $\pi_1^{\mathrm{ab}}(U)$ *induces an isomorphism of topological pro-abelian groups*

$$\theta^{\bullet}_{X|D} \colon \{\pi_1^{\mathrm{adiv}}(X, nD)\}_{n\in\mathbb{N}} \xrightarrow{\cong} \{\pi_1^{\mathrm{ab}}(X, nD)\}_{n\in\mathbb{N}}$$

making the diagram

$$\begin{tikzcd}
\widetilde{C}_{U/X} \arrow[r] \arrow[d, "\widetilde{\rho}_{U/X}"'] & \{C(X, nD)\} \arrow[d, "\{\rho^{\mathrm{div}}_{X|nD}\}"] \arrow[dr, "\rho^{\bullet}_{X|D}"] & \\
\pi_1^{\mathrm{ab}}(U) \arrow[r] & \{\pi_1^{\mathrm{adiv}}(X, nD)\} \arrow[r, "\theta^{\bullet}_{X|D}"] & \{\pi_1^{\mathrm{ab}}(X, nD)\}
\end{tikzcd}$$

commute.

Proof. The commutativity of the diagram is clear from the construction of various reciprocity maps. We fix n and let $n' = \lambda(n)$. Let F denote the kernel of the map $\pi_1^{\mathrm{ab}}(U) \to \pi_1^{\mathrm{adiv}}(X, n'D)$. Then F is same as the kernel of the map $\pi_1^{\mathrm{ab}}(U)_0 \to \pi_1^{\mathrm{adiv}}(X, n'D)_0 \cong \pi_1^{\mathrm{adiv}}(X, n'D)_0/m$ for all $m \gg 0$ (see Theorem 1.3).

By Corollary 5.11, $\rho^{\infty}_{U/X}$ is an isomorphism and it induces an isomorphism between $(\widetilde{C}_{U/X})_0$ and $\pi_1^{\mathrm{ab}}(U)_0$ by Corollary 5.10. Hence, we conclude from Theorem 1.3 that $\rho^{\infty}_{X|D}$ induces an isomorphism $\mathrm{Ker}((\widetilde{C}_{U/X})_0 \to C(X, n'D)_0) \xrightarrow{\cong} F$. However, the kernel on the left hand side dies in $\pi_1^{\mathrm{ab}}(X, nD)_0$. It follows that F dies in $\pi_1^{\mathrm{ab}}(X, nD)_0$.

To prove that $\theta^{\bullet}_{X|D}$ is an isomorphism, we only need to show it is injective. To prove this, it is equivalent to show that the map $\{\pi_1^{\mathrm{adiv}}(X, nD)_0\} \to \{\pi_1^{\mathrm{ab}}(X, nD)_0\}$ is injective. But this follows from Theorem 1.3 and Corollary 6.6. □

We conclude this section by proving the following property of $\mathrm{fil}^c_D H^1(K)$ which will be used in the proof of Theorem 1.4 in § 9.2.

Proposition 6.9. *Let k be any field. Assume that* $\dim(X) \geq 2$ *and* $A \subset X$ *is a closed subscheme such that* $A_{\mathrm{red}} \subset C$ *and* $\dim(A) \leq \dim(X) - 2$*. Let* $X' = X \setminus A$*,* $C' = C \setminus A$

and $D' = D \setminus A$. Then there is an inclusion $\mathrm{fil}^c_D H^1(K) \subset \mathrm{fil}^c_{D'} H^1(K)$ of subgroups of $H^1(K)$.

Proof. Suppose $\chi \in \mathrm{fil}^c_D H^1(K)$ and let $Y' \subset X'$ be an integral curve not contained in D'. Let $Y \subset X$ be the scheme theoretic closure of Y' in X. Then Y is an integral curve in X not contained in D. Let $\nu\colon Y_n \to X$ be induced map from the normalization of Y. We let $D_{Y_n} = D \times_X Y_n$ and $D'_{Y'_n} = D \times_X Y'_n$. We then get a commutative diagram

$$\begin{tikzcd} D'_{Y'_n} \arrow[r] \arrow[d] & Y'_n \arrow[r, "\nu'"] \arrow[d] & X' \arrow[d] \\ D_{Y_n} \arrow[r] & Y_n \arrow[r, "\nu"] & X \end{tikzcd} \tag{6.4}$$

in which the two squares are Cartesian, vertical arrows are open immersions and horizontal arrows are finite.

We write $D_{Y_n} = \sum\limits_{x \in \Sigma} m_x[x]$, where Σ is the support of $\nu^{-1}(C)$. Then the above diagram says that $D'_{Y'_n} = \sum\limits_{x \in \Sigma'} m_x[x]$, where $\Sigma' = \nu^{-1}(C')$. In particular, $D'_{Y'_n}$ is an effective Weil divisor on Y_n such that $D'_{Y'_n} \le D_{Y_n}$ and every $x \in \Sigma'$ (note that $D'_{Y'_n}$ will be zero if $Y \cap C \subset A$) has the property that the multiplicity of $D'_{Y'_n}$ at x is same is that of D_{Y_n}. But this implies that $\nu'^*(\chi) \in H^1(\nu'^{-1}(U)) = H^1(\nu^{-1}(U))$ has the property that its image in $H^1(k(Y)_x)$ lies in $\mathrm{fil}^{\mathrm{ms}}_{m_x} H^1(k(Y)_x)$, for all $x \in \Sigma'$. This proves the proposition. $\square$

7. A moving lemma for 0-cycles with modulus

Our next goal is to prove a moving lemma for the Chow group of 0-cycles with modulus which will be the key ingredient in the proof of Theorem 1.4. In this section, we shall recall the Chow group of 0-cycles with modulus and prove some intermediate results.

7.1. Chow group with modulus

We recall the definition of the Chow group of 0-cycles with modulus from [3] and [28]. Let k be any field and X a reduced quasi-projective scheme over k of dimension $d \ge 1$. Let $D \subset X$ be an effective Cartier divisor and U its complement. Let $D' = D_{\mathrm{red}}$. Let $\mathcal{Z}_0(U)$ denote the free abelian group on the set of closed points in U. Suppose that $Y \subset X$ is an integral curve not contained in D' and $\nu\colon Y_n \to X$ is the projection map from the normalization of Y. We let $D'_Y = \nu^{-1}(D')$. Let I_{D_Y} denote the ideal of $\nu^*(D)$ in the semilocal ring $\mathcal{O}_{Y_n, D'_Y}$. We have the divisor map $\mathrm{div}\colon K^M_1(\mathcal{O}_{Y_n, D'_Y}, I_{D_Y}) \to \mathcal{Z}_0(\nu^{-1}(U))$. We let $\mathcal{R}_0(Y|D)$ denote the image of the composite map

$$K^M_1(\mathcal{O}_{Y_n, D'_Y}, I_{D_Y}) \xrightarrow{\mathrm{div}} \mathcal{Z}_0(\nu^{-1}(U)) \xrightarrow{\nu_*} \mathcal{Z}_0(U). \tag{7.1}$$

We let $\mathcal{R}_0(X|D)$ be the image of the map $\bigoplus_{Y} \mathcal{R}_0(Y|D) \to \mathcal{Z}_0(U)$, where Y runs through the set of curves as above. An element $\alpha \in \mathcal{R}_0(X|D)$ is said to be a 0-cycle rationally equivalent to zero. The Chow group of 0-cycles with modulus is the quotient

$$\mathrm{CH}_0(X|D) = \frac{\mathcal{Z}_0(U)}{\mathcal{R}_0(X|D)}. \tag{7.2}$$

The functor $(X, D) \mapsto \mathrm{CH}_0(X|D)$ has appropriate covariant functorial property for proper maps and contravariant functorial property for flat maps. It is also clear from the definition that for $D_1 \le D_2$, there is a canonical map $\mathrm{CH}_0(X|D_2) \to \mathrm{CH}_0(X|D_1)$. In particular, there is a canonical map $\mathrm{CH}_0(X|D) \to \mathrm{CH}_0^F(X)$. If X is projective over k, composing with the classical degree map $\mathrm{CH}_0^F(X) \to \mathbb{Z}$, we get the degree map

$$\deg\colon \mathrm{CH}_0(X|D) \to \mathbb{Z}, \tag{7.3}$$

whose image coincides with the image of the composite map $\mathcal{Z}_0(U) \hookrightarrow \mathcal{Z}_0(X) \to \mathbb{Z}$. Hence, there exists an integer $n \ge 1$ which depends only on U such that the sequence

$$0 \to \mathrm{CH}_0(X|D)_0 \to \mathrm{CH}_0(X|D) \xrightarrow{\deg/n} \mathbb{Z} \to 0 \tag{7.4}$$

is exact. Furthermore, we have $n = 1$ if either k is finite and X is geometrically irreducible or k is separably closed. We shall consider $\mathrm{CH}_0(X|D)$ a topological group with discrete topology.

We let $C(U) = \varprojlim_{E} \mathrm{CH}_0(X|E)$ and $C(U)_0 = \mathrm{Ker}(C(U) \xrightarrow{\deg} \mathbb{Z})$, where the limit is over all effective Cartier divisors E with support D'. Since we have $\mathrm{CH}_0(X|D_2)_0 \twoheadrightarrow \mathrm{CH}_0(X|D_1)_0$ for every $D_1 \le D_2$, it follows that

$$C(U)_0 \xrightarrow{\cong} \varprojlim_{E} \mathrm{CH}_0(X|E)_0. \tag{7.5}$$

We shall consider $C(U)$ and $C(U)_0$ as topological groups with their inverse limit topology. It is then clear that $C(U)_0$ is a closed subgroup of $C(U)$.

The moving lemma we wish to prove for $\mathrm{CH}_0(X|D)$ is the following. This result is of independent interest in the theory of cycles with modulus aside from its current application.

Theorem 7.1. *Let X be a smooth quasi-projective scheme of dimension $d \ge 2$ over a perfect field k and let $D \subset X$ be an effective Cartier divisor. Let $A \subset D'$ be a closed subscheme such that* $\dim(A) \le d - 2$. *Then $\mathcal{R}_0(X|D)$ is generated by the images of $\mathcal{R}_0(Y|D)$ as in (7.1), where $Y \subset X$ has the additional property that $Y \cap A = \emptyset$.*

7.2. The Levine-Weibel Chow group of the double

The proof of Theorem 7.1 is similar to that of the fundamental exact sequence [3, (6.3)], which relates the Chow group with modulus with an improved version of the Levine-Weibel Chow group of a singular variety. However, we need to introduce several modifications to prove Theorem 7.1 and shall therefore present a complete proof.

Throughout the rest of § 7, we shall follow the notations and assumptions of Theorem 7.1. Recall from [3, § 2.1] that the double of X along D is the quasi-projective scheme S_X such that the square

$$\begin{tikzcd} D \arrow[r, "\iota"] \arrow[d, "\iota"'] & X \arrow[d, "\iota_+"] \\ X \arrow[r, "\iota_-"] & S_X \end{tikzcd} \tag{7.6}$$

is co-Cartesian, i.e., $S_X = X_+ \coprod_D X_-$. The arrows $\iota_\pm$ are closed immersions and $\iota \colon D \hookrightarrow X$ is the inclusion of D in X. There is a projection map $\Delta \colon S_X \to X$ which is finite and flat. Moreover, $\Delta \circ \iota_\pm = \mathrm{Id}_X$. We also have $D' = (S_X)_{\mathrm{sing}}$ via the inclusion $\iota_\pm \circ \iota$. Let us denote this by ι'. The double S_X is a reduced scheme with two irreducible components $X_\pm$, each a copy of X.

Let $\mathrm{CH}_0^{LW}(S_X)$ denote the Levine-Weibel Chow group of S_X (see [3, § 3.4]). Recall that this is the quotient of $\mathcal{Z}_0(S_X \setminus D)$ by the subgroup of rational equivalences, denoted by $\mathcal{R}_0^{LW}(S_X)$. This subgroup is generated by the divisors of rational functions on certain 'Cartier curves on X relative to D'. We refer to [3, § 3.4] for these terms.

In this subsection, we shall prove some lemmas which will allow us to choose some refined Cartier curves in order to define the rational equivalence of 0-cycles on S_X. We shall consider A as a closed subscheme of S_X via the inclusions $A \hookrightarrow D \hookrightarrow S_X$. We shall assume in this subsection that k is infinite and perfect.

Lemma 7.2. *Assume $d \geq 3$. Then $\mathcal{R}_0^{LW}(S_X)$ is generated by the divisors of functions on (possibly non-reduced) Cartier curves $C \hookrightarrow S_X$ relative to D, where C satisfies the following:*

(1) *There is a locally closed embedding $S_X \hookrightarrow \mathbb{P}^N_k$ and distinct hypersurfaces*

$$H_1, \cdots, H_{d-2} \hookrightarrow \mathbb{P}^N_k$$

such that $Y = S_X \cap H_1 \cap \cdots \cap H_{d-2}$ is a complete intersection which is reduced;

(2) *$X_\pm \cap Y = X_\pm \cap H_1 \cap \cdots \cap H_{d-2} := Y_\pm$ are integral;*

(3) *No component of Y is contained in D;*

(4) *$Y \cap A$ is finite;*

(5) *$C \subset Y$;*

(6) *C is a Cartier divisor on Y;*

(7) *$Y_\pm$ are smooth away from C.*

Proof. The proof is identical to that of [3, Lemma 5.4]: we obtain Y and C using the Bertini theorems of Altman-Kleiman [2, Theorem 1] and Jouanolou [19]. Since k is infinite, these Bertini theorems also allow us to ensure that a general hypersurface of a given degree in $\mathbb{P}^n_k$ will intersect A properly under any chosen locally closed embedding $X \hookrightarrow \mathbb{P}^N_k$. Since (4) is an open condition on the linear system of hypersurfaces of a given degree in $\mathbb{P}^N_k$, we can also include it in the proof of [3, Lemma 5.4] because $\dim(A) \leq d-2$. $\square$

Lemma 7.3. *Let $d \geq 2$. Let $\nu\colon C \hookrightarrow S_X$ be a (possibly non-reduced) Cartier curve relative to $D \subset S_X$. Assume that either $d = 2$ or there are inclusions $C \subset Y \subset S_X$, where Y is a reduced complete intersection surface and C is a Cartier divisor on Y, as in Lemma 7.2. Let $f \in \mathcal{O}^{\times}_{C,\nu^*D} \subset k(C)^{\times}$, where $k(C)$ is the total quotient ring of $\mathcal{O}_{C,\nu^*D}$.*

We can then find two Cartier curves $\nu'\colon C' \hookrightarrow S_X$ and $\nu''\colon C'' \hookrightarrow S_X$ relative to D satisfying the following:

1. *There are very ample line bundles $\mathcal{L}', \mathcal{L}''$ on S_X and sections $t' \in H^0(S_X, \mathcal{L}')$, $t'' \in H^0(S_X, \mathcal{L}'')$ such that $C' = Y \cap (t')$ and $C'' = Y \cap (t'')$ (with the convention $Y = S_X$ if $d = 2$);*
2. *C' and C'' are reduced;*
3. *$C' \cap A = C'' \cap A = \emptyset$;*
4. *The restrictions of both C' and C'' to X via the two closed immersions $\iota_{\pm}$ are integral curves in X, which are Cartier and smooth along D;*
5. *There are functions $f' \in \mathcal{O}^{\times}_{C',(\nu')^*D}$ and $f'' \in \mathcal{O}^{\times}_{C'',(\nu'')^*D}$ such that $\nu'_*(\mathrm{div}(f')) + \nu''_*(\mathrm{div}(f'')) = \nu_*(\mathrm{div}(f))$ in $\mathcal{Z}_0(S_X \setminus D)$.*

Proof. In this proof, we shall assume that $Y = S_X$ if $d = 2$. In the latter case, a Cartier curve on S_X along $D' = (S_X)_{\mathrm{sing}}$ must be an effective Cartier divisor on S_X. Hence, we can assume that C is an effective Cartier divisor on Y for any $d \geq 2$. Note that $Y = Y_+ \coprod_D Y_-$.

Since Y is quasi-projective over k and C is an effective Cartier divisor on Y, we can add some very ample effective divisor to C to get another effective Cartier divisor C' (see the proof of [34, Lemma 1.4]) on Y such that

1. $C \subset C'$;
2. $\overline{C' \setminus C} \cap (C \cap D) = \emptyset$;
3. $\mathcal{O}_Y(C')$ is very ample on Y.

Setting $g = f$ on C and $g = 1$ on $C' \setminus C$, we then see that g defines an element in $\mathcal{O}^{\times}_{C',C'\cap D}$ such that $\mathrm{div}(g) = \mathrm{div}(f)$. We can thus assume that $\mathcal{O}_Y(C)$ is very ample on Y.

We now choose $t_0 \in H^0(Y, \mathcal{O}_Y(C))$ such that $C = (t_0)$, where for a line bundle $\mathcal{L}$ and a meromorphic section h of $\mathcal{L}$, (h) denotes the (effective) divisor of zeros of h. Since $\mathcal{O}_Y(C)$ is very ample, and Y is reduced, we can find a new section $t_\infty \in H^0(Y, \mathcal{O}_Y(C))$, sufficiently general so that:

(1) (t_∞) is reduced;
(2) $(t_\infty) \cap (t_0) \cap D = \emptyset$;
(3) $(t_\infty) \cap A = \emptyset$;
(4) (t_∞) contains no component of (t_0);
(5) D contains no component of (t_∞).

Note that we can achieve (3) because $Y \cap A$ is finite by Lemma 7.2. Denote by C_∞ the divisor (t_∞). Notice that the function $h = (1, f)$ is meromorphic on $C_\infty \cup C$ and regular invertible in a neighborhood of $(C_\infty \cup C) \cap D$ by (2). Since $Y \cap A$ is a finite closed set of S_X which does not meet C_∞, we can write $Y \cap A = S_1 \coprod S_2$, where $S_1 \subset C \cap A$ and $S_2 \cap (C_\infty \cup C) = \emptyset$. By setting $h = 1$ on S_2, we can extend the function h to a meromorphic function on $T := (Y \cap A) \cup C_\infty \cup C$ which is regular invertible in a neighborhood of $T' := (Y \cap A) \bigcup ((C_\infty \cup C) \cap D)$. Let S denote the finite set of closed points on $C_\infty \cup C$, which either lie on $C_\infty \cap C$ or where h is not regular. In particular, $S \supset \{\text{poles of } f\} \cup (C \cap C_\infty)$.

Since S_X is reduced quasi-projective and $X_\pm$ (as well as $Y_\pm$) are integral closed subschemes of S_X, we can find a very ample line bundle $\mathcal{L}$ on S_X and a section $s_\infty \in H^0(S_X, \mathcal{L})$ (see again [34, Lemma 1.4]) so that:

a) (s_∞) and $(s_\infty) \cap Y$ are reduced;
b) $Y \not\subset (s_\infty)$;
c) $(s_\infty) \cap X_\pm$ and $(s_\infty) \cap Y_\pm$ are integral;
d) $(s_\infty) \cap T' = \emptyset$;
e) $(s_\infty) \supset S$;
f) $C_\infty \cup C$ contains no component of $(s_\infty) \cap Y$.

If $\overline{S_X}$ is the scheme-theoretic closure of S_X in the projective embedding given by $\mathcal{L} = \mathcal{O}_{S_X}(1)$ and if $\mathcal{I}$ is the ideal sheaf of $\overline{S_X} \setminus S_X$ in the ambient projective space, then we can find a section s'_∞ of the sheaf $\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^N_k}(m)$ for some $m \gg 0$, which restricts to a section s_∞ on S_X satisfying the properties (a) - (f) on $S_X = \overline{S_X} \setminus V(\mathcal{I})$. This implies in particular that $S_X \setminus (s_\infty) = \overline{S_X} \setminus (s'_\infty)$ is affine. Set $\mathcal{L}' = \mathcal{L}^m$.

We now know that Y_+ is smooth away from C, and (d) tells us that (s_∞) intersects Y_+ along D at only those points which are away from $C_\infty \cup C$. It follows that $(s_\infty) \cap Y_+ \cap D \subset (s_\infty) \cap (Y_+)_{\text{reg}}$, where $(Y_+)_{\text{reg}}$ denotes the smooth locus of Y_+. On the other hand, we can use the Bertini theorem of Altman and Kleiman [2, Theorem 1] to ensure that $(s_\infty) \cap (Y_+)_{\text{reg}}$ is smooth. The same holds for Y_- as well. We conclude that we can choose $\mathcal{L}'$ and $s_\infty \in H^0(S_X, \mathcal{L}')$ such that (a) - (f) above as well as the following hold.
g) $(s_\infty) \cap Y_\pm$ are smooth along D.
h) $S_X \setminus (s_\infty)$ is affine.

Now, because h is a meromorphic function on T which is regular outside S, and the latter is contained in (s_∞), the property (h) implies that $h|_{T \setminus (s_\infty)}$ extends to a regular function H on $U = S_X \setminus (s_\infty)$. Since H is a meromorphic function on S_X which has poles only along (s_∞), it follows that Hs_∞^N is an element of $H^0(U, (\mathcal{L}')^N)$ which extends to a section s_0 of $(\mathcal{L}')^N$ on all of S_X, if we choose $N \gg 0$.

Since s_∞ and h are both invertible on T in a neighborhood of T', we see that s_0 is invertible on T in a neighborhood of T'. In particular, $s_0 \notin H^0(S_X, (\mathcal{L}')^N \otimes \mathcal{I}_T)$. Applying [34, Lemma 1.4] and [2, Theorem 1] (see their proofs), we can thus find $\alpha \in H^0(S_X, (\mathcal{L}')^N \otimes \mathcal{I}_T) \subset H^0(S_X, (\mathcal{L}')^N)$ such that $s'_0 := s_0 + \alpha$ has the following properties:

a') (s'_0) and $(s'_0) \cap Y$ are reduced;
b') $Y \not\subset (s'_0)$;
c') $(s'_0) \cap X_\pm$ and $(s'_0) \cap Y_\pm$ are integral;
d') $(s'_0) \cap T' = \emptyset$;
e') $C_\infty \cup C$ contains no component of $(s'_0) \cap Y$;
f') $(s'_0) \cap Y_\pm$ are smooth along D.

We then have

$$\frac{s'_0}{s_\infty^N} = \frac{Hs_\infty^N + (\alpha s_\infty^{-N})s_\infty^N}{s_\infty^N} = H + \alpha s_\infty^{-N} = H', \ \text{(say)}.$$

Since α vanishes along $C_\infty \cup C$ and s_∞ is invertible along U, it follows that $H'_{|(C_\infty \cup C)\cap U} = H_{|(C\cup C_\infty)\cap U} = h_{|U}$. In other words, we have $s'_0/s_\infty^N = h$ as rational functions on $C_\infty \cup C$. We can now compute:

$$\nu_*(\mathrm{div}(f)) = (s'_0)\cdot C - N(s_\infty)\cdot C$$
$$0 = \mathrm{div}(1) = (s'_0)\cdot C_\infty - N(s_\infty)\cdot C_\infty.$$

Setting $(s_\infty^Y) = (s_\infty) \cap Y$ and $({s'_0}^Y) = (s'_0) \cap Y$, we get

$$\begin{aligned}\nu_*(\mathrm{div}(f)) &= (s'_0)\cdot(C - C_\infty) - N(s_\infty)(C - C_\infty)\\ &= ({s'_0}^Y)\cdot(\mathrm{div}(t_0/t_\infty)) - N(s_\infty^Y)\cdot(\mathrm{div}(t_0/t_\infty))\\ &= \iota_{{s'_0}^Y,*}(\mathrm{div}(f')) - N\iota_{s_\infty^Y,*}(\mathrm{div}(f'')),\end{aligned}$$

where $f' = (t_0/t_\infty)|_{({s'_0}^Y)} \in \mathcal{O}^\times_{({s'_0}^Y), D\cap({s'_0}^Y)}$ (by (d')) and $f'' = (t_0/t_\infty)|_{(s_\infty^Y)} \in \mathcal{O}^\times_{(s_\infty^Y), D\cap(s_\infty^Y)}$ (by (d)).

It follows from (g) and (f') that $({s'_0}^Y)_{|X_+}$, $({s'_0}^Y)_{|X_-}$, $(s_\infty^Y)_{|X_+}$ and $(s_\infty^Y)_{|X_-}$ are all smooth along D. Setting $\mathcal{L}'' = (\mathcal{L}')^N$, $t'' = s'_0$ and $t' = s_\infty$, we see that the curves $C' = (t') \cap Y$ and $C'' = (t'') \cap Y$ together with the functions f' and f'' satisfy the conditions of the Lemma. $\square$

7.3. The map τ_X^*

Let $\mathrm{CH}_0^A(X|D)$ be the quotient of the free abelian group $\mathcal{Z}_0(X \setminus D)$ on the closed points on $X \setminus D$ by the subgroup of rational equivalences $\mathcal{R}_0^A(X|D)$ generated by the images of $\mathcal{R}_0(Y|D)$ as in (7.1), where $Y \subset X$ has an additional property that $Y \cap A = \emptyset$. It is clear that there is a canonical surjection $\mathrm{CH}_0^A(X|D) \twoheadrightarrow \mathrm{CH}_0(X|D)$. Our goal is to prove that this is an isomorphism.

We let $\tau_X^*\colon \mathcal{Z}_0(S_X \setminus D) \to \mathcal{Z}_0(X \setminus D)$ be the map $\tau_X^* = \iota_+^* - \iota_-^*$ under the embeddings $X \xrightarrow{\iota_\pm} S_X$. We want to show that τ_X^* preserves the subgroups of rational equivalences. We continue to assume that k is infinite and perfect.

Lemma 7.4. *Assume $d = 2$. Then the map τ_X^* descends to a group homomorphism $\mathrm{CH}_0^{LW}(S_X) \to \mathrm{CH}_0^A(X|D)$.*

Proof. The proof is essentially identical to that of [3, Proposition 5.7], but subtle changes are required at several places. So we provide the details.

We have shown in Lemma 7.3 that in order to prove that τ_X^* preserves the subgroups of rational equivalences, it suffices to show that $\tau_X^*(\mathrm{div}(f)) \in \mathcal{R}_0^A(X|D)$, where f is a rational function (which is regular and invertible along D) on a Cartier curve $\nu\colon C \hookrightarrow S_X$ that we can choose in the following way.

(1) There is a very ample line bundle $\mathcal{L}$ on S_X and sections $t \in H^0(S_X, \mathcal{L}), t_\pm = \iota_\pm^*(t) \in H^0(X, \iota_\pm^*(\mathcal{L}))$ such that $C = (t)$ and $C_\pm = (t_\pm)$.

(2) C is a reduced Cartier curve of the form $C = C_+ \coprod_E C_-$, where $E = \nu^*(D)$ such that $C_\pm$ are integral curves on X, none of which is contained in D, none of which meets A and each of which is smooth along D (see [3, Remark 5.6]).

If $E = \emptyset$, then $C_\pm$ are two integral curves on X away from D and $\tau_X^*(\mathrm{div}(f)) = \mathrm{div}(f|_{C_+}) - \mathrm{div}(f|_{C_-}) \in \mathcal{R}_0^A(X|D)$. We can thus assume that $E \neq \emptyset$. Then (1) implies that

$$(t_+)_{|D} = \iota'^*(t) = (t_-)_{|D}, \tag{7.7}$$

where recall that $\iota' = \iota_+ \circ \iota = \iota_- \circ \iota\colon D \hookrightarrow S_X$ denotes the inclusion map.

Let (f_+, f_-) be the image of f in $\mathcal{O}_{C_+,E}^\times \times \mathcal{O}_{C_-,E}^\times \hookrightarrow k(C_+) \times k(C_-)$. It follows from [3, Lemma 2.2] that there is an exact sequence

$$0 \to \mathcal{O}_{C,E} \to \mathcal{O}_{C_+,E} \times \mathcal{O}_{C_-,E} \to \mathcal{O}_E \to 0.$$

In particular, we have

$$(f_+)_{|E} = (f_-)_{|E} \in \mathcal{O}_E^\times. \tag{7.8}$$

Let us first assume that $C_+ = C_-$ as curves on X. Let C denote this curve and let C_n denote its normalization. Let $\pi\colon C_n \to C \hookrightarrow X$ denote the composite map. Since C is regular along E by (2), we get $f_+, f_- \in \mathcal{O}_{C_n,E}^\times$. Setting $g := f_+ f_-^{-1} \in \mathcal{O}_{C_n,E}^\times$, it follows from (7.8) that $g \in \mathrm{Ker}(\mathcal{O}_{C_n,E}^\times \to \mathcal{O}_E^\times)$. Moreover, $\tau_X^*(\mathrm{div}(f)) = \iota_+^*(\mathrm{div}(f)) - \iota_-^*(\mathrm{div}(f)) = \mathrm{div}(f_+) - \mathrm{div}(f_-) = \pi_*(\mathrm{div}(g))$. Since $C \cap A = \emptyset$ by (2), we conclude that $\tau_X^*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^A(X|D)$.

We now assume that $C_+ \neq C_-$. Since $C_+ \cap D = C_- \cap D = E$ as closed subschemes, we see that the support of $(C_+ \cup C_-) \cap D$ is same as E_{red}, where $C_+ \cup C_- \subset X$ is the closed subscheme defined by the ideal sheaf $\mathcal{I}_{C_+} \cap \mathcal{I}_{C_-}$. Note that since $C_\pm$ are integral, $C_+ \cup C_-$ is a reduced closed subscheme of X with irreducible components $C_\pm$.

Since A is a finite closed subscheme of X and $(C_+ \cup C_-) \cap A = \emptyset$, the functions $f_\pm$ extend to meromorphic functions on $T_\pm = A \cup C_\pm$ which are regular invertible in a neighborhood of $T' = A \cup E$ by letting $f_\pm = 1$ on A. Let $S_\pm$ denote the set of closed points on $C_\pm$, where $f_\pm$ have poles. We let $T = T_+ \cup T_-$ and $S = S_+ \cup S_-$. It is clear that $S \cap D = \emptyset$.

We now repeat the constructions in the proof of Lemma 7.3 to find a very ample line bundle $\mathcal{L}$ on X and a section $s_\infty \in H^0(X, \mathcal{L})$ (see [34, Lemma 1.4] and [2, Theorem 1]) such that:

a) (s_∞) is integral (because X is integral);
b) $(s_\infty) \cap T' = \emptyset$;
c) $(s_\infty) \supset S$;
d) $(s_\infty) \not\subset C_+ \cup C_-$;
e) (s_∞) is smooth away from S;
f) $X \setminus (s_\infty)$ is affine.

It follows that $f_\pm|_{T_\pm \setminus (s_\infty)}$ extend to regular functions $F_\pm$ on $X \setminus (s_\infty)$. Since $F_\pm$ are meromorphic functions on X which have poles only along (s_∞), it follows that $F_\pm s_\infty^N$ are elements of $H^0(X \setminus (s_\infty), \mathcal{L}^N)$ which extend to sections $(s_0)_\pm$ of $\mathcal{L}^N$ on all of X, if we choose $N \gg 0$.

Since the functions s_∞ and $F_\pm$ are all meromorphic functions on X which are regular on $X \setminus (s_\infty)$, it follows that each of them restricts to a meromorphic function on T which is regular on $T \setminus (s_\infty)$ and $F_\pm|_{T\pm\setminus(s_\infty)} = f_\pm|_{T_\pm\setminus(s_\infty)}$. Since s_∞ and $F_\pm$ are invertible on $T_\pm$ in some neighborhood of T', we see that $(s_0)_\pm$ are invertible on $T_\pm$ in some neighborhood of T'. In particular, $(s_0)_\pm \notin H^0(X, \mathcal{L}^N \otimes \mathcal{I}_T)$. As before, using [34, Lemma 1.4] and [2, Theorem 1], we can moreover find $\alpha_\pm \in H^0(X, \mathcal{L}^N \otimes \mathcal{I}_T) \subset H^0(X, \mathcal{L}^N)$ such that $(s'_0)_\pm := (s_0)_\pm + \alpha_\pm$ have the following properties:

a') $((s'_0)_\pm)$ are integral;
b') $((s'_0)_\pm) \not\subset C_+ \cup C_-$;
c') $((s'_0)_\pm) \cap T' = \emptyset$;
d') $((s'_0)_\pm)$ are smooth away from S.

Let $H_\pm := \frac{(s'_0)_\pm}{s_\infty^N}$. We then have

$$H_\pm = \frac{F_\pm s_\infty^N + (\alpha_\pm s_\infty^{-N}) s_\infty^N}{s_\infty^N} = F_\pm + \alpha_\pm s_\infty^{-N}. \tag{7.9}$$

We can now find a dense open subscheme $U' \subset X \setminus ((s_\infty) \cup ((s'_0)_+) \cup ((s'_0)_-))$ which contains T' and where $F_\pm, \alpha_\pm, s_\infty$ and $(s'_0)_\pm$ are all regular. In particular, $H_\pm$ are rational functions on X which are regular on U'.

Since $T' \subset U'$ and $T' \neq \emptyset$ (because $E \neq \emptyset$), it follows that $T_\pm \cap U'$ are dense open in $T_\pm$. It follows that s_∞ and $(s'_0)_\pm$ restrict to regular functions on $T_\pm \cap U'$ which are invertible in a neighborhood of T'. Since $\alpha_\pm$ vanish along T and s_∞ is invertible on U', it follows that $H_{\pm|T_\pm\cap U'} = F_{\pm|T_\pm\cap U'}$. Since $F_\pm$ restrict to regular functions on $T_\pm \cap U'$ and are invertible along T', it follows that $H_\pm$ restrict to regular functions on $T_\pm \cap U'$ and are invertible along T'.

As H_+ (resp. H_-) is regular on U', it restricts to a regular function on the dense open subset $C_- \cap U'$ (resp. $C_+ \cap U'$) of C_- (resp. C_+). Furthermore, we have

$$H_{+|E} = F_{+|E} = f_{+|E} \ \stackrel{\dagger}{=} \ f_{-|E} = F_{-|E} = H_{-|E}, \tag{7.10}$$

where † follows from (7.8).

We thus saw above that H_+ and H_- are both regular functions on $C_- \cap U'$ such that $H_- \neq 0$. In particular, H_+/H_- is a rational function on C_-. Since $(s'_0)_+$ and $(s'_0)_-$ are both invertible functions on $C_- \cap U'$, it follows that the restriction of $(s'_0)_+/(s'_0)_-$ on C_- is a rational function on C_-, which is regular and invertible on the dense open $C_- \cap U'$. On the other hand, we have

$$\frac{(s'_0)_+}{(s'_0)_-} = \frac{(s'_0)_+/s_\infty^N}{(s'_0)_-/s_\infty^N} = \frac{H_+}{H_-}, \tag{7.11}$$

as rational functions on X. In particular, $(s'_0)_+ \cdot H_- = (s'_0)_- \cdot H_+$ as regular functions on U'. In particular, this identity holds after restricting these regular functions to $C_- \cap U'$. We thus get

$$\frac{(s'_0)_+}{(s'_0)_-} = \frac{(s'_0)_+/s_\infty^N}{(s'_0)_-/s_\infty^N} = \frac{H_+}{H_-}, \tag{7.12}$$

as rational functions on C_-. Note that H_- is non-zero on C_-. Since $\frac{(s'_0)_+}{(s'_0)_-}$ restricts to a rational function on C_- which is regular and invertible in the dense open $C_- \cap U'$, we conclude that H_+/H_- restricts to an identical rational function on C_- which is regular and invertible on $C_- \cap U'$.

We now compute

$$\begin{aligned}
\tau_X^*(\mathrm{div}(f)) &= \iota_+^*(\mathrm{div}(f)) - \iota_-^*(\mathrm{div}(f)) \\
&= \mathrm{div}(f_+) - \mathrm{div}(f_-) \\
&= \big[((s'_0)_+) \cdot C_+ - (s_\infty^N) \cdot C_+\big] - \big[((s'_0)_-) \cdot C_- - (s_\infty^N) \cdot C_-\big] \\
&= [((s'_0)_+) \cdot C_+ - ((s'_0)_+) \cdot C_-] + [((s'_0)_+) \cdot C_- - ((s'_0)_-) \cdot C_-] \\
&\quad - \big[(s_\infty^N) \cdot C_+ - (s_\infty^N) \cdot C_-\big] \\
&= [((s'_0)_+) \cdot (C_+ - C_-)] + [C_- \cdot (((s'_0)_+) - ((s'_0)_-))] - \big[(s_\infty^N) \cdot (C_+ - C_-)\big] \\
&= ((s'_0)_+) \cdot (\mathrm{div}(t_+/t_-)) + C_- \cdot (\mathrm{div}((s'_0)_+/(s'_0)_-)) - N(s_\infty) \cdot (\mathrm{div}(t_+/t_-)) \\
&= ((s'_0)_+) \cdot (\mathrm{div}(t_+/t_-)) + C_- \cdot (\mathrm{div}(H_+/H_-)) - N(s_\infty) \cdot (\mathrm{div}(t_+/t_-)).
\end{aligned}$$

It follows from (b) and (c') that $t_\pm$ restrict to regular invertible functions on $((s'_0)_+)$ and (s_∞) along D. We set $h_1 = (\frac{t_+}{t_-})_{|((s'_0)_+)}$, $h_2 = (\frac{H_+}{H_-})_{|C_-}$ and $h_3 = (\frac{t_+}{t_-})_{|s_\infty}$. Let

$((s_0')_+)_n \to ((s_0')_+)$, $(C_-)_n \to C_-$ and $(s_\infty)_n \to (s_\infty)$ denote the normalization maps. Let $\nu_1 \colon ((s_0')_+)_n \to X$, $\nu_2 \colon (C_-)_n \to X$ and $\nu_3 \colon (s_\infty)_n \to X$ denote the composite maps. We now note that $((s_0')_+)$, C_- and (s_∞) are all regular along D by (2), (e) and (d'). Furthermore, none of these meets A by (2), (b) and (c'). It follows from (7.7) and (7.10) that $(\nu_1)_*(h_1)$, $(\nu_2)_*(h_2)$ and $(\nu_3)_*(h_3)$ die in $\mathrm{CH}_0^A(X|D)$. We conclude that $\tau_X^*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^A(X|D)$. In particular, τ_X^* descends to a map $\tau_X^* \colon \mathrm{CH}_0^{LW}(S_X) \to \mathrm{CH}_0^A(X|D)$. This finishes the proof. □

The following result generalizes Lemma 7.4 to higher dimensions.

Proposition 7.5. *Assume $d \geq 2$. Then the map τ_X^* descends to a group homomorphism $\tau_X^* \colon \mathrm{CH}_0^{LW}(S_X) \to \mathrm{CH}_0^A(X|D)$.*

Proof. We can assume $d \geq 3$ by Lemma 7.4. Let $\nu \colon C \hookrightarrow S_X$ be a reduced Cartier curve relative to D and let $f \in \mathcal{O}_{C,E}^{\times}$, where $E = \nu^*(D)$. By Lemma 7.2, we can assume that there are inclusions $C \hookrightarrow Y \hookrightarrow S_X$ satisfying the conditions (1) - (7) of Lemma 7.2. The only price we pay by doing so is that C may no longer be reduced. We achieve its reducedness using Lemma 7.3 as follows.

We replace C by a reduced Cartier curve (which we also denote by C) that is of the form given in Lemma 7.3. We shall now continue with the notations of the proof of Lemma 7.3.

We write $C = (t) \cap Y$, where $t \in H^0(S_X, \mathcal{L})$ such that $\mathcal{L}$ is a very ample line bundle on S_X. Let $t_\pm = \iota_\pm^*(t) \in H^0(X, \iota_\pm^*(\mathcal{L}))$ and let $C_\pm = (t_\pm) \cap Y = (t_\pm) \cap Y_\pm$. Let $\nu_\pm \colon C_\pm \hookrightarrow X$ denote the inclusions. It follows from our choice of the section that $(t_\pm)$ are integral. If $C_+ = C_-$, exactly the same argument as in the case of surfaces applies to show that $\tau_X^*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^A(X|D)$. So we assume $C_+ \neq C_-$. We can also assume $E \neq \emptyset$.

Let $\Delta(C) = C_+ \cup C_-$ denote the scheme theoretic image in X under the finite map Δ. Since X is smooth and integral, we can find a complete intersection integral surface $Z \subset X$ satisfying the following:

(1) $Z \supset \Delta(C)$;
(2) $Z \cap A$ is finite;
(3) $Z \cap (t_\pm)$ are integral curves;
(4) Z is smooth away from $\Delta(C)$.

Set $t_\pm^Z = (t_\pm)_{|Z}$. Since $C_\pm$ are integral and contained in $Z \cap (t_\pm)$, it follows that

$$(t_\pm^Z) = C_\pm. \tag{7.13}$$

We let $f_\pm = \nu_\pm^*(f)$. As in the proof of Lemma 7.4, the functions $f_\pm$ extend to meromorphic functions on $T_\pm = (Z \cap A) \cup C_\pm = (Z \cap A) \coprod C_\pm$ which are regular invertible in a neighborhood of $T' = (Z \cap A) \cup E$ by letting $f_\pm = 1$ on $Z \cap A$. Moreover, $f_+|_E = f_-|_E$, where $E = \nu^*(D)$. Let $S_\pm$ denote the set of closed points on $C_\pm$, where $f_\pm$ have poles. We let $T = T_+ \cup T_-$ and $S = S_+ \cup S_-$. It is clear that $S \cap D = \emptyset$.

We now choose another very ample line bundle $\mathcal{M}$ on X and $s_\infty \in H^0(X, \mathcal{M})$ (see the proof of Lemma 7.3) such that:

i) (s_∞) is integral;
ii) $(s_\infty) \cap Z$ and $(s_\infty) \cap (t_\pm)$ are proper and integral;
iii) $(s_\infty) \supset S$;
iv) $(s_\infty) \cap T' = \emptyset$;
v) $X \setminus (s_\infty)$ is affine;
vi) (s_∞) is smooth away from S;
vii) $(s_\infty) \cap Z$ is smooth away from $\Delta(C)$;
viii) $(s_\infty) \cap Z \not\subset \Delta(C)$.

As shown in the proof of Lemma 7.3, it follows from (3), (iv) and (vii) above that $(s_\infty^Z) := (s_\infty) \cap Z$ is smooth along D. Using (v), we can lift $f_\pm \in k(T_\pm)^\times$ to regular functions $F_\pm$ on $X \setminus (s_\infty)$. Using an argument identical to that given in the proof of Lemma 7.4, we can extend $(s_0)_\pm = s_\infty^N F_\pm$ (for some $N \gg 0$) to global sections $(s_0)_\pm$ of $\mathcal{M}^N$ on X so that:

a) $((s_0)_\pm)$ and $((s_0)_\pm) \cap Z$ are integral;
b) $((s_0)_\pm) \cap T' = \emptyset$;
c) $((s_0)_\pm) \cap Z \not\subset \Delta(C)$;
d) $((s_0)_\pm) \cap Z$ are smooth away from $\Delta(C)$.

As we argued in the proof of Lemma 7.3, it follows from (iv), (vii), (viii), (c) and (d) that (s_∞^Z) and $((s_0^Z)_\pm) := ((s_0)_\pm) \cap Z$ are smooth along D.

Setting $H_\pm = (s_0)_\pm / s_\infty^N$ and using the argument of the proof of Lemma 7.4, we get $H_\pm \in k(X)^\times$ and they restrict to rational functions on $C_\pm$ which are regular and invertible along D. Moreover, H_+/H_- restricts to a rational function on C_- which is regular and invertible along D. Since

$$H_{+|E} = F_{+|E} = f_{+|E} \ =^\dagger \ f_{-|E} = F_{-|E} = H_{-|E}, \tag{7.14}$$

where $\dagger$ follows from (7.8), we have $H_+/H_- = 1$ on E.

We now compute

$$\begin{aligned}
&\tau_X^*(\mathrm{div}(f)) \\
&\quad = \iota_+^*(\mathrm{div}(f)) - \iota_-^*(\mathrm{div}(f)) \\
&\quad = \mathrm{div}(f_+) - \mathrm{div}(f_-) \\
&\quad = \left[((s_0^Z)_+) \cdot C_+ - N(s_\infty^Z) \cdot C_+\right] - \left[((s_0^Z)_-) \cdot C_- - N(s_\infty^Z) \cdot C_-\right] \\
&\quad = \left[((s_0^Z)_+) \cdot C_+ - ((s_0^Z)_+) \cdot C_-\right] + \left[((s_0^Z)_+) \cdot C_- - ((s_0^Z)_-) \cdot C_-\right] \\
&\qquad - N\left[(s_\infty^Z) \cdot C_+ - (s_\infty^Z) \cdot C_-\right] \\
&\quad = \left[((s_0^Z)_+) \cdot (C_+ - C_-)\right] + \left[C_- \cdot (((s_0^Z)_+) - ((s_0^Z)_-))\right] - N\left[(s_\infty^Z) \cdot (C_+ - C_-)\right] \\
&\quad =^\dagger \left[((s_0^Z)_+) \cdot ((t_+^Z) - (t_-^Z))\right] + \left[C_- \cdot (((s_0^Z)_+) - ((s_0^Z)_-))\right] \\
&\qquad - N\left[(s_\infty^Z) \cdot ((t_+^Z) - (t_-^Z))\right] \\
&\quad = ((s_0^Z)_+) \cdot (\mathrm{div}(t_+^Z/t_-^Z)) - C_- \cdot (\mathrm{div}((s_0)_+/(s_0)_-)) - N(s_\infty^Z) \cdot (\mathrm{div}(t_+^Z/t_-^Z)) \\
&\quad = ((s_0^Z)_+) \cdot (\mathrm{div}(t_+^Z/t_-^Z)) - C_- \cdot (\mathrm{div}(H_+/H_-)) - N(s_\infty^Z) \cdot (\mathrm{div}(t_+^Z/t_-^Z)),
\end{aligned}$$

where $=^{\dagger}$ follows from (7.13).

It follows from (iv) and (b) that t^Z_+/t^Z_- restricts to regular and invertible functions on $((s^Z_0)_+)$ and (s^Z_∞) along D. Since $t \in H^0(S_X, \mathcal{L})$ and $t_\pm = \iota^*_\pm(t) \in H^0(X, \iota^*_\pm(\mathcal{L}))$, it follows that $(t_+)_{|D} = \iota'^*(t) = (t_-)_{|D}$. In particular, $(t^Z_+/t^Z_-)_{|E} = 1$. We have seen before that $(\frac{H_-}{H_+})_{|C_-}$ is a regular and invertible function on C_- along D and $(\frac{H_+}{H_-})_{|E} = 1$.

We set $h_1 = (\frac{t^Z_+}{t^Z_-})_{|((s^Z_0)_+)}$, $h_2 = (\frac{H_+}{H_-})_{|C_-}$ and $h_3 = (\frac{t^Z_+}{t^Z_-})_{|s^Z_\infty}$. Let $((s^Z_0)_+)_n \to ((s^Z_0)_+)$, $(C_-)_n \to C_-$ and $(s^Z_\infty)_n \to (s^Z_\infty)$ denote the normalization maps. Let $\nu_1 \colon ((s^Z_0)_+)_n \to X$, $\nu_2 \colon (C_-)_n \to X$ and $\nu_3 \colon (s^Z_\infty)_n \to X$ denote the composite maps. The curves $((s^Z_0)_+)$ and (s^Z_∞) are all smooth along D, and C_- is smooth along D by Lemma 7.3. Since none of these curves meets A by (iv), (b) and Lemma 7.3, we see that $(\nu_1)_*(h_1)$, $(\nu_2)_*(h_2)$ and $(\nu_3)_*(h_3)$ all die in $\mathrm{CH}^A_0(X|D)$. It follows that $\tau^*_X(\mathrm{div}(f))$ dies in $\mathrm{CH}^A_0(X|D)$. This finishes the proof. $\square$

8. Proof of the moving lemma

In this section, we shall finish the proof of Theorem 7.1. We need to recall the definition of the Chow group of 0-cycles on singular varieties introduced by Binda-Krishna [3]. This is an improved version of the Levine-Weibel Chow group.

Let k be a field and X a reduced quasi-projective scheme of dimension $d \geq 2$ over k. A good curve over X is a reduced scheme C of pure dimension one together with a finite map $\nu \colon C \to X$ whose image is not contained in X_{sing} and which is a local complete intersection (l.c.i.) morphism (see [3, § 2.3] for the definition of such a morphism) over a neighborhood of $\nu(C) \cap X_{\mathrm{sing}}$ in X. We let $\mathcal{R}_0(X)$ be the subgroup of $\mathcal{Z}_0(X_{\mathrm{reg}})$ generated by $\nu_*(\mathrm{div}(f))$, where $\nu \colon C \to X$ is a good curve over X and $f \in \mathcal{O}^\times_{C,\nu^{-1}(X_{\mathrm{sing}})}$. We let $\mathrm{CH}_0(X) = \mathcal{Z}_0(X)/\mathcal{R}_0(X)$. One can in fact consider only those good curves C in this definition which are regular away from $\nu^{-1}(X_{\mathrm{sing}})$ (see [3, Lemma 3.5]). It is known that the identity map of $\mathcal{Z}_0(X_{\mathrm{reg}})$ induces a surjection $\mathrm{CH}^{LW}_0(X) \twoheadrightarrow \mathrm{CH}_0(X)$. In certain cases, this map is known to be an isomorphism if k is algebraically closed. Otherwise, $\mathrm{CH}_0(X)$ has better behavior than $\mathrm{CH}^{LW}_0(X)$. Later in this section, we shall also prove a moving lemma for $\mathrm{CH}_0(X)$.

8.1. *Factorization of* τ^*_X *through* $\mathrm{CH}_0(S_X)$

Let k be a field and X a smooth quasi-projective scheme of pure dimension $d \geq 2$ over k. Let $D \subset X$ be an effective Cartier divisor with $D' = D_{\mathrm{red}}$. Let $A \subset D$ be a closed subscheme such that $\dim(A) \leq d - 2$. We shall now show that the map τ^*_X that we constructed in Proposition 7.5 actually factors through $\mathrm{CH}_0(S_X)$.

Before we do this, recall that if $f \colon X_1 \to X_2$ is a proper map and D_1 (resp. D_2) is an effective Cartier divisor on X_1 (resp. X_2) such that $f^*(D_2) \leq D_1$, then there is a push-forward map $f_* \colon \mathrm{CH}_0(X_1|D_1) \to \mathrm{CH}_0(X_2|D_2)$ (see [5, Lemma 2.7] or [32, Proposition 2.10]). But the existing proofs of this use a different (but equivalent) definition of the Chow group of 0-cycles with modulus from the one presented in § 7.1. In particular, if we are given a closed subscheme $A_2 \subset D_2$ such that $\dim(A_2) \leq \dim(X_2) - 2$ and $A_1 = f^*(A_2)$, then it is not immediately clear from our definition that the push-forward map $f_*\colon \mathrm{CH}_0^{A_1}(X_1|D_1) \to \mathrm{CH}_0^{A_2}(X_2|D_2)$ exists. The next lemma shows that this map actually exists.

Lemma 8.1. *The map $f_*\colon \mathcal{Z}_0(X_1 \setminus D_1) \to \mathcal{Z}_0(X_2 \setminus D_2)$ induces a push-forward map*

$$f_*\colon \mathrm{CH}_0^{A_1}(X_1|D_1) \to \mathrm{CH}_0^{A_2}(X_2|D_2).$$

Proof. It suffices to consider the case when $D_1 = f^*(D_2)$. Let $Y_1 \subset X_1$ be an integral curve not contained in D_1 and not meeting A_1. Let $\nu_1\colon (Y_1)_n \to X_1$ be the induced finite map from the normalization of Y_1. Let $Y_2 = f(Y_1)$. As f is proper, $Y_2 \subset X_2$ is closed and $Y_2 \cap A = \emptyset$. If Y_2 is a closed point, the proof is straightforward. So we assume Y_2 is an integral curve. Then it can not be contained in D_2. We let $\nu_2\colon (Y_2)_n \to X_2$ be the induced map from the normalization of Y_2. This gives rise to a finite dominant map $f'\colon (Y_1)_n \to (Y_2)_n$.

We let $g \in K_1^M(\mathcal{O}_{(Y_1)_n, \nu_1^{-1}(D_1)}, I_{D_1})$. We know from [12, Chapter 1] that $f_*(\mathrm{div}(g)) = \mathrm{div}(N(g))$, where $N\colon K_1^M(k(Y_1)) \to K_1^M(k(Y_2))$ is the norm map. So all we need to show to finish the proof is that

$$N(K_1^M(\mathcal{O}_{(Y_1)_n, \nu_1^{-1}(D_1)}, I_{D_1})) \subset K_1^M(\mathcal{O}_{(Y_2)_n, \nu_1^{-1}(D_2)}, I_{D_2}).$$

But this is well known (e.g., see [38, Lemma 6.19]) since $I_{D_1} = f^*(I_{D_2})$. $\square$

Using Lemma 8.1, we can prove the following result which will be used later in this section. This shows that $\mathrm{CH}_0^A(X|D)$ can also be defined in the style of the definition of $\mathrm{CH}_0(X|D)$ given in [5]. For an integral curve $C \subset \mathbb{P}^1_X$ and a point $t \in \mathbb{P}^1_k(k)$ such that $C \not\subset (X \times \{t\})$, we let $[C_t] = \pi_*([C] \cdot (X \times \{t\}))$, where $\pi\colon \mathbb{P}^1_X \to X$ is the projection and $[C] \cdot (X \times \{t\})$ is the 0-cycle associated to the scheme theoretic intersection of C and $X \times \{t\}$.

Lemma 8.2. *Let X and D be as above. Let $\mathcal{R}'_0(X|D) \subset \mathcal{Z}_0(X \setminus D)$ be the subgroup generated by the 0-cycles $[C_0] - [C_\infty]$, where $C \subset \mathbb{P}^1_X$ is an integral curve satisfying the following properties:*

1. *$C \cap \mathbb{P}^1_D$ is finite;*
2. *$C \cap (D \times_k \{0, \infty\}) = \emptyset$;*
3. *$C \cap \mathbb{P}^1_A = \emptyset$;*
4. *The Weil divisor $\nu^*(X \times \{1\}) - \nu^*(\mathbb{P}^1_D)$ is effective, where $\nu : C_n \to C \hookrightarrow \mathbb{P}^1_X$ is the composite finite map.*

Then $\mathrm{CH}_0^A(X|D) = \mathrm{Coker}(\mathcal{R}'_0(X|D) \to \mathcal{Z}_0(X \setminus D))$.

Proof. Let $\mathcal{R}_0^A(X|D) = \mathrm{Ker}(\mathcal{Z}_0(X \setminus D) \twoheadrightarrow \mathrm{CH}^A(X|D))$. Let $C \subset X$ be an integral curve not contained in D and not meeting A. Let $\nu\colon C_n \to X$ be the induced map and let $f \in K_1^M(\mathcal{O}_{C_n,\nu^{-1}(D)}, I_D)$. Taking the closure Γ_f of the graph of $f\colon C \dashrightarrow \mathbb{P}^1_k$ in $\mathbb{P}^1_X$, one easily sees that $\mathrm{div}(f) \in \mathcal{R}'_0(X|D)$.

Conversely, suppose $C \hookrightarrow \mathbb{P}^1_X$ is an integral curve defining an element in $\mathcal{R}'_0(X|D)$. Let $\pi\colon C_n \to \mathbb{P}^1_X \to X$ be the composite map. The projection $C \to \mathbb{P}^1_k$ defines an element $g \in k(C)^\times$ which lies in $K_1^M(\mathcal{O}_{C_n,\pi^{-1}(D)}, I_D)$ by (1) - (4). In particular, $\mathrm{div}(g) \in \mathcal{R}_0(C_n|\pi^*(D)) = \mathcal{R}_0^{\pi^{-1}(A)}(C_n|\pi^*(D))$.

We let C' be the image of C under the projection to X. If C' is a closed point, the proof is straightforward. So we can assume that C' is a curve, not contained in D and not meeting A by (1) and (3). Moreover, projection to X defines a finite dominant map $C \to C'$. This in turn induces a finite dominant map $f'\colon C_n \to C'_n$. Furthermore, π is same as the composition $C_n \xrightarrow{f'} C'_n \xrightarrow{\nu} X$. Since $[C_0] - [C_\infty] = \nu_* \circ f'_*(\mathrm{div}(g)) = (\nu \circ f')_*(\mathrm{div}(g))$, it suffices therefore to show that $\pi_*(\mathrm{div}(g)) = (\nu \circ f')_*(\mathrm{div}(g)) \in \mathcal{R}_0^A(X|D)$. But this follows directly from Lemma 8.1. $\square$

Lemma 8.3. *Assume that for every integer $n \geq 0$, the map $\tau^*_{\mathbb{P}^n_X}$ descends to a well defined homomorphism*

$$\tau^*_{\mathbb{P}^n_X} : \mathrm{CH}_0^{LW}(S_{\mathbb{P}^n_X}) \to \mathrm{CH}_0^{\mathbb{P}^n_A}(\mathbb{P}^n_X|\mathbb{P}^n_D).$$

Then the map $\tau^*_X\colon \mathrm{CH}_0^{LW}(S_X) \to \mathrm{CH}_0^A(X|D)$ factors through $\mathrm{CH}_0(S_X)$, giving a well defined homomorphism

$$\tau^*_X\colon \mathrm{CH}_0(S_X) \to \mathrm{CH}_0^A(X|D).$$

Proof. Let $\delta\colon \mathcal{Z}_0(S_X \setminus D) \to \mathrm{CH}_0^A(X|D)$ be the composition $\mathcal{Z}_0(S_X \setminus D) \to \mathrm{CH}_0^{LW}(S_X) \xrightarrow{\tau^*_X} \mathrm{CH}_0^A(X|D)$. We have to show that δ factors through $\mathrm{CH}_0(S_X)$. Using [3, Lemma 3.5], we have to show more precisely that $\delta(\nu_*(\mathrm{div}(f))) = 0$ for every finite l.c.i. morphism $\nu\colon C \to S_X$ from a reduced curve C whose image is not contained in D and for every rational function f on C that is regular and invertible along $\nu^{-1}(D)$.

Since ν is a finite l.c.i. morphism, we can factor it as a composition $\nu = \pi \circ \mu$, where $\mu\colon C \hookrightarrow \mathbb{P}^n_{S_X} = S_{\mathbb{P}^n_X}$ (using [3, Proposition 2.3]) is a regular embedding (see [9, Lemma 37.55.3]) and $\pi\colon \mathbb{P}^n_{S_X} \to S_X$ is the projection. In particular, $\mu(C) = C$ is a Cartier curve on the double $S_{\mathbb{P}^n_X}$. We let $Y = S_{\mathbb{P}^n_X}$ and $E = \mathbb{P}^n_D$. It is then clear that $\pi^{-1}(A) = \mathbb{P}^n_A$ is a closed subscheme of Y_{sing} and $\dim(\mathbb{P}^n_A) \leq \dim(Y) - 2$.

It follows from the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_0(Y \setminus E) \arrow[r, "\iota_\pm^*"] \arrow[d, "\pi_*"'] & \mathcal{Z}_0(\mathbb{P}^n_X \setminus E) \arrow[d, "\pi_*"] \\ \mathcal{Z}_0(S_X \setminus D) \arrow[r, "\iota_\pm^*"'] & \mathcal{Z}_0(X \setminus D) \end{tikzcd} \tag{8.1}$$

and the formula $\delta = \iota_+^* - \iota_-^*$ that the square

$$\begin{tikzcd} \mathcal{Z}_0(Y \setminus E) \arrow[r, "\delta_Y"] \arrow[d, "\pi_*"'] & \mathrm{CH}_0^{\mathbb{P}^n_A}(\mathbb{P}^n_X|E) \arrow[d, "\pi_*"] \\ \mathcal{Z}_0(S_X \setminus D) \arrow[r, "\delta"'] & \mathrm{CH}_0^A(X|D) \end{tikzcd}$$

commutes. That is, $\delta(\nu_*(\mathrm{div}(f))) = \delta(\pi_*(\mu_*(\mathrm{div}(f)))) = \pi_*(\delta_Y(\mu_*(\mathrm{div}(f))))$. Note that the push-forward map π_* on the right exists by Lemma 8.1.

By assumption, we have $\delta_Y(\mu_*(\mathrm{div}(f))) = 0 \in \mathrm{CH}_0^{\mathbb{P}^n_A}(\mathbb{P}^n_X|E)$, where δ_Y is the composition $\mathcal{Z}_0(Y \setminus E) \to \mathrm{CH}_0^{LW}(Y) \xrightarrow{\tau^*_{\mathbb{P}^n_X}} \mathrm{CH}_0^{\mathbb{P}^n_A}(\mathbb{P}^n_X|E)$. Since the push-forward map $\pi_*\colon \mathrm{CH}_0^{\mathbb{P}^n_A}(\mathbb{P}^n_X|E) \to \mathrm{CH}_0^A(X|D)$ is well defined as we saw above, we are done. $\square$

Lemma 8.4. *Let k be an infinite perfect field. Then τ_X^* factorizes through a homomorphism*

$$\tau_X^*\colon \mathrm{CH}_0(S_X) \to \mathrm{CH}_0^A(X|D).$$

Proof. Combine Proposition 7.5 and Lemma 8.3. $\square$

8.2. Behavior of $\mathrm{CH}_0^A(X|D)$ under the change of base field

Let k now be any field and X a smooth quasi-projective scheme of pure dimension $d \geq 2$ over k. Let $D \subset X$ be an effective Cartier divisor. Let $A \subset D$ be a closed subscheme such that $\dim(A) \leq d-2$. We shall need the following base change property of $\mathrm{CH}_0^A(X|D)$.

Proposition 8.5. *Let $k \hookrightarrow k'$ be a separable algebraic (possibly infinite) extension of fields. Let $X' = X_{k'}$, $D' = D_{k'}$ and $A' = A_{k'}$ denote the base change of X, D and A, respectively. Let $\mathrm{pr}_{k'/k} : X' \to X$ be the projection map. Then the following hold.*

(1) *There exists a pull-back $\mathrm{pr}^*_{k'/k} : \mathrm{CH}_0^A(X|D) \to \mathrm{CH}_0^{A'}(X'|D')$.*

(2) *If there exists a sequence of separable field extensions* $k = k_0 \subset k_1 \subset \cdots \subset k'$ *with* $k' = \cup_i k_i$, *then we have* $\varinjlim_i \mathrm{CH}_0^{A_{k_i}}(X_{k_i}|D_{k_i}) \xrightarrow{\simeq} \mathrm{CH}_0^{A'}(X'|D')$.

(3) *If* $k \hookrightarrow k'$ *is finite, then there exists a push-forward* $\mathrm{pr}_{k'/k\,*} : \mathrm{CH}_0^{A'}(X'|D') \to \mathrm{CH}_0^{A}(X|D)$ *such that* $\mathrm{pr}_{k'/k\,*} \circ \mathrm{pr}^*_{k'/k}$ *is multiplication by* $[k' : k]$.

Proof. Let $x \in X \setminus D$ be a closed point. Since $\mathrm{pr}_{k'/k}$ is smooth (ind-smooth to be precise), it follows from our hypothesis that $\mathrm{pr}^*_{k'/k}([x])$ is a well defined 0-cycle in $\mathcal{Z}_0(X'|D')$. We thus have a pull-back map $\mathrm{pr}^*_{k'/k} : \mathcal{Z}_0(X \setminus D) \to \mathcal{Z}_0(X' \setminus D')$. Let $\mathcal{R}'_0(X|D) \subset \mathcal{Z}_0(X \setminus D)$ be as in Lemma 8.2. To show that $\mathrm{pr}^*_{k'/k}$ preserves rational equivalence, it suffices to show that it takes $\mathcal{R}'_0(X|D)$ to $\mathcal{R}'_0(X'|D')$ by Lemma 8.2.

Let $C \subset X \times_k \mathbb{P}^1_k$ be an integral curve as in Lemma 8.2. It follows from the smoothness of $\mathrm{pr}_{k'/k}$ that $C' = \mathrm{pr}^*_{k'/k}(C) = C_{k'} \hookrightarrow (X \times_k \mathbb{P}^1_k)_{k'} = X' \times_{k'} \mathbb{P}^1_{k'}$ is a reduced curve whose irreducible components $C'_1, \ldots, C'_r$ satisfy conditions (1)-(4) of Lemma 8.2. Furthermore, the flat pull-back property of Bloch's cycle complex (see [8, Proposition 1.3]) says that

$$\mathrm{pr}^*_{k'/k}([C_0] - [C_\infty]) = [C'_0] - [C'_\infty] = \sum_{i=1}^{r} ([(C'_i)_0] - [(C'_i)_\infty]).$$

In particular, $\mathrm{pr}^*_{k'/k}([C_0] - [C_\infty])$ dies in $\mathrm{CH}_0^{A'}(X'|D')$. This proves (1).

It is clear that the map $\varinjlim_i \mathrm{CH}_0^{A_i}(X_i|D_i) \to \mathrm{CH}_0^{A'}(X'|D')$ is surjective. To show injectivity, suppose there is some $i \geq 0$ and $\alpha \in \mathcal{Z}_0(X_i|D_i)$ such that $\mathrm{pr}^*_{k'/k_i}(\alpha) \in \mathcal{R}'_0(X'|D')$. We can replace k by k_i and assume $i = 0$.

Let $C^j \hookrightarrow X' \times_{k'} \mathbb{P}^1_{k'} = (X \times_k \mathbb{P}^1_k)_{k'}$ for $j = 1, \cdots, r$ be a collection of integral curves as in the proof of (1) so that $\mathrm{pr}^*_{k'/k}(\alpha) = (\sum_{j=1}^{r} [C^j_0]) - (\sum_{j=1}^{r} [C^j_\infty])$. Let $\nu^j : C^j_n \to X' \times_{k'} \mathbb{P}^1_{k'}$ denote the maps from the normalizations of the above curves.

We can then find some $i \gg 0$ and integral curves $W^j \hookrightarrow X_i \times_{k_i} \mathbb{P}^1_{k_i}$ such that $C^j = W^j \times_{k_i} k'$ for each $j = 1, \cdots, r$. In particular, we have $C^j_0 = \mathrm{pr}^*_{k'/k_i}(W^j_0)$ and $C^j_\infty = \mathrm{pr}^*_{k'/k_i}(W^j_\infty)$ for $j = 1, \cdots, r$. Since pr_{k'/k_i} is smooth, it follows that $C^j_n = (W^j_n)_{k'}$ for each j. Moreover, it follows from [32, Lemma 2.2] that condition (4) above holds on each W^j_n. It follows that each W^j defines a rational equivalence for 0-cycles with modulus D_i on X_i.

We now set $\alpha_i = \mathrm{pr}^*_{k_i/k}(\alpha)$ and let $\beta = \alpha_i - (\sum_{j=1}^{r} ([W^j_0] - [W^j_\infty])) \in \mathcal{Z}_0(X_i \setminus D_i)$. It then follows that $\mathrm{pr}^*_{k'/k_i}(\beta) = \mathrm{pr}^*_{k'/k_i}(\alpha_i) - \sum_{j=1}^{r} ([C^j_0] - [C^j_\infty]) = \mathrm{pr}^*_{k'/k}(\alpha) - \sum_{j=1}^{r} ([C^j_0] - [C^j_\infty]) = 0$ in $\mathcal{Z}_0(X' \setminus D')$. Since the map $\mathrm{pr}^*_{k'/k_i} : \mathcal{Z}_0(X_i \setminus D_i) \to \mathcal{Z}_0(X' \setminus D')$ of free abelian groups is clearly injective, we get $\beta = 0$, which means that $\alpha_i \in \mathcal{R}_0(X_i|D_i)$. This proves (2). The existence of push-forward follows from Lemma 8.1 and the formula $\mathrm{pr}_{k'/k\,*} \circ \mathrm{pr}^*_{k'/k} = [k' : k]$ is obvious from the definitions. This proves (3). $\square$

8.3. *Proof of Theorem 7.1*

We shall now prove Theorem 7.1. This is equivalent to showing that the canonical surjection $\mathrm{CH}_0^A(X|D) \twoheadrightarrow \mathrm{CH}_0(X|D)$ is also a monomorphism. Using Proposition 8.5 and the standard pro-ℓ extension trick, we easily reduce to the case when k is infinite and perfect. We assume this to be the case in the rest of the proof.

By Lemma 8.4, the map $(\iota_+^* - \iota_-^*)\colon \mathcal{Z}_0(S_X \setminus D) \to \mathcal{Z}_0(X \setminus D)$ defines a homomorphism $\tau_X^*\colon \mathrm{CH}_0(S_X) \to \mathrm{CH}_0^A(X|D)$. We now define two maps in opposite direction,

$$p_{\pm,*}\colon \mathcal{Z}_0(X \setminus D) \rightrightarrows \mathcal{Z}_0(S_X \setminus D)$$

by $p_{+,*}([x]) = \iota_{+,*}([x])$ (resp. by $p_{-,*}([x]) = \iota_{-,*}([x])$) for a closed point $x \in X \setminus D$. Concretely, the two maps $p_{+,*}$ and $p_{-,*}$ copy a cycle α in one of the two components of the double S_X (the X_+ or the X_- copy). Since α is supported outside D (by definition of $\mathcal{R}_0(X|D)$), the cycles $p_{+,*}(\alpha)$ and $p_{-,*}(\alpha)$ give classes in $\mathrm{CH}_0(S_X)$. By [3, Proposition 5.9], the maps $p_{\pm,*}$ descend to group homomorphisms $p_{\pm,*}\colon \mathrm{CH}_0(X|D) \to \mathrm{CH}_0(S_X)$. Composing with the canonical surjection $\mathrm{CH}_0^A(X|D) \twoheadrightarrow \mathrm{CH}_0(X|D)$, we get maps $p^A_{\pm,*}\colon \mathrm{CH}_0^A(X|D) \to \mathrm{CH}_0(S_X)$. Furthermore, it is clear from the definitions of $p^A_{\pm,*}$ and τ_X^* that $\tau_X^* \circ p^A_{\pm,*} = \mathrm{Id}$. We thus get maps

$$\mathrm{CH}_0^A(X|D) \twoheadrightarrow \mathrm{CH}_0(X|D) \xrightarrow{p_{\pm,*}} \mathrm{CH}_0(S_X) \xrightarrow{\tau_X^*} \mathrm{CH}_0^A(X|D),$$

whose composition is identity. It follows that the first arrow from the left is injective. This finishes the proof. □

8.4. *Moving lemma for* $\mathrm{CH}_0(X)$

We shall end this section with a moving lemma for the Chow group of 0-cycles $\mathrm{CH}_0(X)$ for a singular scheme X. This result is of independent interest in the study of 0-cycles and algebraic K-theory on singular varieties. We remark that this moving lemma for $\mathrm{CH}_0^{LW}(X)$ is yet unknown over finite fields.

Let k be a field and X a reduced quasi-projective scheme of dimension $d \geq 2$ over k. Suppose that $A \subset X_{\mathrm{sing}}$ is a closed subscheme such that $\dim(A) \leq d-2$. We can then define $\mathrm{CH}_0^A(X)$ by repeating the definition of $\mathrm{CH}_0(X)$, except that we allow only those good curves $\nu\colon C \to X$ which satisfy the additional condition that $\nu(C) \cap A = \emptyset$. We clearly have a canonical surjection $\mathrm{CH}_0^A(X) \twoheadrightarrow \mathrm{CH}_0(X)$.

Proposition 8.6. *Let k be any perfect field. Then the map $\mathrm{CH}_0^A(X) \to \mathrm{CH}_0(X)$ is an isomorphism.*

Proof. We only need to show injectivity. To reduce it to the case when k is infinite, we note that an analogue of Proposition 8.5 is proven for $\mathrm{CH}_0(X)$ in [3, Proposition 6.1].

The reader may check that this proof works verbatim for $\mathrm{CH}_0^A(X)$. This allows us to use the pro-ℓ extension trick as we did in Theorem 7.1 to reduce the proof of the proposition when k is infinite and perfect.

Let $\nu\colon C \to X$ be a good curve over X and let f be a rational function on C which is regular and invertible in a neighborhood of $\nu^{-1}(X_{\mathrm{sing}})$. It suffices to show that $\nu_*(\mathrm{div}(f))$ belongs to the subgroup $\mathcal{R}_0^A(X)$ of rational equivalences that define $\mathrm{CH}_0^A(X)$.

First of all, we can assume by [3, Lemma 3.5] that $\nu\colon C \to X$ is an l.c.i. morphism. As a consequence, we have a factorization $\nu = \pi \circ \mu$, where $\mu\colon C \hookrightarrow \mathbb{P}^n_X$ is a regular embedding and $\pi\colon \mathbb{P}^n_X \to X$ is the projection. In particular, $\mu(C) = C$ is a Cartier curve on $\mathbb{P}^n_X$. The smoothness of π implies that $(\mathbb{P}^n_X)_{\mathrm{sing}} = \mathbb{P}^n_{X_{\mathrm{sing}}}$. Furthermore, $\pi^{-1}(A) = \mathbb{P}^n_A$ is a closed subscheme of $(\mathbb{P}^n_X)_{\mathrm{sing}}$ and $\dim(\mathbb{P}^n_A) \le \dim(\mathbb{P}^n_X) - 2$. It is also clear that $\nu_*(\mathrm{div}(f)) = \pi_*(\mu_*(\mathrm{div}(f)))$.

We can now apply [10, Lemma 1.3] to find a reduced Cartier curve $\mu'\colon C' \hookrightarrow \mathbb{P}^n_X$ such that $C' \cap \mathbb{P}^n_A = \emptyset$, and a rational function f' on C' which is regular and invertible along $C' \cap \mathbb{P}^n_{X_{\mathrm{sing}}}$ and $\mu_*(\mathrm{div}(f)) = \mathrm{div}(f')$. This can also be easily deduced from the proof of Proposition 7.5. We let $\nu'\colon C' \xrightarrow{\mu'} \mathbb{P}^n_X \xrightarrow{\pi} X$ denote the composite map. Since μ' is a regular immersion along $\mathbb{P}^n_{X_{\mathrm{sing}}}$, it follows that ν' is an l.c.i. morphism along X_{sing}. In particular, $\nu'\colon C' \to X$ is a good curve over X. Furthermore, we have $\nu'(C') \cap A = \pi(C' \cap \mathbb{P}^n_A) = \emptyset$. Since $\nu'_*(\mathrm{div}(f')) = \pi_*(\mathrm{div}(f')) = \pi_* \circ \mu_*(\mathrm{div}(f)) = \nu_*(\mathrm{div}(f))$, it follows that $\mathrm{div}(f)$ dies in $\mathrm{CH}_0^A(X)$. This finishes the proof. $\square$

9. Comparison of the idele class groups

We shall work under the following assumptions in this section. We let k be a finite field and X a normal projective integral scheme of dimension $d \ge 1$ over k. Let $C \subset X$ be a reduced closed subscheme of pure codimension one whose complement U is regular. We let $A = C_{\mathrm{sing}}$ with reduced induced closed subscheme structure. Note that $\dim(A) \le d - 2$. In particular, $A = \emptyset$ when $d = 1$. Recall that $\mathrm{Div}_C(X)$ is the directed set of closed subschemes of X with support C. Let K denote the function field of X. We fix a separable closure $\overline{K}$ of K and let $K^{\mathrm{ab}} \subset \overline{K}$ be the maximal abelian extension of K.

We fix a closed subscheme $D \in \mathrm{Div}_C(X)$. If $x \in U$ is a closed point, then it defines a Parshin chain on $(U \subset X)$ of length zero. Hence, we have the injection $\mathbb{Z} = K_0^M(k(x)) \hookrightarrow I_{U/X}$. Extending it linearly over $U_{(0)}$, we see that there is canonical inclusion $\mathcal{Z}_0(U) \hookrightarrow I_{U/X}$. Composing with the canonical maps $I_{U/X} \twoheadrightarrow C_{U/X} \to \widetilde{C}_{U/X}$, we get a map

$$\mathrm{cyc}_{U/X}\colon \mathcal{Z}_0(U) \to \widetilde{C}_{U/X}. \tag{9.1}$$

Composing with the surjection $\widetilde{C}_{U/X} \twoheadrightarrow C(X, D)$, we get our cycle class map

$$\mathrm{cyc}_{X|D}\colon \mathcal{Z}_0(U) \to C(X, D). \tag{9.2}$$

Recall that Bloch's formula for the 0-cycles with modulus asks whether the following hold if D is an effective Cartier divisor.

(1) $\mathrm{cyc}_{X|D}$ annihilates the subgroup of rational equivalences.
(2) The resulting map $\mathrm{cyc}_{X|D}\colon \mathrm{CH}_0(X|D) \to C(X, D)$ is an isomorphism.

These conditions together are equivalent to asking whether the idele class groups $\mathrm{CH}_0(X|D)$ and $C_{KS}(X, D)$ are isomorphic. The goal of this section is to prove this to be the case when X is regular.

9.1. Reciprocity maps for 0-cycles

At any rate, we see using (7.4) and [18, Proposition 4.8, Lemma 8.4] that we have a commutative diagram of short exact sequences of topological abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Z}_0(U)_0 \arrow[r] \arrow[d] & \mathcal{Z}_0(U) \arrow[r, "\deg/n"] \arrow[d, "\mathrm{cyc}_{X|D}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C(X,D)_0 \arrow[r] \arrow[d] & C(X,D) \arrow[r, "\deg/n"] \arrow[d, "\rho_{X|D}"] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \pi_1^{\mathrm{adiv}}(X,D)_0 \arrow[r] & \pi_1^{\mathrm{adiv}}(X,D) \arrow[r, "\deg/n"] & \widehat{\mathbb{Z}} \arrow[r] & 0,
\end{tikzcd} \tag{9.3}$$

where $n \geq 1$ is an integer depending only on U.

We also have a commutative diagram of short exact sequences of topological abelian groups (see § 6 for the definition of D_ρ)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Z}_0(U)_0 \arrow[r] \arrow[d] & \mathcal{Z}_0(U) \arrow[r, "\deg/n"] \arrow[d, "\mathrm{cyc}_{X|D_\rho}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C(X,D_\rho)_0 \arrow[r] \arrow[d] & C(X,D_\rho) \arrow[r, "\deg/n"] \arrow[d, "\rho^{\mathrm{c}}_{X|D}"] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \pi_1^{\mathrm{ab}}(X,D)_0 \arrow[r] & \pi_1^{\mathrm{ab}}(X,D) \arrow[r, "\deg/n"] & \widehat{\mathbb{Z}} \arrow[r] & 0.
\end{tikzcd} \tag{9.4}$$

Let $\vartheta_{U/X} = \widetilde{\rho}_{U/X} \circ \mathrm{cyc}_{U/X}$, $\vartheta_{X|D} = \rho_{X|D} \circ \mathrm{cyc}_{X|D}$ and $\vartheta^{\mathrm{c}}_{X|D} = \rho^{\mathrm{c}}_{X|D} \circ \mathrm{cyc}_{X|D_\rho}$. It is then clear from the definition of the reciprocity map $\rho_{U/X}$ (see § 5.2) and the cycle class map $\mathrm{cyc}_{U/X}$ that the composition $\vartheta_{U/X}$ is the Frobenius substitution which takes a closed point $x \in U$ to the image of the Frobenius automorphism under the canonical map $\mathrm{Gal}(\overline{k(x)}/k(x)) \to \pi_1^{\mathrm{ab}}(U)$. Hence, the same holds for $\rho_{X|D}$ and $\vartheta^{\mathrm{c}}_{X|D}$.

It follows from the classical ramified class field theory for curves that for every effective Cartier divisor $D \in \mathrm{Div}_C(X)$, the map $\vartheta^{\mathrm{c}}_{X|D}$ factors through rational equivalences and defines a map (obviously continuous) $\vartheta^c_{X|D}\colon \mathrm{CH}_0(X|D) \to \pi_1^{\mathrm{ab}}(X,D)$ (see [28, Proposition 3.2] and [4, § 8]). Taking the limit and using Proposition 5.4, we see that $\vartheta_{U/X}$ descends to a continuous homomorphism

$$\vartheta_{U/X}\colon C(U) \to \pi_1^{\mathrm{ab}}(U). \tag{9.5}$$

It follows from the generalized Chebotarev-Lang density theorem (e.g., see [42, Theorem 5.8.16]) that this map has dense image.

9.2. Bloch's formula for regular schemes

Throughout this subsection, we shall assume that X is regular. Note that every $D \in \mathrm{Div}_C(X)$ is then an effective Cartier divisor on X. Under the extra assumption, we would like to work with a new subgroup of $H^1(K)$ which we introduce below.

Definition 9.1. Let $\mathrm{fil}^A_D H^1(K)$ be the subgroup of characters $\chi \in H^1(U)$ such that for every integral curve $Y \subset X$ not contained in C and not meeting A, the finite map $\nu\colon Y_n \to X$ has the property that the image of χ under $\nu^*\colon H^1(U) \to H^1(\nu^{-1}(U))$ lies in $\mathrm{fil}_{\nu^*(D)} H^1(k(Y))$, where Y_n is the normalization of Y (see Definition 5.1). We let $\pi_1^A(X,D)$ denote the Pontryagin dual of $\mathrm{fil}^A_D H^1(K)$ and let $\pi_1^A(X,D)_0$ be the kernel of the map $\pi_1^A(X,D) \to \mathrm{Gal}(\overline{k}/k)$.

We have the canonical inclusions

$$\mathrm{fil}^c_D H^1(K) \subset \mathrm{fil}^A_D H^1(K) \subset H^1(U) \subset H^1(K). \tag{9.6}$$

These give rise to the surjective continuous homomorphisms of profinite groups

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^A(X,D) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X,D). \tag{9.7}$$

Lemma 9.2. *The composite map $C(U) \xrightarrow{\vartheta_{U/X}} \pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^A(X,D)$ factors through a continuous homomorphism*

$$\vartheta^A_{X|D}\colon \mathrm{CH}^A(X|D) \to \pi_1^A(X,D).$$

Proof. Since $C(U)$ is the limit of the pro-abelian group $\{\mathrm{CH}^A_0(X|D)\}_{D\in \mathrm{Div}_C(X)}$ by Theorem 7.1, it suffices to show that the composite map $\mathcal{Z}_0(U) \to \pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^A(X,D)$ kills $\mathcal{R}^A_0(X|D)$. Equivalently, we need to show that for every character $\chi \in \mathrm{fil}^A_D H^1(K)$, the induced character $\vartheta^*_{U/X}(\chi)$ annihilates $\mathcal{R}^A_0(X|D)$.

Let $Y \subset X$ be an integral curve not contained in D and not meeting A and let $\nu\colon Y_n \to X$ be the induced finite map. Let $V = \nu^{-1}(U)$ and $E = \nu^*(D)$. Then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{fil}^A_D H^1(U) & \xrightarrow{\vartheta^*_{U/X}} & \mathrm{Hom}_{\mathrm{cont}}(\mathcal{Z}_0(U), \mathbb{Q}/\mathbb{Z}) \\ \downarrow{\scriptstyle \nu^*} & & \downarrow{\scriptstyle \nu^*} \\ \mathrm{fil}_E H^1(V) & \xrightarrow{\vartheta^*_{V/Y_n}} & \mathrm{Hom}_{\mathrm{cont}}(\mathcal{Z}_0(V), \mathbb{Q}/\mathbb{Z}), \end{array} \tag{9.8}$$

where the right vertical arrow is induced by the push-forward map $\nu_* \colon \mathcal{Z}_0(V) \to \mathcal{Z}_0(U)$ and the left vertical arrow is induced by the definition of $\mathrm{fil}^A_D H^1(U)$. We need to show that $\vartheta^*_{V/Y_n} \circ \nu^*(\chi)$ annihilates $\mathcal{R}_0(Y_n|E)$. But this follows from the classical ramified class field theory for curves (e.g., see [41]). □

It is clear from the above definitions that there is a commutative diagram

$$\begin{array}{ccccc} C(U) & \twoheadrightarrow & \mathrm{CH}^A_0(X|D) & \twoheadrightarrow & \mathrm{CH}_0(X|D) \\ \downarrow{\scriptstyle \vartheta_{U/X}} & & \downarrow{\scriptstyle \vartheta^A_{X|D}} & & \downarrow{\scriptstyle \vartheta^c_{X|D}} \\ \pi^{\mathrm{ab}}_1(U) & \twoheadrightarrow & \pi^A_1(X, D) & \twoheadrightarrow & \pi^{\mathrm{ab}}_1(X, D). \end{array} \tag{9.9}$$

Lemma 9.3. *The map $\vartheta^A_{X|D} \colon \mathrm{CH}^A_0(X|D) \to \pi^A_1(X, D)$ is injective with dense image. Moreover, the induced map $\vartheta^A_{X|D} \colon \mathrm{CH}^A_0(X|D)_0 \to \pi^A_1(X, D)_0$ is an isomorphism.*

Proof. The density of the image follows from (9.5). We show the other assertions. We first show that $\vartheta^A_{X|D} \colon \mathrm{CH}^A_0(X|D)_0 \to \pi^A_1(X, D)_0$ is surjective. It follows from [4, Lemma 8.4, Theorem 8.5] that the map $\vartheta^c_{X|D} \colon \mathrm{CH}_0(X|D)_0 \to \pi^{\mathrm{ab}}_1(X, D)_0$ is an isomorphism for all $D \in \mathrm{Div}_C(X)$. Taking the limit and noting that $C(U)_0 \cong \varprojlim_D \mathrm{CH}_0(X|D)_0$ by (7.5), it follows that $\vartheta_{U/X} \colon C(U)_0 \to \pi^{\mathrm{ab}}_1(U)_0$ is an isomorphism. We now use the commutative diagram

$$\begin{array}{ccc} C(U)_0 & \xrightarrow{\vartheta_{U/X}} & \pi^{\mathrm{ab}}_1(U)_0 \\ \downarrow & & \downarrow \\ \mathrm{CH}^A_0(X|D)_0 & \xrightarrow{\vartheta^A_{X|D}} & \pi^A_1(X, D)_0. \end{array} \tag{9.10}$$

We have shown above that the top horizontal arrow is an isomorphism. The right vertical arrow is surjective. The surjectivity of $\vartheta^A_{X|D}$ on degree zero subgroups follows.

To show injectivity, we consider the diagram

$$\begin{array}{ccc} \mathrm{CH}^A_0(X|D) & \xrightarrow{\vartheta^A_{X|D}} & \pi^A_1(X, D) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathrm{CH}_0(X|D) & \xrightarrow{\vartheta^c_{X|D}} & \pi^{\mathrm{ab}}_1(X, D). \end{array} \tag{9.11}$$

The left vertical arrow is an isomorphism by Theorem 7.1 and the bottom horizontal arrow is injective by [4, Lemma 8.4, Theorem 8.5]. It follows that the top horizontal arrow is injective. □

Corollary 9.4. *The canonical map $\pi_1^A(X, D) \to \pi_1^{\mathrm{ab}}(X, D)$ is an isomorphism. In particular, the canonical inclusion $\mathrm{fil}^c_D H^1(K) \hookrightarrow \mathrm{fil}^A_D H^1(K)$ is a bijection.*

Proof. We only need to show that $\pi_1^A(X, D)_0 \to \pi_1^{\mathrm{ab}}(X, D)_0$ is injective. But this follows immediately by restricting the diagram (9.9) to degree zero subgroups and noting that all arrows in the right square except the bottom horizontal arrow are isomorphisms by [4, Lemma 8.4, Theorem 8.5], Theorem 7.1 and Lemma 9.3. □

Proof of Theorem 1.4. By Proposition 5.2 and Definition 5.3, the theorem is equivalent to the statement that $\mathrm{fil}_D H^1(K) = \mathrm{fil}^c_D H^1(K)$ as subgroups of $H^1(K)$.

We first show that $\mathrm{fil}_D H^1(K) \subset \mathrm{fil}^c_D H^1(K)$. In order to show this, we can replace $\mathrm{fil}^c_D H^1(K)$ by $\mathrm{fil}^A_D H^1(K)$ using Corollary 9.4. We now let $\chi \in \mathrm{fil}_D H^1(K)$ and let $Y \subset X$ be an integral curve not contained in D and not meeting A. Let $\nu \colon Y_n \to X$ be the induced map from the normalization of Y. We need to show that $\nu^*(\chi) \in \mathrm{fil}_{\nu^*(D)} H^1(k(Y))$.

Since $Y \cap C_{\mathrm{sing}} = Y \cap A = \emptyset$, we can replace X (resp. C) by $X \setminus A$ (resp. C_{reg}) to show the above assertion. Since C_{reg} is a smooth divisor inside the smooth variety $X \setminus A$, [28, Corollary 2.8] (which does not require $X \setminus A$ to be projective) applies. This yields $\nu^*(\chi) \in \mathrm{fil}_{\nu^*(D)} H^1(k(Y))$.

We now show the reverse inclusion. Let $\chi \in \mathrm{fil}^c_D H^1(K) \subset H^1(U)$. We let $X' = X \setminus A$, $D' = D \cap X'$ and $C' = C \cap X'$. Note that $U = X' \setminus C'$. By [18, Theorem 7.19], it suffices to show that $\chi \in \mathrm{fil}_{D'} H^1(K)$. To prove this, we first deduce from Proposition 6.9 that $\chi \in \mathrm{fil}^c_{D'} H^1(K)$. Since X' is regular and C' is a regular divisor on X' with complement U, it follows from [28, Corollary 2.8] that $\chi \in \mathrm{fil}_{D'} H^1(K)$. This finishes the proof. □

Remark 9.5. With (X, D, U) as in Theorem 1.4 and $r \geq 1$ an integer, let $\mathcal{R}_r(U)$ be the set of isomorphism classes of lisse $\overline{\mathbb{Q}}_\ell$-Weil sheaves on U of rank r up to semi-simplification. For a given $V \in \mathcal{R}_r(U)$, one can define the Swan conductor $\mathrm{Sw}(V)$ as $\mathrm{Sw}(V) = \sum\limits_E \mathrm{Sw}_E(V)[E] \in \mathrm{Div}(X)$, where E runs through integral divisors on X and $\mathrm{Sw}_E(V)$ is the Swan conductor of V at the generic point of E. The latter was defined in [11, Definition 3.1] when X is a curve. But $\mathrm{Sw}_E(V)$ makes sense in any dimension if we use the Abbes-Saito filtration $G^{(\bullet)}_{K_E}$ of $G(K_E)$, where K_E is the Henselization of K at the generic point of E (see [1]) instead of the classical ramification filtration $I^{(\bullet)}_{K_E}$ of $G(K_E)$ for curves, used in [11]. One says that $V \in \mathcal{R}^{\mathrm{div}}_r(X, D) \subset \mathcal{R}_r(U)$ if $\mathrm{Sw}(V) \leq D$.

We can now ask if $\mathcal{R}^{\mathrm{div}}_r(X, D) = \mathcal{R}_r(X, D)$, where the latter group is as in [11, Definition 3.6]. Theorem 1.4 can be viewed as an answer to the $r = 1$ case of this question. An attempt to answer this question may also lead one to a proof of Theorem 1.4 using only ramification theory.

Proof of Theorem 1.1. We first assume k to be finite. We need to show that $\mathrm{cyc}_{X|D} \colon \mathcal{Z}_0(U) \to C_{KS}(X,D)$ induces an isomorphism $\mathrm{cyc}_{X|D} \colon \mathrm{CH}_0(X|D) \xrightarrow{\cong} C_{KS}(X,D)$.

The theorem is already known when $d \le 2$ by [4]. We shall therefore assume that $d \ge 3$. We can also assume that X is integral. Since the composite map

$$\mathcal{Z}_0(U) \xrightarrow{\mathrm{cyc}_{X|D}} C(X,D) \xrightarrow{\psi_{X|D}} C_{KS}(X,D) \tag{9.12}$$

is the cycle class map of (1.1), we can replace $C_{KS}(X,D)$ by $C(X,D)$ using [18, Theorem 3.8].

Let $s_{X|D} \colon \mathcal{Z}_0(U) \twoheadrightarrow \mathrm{CH}_0(X|D)$ be the canonical quotient map. We denote the quotient maps $C(U) \twoheadrightarrow \mathrm{CH}_0(X|D)$ and $\pi_1^{\mathrm{ab}}(U) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X,D)$ by $t_{X|D}$ and $q'_{X|D}$, respectively. Let $\widehat{\rho}_{U/X} \colon \mathcal{Z}_0(U) \to C(U)$ be the canonical map induced by taking the limit of the maps $s_{X|D'}$ as D' runs through $\mathrm{Div}_C(X)$.

We now consider the commutative diagram

$$\begin{tikzcd}
\mathcal{Z}_0(U) \arrow[r, "\widehat{\rho}_{U/X}"] \arrow[dr, "s_{X|D}"'] \arrow[ddd, "\mathrm{cyc}_{U/X}"'] & C(U) \arrow[r, "\vartheta_{U/X}"] \arrow[d, "t_{X|D}"] & \pi_1^{\mathrm{ab}}(U) \arrow[d, "q'_{X|D}"] \\
& \mathrm{CH}_0(X|D) \arrow[r, "\vartheta^{c}_{X|D}"] \arrow[d, dotted, "\mathrm{cyc}_{X|D}"] & \pi_1^{\mathrm{ab}}(X,D) \arrow[d, "r_{X|D}"] \\
& C(X,D) \arrow[r, "\rho_{X|D}"] & \pi_1^{\mathrm{adiv}}(X,D) \\
\widetilde{C}_{U/X} \arrow[ur, "p_{X|D}"] \arrow[rr, "\widetilde{\rho}_{U/X}"] & & \pi_1^{\mathrm{ab}}(U). \arrow[u, "q_{X|D}"']
\end{tikzcd} \tag{9.13}$$

The triangle on the top left commutes by the definitions of its various arrows. The square on the top right commutes by (9.9). The bottom trapezium commutes by (5.8). The big outer square is commutative because we have seen before that for any closed point $x \in U$, both the maps $r_{X|D} \circ q'_{X|D} \circ \vartheta_{X|D} \circ \widehat{\rho}_{U/X}$ and $q_{X|D} \circ \widetilde{\rho}_{X|D} \circ \mathrm{cyc}_{U/X}$ coincide with the Frobenius substitution at x. The map $\vartheta^{c}_{X|D}$ is injective by Theorem 7.1, Lemma 9.3 and Corollary 9.4. The map $r_{X|D}$ is an isomorphism by Theorem 1.4 and $\rho_{X|D}$ is injective by Theorem 1.3. By a diagram chase, it follows that the composite map $p_{X|D} \circ \mathrm{cyc}_{U/X}$ annihilates $\mathcal{R}_0(X|D)$ and induces an honest map

$$\mathrm{cyc}_{X|D} \colon \mathrm{CH}_0(X|D) \to C(X,D) \tag{9.14}$$

such that the middle square in (9.13) commutes. Moreover, this map is injective. It is surjective by (2.10) and [22, Theorem 2.5]. This proves the theorem for finite base field case.

When k is the algebraic closure of a finite field, an easy descent argument shows that there exists a finite field $k' \subset k$ and a geometrically integral and smooth projective variety X' over k' together with an effective Cartier divisor $D' \subset X'$ such that $X = X'_k$ and $D = D'_k$. Since $cyc_{X|D}$ is clearly compatible with the pull-back via field extensions, and since $\mathrm{CH}_0(X|D)$ and $C_{KS}(X,D)$ are continuous functors (see [3, Proposition 6.2] and [24, Sublemma 7.3] and recall that K-theory is also a continuous functor), we conclude from the case of finite base field. □

9.3. Bloch's formula without regularity

In this subsection, we shall prove Theorem 1.2, Bloch's formula in the pro-setting when X is not regular. We remark that it is not clear if one should even expect Bloch's formula for each D when X is singular. Our assumption now is the following. We let X be a normal projective integral scheme of dimension $d \geq 1$ over a finite field k and let D be an effective Cartier divisor on X. We assume that $U = X \setminus D$ is regular.

We shall prove Theorem 1.2 after the following lemma. We shall ignore to write the indexing set $\mathbb{N}$ of the pro-abelian groups used in the theorem in what follows.

Lemma 9.6. *The cycle class map* $\mathrm{cyc}_{U/X} \colon \mathcal{Z}_0(U) \to \widetilde{C}_{U/X}$ *descends to a continuous homomorphism of pro-abelian groups*

$$\mathrm{cyc}^{\bullet}_{X|D} \colon \{\mathrm{CH}_0(X|nD)\} \to \{C(X,nD)\}.$$

Proof. We need to show that the map of pro-abelian groups $\{\mathrm{cyc}_{X|D}\} \colon \{\mathcal{R}_0(X|nD)\} \to \{C(X,nD)\}$ is zero. Since $\mathcal{R}_0(X|nD) \subset \mathcal{Z}_0(U)_0$, it suffices to show using (9.3) that the map of pro-abelian groups $\{\mathrm{cyc}_{X|nD}\} \colon \{\mathcal{R}_0(X|nD)\} \to \{C(X,nD)_0\}$ is zero. To show this, we look at the sequence of maps

$$\{\mathcal{R}_0(X|nD)\} \xrightarrow{\{\mathrm{cyc}_{X|nD}\}} \{C(X,nD)_0\} \xrightarrow{\{\rho_{X|nD}\}} \{\pi_1^{\mathrm{ab}}(X,nD)_0\}, \tag{9.15}$$

where the second map $\{\rho_{X|nD}\} = \rho^{\bullet}_{X|D}$ exists by Theorem 6.4.

We have seen just above (9.5) that the map $\mathcal{Z}_0(U) \to \pi_1^{\mathrm{ab}}(X,nD)$ factors through $\vartheta^{c}_{X|nD} \colon \mathrm{CH}_0(X|nD) \to \pi_1^{\mathrm{ab}}(X,nD)$ for every n. It follows that the composite arrow in (9.15) is zero. The lemma now follows from Theorem 6.5. □

Using Lemma 9.6, we get commutative diagrams

$$\begin{tikzcd} C(U) \arrow[r, "\mathrm{cyc}_{U/X}"] \arrow[dr, "\vartheta_{U/X}"'] & \widetilde{C}_{U/X} \arrow[d, "\widetilde{\rho}_{U/X}"] \\ & \pi_1^{\mathrm{ab}}(U) \end{tikzcd} \qquad \begin{tikzcd} \{\mathrm{CH}_0(X|nD)\} \arrow[r, "\mathrm{cyc}^{\bullet}_{X|D}"] \arrow[dr, "\vartheta^{\bullet}_{X|D}"'] & \{C(X,nD)\} \arrow[d, "\rho^{\bullet}_{X|D}"] \\ & \{\pi_1^{\mathrm{ab}}(X,nD)\}, \end{tikzcd} \tag{9.16}$$

where the triangle on the left is the limit of the one on the right.

Proof of Theorem 1.2. The composite map $\mathcal{Z}_0(U) \to \{\mathrm{CH}_0(X|nD)\} \to \{C_{KS}(X,nD)\}$ is surjective by [22, Theorem 2.5]. So we only need to show that $\mathrm{cyc}^\bullet_{X|D}$ is injective. For this, we can replace $\{C_{KS}(X,nD)\}$ by $\{C(X,nD)\}$ using (2.10).

Using the commutative diagram of exact sequences of pro-abelian groups

$$\begin{array}{ccccccc} 0 & \to & \{\mathrm{CH}_0(X|nD)_0\} & \to & \{\mathrm{CH}_0(X|nD)\} & \xrightarrow{\deg} & \mathbb{Z} \\ & & \downarrow{\scriptstyle \mathrm{cyc}^\bullet_{X|D}} & & \downarrow{\scriptstyle \mathrm{cyc}^\bullet_{X|D}} & & \| \\ 0 & \longrightarrow & \{C(X,nD)_0\} & \longrightarrow & \{C(X,nD)\} & \xrightarrow{\deg} & \mathbb{Z}, \end{array} \tag{9.17}$$

it suffices to show that the left vertical arrow is injective.

For this, we look at the sequence of maps (see (9.16))

$$\{\mathrm{CH}_0(X|nD)_0\} \xrightarrow{\mathrm{cyc}^\bullet_{X|D}} \{C(X,nD)_0\} \xrightarrow{\rho^\bullet_{X|D}} \{\pi_1^{\mathrm{ab}}(X,nD)_0\}. \tag{9.18}$$

We have seen in the proof of Theorem 1.1 that the composite map is an (in fact level wise) isomorphism (see [4, Lemma 8.4, Theorem 8.5]). The map $\rho^\bullet_{X|D}$ is an isomorphism by Corollary 6.6. We conclude that $\mathrm{cyc}^\bullet_{X|D}$ is an isomorphism. This finishes the proof. □

As a consequence of Theorem 1.2, we have the following.

Corollary 9.7. *The inverse limit idele class group $\widetilde{C}_{U/X}$ is independent of the normal compactification X of U.*

Proof. Follows from Theorem 1.2 and [28, Lemma 3.1]. □

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