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$\mathbb{K}$-HOMOGENEOUS TUPLE OF OPERATORS ON BOUNDED SYMMETRIC DOMAINS

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ABSTRACT

Let Ω be an irreducible bounded symmetric domain of rank r in $\mathbb{C}^d$. Let $\mathbb{K}$ be the maximal compact subgroup of the identity component G of the biholomorphic automorphism group of the domain Ω. The group $\mathbb{K}$ consisting of linear transformations acts naturally on any d-tuple $\boldsymbol{T}=(T_1,\ldots,T_d)$ of commuting bounded linear operators. If the orbit of this action modulo unitary equivalence is a singleton, then we say that $\boldsymbol{T}$ is $\mathbb{K}$-homogeneous. In this paper, we obtain a model for a certain class of $\mathbb{K}$-homogeneous d-tuple $\boldsymbol{T}$ as the operators of multiplication by the coordinate functions $z_1,\ldots,z_d$ on a reproducing kernel Hilbert space of holomorphic functions defined on Ω. Using this model we obtain a criterion for (i) boundedness, (ii) membership in the Cowen–Douglas class, (iii) unitary equivalence and similarity of these d-tuples. In particular, we show that the adjoint of the d-tuple of multiplication by the coordinate functions on the weighted Bergman spaces are in the Cowen–Douglas class $B_1(\Omega)$. For an irreducible bounded symmetric domain Ω of rank 2, an explicit description of the operator $\sum_{i=1}^d T_i^*T_i$ is given. In general, based on this formula, we make a conjecture giving the form of this operator.

1. Introduction

The theory of circular operators on a Hilbert space is well studied by several authors [27],[15], [4]. It was noted in [27, Corollary 2] that the weighted shift operators are circular. In [8], Chavan and Yakubovich generalized this notion to a spherical tuple of operators. A d-tuple $\boldsymbol{T} = (T_1,\ldots,T_d)$ of commuting operators is said to be **spherical** if $U\cdot\boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{T}$ for all unitary matrix U in the group $\mathcal{U}(d)$ of $d\times d$ unitary matrices. Here $U\cdot\boldsymbol{T}$ is the natural action of $\mathcal{U}(d)$ on the d-tuple $\boldsymbol{T}$. Chavan and Yakubovich proved that under some mild hypothesis, every spherical d-tuple is unitarily equivalent to the d-tuple $\boldsymbol{M} = (M_1,\ldots,M_d)$ of multiplication operators by the coordinate function $z_1,\ldots,z_d$ on a reproducing kernel Hilbert space determined by a $\mathcal{U}(d)$-invariant kernel function

$$\sum_{j=0}^{\infty} a_j\langle z,\, w\rangle^j$$

defined on the open Euclidean unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. One of our main objectives in this paper is to explore a notion analogous to that of spherical operator tuples in the context of a bounded symmetric domain.

Bounded symmetric domains are the natural generalization of an open unit disc in one complex variable and an open Euclidean unit ball in several complex variables. A bounded domain $\Omega \subset \mathbb{C}^d$ is said to be **symmetric** if for every $z \in \Omega$, there exists a biholomorphic automorphism of Ω of period two, having z as isolated fixed point. The domain Ω is said to be **irreducible** if it is not biholomorphically equivalent to a product of two non-trivial domains. We refer to [20], [1] for the definition and basic properties of bounded symmetric domains.

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$ and let $\mathrm{Aut}(\Omega)$ denote the group of biholomorphic automorphisms of Ω, equipped with the topology of uniform convergence on compact subsets of Ω. Let G denote the connected component of identity in $\mathrm{Aut}(\Omega)$. It is known that G acts transitively on Ω. Let $\mathbb{K}$ be the subgroup of linear automorphisms in G. By Cartan's theorem [25, Proposition 2, p. 67],

$$\mathbb{K} = \{\phi \in G : \phi(0) = 0\}$$

is a maximal compact subgroup of G and Ω is ismorphic to $G/\mathbb{K}$. Note that $\mathcal{U}(d)$ is the subgroup of linear biholomorphic automorphisms of $\mathrm{Aut}(\mathbb{B}^{\mathrm{d}})$. Therefore, it is natural to replace $\mathcal{U}(d)$ with the subgroup $\mathbb{K}$ of linear biholomorphic automorphisms of an irreducible bounded symmetric domain Ω and study all commuting d-tuples $\boldsymbol{T}$ such that $k \cdot \boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{T}$ for all $k \in \mathbb{K}$. The action of the group $\mathbb{K}$ on the d-tuples is defined below. The group $\mathbb{K}$ acts on Ω by the rule

$$k \cdot \boldsymbol{z} := (k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})), \quad k \in \mathbb{K} \text{ and } \boldsymbol{z} \in \Omega.$$

Note that $k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})$ are linear polynomials. Thus $k \in \mathbb{K}$ acts on any commuting d-tuple of bounded linear operators $\boldsymbol{T} = (T_1, \ldots, T_d)$, defined on a complex separable Hilbert space $\mathcal{H}$, naturally, via the map

$$k \cdot \boldsymbol{T} := (k_1(T_1, \ldots, T_d), \ldots, k_d(T_1, \ldots, T_d)).$$

Definition 1.1: A d-tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ of commuting bounded linear operators on $\mathcal{H}$ is said to be **$\mathbb{K}$-homogeneous** if for all k in $\mathbb{K}$ the operators $\boldsymbol{T}$ and $k \cdot \boldsymbol{T}$ are unitarily equivalent, that is, for all k in $\mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathcal{H}$ such that

$$T_j\Gamma(k) = \Gamma(k)k_j(T_1, \ldots, T_d), \quad j = 1, 2, \ldots, d. \tag{1.1}$$

For brevity, we will write

$$\boldsymbol{T}\Gamma(k) = \Gamma(k)(k \cdot \boldsymbol{T}).$$

While a d-tuple of a $\mathbb{K}$-homogeneous operator is clearly modeled after that of a spherical tuple, it is a much more intricate notion, in general. For instance, spherical tuples in the class $B_1(\mathbb{B}^d)$, introduced by Cowen and Douglas in the very influential paper [9], are necessarily joint weighted shifts. On the other hand, the structure of $\mathbb{K}$-homogeneous operator tuples in $B_1(\Omega)$, where Ω is a bounded symmetric domain of rank > 1, is much more complex. In particular, they are not joint weighted shifts. Also, recall that the commuting operator tuples $\boldsymbol{T} = (T_1, \ldots, T_d)$ such that $\boldsymbol{T}$ and $g(\boldsymbol{T})$ are unitarily equivalent for all g in G, called **homogeneous** tuples, have been studied extensively over the past few years, see [22], [23], [18]. In the case of an open unit disc $\mathbb{D}$, all homogeneous operators in $B_1(\mathbb{D})$ were classified by Misra in [21]. As a corollary of his abstract classification theorem, Wilkins provided an explicit model for all homogeneous operators in $B_2(\mathbb{D})$; see [31]. Later in 2011, using techniques from complex geometry and representation theory, a complete classification of homogeneous operators in the Cowen–Douglas class $B_n(\mathbb{D})$ was obtained by Misra and Korányi in [17]. Homogeneous operators on an irreducible bounded symmetric domain of type I, discussed below, were studied by Misra and Bagchi in [6]. Later in [2], their results were generalized for an arbitrary irreducible bounded symmetric domain by Arazy and Zhang. A comparison of the class of d-tuples of homogeneous operators with $\mathbb{K}$-homogeneous operator tuples might reveal interesting connections with the inducing construction, which we intend to study in future.

Every irreducible bounded symmetric domain Ω of rank r can be realized as an open unit ball of a **Cartan factor** $Z = \mathbb{C}^d$. For a fixed frame $e_1, \ldots, e_r$ of pairwise orthogonal minimal tripotents, let

$$Z = \sum_{0 \le i \le j \le r} Z_{ij}$$

be the **joint Peirce decomposition** of Z (see [29, p. 57]). Note that $Z_{00} = \{0\}$ and $Z_{ii} = \mathbb{C} e_i$ for all $i = 1, \ldots, r$. Moreover,

$$a := \dim Z_{ij}, \quad 1 \le i < j \le r$$

is independent of i, j and

$$b := \dim Z_{0j}, \quad 1 \le j \le r$$

is independent of j. The parameters a, b are known to be the **characterstic multiplicities** of Z and the numerical invariants (r, a, b) determine the domain Ω uniquely up to biholomorphic equivalence (see [1]). The dimension d is related to the numerical invariants (r, a, b) as follows:

$$d = r + \frac{a}{2}r(r-1) + rb.$$

According to the classification due to É. Cartan [7], there are six types of irreducible bounded symmetric domains up to biholomorphic equivalence (see also [20]). The first four types of these domains are called the classical Cartan domains, while the other two types are known as the exceptional domains. In what follows, we consider only the classical domains, that is, an irreducible bounded symmetric domain of one of the following four types:

(i) **Type I**: $n \times m$ $(m \geq n)$ complex matrices $\boldsymbol{z}$ with $\|\boldsymbol{z}\| < 1$. These domains are determined by the numerical invariants $(n, 2, m-n)$.

(ii) **Type II**: symmetric complex matrices $\boldsymbol{z}$ of order n with $\|\boldsymbol{z}\| < 1$. In this case, the numerical invariants $(n, 1, 0)$ are complete biholomorphic invariant.

(iii) **Type III**: anti-symmetric complex matrices $\boldsymbol{z}$ of order n with $\|\boldsymbol{z}\| < 1$. Here $r = [\frac{n}{2}]$, $a = 4$ and $b = 0$ if n is even and $b = 2$ if n is odd.

(iv) **Type IV (the Lie ball)**: all $\boldsymbol{z} \in \mathbb{C}^d$ $(d \geq 5)$ such that $1+|\frac{1}{2}\boldsymbol{z}^t\boldsymbol{z}|^2 > \overline{\boldsymbol{z}}^t\boldsymbol{z}$ and $\overline{\boldsymbol{z}}^t\boldsymbol{z} < 2$, where $\overline{\boldsymbol{z}}^t$ is the complex conjugate of the transpose $\boldsymbol{z}^t$. The numerical invariants $(2, d-2, 0)$ are complete biholomorphic invariant for these domains.

Throughout the paper, let $\mathbb{N}_0$ denote the set of all non-negative integers. Let $\mathcal{P}$ be the space of all analytic polynomials on Z, and let $\mathcal{P}_n$, $n \in \mathbb{N}_0$, denote the subspace of $\mathcal{P}$ consisting of all homogeneous polynomials of degree n. Clearly, as a vector space, $\mathcal{P}$ can be written as the direct sum $\sum_{n=0}^{\infty} \mathcal{P}_n$. The group $\mathbb{K}$ acts on the space $\mathcal{P}$ by composition, that is,

$$(k \cdot p)(\boldsymbol{z}) = p(k^{-1}\boldsymbol{z}), \quad k \in \mathbb{K}, p \in \mathcal{P}.$$

Below we describe the irreducible components of this action. An r-tuple $\underline{s} = (s_1, \ldots, s_r)$ is called a **signature** if $s_1 \geq \cdots \geq s_r \geq 0$. Let $\overrightarrow{\mathbb{N}}_0^r$ denote the set of all signatures. For all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$, we associate the **conical polynomial** $\Delta_{\underline{s}}$, see [29, p. 128] for the definition, where

$$\Delta_{\underline{s}}(\boldsymbol{z}) = \Delta_1^{s_1-s_2}(\boldsymbol{z}) \cdots \Delta_{r-1}^{s_{r-1}-s_r}(\boldsymbol{z})\Delta_r^{s_r}(\boldsymbol{z})$$

and the polynomial space $\mathcal{P}_{\underline{s}}$ is the linear span of $\{\Delta_{\underline{s}} \circ k : k \in \mathbb{K}\}$. It is known that the polynomial spaces $\{\mathcal{P}_{\underline{s}}\}_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r}$ are precisely the $\mathbb{K}$-invariant, irreducible subspaces of $\mathcal{P}$ which are mutually $\mathbb{K}$-inequivalent, and

$$\mathcal{P} = \sum_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathcal{P}_{\underline{s}}.$$

The Fischer–Fock inner product on $\mathcal{P}$, defined by

$$\langle p, q \rangle_F := \frac{1}{\pi^d} \int_{\mathbb{C}^d} p(\boldsymbol{z}) \overline{q(\boldsymbol{z})} e^{-|\boldsymbol{z}|^2} dm(\boldsymbol{z}),$$

is $\mathbb{K}$-invariant. The reproducing kernel of the space $\mathcal{P}_{\underline{s}}$ with respect to the Fischer–Fock inner product is denoted by $K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$. Note that $K_{\underline{s}}$ is $\mathbb{K}$-invariant and

$$\sum_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}) = e^{\boldsymbol{z} \cdot \overline{\boldsymbol{w}}}.$$

Further, any $\mathbb{K}$-invariant Hilbert space $\mathcal{H}$ of analytic functions on Ω has the decomposition

$$\mathcal{H} = \bigoplus_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathcal{P}_{\underline{s}}.$$

This decomposition is called **Peter–Weyl decomposition** [28].

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting d-tuple of bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$. Also, let

$$D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$$

be the operator

$$D_{\boldsymbol{T}} h := (T_1 h, \ldots, T_d h), \quad h \in \mathcal{H}.$$

We note that

$$\ker D_{\boldsymbol{T}} = \bigcap_{i=1}^{d} \ker T_i$$

is the **joint kernel** and

$$\sigma_p(\boldsymbol{T}) = \{\boldsymbol{w} \in \mathbb{C}^d : \ker D_{\boldsymbol{T} - \boldsymbol{w} I} \neq \boldsymbol{0}\}$$

is the **joint point spectrum** of the d-tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$. Throughout this paper we will study a class of $\mathbb{K}$-homogeneous d-tuples, which is defined below.

Definition 1.2: A commuting d-tuple of $\mathbb{K}$-homogeneous operators $\boldsymbol{T}$ possessing the following properties

(i) $\dim \ker D_{\boldsymbol{T}^*} = 1$,
(ii) any non-zero vector e in $\ker D_{\boldsymbol{T}^*}$ is cyclic for $\boldsymbol{T}$,
(iii) $\Omega \subseteq \sigma_p(\boldsymbol{T}^*)$,

is said to be in the class $\mathcal{A}\mathbb{K}(\Omega)$.

In this paper, we provide a concrete model for all the commuting d-tuples $\boldsymbol{T}$ (which are necessarily $\mathbb{K}$-homogeneous) in the class $\mathcal{A}\mathbb{K}(\Omega)$ as multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space of holomorphic functions $\mathcal{H}_K$ defined on Ω. We describe the kernel K in terms of the $\mathbb{K}$-invariant kernels $K_{\underline{s}}$ of the spaces $\mathcal{P}_{\underline{s}}$.

Having described the model, we obtain a criterion for boundedness of these operators. Using this criterion, we determine which d-tuple of multiplication operators on the weighted Bergman spaces are bounded. The boundedness criterion for the multiplication operators on the weighted Bergman spaces has appeared before in [6] and [2].

We also obtain a criterion for the adjoint of the d-tuple of operators in $\mathcal{A}\mathbb{K}(\Omega)$ to be in the Cowen–Douglas class $B_1(\Omega_0)$ for some neighbourhood $\Omega_0 \subset \Omega$ of $0 \in \Omega$. In case of weighted Bergman spaces $\mathcal{H}^{(\nu)}$, we prove that the adjoint of the d-tuple of multiplication operators by the coordinate functions are in the Cowen–Douglas class $B_1(\Omega)$.

For any $\boldsymbol{T}$ in the class $\mathcal{A}\mathbb{K}(\Omega)$, we point out that the operators $\sum_{i=1}^d T_i^* T_i$ and $\sum_{i=1}^d T_i T_i^*$ restricted to the subspace $\mathcal{P}_{\underline{s}}$ are scalar times the identity. In particular, for the weighted Bergman spaces $\mathcal{H}^{(\nu)}$, [2, Proposition 4.4] provides an explicit form for the operator $\sum_{i=1}^d T_i T_i^*$. We extend this formula for any $\boldsymbol{T}$ in the class $\mathcal{A}\mathbb{K}(\Omega)$. Moreover, for the Hardy space of the Shilov boundary S of Ω, we show that $\sum_{i=1}^d M_i^* M_i$ is the rank times identity, see also [5]. Also, for any $\boldsymbol{T}$ in $\mathcal{A}\mathbb{K}(\Omega)$, we have computed the operator $\sum_{i=1}^d T_i^* T_i$ on certain subspaces of $\mathcal{H}$, and as a consequence, it is shown that the commutators $[M_i^*, M_i]$, $i = 1, \ldots, d$, on the weighted Bergman spaces are compact if and only if $r = 1$. For any domain Ω of rank 2, we obtained an explicit description of the operator $\sum_{i=1}^d T_i^* T_i$ and conjectured the form of this operator for a domain of any rank $r > 2$. This conjecture was proved by Upmeier; see [30].

Finally, we study the question of unitary equivalence and similarity of d-tuples of operators in the class $\mathcal{A}\mathbb{K}(\Omega)$.

2. Model for operators in $\mathcal{AK}(\Omega)$

We begin this section by providing a well known family of examples, namely, the d-tuple of multiplication by the coordinate functions on the weighted Bergman spaces, which belongs to the class $\mathcal{AK}(\Omega)$.

For $\nu \in \{0, \ldots, \frac{a}{2}(r-1)\} \cup (\frac{a}{2}(r-1), \infty)$, the so-called **Wallach set** of Ω (see [14]), consider the weighted Bergman kernel

$$K^{(\nu)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s}} (\nu)_{\underline{s}} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}), \quad \boldsymbol{z}, \boldsymbol{w} \in \Omega,$$

where $(\nu)_{\underline{s}}$ is the generalized Pochhammer symbol

$$(\nu)_{\underline{s}} := \prod_{j=1}^{r} \Big(\nu - \frac{a}{2}(j-1)\Big)_{s_j} = \prod_{j=1}^{r} \prod_{l=1}^{s_j} \Big(\nu - \frac{a}{2}(j-1) + l - 1\Big).$$

Let $\mathcal{H}^{(\nu)}$ denote the weighted Bergman space of holomorphic functions on Ω determined by the reproducing kernel $K^{(\nu)}$. If $\nu = \frac{d}{r}$ and $\nu = \frac{a}{2}(r-1) + \frac{d}{r}$, then the weighted Bergman spaces $\mathcal{H}^{(\nu)}$ coincide with the Hardy space $H^2(S)$ over the **Shilov boundary** S of Ω and the classical Bergman space $\mathbb{A}^2(\Omega)$ respectively. For $\nu > \frac{a}{2}(r-1)$, the multiplication d-tuple

$$\boldsymbol{M}^{(\nu)} = (M_1^{(\nu)}, \ldots, M_d^{(\nu)})$$

on $\mathcal{H}^{(\nu)}$ is bounded and homogeneous (cf. [6], [2]). One can also verify that $\boldsymbol{M}^{(\nu)}$ is in $\mathcal{AK}(\Omega)$. Replacing $(\nu)_{\underline{s}}$ by any arbitrary positive number $a_{\underline{s}}$ with some boundedness condition, we get a large class of operator tuples in $\mathcal{AK}(\Omega)$ and we prove that upto unitary equivalence every operator tuple in $\mathcal{AK}(\Omega)$ is of this form.

To facilitate the study of $\mathbb{K}$-homogeneous operators, we recall the following result from [1] describing all the $\mathbb{K}$-invariant kernels on Ω.

Proposition 2.1 (Proposition 3.4, [1]): *For any $\mathbb{K}$-invariant semi-inner product $\langle \cdot, \cdot \rangle$ on the the space of polynomials $\mathcal{P}$, the following statements hold:*

(i) *$\mathcal{P}_{\underline{s}}$ is orthogonal to $\mathcal{P}_{\underline{s}'}$ whenever $\underline{s} \neq \underline{s}'$.*

(ii) *There exists a constant $b_{\underline{s}} \geq 0$ associated to each $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ such that*

$$\langle p, q \rangle = b_{\underline{s}} \langle p, q \rangle_{\mathcal{F}}, \quad \textit{for all } p, q \in \mathcal{P}_{\underline{s}}.$$

(iii) *$b_{\underline{s}} > 0$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ if and only if $\langle \cdot, \cdot \rangle$ is an inner product.*

(iv) *If the evaluation map at each point of* Ω *is continuous on* $(\mathcal{P}, \langle\cdot,\cdot\rangle)$, *then the completion* $\mathcal{H}$ *of* $(\mathcal{P}, \langle\cdot,\cdot\rangle)$ *is a reproducing kernel Hilbert space. Moreover, the kernel* $K(\boldsymbol{z},\boldsymbol{w})$ *is of the form*

$$K(\boldsymbol{z},\boldsymbol{w}) = \sum_{\underline{s}\in\overrightarrow{\mathbb{N}}_0^r} b_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z},\boldsymbol{w}),$$

where convergence is both uniformly on compact subsets of $\Omega\times\Omega$ *and in norm.*

The following result is a generalization of [8, Lemma 2.10] which is necessary for the proof of Theorem 2.3 giving a model for commuting a d-tuple of operators in the class $\mathcal{A}\mathbb{K}(\Omega)$.

Lemma 2.2: *Let* $\boldsymbol{T} = (T_1,\dots,T_d)$ *be a* $\mathbb{K}$*-homogeneous* d*-tuple of commuting operators on* $\mathcal{H}$. *Suppose that* $\ker D_{\boldsymbol{T}^*}$ *is one-dimensional and is spanned by a vector* $e\in\mathcal{H}$ *which is cyclic for* $\boldsymbol{T}$. *Then there exists a sequence* $\{a_{\underline{s}}\}_{\underline{s}\in\overrightarrow{\mathbb{N}}_0^r}$ *of non-negative real numbers such that for any polynomial* $p\in\mathcal{P}$,

$$||p(\boldsymbol{T})e||_{\mathcal{H}}^2 = \sum_{k=0}^{\deg p}\sum_{|\underline{s}|=k} a_{\underline{s}} \|p_{\underline{s}}\|_{\mathcal{F}}^2, \tag{2.2}$$

where $\deg p$ *is the degree of* p *and*

$$p = \sum_{k=0}^{\deg p}\sum_{|\underline{s}|=k} p_{\underline{s}}$$

is the Peter–Weyl decomposition.

Proof. Since $\boldsymbol{T}$ is $\mathbb{K}$-homogeneous, for each $k\in\mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathcal{H}$ such that

$$T_j\Gamma(k) = \Gamma(k)k_j(\boldsymbol{T}), \quad j=1,\dots,d.$$

Hence

$$T_j^*\Gamma(k) = \Gamma(k)k_j(\boldsymbol{T})^*, \quad j=1,\dots,d.$$

Since $k_j(\boldsymbol{T})$ is a linear combination of $T_1,\dots,T_d$ and $e\in\ker D_{\boldsymbol{T}^*}$, it follows that $\Gamma(k)e$ belongs to $\ker D_{\boldsymbol{T}^*}$ for all $k\in\mathbb{K}$. Furthermore, since $\ker D_{\boldsymbol{T}^*}$ is one-dimensional and spanned by e, we obtain that

$$\Gamma(k)e = \eta(k)e$$

for some $\eta(k)$ such that $|\eta(k)| = 1$. We now define a semi-inner product on $\mathcal{P}_{\underline{s}}$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ by the formula

$$\langle p_{\underline{s}}, q_{\underline{s}}\rangle_{\mathcal{P}_{\underline{s}}} := \langle p_{\underline{s}}(\boldsymbol{T})e, q_{\underline{s}}(\boldsymbol{T})e\rangle_{\mathcal{H}}, \quad p_{\underline{s}}, q_{\underline{s}} \in \mathcal{P}_{\underline{s}}.$$

Now for any $k \in \mathbb{K}$ we have

$$\begin{aligned}
\langle p_{\underline{s}}(k \cdot \boldsymbol{z}), q_{\underline{s}}(k \cdot \boldsymbol{z})\rangle_{\mathcal{P}_{\underline{s}}} &= \langle p_{\underline{s}}(k \cdot \boldsymbol{T})e, q_{\underline{s}}(k \cdot \boldsymbol{T})e\rangle_{\mathcal{H}} \\
&= \langle \Gamma(k)^* p_{\underline{s}}(\boldsymbol{T})\Gamma(k)e, \Gamma(k)^* q_{\underline{s}}(\boldsymbol{T})\Gamma(k)e\rangle_{\mathcal{H}} \\
&= \langle p_{\underline{s}}(\boldsymbol{T})\Gamma(k)e, q_{\underline{s}}(\boldsymbol{T})\Gamma(k)e\rangle_{\mathcal{H}} \\
&= \langle p_{\underline{s}}(\boldsymbol{T})\eta(k)e, q_{\underline{s}}(\boldsymbol{T})\eta(k)e\rangle_{\mathcal{H}} \\
&= |\eta(k)|^2 \langle p_{\underline{s}}(\boldsymbol{T})e, q_{\underline{s}}(\boldsymbol{T})e\rangle_{\mathcal{H}} \\
&= \langle p_{\underline{s}}(\boldsymbol{T})e, q_{\underline{s}}(\boldsymbol{T})e\rangle_{\mathcal{H}} \\
&= \langle p_{\underline{s}}, q_{\underline{s}}\rangle_{\mathcal{P}_{\underline{s}}}.
\end{aligned}$$

So $\langle \cdot, \cdot\rangle_{\mathcal{P}_{\underline{s}}}$ is a $\mathbb{K}$-invariant semi-inner product on $\mathcal{P}_{\underline{s}}$ for each $\underline{s}$. Therefore, on $\mathcal{P}$,

$$\langle p,\, q\rangle := \sum_{k=0}^{\ell} \sum_{|\underline{s}|=k} \langle p_{\underline{s}}, q_{\underline{s}}\rangle_{\mathcal{P}_{\underline{s}}},$$

where p and q have the Peter–Weyl decomposition

$$\sum_{k=0}^{\deg p} \sum_{|\underline{s}|=k} p_{\underline{s}} \quad \text{and} \quad \sum_{k=0}^{\deg q} \sum_{|\underline{s}|=k} q_{\underline{s}}$$

respectively and $\ell = \min\{\deg p, \deg q\}$, defines a $\mathbb{K}$-invariant semi-inner product. Thus by Proposition 2.1, there exists a sequence of non-negative real numbers $a_{\underline{s}}$ such that

$$\langle p, q\rangle = \sum_{k=0}^{\ell} \sum_{|\underline{s}|=k} a_{\underline{s}} \langle p_{\underline{s}}, q_{\underline{s}}\rangle_{\mathcal{F}}.$$

This completes the proof. ∎

For all the classical bounded symmetric domains, it can be easily verified that

$$\Omega = \{\boldsymbol{w} \in \mathbb{C}^d : \overline{\boldsymbol{w}} \in \Omega\}.$$

Consequently, in the following theorem, the Hilbert space that we construct consists of holomorphic functions on Ω rather than $\{\boldsymbol{w} \in \mathbb{C}^d : \overline{\boldsymbol{w}} \in \Omega\}$. The next result provides an analytic model for any d-tuple of operators $\boldsymbol{T}$ in $\mathcal{A}\mathbb{K}(\Omega)$.

THEOREM 2.3: *If $\boldsymbol{T}$ is a d-tuple of operators in $\mathcal{A}\mathbb{K}(\Omega)$, then $\boldsymbol{T}$ is unitarily equivalent to a d-tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space H_K of holomorphic functions defined on Ω with $K(\boldsymbol{z}, \boldsymbol{w}) = \sum a_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$ for all $\boldsymbol{z}, \boldsymbol{w} \in \Omega$, for some choice of positive real numbers $a_{\underline{s}}$ with $a_{\underline{0}} = 1$.*

Proof. Since $\Omega \subseteq \sigma_p(\boldsymbol{T}^*)$, for each $\boldsymbol{w} \in \Omega$ there exists a non-zero vector $x \in \mathcal{H}$, such that $T_j^* x = \bar{w}_j x$ for all $j = 1, \ldots, d$. Thus for any polynomial $p \in \mathcal{P}$, we have $p(\boldsymbol{T}^*)x = p(\overline{\boldsymbol{w}})x$. Let $e \in kerD_{\boldsymbol{T}^*}$ be a cyclic vector for T of norm 1. Then

$$p(\boldsymbol{w})\langle e, x\rangle_{\mathcal{H}} = \langle e, \overline{p(\boldsymbol{w})}x\rangle_{\mathcal{H}} = \langle e, \bar{p}(\boldsymbol{T}^*)x\rangle_{\mathcal{H}} = \langle p(\boldsymbol{T})e, x\rangle_{\mathcal{H}},$$

where $\bar{p}(z) = \overline{p(\bar{z})}$, $z \in \Omega$. Since $x \neq 0$ and e is cyclic for $\boldsymbol{T}$, we get $\langle e, x\rangle_{\mathcal{H}} \neq 0$ and

$$|p(w)| \leq \frac{\|p(\boldsymbol{T})e\|_{\mathcal{H}}\|x\|_{\mathcal{H}}}{|\langle e, x\rangle_{\mathcal{H}}|}.$$

Thus it follows that evaluation at $\boldsymbol{w} \in \Omega$ is bounded and therefore, the semi-inner product defined by the rule $\langle p, q\rangle_{\mathcal{P}_{\underline{s}}} = \langle p(\boldsymbol{T})e, q(\boldsymbol{T})e\rangle_{\mathcal{H}}$ is an inner product on each $\mathcal{P}_{\underline{s}}$. This gives rise to an inner product $\langle\cdot, \cdot\rangle$ on the space of polynomials $\mathcal{P}$. The sequence $\{a_{\underline{s}}\}_{\underline{s} \in \vec{\mathbb{N}}_0^r}$ of Lemma 2.2, using Proposition 2.1(iii), is now evidently positive. Moreover, since $\|e\| = 1$, it follows from (2.2) that $a_{\underline{0}} = 1$. Thus, by Proposition 2.1(iv), the completion of $(\mathcal{P}, \langle\cdot, \cdot\rangle)$, say H_K, is a reproducing kernel Hilbert space, where

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum a_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}), \quad \boldsymbol{z}, \boldsymbol{w} \in \Omega.$$

Clearly, the map $p \mapsto p(\boldsymbol{T})e$ extends to a unitary from H_K to $\mathcal{H}$, which intertwines $\boldsymbol{T}$ with the multiplication d-tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ on H_K. ∎

If $\boldsymbol{T}$ is a K-homogeneous d-tuple of operators, then, in general, the map $k \mapsto \Gamma(k)$ of (1.1) need not be a homomorphism. The next proposition assures that if $\boldsymbol{T}$ is in the class $\mathcal{A}\mathbb{K}(\Omega)$, then there exists a choice of $\Gamma(k)$ for which the map $k \mapsto \Gamma(k)$ is a homomorphism.

PROPOSITION 2.4: *If $\boldsymbol{T}$ is a d-tuple of operators in $\mathcal{A}\mathbb{K}(\Omega)$, then there exists a unitary representation $\Gamma : \mathbb{K} \to \mathcal{U}(\mathcal{H})$ such that*

$$\boldsymbol{T}\Gamma(k) = \Gamma(k)(k \cdot \boldsymbol{T}).$$

Proof. By Theorem 2.3, $\boldsymbol{T}$ is unitarily equivalent to the d-tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ of multiplication operators on a reproducing kernel Hilbert space H_K of holomorphic functions defined on Ω with a kernel $K(\boldsymbol{z}, \boldsymbol{w})$ which is $\mathbb{K}$-invariant. Clearly, the map Γ on $\mathcal{H}_K$ given by $\Gamma(k)(f) = f \circ k^{-1}(\cdot)$ is a unitary representation of $\mathbb{K}$ satisfying the intertwining condition. ∎

Remark 2.5: Since $\mathbb{K}$ is a subgroup of the group $\mathcal{U}(d)$ of unitary linear transformations on $\mathbb{C}^d$, every spherical d-tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ is $\mathbb{K}$-homogeneous. Conversely, a $\mathbb{K}$-homogeneous d-tuple of Theorem 2.3 is spherical if and only if $a_{\underline{s}} = a_{\underline{s}'}$ for all $\underline{s}, \underline{s}' \in \overrightarrow{\mathbb{N}}_0^r$ with $|\underline{s}| = |\underline{s}'|$.

Remark 2.6: We also point out that, by the spectral mapping theorem, the Taylor joint spectrum $\sigma(\boldsymbol{T})$ of a $\mathbb{K}$-homogeneous operator $\boldsymbol{T}$ is $\mathbb{K}$-invariant, that is, if $\boldsymbol{w}$ belongs to $\sigma(\boldsymbol{T})$, then $k.\boldsymbol{w}$ also belongs to $\sigma(\boldsymbol{T})$ for all $k \in \mathbb{K}$.

3. Boundedness of the multiplication tuple

Throughout the rest of the paper, let $K^{(a)} : \Omega \times \Omega \to \mathbb{C}$ denote the kernel function given by the formula $K^{(a)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$, $\boldsymbol{z}, \boldsymbol{w} \in \Omega$, for some choice of positive real numbers $a_{\underline{s}}$. The positivity of the sequence $a_{\underline{s}}$ ensures that $K^{(a)}$ is a positive definite kernel. Thus it determines a unique Hilbert space $\mathcal{H}^{(a)} \subseteq \mathrm{Hol}(\Omega)$ with the reproducing property:

$$\langle f, K^{(a)}(\cdot, \boldsymbol{w}) \rangle = f(\boldsymbol{w}), \quad f \in \mathcal{H}^{(a)},\ \boldsymbol{w} \in \Omega.$$

It follows from Proposition 2.1 that the polynomial ring $\mathcal{P}$ is dense in $\mathcal{H}^{(a)}$ and $\mathcal{P}_{\underline{s}}$ is orthogonal to $\mathcal{P}_{\underline{s}'}$ whenever $\underline{s} \neq \underline{s}'$, that is, $\mathcal{H}^{(a)} = \bigoplus_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathcal{P}_{\underline{s}}$. In this section, we discuss the boundedness of the d-tuple $\boldsymbol{M}^{(a)} := (M_1^{(a)}, \ldots, M_d^{(a)})$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on $\mathcal{H}^{(a)}$. We begin with the following basic lemma, which is surely known to the experts, but we provide a proof for the sake of completeness.

Lemma 3.1: *The operators*

$$\sum_{i=1}^{d} M_i^{(a)^*} M_i^{(a)} \quad \text{and} \quad \sum_{i=1}^{d} M_i^{(a)} M_i^{(a)^*},$$

acting on $\mathcal{H}^{(a)}$, are block diagonal with respect to the decomposition $\bigoplus_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathcal{P}_{\underline{s}}$, where each block is a non-negative scalar multiple of the identity operator.

Proof. It is enough to give the proof for the operator $\sum_{i=1}^d M_i^{(a)^*} M_i^{(a)}$ since the proof for the operator $\sum_{i=1}^d M_i^{(a)} M_i^{(a)^*}$ follows in exactly the same way. First, note that

$$\Gamma(k)^* M_i^{(a)} \Gamma(k) = M_{z_i \circ k^{-1}}^{(a)} \quad \text{for } k \in \mathbb{K}.$$

Let $e_1, \ldots, e_d$ be the standard basis vectors in $\mathbb{C}^d$. Note that

$$M_{z_i \circ k^{-1}}^{(a)} = \sum_{j=1}^d \langle k^{-1} e_j, e_i \rangle M_j^{(a)}.$$

In consequence, we have

$$\begin{aligned}
\Gamma(k)^* \Big(\sum_{i=1}^d M_i^{(a)^*} M_i^{(a)} \Big) \Gamma(k) &= \sum_{i=1}^d \Gamma(k)^* M_i^{(a)^*} \Gamma(k) \Gamma(k)^* M_i^{(a)} \Gamma(k) \\
&= \sum_{i=1}^d {M_{z_i \circ k^{-1}}^{(a)}}^* M_{z_i \circ k^{-1}}^{(a)} \\
&= \sum_{i=1}^d \sum_{p,q=1}^d \langle e_i, k^{-1} e_p \rangle \langle k^{-1} e_q, e_i \rangle M_p^{(a)^*} M_q^{(a)} \\
&= \sum_{p,q=1}^d \langle k^{-1} e_q, k^{-1} e_p \rangle M_p^{(a)^*} M_q^{(a)} \\
&= \sum_{i=1}^d M_i^{(a)^*} M_i^{(a)}.
\end{aligned}$$

Here the last equality follows from the fact that the subgroup $\mathbb{K}$ is contained in the group $\mathcal{U}(d)$ of unitary linear transformations on $\mathbb{C}^d$. Since $\{\mathcal{P}_{\underline{s}}\}_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r}$ are $\mathbb{K}$-irreducible, mutually $\mathbb{K}$-inequivalent subspaces of $\mathcal{H}^{(a)}$ and $\mathcal{H}^{(a)} = \bigoplus_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathcal{P}_{\underline{s}}$, the conclusion follows from Schur's lemma. ∎

For any $\underline{s}$ in $\overrightarrow{\mathbb{N}}_0^r$, let $I^+(\underline{s})$ and $I^-(\underline{s})$ denote the sets given by

$$\begin{aligned}
I^+(\underline{s}) &:= \{j : 1 \le j \le r,\ \underline{s} + \varepsilon_j \in \overrightarrow{\mathbb{N}}_0^r\}, \\
I^-(\underline{s}) &:= \{j : 1 \le j \le r,\ \underline{s} - \varepsilon_j \in \overrightarrow{\mathbb{N}}_0^r\}.
\end{aligned}$$

Further, in the remaining portion of this paper, we set

$$c_{\underline{s}}(j) = \prod_{k \ne j} \frac{s_j - s_k + \frac{a}{2}(k - j + 1)}{s_j - s_k + \frac{a}{2}(k - j)}, \quad j = 1, \ldots, r,$$

and

$$c'_{\underline{s}}(j) = \prod_{k \neq j} \frac{s_j - s_k + \frac{a}{2}(k-j-1)}{s_j - s_k + \frac{a}{2}(k-j)}, \quad j = 1, \ldots, r.$$

If $j \in I^+(\underline{s})$, then it is easy to see that $c_{\underline{s}}(j) > 0$. Otherwise, $c_{\underline{s}}(j) = 0$. Similarly, if $j \in I^-(\underline{s})$, then $c'_{\underline{s}}(j) > 0$. Otherwise, $c'_{\underline{s}}(j) = 0$ for $1 \leq j \leq r-1$ and $c'_{\underline{s}}(r) > 0$.

The following lemma, which describes the operator $\sum_{i=1}^d M_i^{(a)} M_i^{(a)^*}$, generalizes a known result [2, Proposition 4.4] for the weighted Bergman spaces.

Lemma 3.2: *For $f \in \mathcal{P}_{\underline{s}}$, we have*

$$\sum_{i=1}^d M_i^{(a)} M_i^{(a)^*} f = \tau(\underline{s}) f,$$

where

$$\tau(\underline{s}) = \begin{cases} \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j) & \text{if } \underline{s} \neq 0, \\ 0 & \text{if } \underline{s} = 0. \end{cases}$$

The proof of the preceding lemma is very similar to the proof of [2, Proposition 4.4] and therefore it is omitted.

For any finite set A, let $|A|$ denote the cardinality of A.

Lemma 3.3: *For any fixed but arbitrary $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$, we have*

$$\sum_{j=1}^r c'_{\underline{s}}(j) = \sum_{j=1}^r c_{\underline{s}}(j) = r.$$

Proof. Evidently, we have

$$\begin{aligned} \sum_{j=1}^r c'_{\underline{s}}(j) &= \sum_{j=1}^r \prod_{k \neq j} \frac{s_j - s_k + \frac{a}{2}(k-j-1)}{s_j - s_k + \frac{a}{2}(k-j)} \\ &= \sum_{j=1}^r \prod_{k \neq j} \Big(1 - \frac{\frac{a}{2}}{s_j - s_k + \frac{a}{2}(k-j)}\Big) \\ &= \sum_{j=1}^r \prod_{k \neq j} \Big(1 - \frac{\frac{a}{2}}{(s_j - \frac{a}{2}j) - (s_k - \frac{a}{2}k)}\Big). \end{aligned}$$

Setting $s'_j = \frac{s_j - \frac{a}{2}j}{\frac{a}{2}}$, we see that $s'_1 > s'_2 > \cdots > s'_r$, and

$$\begin{aligned}\sum_{j=1}^{r} c'_{\underline{s}}(j) &= \sum_{j=1}^{r} \prod_{k\neq j} \Big(1 - \frac{1}{s'_j - s'_k}\Big)\\ &= r + \sum_{j=1}^{r} \sum_{\substack{A\subseteq\{1,\ldots,j-1,j+1,\ldots,r\}\\ A\neq\phi}} (-1)^{|A|} \prod_{k\in A} \frac{1}{s'_j - s'_k}\\ &= r + \sum_{\substack{A\subseteq\{1,\ldots,r\}\\ |A|\geq 2}} (-1)^{|A|-1} \sum_{j\in A} \prod_{\substack{k\in A\\ k\neq j}} \frac{1}{s'_j - s'_k}.\end{aligned}$$

Now, by [6, Corollary 2.3], it follows that

$$\sum_{j\in A} \prod_{\substack{k\in A\\ k\neq j}} \frac{1}{s'_j - s'_k} = 0$$

for all $A \subseteq \{1, \ldots, r\}$ with $|A| \geq 2$. Therefore, $\sum_{j=1}^{r} c'_{\underline{s}}(j) = r$. The proof of the other part follows in exactly the same way. ∎

Theorem 3.4: *The d-tuple $\boldsymbol{M}^{(a)} = (M_1^{(a)}, \ldots, M_d^{(a)})$ of multiplication operators on $\mathcal{H}^{(a)}$ is bounded if and only if*

$$A := \sup\Big\{\frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} : \underline{s},\ \underline{s}-\varepsilon_j \in \overrightarrow{\mathbb{N}}_0^r,\ j = 1, \ldots, r\Big\}$$

is finite.

Proof. Clearly, the multiplication d-tuple $\boldsymbol{M}^{(a)}$ on $\mathcal{H}^{(a)}$ is bounded if and only if the operator $\sum_{i=1}^{d} M_i^{(a)} M_i^{(a)^*}$ is bounded. Therefore, using Lemma 3.2, it is enough to show that $\tau(\underline{s})$ is bounded for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ if and only if A is finite. First assume that A is finite. Then

$$\begin{aligned}\tau(\underline{s}) &= \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \frac{\frac{a}{2}(r-j) + s_j}{b + \frac{a}{2}(r-j) + s_j} c'_{\underline{s}}(j)\\ &\leq A \sum_{j=1}^{r} \frac{\frac{a}{2}(r-j) + s_j}{b + \frac{a}{2}(r-j) + s_j} c'_{\underline{s}}(j)\\ &\leq A \sum_{j=1}^{r} c'_{\underline{s}}(j)\\ &= Ar\end{aligned}$$

for any $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$. Here, the last equality follows from Lemma 3.3. To prove the other direction, assume that $\tau(\underline{s})$ is bounded, that is, $\tau(\underline{s}) \leq B$ for some positive real number B and for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$. Thus

$$\frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j) \leq \tau(\underline{s}) \leq B, \quad j \in I^-(\underline{s}).$$

Now, note that if $j \in I^-(\underline{s})$, then

$$\begin{aligned}\frac{1}{c'_{\underline{s}}(j)} &= \prod_{k\neq j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \\ &= \prod_{k<j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \prod_{k>j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \\ &\leq \prod_{k>j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \\ &\leq \prod_{k>j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k} \\ &\leq \prod_{k>j} (1 + \frac{\frac{a}{2}(k-j)}{s_j - s_k}) \\ &\leq \Big(1 + \frac{a}{2}(r-1)\Big)^r.\end{aligned} \tag{3.3}$$

Here the third inequality holds since $\frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \leq 1$ for $k < j$. Now, it follows that

$$\frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \leq \frac{B}{c'_{\underline{s}}(j)} \frac{b+\frac{a}{2}(r-j)+s_j}{\frac{a}{2}(r-j)+s_j} \leq B\Big(1+\frac{a}{2}(r-1)\Big)^r (1+b).$$

This completes the proof. ∎

Corollary 3.5: *The multiplication d-tuple $\boldsymbol{M}^{(\nu)}$ on $\mathcal{H}^{(\nu)}$ is bounded if*

$$\nu > \frac{a}{2}(r-1).$$

Proof. If $\nu > \frac{a}{2}(r-1)$, then

$$\frac{(\nu)_{\underline{s}-\varepsilon_j}}{(\nu)_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} = \frac{\frac{d}{r} - \frac{a}{2}(j-1) + s_j - 1}{\nu - \frac{a}{2}(j-1) + s_j - 1} \leq \max\Big\{1, \frac{1+b}{\nu - \frac{a}{2}(r-1)}\Big\}.$$

Therefore, from Theorem 3.4, it follows that $\boldsymbol{M}^{(\nu)}$ is bounded. ∎

Having (a) determined the condition for boundedness of the operator $\boldsymbol{M}^{(a)}$, (b) noting that each $\boldsymbol{w}$ in Ω is a joint eigenvalue for the multiplication d-tuple $\boldsymbol{M}^{(a)*}$ and finally since the constant vector 1 is cyclic for $\boldsymbol{M}^{(a)}$, it is natural to investigate the question of which of these are in the Cowen–Douglas class $\mathrm{B}_1(\Omega)$; see [9], [10] for the definition of this very important class of operators. As shown in [12, p. 285], the cyclicity implies that the dimension of the joint eigenspace at each $\boldsymbol{w}$ in Ω is 1. Thus to determine the membership in the Cowen–Douglas class in a neighbourhood of the origin contained in Ω, we only need to find when $\operatorname{ran} D_{\boldsymbol{M}^{(a)*}}$ is closed. The following theorem provides the precise condition for this.

Theorem 3.6: *For a multiplication d-tuple $\boldsymbol{M}^{(a)} = (M_1^{(a)}, \ldots, M_d^{(a)})$ on $\mathcal{H}^{(a)}$, $\operatorname{ran} D_{\boldsymbol{M}^{(a)*}}$ is closed if and only if*

$$B := \inf\Big\{ \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} : \underline{s} \in \overrightarrow{\mathbb{N}}_0^r \Big\}$$

is non-zero positive.

Proof. It is elementary to see that $\operatorname{ran} D_{\boldsymbol{M}^{(a)*}}$ is closed if and only if

$$\sum_{i=1}^{d} M_i^{(a)} M_i^{(a)^*}$$

is bounded below on $(\ker D_{\boldsymbol{M}^{(a)*}})^{\perp}$. Also, for the d-tuple $\boldsymbol{M}^{(a)}$ on $\mathcal{H}^{(a)}$, we have $\ker D_{\boldsymbol{M}^{(a)*}} = \mathcal{P}_0$, the space of constant functions. Therefore, in view of Lemma 3.2, it suffices to show that B is non-zero positive if and only if $\inf\{\tau(\underline{s}) : \underline{s} \neq 0,\ \underline{s} \in \overrightarrow{\mathbb{N}}_0^r\}$ is non-zero positive. Suppose that B is a non-zero positive number. Now, for any non-zero $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$, we have

$$\begin{aligned}
\tau(\underline{s}) &= \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j) \\
&\geq \frac{1}{b+1} \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} c'_{\underline{s}}(j) \\
&\geq \frac{1}{b+1} \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \frac{1}{(1+\frac{a}{2}(r-1))^r} \\
&\geq \frac{B}{(b+1)(1+\frac{a}{2}(r-1))^r}.
\end{aligned}$$

Here the third inequality follows from (3.3). Conversely, assume that $\inf\{\tau(\underline{s}) : \underline{s} \neq 0,\ \underline{s} \in \overrightarrow{\mathbb{N}}_0^r\}$ is a non-zero positive number, say C. Thus for each non-zero $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$,

$$\sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c_s'(j) \geq C. \tag{3.4}$$

Hence, noting that $c_s'(j) \leq r$ by Lemma 3.3 and $\frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)} \leq 1$, it follows that

$$\sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} \geq \frac{C}{r}. \qquad ∎$$

Corollary 3.7: *The range of $D_{\boldsymbol{M}^{(\nu)*}}$ is closed if $\nu > \frac{a}{2}(r-1)$.*

Proof. Suppose $\nu = \frac{a}{2}(r-1)+\varepsilon$ for some $\varepsilon > 0$. Then

$$\sum_{j\in I^-(\underline{s})} \frac{(\nu)_{\underline{s}-\varepsilon_j}}{(\nu)_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\varepsilon_j}} = \sum_{j\in I^-(\underline{s})} \frac{b+\frac{a}{2}(r-j)+s_j}{\frac{a}{2}(r-j)+s_j+\varepsilon-1}$$

which is always bounded below by 1 if $\varepsilon \leq b+1$. On the other hand, for $\varepsilon \geq b+1$, it is bounded below by $\frac{1}{\varepsilon}$. Hence, by Theorem 3.6, $\operatorname{ran} D_{\boldsymbol{M}^{(\nu)*}}$ is closed. ∎

Now, we wish to show that the adjoint $\boldsymbol{M}^{(\nu)^*}$ of the d-tuple of multiplication operators on $\mathcal{H}^{(\nu)}$ is in the Cowen–Douglas class $\mathrm{B}_1(\Omega)$ for $\nu > \frac{a}{2}(r-1)$.

Recall that the left essential spectrum $\pi_e^{\ell,0}(\boldsymbol{T})$ of a commuting d-tuple of operators $\boldsymbol{T}$ is defined to be the complement of the set of all $\boldsymbol{w} \in \mathbb{C}^d$ with the property:

(1) $\dim\ker D_{(\boldsymbol{T}-\boldsymbol{w}I)}$ is finite,
(2) $\operatorname{ran} D_{(\boldsymbol{T}-\boldsymbol{w}I)}$ is closed.

If $0 \notin \pi_e^{\ell,0}(\boldsymbol{T})$, then the d-tuple $\boldsymbol{T}$ is said to be left semi-Fredholm.

The essential ingredient of the proof of the following theorem is based on the spectral mapping property of the left essential spectrum, which appears in [13] and was pointed out to G. Misra by J. Eschmeier during a conversation at University of Saarbrucken in February 2014.

Theorem 3.8: *The adjoint $\boldsymbol{M}^{(\nu)^*}$ of the multiplication d-tuple on $\mathcal{H}^{(\nu)}$ is in the Cowen–Douglas class $B_1(\Omega)$ whenever $\nu > \frac{a}{2}(r-1)$.*

Proof. Since the set of polynomials is dense in the Hilbert space $\mathcal{H}^{(\nu)}$, it follows that $\dim\ker D_{\boldsymbol{M}^{(\nu)*}}$ is 1. By Corollary 3.7, we also have that ran $D_{\boldsymbol{M}^{(\nu)*}}$ is closed. Therefore, $D_{\boldsymbol{M}^{(\nu)*}}$ is left semi-Fredholm and hence there is a $\varepsilon > 0$ such that for $\boldsymbol{w} \in \Omega$ with $\sum_{i=1}^{d} |w_i|^2 < \varepsilon$, the operators $D_{(\boldsymbol{M}-\boldsymbol{w}I)^*}$ are left semi-Fredholm. Thus ${\boldsymbol{M}^{(\nu)}}^*$ is in the Cowen–Douglas class $\mathrm{B}_1(\Omega_\varepsilon)$, where

$$\Omega_\varepsilon = \Big\{\boldsymbol{w} \in \Omega : \sum_{i=1}^{d} |w_i|^2 < \varepsilon\Big\}.$$

Now, using the homogeneity of $\boldsymbol{M}^{(\nu)}$ and the spectral mapping property of the left essential spectrum, we show that ${\boldsymbol{M}^{(\nu)}}^*$ is actually in $\mathrm{B}_1(\Omega)$.

To complete the proof, first note that if $\boldsymbol{w} \in \Omega$ is any fixed but arbitrary point, then there exists a biholomorphic automorphism φ of Ω with the property: $\varphi(0) = \boldsymbol{w}$. We have seen that $0 \notin \pi_e^{\ell,0}({\boldsymbol{M}^{(\nu)}}^*)$. An analytic spectral mapping property for the left essential spectrum is ensured by [13, Corollary 2.6.9]. It follows that

$$\boldsymbol{w} = \varphi(0) \notin \varphi(\pi_e^{\ell,0}({\boldsymbol{M}^{(\nu)}}^*)) = \pi_e^{\ell,0}(\varphi({\boldsymbol{M}^{(\nu)}}^*)) = \pi_e^{\ell,0}({\boldsymbol{M}^{(\nu)}}^*).$$

Here the last equality follows from the homogeneity assumption. ∎

4. Computation of the operator $\sum M_i^* M_i$ on $\mathcal{H}^{(a)}$

In this section, we wish to compute the operator

$${\boldsymbol{M}^{(a)}}^* \boldsymbol{M}^{(a)} := \sum_{i=1}^{d} {M_i^{(a)}}^* M_i^{(a)}$$

on the Hilbert space $\mathcal{H}^{(a)}$. First, we note that the bounded symmetric domain Ω sits inside a linear space of dimension d in its Harish-Chandra realization. The type I domains are realized as the open unit ball, with respect to the operator norm, in the linear space of $n \times m$ matrices. The situation becomes somewhat different when we consider domains of type II. Pick one of these domains of dimension $\frac{n(n+1)}{2}$. It is convenient to put $\frac{n(n+1)}{2}$ variables in the form of a symmetric matrix, where the inner product is given by $\mathrm{tr}(AB^*)$. Now, in the space of these symmetric matrices of size n, the matrices E_{ii}, $i = 1, \ldots, n$ together with $\frac{E_{ij}+E_{ji}}{\sqrt{2}}$, $1 \leq i \neq j \leq n$, form an orthonormal basis. Consequently, the coordinates of this domain is of the form

$$z_{11}, \sqrt{2}z_{12}, \ldots, \sqrt{2}z_{1n}, z_{22} \ldots \sqrt{2}z_{2n}, \ldots, z_{n-1n-1}, \sqrt{2}z_{n-1n}, z_{nn},$$

see [16, p. 130]. One may pick coordinates similarly for the type III domains consisting of the $n \times n$ anti-symmetric matrices of norm at most 1. Finally, the type IV domains, in its Harish-Chandra realization, are described in [24, p. 76]:

$$\Big\{ \boldsymbol{z} := (z_1, \ldots, z_d) : \sum_{i=1}^d |z_i|^2 < 2 \text{ and } \sum_{i=1}^d |z_i|^2 < 1 + \Big| \frac{1}{2} \sum_{i=1}^d z_i^2 \Big|^2 \Big\}.$$

The following theorem appears in [5] in a slightly different form. The difference arises since we take the multiplication by the coordinate functions to be the ones described in the previous paragraph, while in the paper [5], these are the usual coordinates. Thus it makes no difference in the case of the type I domains, while for the other domains, the answer is different.

Theorem 4.1: *Let $\boldsymbol{M}^{(S)} = (M_1^{(S)}, \ldots, M_d^{(S)})$ be the d-tuple of multiplication operators by the coordinate functions $z_1, \ldots, z_d$ on the Hardy space $H^2(S)$. Then*

$$\sum_{i=1}^d {M_i^{(S)}}^* M_i^{(S)} = rI. \tag{4.5}$$

By Lemma 3.1, note that ${\boldsymbol{M}^{(a)}}^* \boldsymbol{M}^{(a)}$ is a block diagonal operator with respect to the decomposition $\bigoplus \mathcal{P}_{\underline{s}}$, where each block is a non-negative scalar multiple of the identity, that is,

$${\boldsymbol{M}^{(a)}}^* \boldsymbol{M}^{(a)} p = \delta(\underline{s}) p, \quad p \in \mathcal{P}_{\underline{s}}$$

for some non-negative real number $\delta(\underline{s})$. Therefore, in order to compute the operator ${\boldsymbol{M}^{(a)}}^* \boldsymbol{M}^{(a)}$, it is sufficient to obtain the constants $\delta(\underline{s})$ for all $\underline{s}$ in $\overrightarrow{\mathbb{N}}_0^r$. Unfortunately, we are only able to find $\delta(\underline{s})$ when $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ and $|I^+(\underline{s})| \leq 2$. In particular, we have the complete answer in case the rank $r = 2$.

The following lemma gives a description of the operator ${M_i^{(a)}}^*$ on $\mathcal{H}^{(a)}$. In case of weighted Bergman spaces, it is described in [29, Lemma 4.12.19].

For any polynomial p and $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$, the Peter–Weyl component of p in $\mathcal{P}_{\underline{s}}$ is denoted by $(p)_{\underline{s}}$.

Lemma 4.2: *If $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ and p is a polynomial in $\mathcal{P}_{\underline{s}}$, then*

$${M_i^{(a)}}^* p = \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}} (\partial_i p)_{\underline{s}-\varepsilon_j},$$

where ∂_i denotes the partial derivative with respect to the variable z_i.

Proof. By [29, Theorem 4.11.86], we have that $z_i\mathcal{P}_{\underline{s}}$ is contained in

$$\bigoplus_{j\in I^+(\underline{s})}\mathcal{P}_{\underline{s}+\varepsilon_j}.$$

Thus, for any polynomial p in $\mathcal{P}_{\underline{s}}$, it follows that M_i^*p belongs to $\bigoplus_{j\in I^-(\underline{s})}\mathcal{P}_{\underline{s}-\varepsilon_j}$. Now for $j\in I^-(\underline{s})$ and $q\in\mathcal{P}_{\underline{s}-\varepsilon_j}$, we have

$$\begin{aligned}\langle M_i^*p,q\rangle_{\mathcal{H}^{(a)}}&=\langle p,z_iq\rangle_{\mathcal{H}^{(a)}}=\langle p,(z_iq)_{\underline{s}}\rangle_{\mathcal{H}^{(a)}}\\&=\frac{1}{a_{\underline{s}}}\langle p,(z_iq)_{\underline{s}}\rangle_{\mathcal{F}}\\&=\frac{1}{a_{\underline{s}}}\langle p,z_iq\rangle_{\mathcal{F}}\\&=\frac{1}{a_{\underline{s}}}\langle \partial_ip,q\rangle_{\mathcal{F}}\\&=\frac{1}{a_{\underline{s}}}\langle (\partial_ip)_{\underline{s}-\varepsilon_j},q\rangle_{\mathcal{F}}\\&=\frac{a_{\underline{s}-\varepsilon_j}}{a_{\underline{s}}}\langle (\partial_ip)_{\underline{s}-\varepsilon_j},q\rangle_{\mathcal{H}^{(a)}}.\end{aligned}$$

Here the equality

$$\langle p,z_iq\rangle_{\mathcal{F}}=\langle\partial_ip,q\rangle_{\mathcal{F}}$$

follows from [29, Proposition 4.11.36]. This completes the proof. ∎

The following theorem describes the operator $\boldsymbol{M}^{(a)^*}\boldsymbol{M}^{(a)}$ on some subspace of $\mathcal{H}^{(a)}$.

Theorem 4.3: *Let $\underline{s}\in\overrightarrow{\mathbb{N}}_0^r$ be such that $|I^+(\underline{s})|\le 2$. Then*

$$\boldsymbol{M}^{(a)^*}\boldsymbol{M}^{(a)}p=\delta(\underline{s})p,\ p\in\mathcal{P}_{\underline{s}},$$

where

$$\delta(\underline{s})=\sum_{j\in I^+(\underline{s})}\frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_j}}\frac{(\frac{d}{r})_{\underline{s}+\varepsilon_j}}{(\frac{d}{r})_{\underline{s}}}c_{\underline{s}}(j).\tag{4.6}$$

Proof. First note that, for $p \in \mathcal{P}_{\underline{s}}$, we have

$$\begin{aligned}\sum_{i=1}^{d} M_i^{(a)^*} M_i^{(a)} p &= \sum_{i=1}^{d} M_i^{(a)^*}(z_i p) = \sum_{i=1}^{d} \Big(M_i^{(a)^*} \Big(\sum_{j \in I^+(\underline{s})} (z_i p)_{\underline{s}+\varepsilon_j} \Big) \Big)_{\underline{s}} \\ &= \sum_{i=1}^{d} \Big(\sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_j}} \partial_i((z_i p)_{\underline{s}+\varepsilon_j}) \Big)_{\underline{s}} \\ &= \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_j}} \sum_{i=1}^{d} (\partial_i((z_i p)_{\underline{s}+\varepsilon_j}))_{\underline{s}},\end{aligned}$$

where the third equality follows from Lemma 4.2. Let Q_j be the linear map on $\mathcal{P}_{\underline{s}}$ given by

$$Q_j(p) = \begin{cases} \sum_{i=1}^{d} (\partial_i((z_i p)_{\underline{s}+\varepsilon_j}))_{\underline{s}}, & \text{if } j \in I^+(\underline{s}), \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly,

$$\delta(\underline{s}) p = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_j}} Q_j(p). \tag{4.7}$$

Note that, for $p \in \mathcal{P}_{\underline{s}}$, Q_j satisfies the following:

$$\begin{aligned}\sum_{j \in I^+(\underline{s})} Q_j(p) &= \sum_{i=1}^{d} \sum_{j \in I^+(\underline{s})} (\partial_i((z_i p)_{\underline{s}+\varepsilon_j}))_{\underline{s}} = \sum_{i=1}^{d} \Big(\partial_i \Big(\sum_{j \in I^+(\underline{s})} (z_i p)_{\underline{s}+\varepsilon_j} \Big) \Big)_{\underline{s}} \\ &= \sum_{i=1}^{d} (\partial_i(z_i p))_{\underline{s}} \\ &= dp + \sum_{i=1}^{d} (z_i \partial_i p)_{\underline{s}}.\end{aligned}$$

Therefore, by Euler's formula, we obtain

$$\sum_{j \in I^+(\underline{s})} Q_j(p) = (d + |\underline{s}|) p. \tag{4.8}$$

Now, assume that $|I^+(\underline{s})| = 1$. Then $\underline{s}$ is necessarily of the form $(s_1, 0, \ldots, 0)$ and $I^+(\underline{s}) = \{1\}$. Thus it follows easily from (4.7) and (4.8) that

$$\delta(\underline{s}) = \frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_1}} r \Big(\frac{d}{r} + s_1 \Big).$$

To complete the proof, assume that $|I^+(\underline{s})| = 2$. Then $I^+(\underline{s}) = \{1, k\}$, where $2 \leq k \leq r$. Note that by (4.7) and Theorem 4.1, we have

$$\frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}+\varepsilon_1}} Q_1(p) + \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}+\varepsilon_k}} Q_k(p) = rp. \tag{4.9}$$

By solving equations (4.8) and (4.9), it is easily verified that

$$Q_1(p) = \frac{(k-1)(\frac{d}{r} + s_1)(s_1 - s_k + \frac{ar}{2})}{(s_1 - s_k + \frac{a}{2}(k-1))} p,$$

and

$$Q_k(p) = \frac{(r-k+1)(\frac{d}{r} - \frac{a}{2}(k-1) + s_k)(s_1 - s_k)}{(s_1 - s_k + \frac{a}{2}(k-1))} p.$$

Now, the proof is completed by (4.7). ∎

As an immediate consequence of Theorem 4.3, we obtain the following corollary giving the complete form of the operator $\boldsymbol{M}^{(a)^*}\boldsymbol{M}^{(a)}$ on $\mathcal{H}^{(a)}$ when the domain Ω is of rank 2.

Corollary 4.4: *Let Ω be an irreducible bounded symmetric domain of rank 2. Then, for any polynomial p in $\mathcal{P}_{\underline{s}}$, $\boldsymbol{M}^{(a)^*}\boldsymbol{M}^{(a)}p = \delta(\underline{s})p$, where*

$$\delta(\underline{s}) = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_j}} \frac{(\frac{d}{r})_{\underline{s}+\varepsilon_j}}{(\frac{d}{r})_{\underline{s}}} c_{\underline{s}}(j).$$

As a consequence of Theorem 4.3, we also obtain the following corollary about the essential normality of the multiplication operators by the coordinate functions on the weighted Bergman spaces.

Corollary 4.5: *Let $\nu > \frac{a}{2}(r-1)$ and $\boldsymbol{M}^{(\nu)} = (M_1^{(\nu)}, \ldots, M_d^{(\nu)})$ be the d-tuple of multiplication operators on $\mathcal{H}^{(\nu)}$. Then the operator $M_i^{(\nu)}$ is essentially normal, that is, the commutator*

$$M_i^{(\nu)^*} M_i^{(\nu)} - M_i^{(\nu)} M_i^{(\nu)^*}$$

is compact for all $i = 1, \ldots, d$ if and only if $r = 1$.

Proof. If $r = 1$, then by a direct computation it is easily verified that each $M_i^{(\nu)}$ is essentially normal. For the converse part, first set $\underline{\boldsymbol{l}}$ to be the signature $(l, 0, \ldots, 0)$, where l is a positive integer. Then, by Lemma 3.2 and Theorem 4.3, we see that

$$\sum_{i=1}^{d}(M_i^{(\nu)^*}M_i^{(\nu)} - M_i^{(\nu)}M_i^{(\nu)^*})p = \eta(\underline{\boldsymbol{l}})p, \quad p \in \mathcal{P}_{\underline{\boldsymbol{l}}},$$

where

$$\eta(\underline{\boldsymbol{l}}) = \frac{(\frac{d}{r}+l)(l+\frac{ar}{2})}{(\nu+l)(l+\frac{ar}{2})} + \frac{l(r-1)(\frac{d}{r}-\frac{a}{2})}{(\nu-\frac{a}{2})(l+\frac{a}{2})} - \frac{l}{\nu+l-1}. \tag{4.10}$$

Suppose that each $M_i^{(\nu)}$ is essentially normal. Then the operator

$$\sum_{i=1}^{d}(M_i^{(\nu)^*}M_i^{(\nu)} - M_i^{(\nu)}M_i^{(\nu)^*})$$

is compact. Hence $\eta(\underline{\boldsymbol{l}})$ must converge to 0 as $l \to \infty$. Thus, from (4.10), we obtain that $\frac{(r-1)(\frac{d}{r}-\frac{a}{2})}{\nu-\frac{a}{2}} = 0$. Finally, since $\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b$, we conclude that $r = 1$. ∎

We finish this section with the following conjecture on the description of the operator $\boldsymbol{M}^{(a)^*}\boldsymbol{M}^{(a)}$ on the Hilbert space $\mathcal{H}^{(a)}$ when the domain Ω is of arbitrary rank.

Conjecture 4.6: Let Ω be an irreducible bounded symmetric domain of rank r. Then, for any polynomial p in $\mathcal{P}_{\underline{s}}$, $\boldsymbol{M}^{(a)^*}\boldsymbol{M}^{(a)}p = \delta(\underline{s})p$ on the Hilbert space $\mathcal{H}^{(a)}$, where

$$\delta(\underline{s}) = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\varepsilon_j}} \frac{(\frac{d}{r})_{\underline{s}+\varepsilon_j}}{(\frac{d}{r})_{\underline{s}}} c_{\underline{s}}(j). \tag{4.11}$$

5. Unitary equivalence and Similarity

In this section, we study the question of unitary equivalence and similarity of two commuting d-tuple of operators in the class $\mathcal{A}\mathbb{K}(\Omega)$. In particular, when $\mathbb{K}$ is the unit circle $\mathbb{T}$, these results were obtained by Shields in [27] and the case when $\mathbb{K}$ is $\mathcal{U}(d)$, the similarity result was obtained in [19, Lemma 2.2].

By Theorem 2.3, any d-tuple of operators $\boldsymbol{T}$ in $\mathcal{A}\mathbb{K}(\Omega)$ is unitarily equivalent to $\boldsymbol{M}^{(a)}$ consisting of multiplication operators by the coordinate functions $z_1, \ldots, z_d$ on the reproducing kernel Hilbert space $\mathcal{H}^{(a)}$ with the reproducing kernel $K^{(a)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$, where $a_{\underline{s}} > 0$ with $a_{\underline{0}} = 1$. Thus we assume, without loss of generality, that $\boldsymbol{T} \sim_u \boldsymbol{M}^{(a)}$.

THEOREM 5.1: *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two operator tuples in $\mathcal{A}\mathbb{K}(\Omega)$. Suppose that $\boldsymbol{T}_1 \sim_u \boldsymbol{M}^{(a)}$ and $\boldsymbol{T}_2 \sim_u \boldsymbol{M}^{(b)}$. Then the following statements are equivalent:*

(i) *$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are unitarily equivalent.*
(ii) *$a_{\underline{s}} = b_{\underline{s}}$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$.*
(iii) *$K^{(a)} = K^{(b)}$.*

Proof. It is easy to see that (ii) and (iii) are equivalent. It is obvious that (iii) implies (i). Therefore it remains to verify that (i) implies (iii). Assume that the d-tuples $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are unitarily equivalent. Then so are the operators $\boldsymbol{M}^{(a)}$ and $\boldsymbol{M}^{(b)}$. By [11, Theorem 3.7], there exists a holomorphic function g on Ω such that

$$K^{(a)}(\boldsymbol{z},\boldsymbol{w}) = g(\boldsymbol{z})K^{(b)}(\boldsymbol{z},\boldsymbol{w})\overline{g(\boldsymbol{w})}, \quad \boldsymbol{z},\boldsymbol{w} \in \Omega.$$

In particular, $K^{(a)}(\boldsymbol{z},0) = g(\boldsymbol{z})K^{(b)}(\boldsymbol{z},0)\overline{g(0)}$, $\boldsymbol{z} \in \Omega$. Therefore, $a_{\underline{0}} = b_{\underline{0}}g(\boldsymbol{z})\overline{g(0)}$, and consequently, $g(z)\overline{g(0)} = 1$ since $a_{\underline{0}} = b_{\underline{0}} = 1$. Hence $K^{(a)} = K^{(b)}$. ∎

Recall that two commuting d-tuples $\boldsymbol{A} = (A_1, \ldots, A_d)$ and $\boldsymbol{B} = (B_1, \ldots, B_d)$, defined on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, are said to be similar if there exists an invertible operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XA_i = B_iX$ for all $i = 1, \ldots, d$. For a non-negative integer n, as before, $\mathcal{P}_n$ denotes the space of homogeneous polynomials of degree n in d variables. For two non-negative definite kernels K and $\tilde{K}$, we write $K \preceq \tilde{K}$ if $\tilde{K} - K$ is a non-negative definite kernel.

THEOREM 5.2: *Let $\Omega \subseteq \mathbb{C}^d$ be any bounded domain (not necessarily symmetric), and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two reproducing kernel Hilbert spaces determined by the positive definite kernels K_1 and K_2 respectively. Suppose that*

(i) *the space of polynomials $\mathcal{P}$ is dense in both $\mathcal{H}_1$ and $\mathcal{H}_2$,*
(ii) *$\mathcal{P}_n$ is orthogonal to $\mathcal{P}_m$ if $m \neq n$ in both $\mathcal{H}_1$ and $\mathcal{H}_2$,*
(iii) *for each $i = 1, 2$, the d-tuple $\boldsymbol{M}^{(i)} = (M_1^{(i)}, \ldots, M_d^{(i)})$ of multiplication operators by the coordinate functions $z_1, \ldots, z_d$ on $\mathcal{H}_i$ is bounded.*

Then the following statements are equivalent:

(i) *$\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$ are similar.*
(ii) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha\|p\|_{\mathcal{H}_1} \leq \|p\|_{\mathcal{H}_2} \leq \beta\|p\|_{\mathcal{H}_1}, \quad p \in \mathcal{P}. \tag{5.12}$$

(iii) *$\mathcal{H}_1 = \mathcal{H}_2$.*
(iv) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha K_1 \preceq K_2 \preceq \beta K_1.$$

Proof. The equivalence of (iii) and (iv) follows from the standard theory of reproducing kernel Hilbert spaces (cf. [3], [26]). Since the polynomials are dense in both $\mathcal{H}_1$ and $\mathcal{H}_2$, by (5.12), it is clear that (ii) implies (iii). If $\mathcal{H}_1 = \mathcal{H}_2$, then the identity operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a bounded invertible operator which intertwines the multiplication d-tuples $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$, and consequently, (iii) implies (i). Now, to complete the proof, it remains to show that (i) implies (ii).

Suppose that $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$ are similar. Then there exists an invertible operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$XM_j^{(1)} = M_j^{(2)}X, \quad j = 1, \dots, d. \tag{5.13}$$

Since the subspaces $\mathcal{P}_n$, $n \geq 0$, are mutually orthogonal , it suffices to show that (5.12) is satisfied for all $p \in \mathcal{P}_n$ and for some $\alpha, \beta > 0$ (which is independent of n). Fix a polynomial p in $\mathcal{P}_n$. Clearly, it follows from (5.13) that

$$XM_p^{(1)} = M_p^{(2)}X, \tag{5.14}$$

where $M_p^{(i)}$ is the operator of multiplication by the polynomial p on $\mathcal{H}_i$ for i=1, 2.

Let $(X_{r,s})_{r,s=0}^{\infty}$ be the matrix representation of X with respect to $\bigoplus_{n=0}^{\infty} \mathcal{P}_n$, that is,

$$X_{r,s} = P_{\mathcal{P}_r} X_{|\mathcal{P}_s}.$$

Similarly, let

$$M_p^{(i)} = ((M_p^{(i)})_{r,s})_{r,s=0}^{\infty}$$

be the matrix representation of $M_p^{(i)}$, $i = 1, 2$. Since $M_p^{(i)}$ maps $\mathcal{P}_s$ into $\mathcal{P}_{s+n}$, $i = 1, 2$, it clear that

$$(M_p^{(i)})_{r,s} = \begin{cases} (M_p^{(i)})_{|\mathcal{P}_s}, & \text{if } r = s + n, \\ 0, & \text{otherwise.} \end{cases} \tag{5.15}$$

Therefore it follows from (5.14) that

$$X_{r,s+n}(M_p^{(1)})_{s+n,s} = \begin{cases} (M_p^{(2)})_{r,r-n}X_{r-n,s}, & \text{if } r - n \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.16}$$

Choosing $r = n$ and $s = 0$, we see that

$$(M_p^{(2)})_{n,0}X_{0,0} = X_{n,n}(M_p^{(1)})_{n,0}. \tag{5.17}$$

Therefore

$$(M_p^{(1)})_{n,0}^* X_{n,n}^* X_{n,n} (M_p^{(1)})_{n,0} = X_{0,0}^* (M_p^{(2)})_{n,0}^* (M_p^{(2)})_{n,0} X_{0,0}. \tag{5.18}$$

Since $\|X_{n,n}\| \le \|X\|$, we have

$$X_{n,n}^* X_{n,n} \le \|X\|^2 I.$$

Hence from (5.18) we obtain

$$X_{0,0}^* (M_p^{(2)})_{n,0}^* (M_p^{(2)})_{n,0} X_{0,0} \le \|X\|^2 (M_p^{(1)})_{n,0}^* (M_p^{(1)})_{n,0}. \tag{5.19}$$

Note that $X_{0,0}$ is a linear map from $\mathcal{P}_0$ to $\mathcal{P}_0$ and $\dim \mathcal{P}_0 = 1$. Hence $X_{0,0} 1 = \eta 1$ for some $\eta \in \mathbb{C}$. Also, taking p to be the polynomial z_j, $1 \le j \le d$, and $r = 0$ in (5.16) we see that

$$X_{0,s+1} (M_{z_j}^{(1)})_{s+1,s} = 0, \quad \text{for all } s \ge 0.$$

Since this is true for all $j = 1, \ldots, d$, it follows that $X_{0,s+1} = 0$ for all $s \ge 0$. Moreover, since X is invertible we must have $X_{0,0} \ne 0$. Otherwise, $X_{0,s} = 0$ for all $s \ge 0$, implying that $\mathcal{P}_0$ is orthogonal to the range of X, which is a contradiction. Hence $X_{0,0} \ne 0$, and consequently $\eta \ne 0$. Therefore, from (5.19), we obtain

$$\langle (M_p^{(2)})_{n,0} X_{0,0} 1, (M_p^{(2)})_{n,0} X_{0,0} 1 \rangle \le \|X\|^2 \langle (M_p^{(1)})_{n,0} 1, (M_p^{(1)})_{n,0} 1 \rangle.$$

Consequently,

$$|\eta|^2 \|p\|_{\mathcal{H}_2}^2 \le \|X\|^2 \|p\|_{\mathcal{H}_1}^2. \tag{5.20}$$

To finish the proof, note that (5.13) implies

$$X^{-1} M_j^{(2)} = M_j^{(1)} X^{-1}, \quad j = 1, \ldots, d.$$

Hence repeating the arguments used to establish (5.20) we obtain that

$$|\zeta|^2 \|p\|_{\mathcal{H}_1}^2 \le \|X^{-1}\|^2 \|p\|_{\mathcal{H}_2}^2,$$

where $(X^{-1})_{0,0} 1 = \zeta 1$, $\zeta \ne 0$. This completes the proof. ∎

Remark 5.3: In the proof given above, we have shown that $X_{0,s} = 0$ for all $s \ge 1$. But using (5.16), it can be easily verified that $X_{r,s} = 0$ for all $s > r$, that is, X is lower triangular with respect to the decomposition $\bigoplus_{n=0}^{\infty} \mathcal{P}_n$. Consequently,

$$\zeta = \frac{1}{\eta}.$$

THEOREM 5.4: *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two operator tuples in $\mathcal{A}\mathbb{K}(\Omega)$. Suppose that $\boldsymbol{T}_1 \sim_u \boldsymbol{M}^{(a)}$ and $\boldsymbol{T}_2 \sim_u \boldsymbol{M}^{(b)}$. Then the following statements are equivalent.*

(i) *$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are similar.*

(ii) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha\|p\|_{\mathcal{H}^{(a)}} \leq \|p\|_{\mathcal{H}^{(b)}} \leq \beta\|p\|_{\mathcal{H}^{(a)}}, \quad p \in \mathcal{P}. \tag{5.21}$$

(iii) $\mathcal{H}^{(a)} = \mathcal{H}^{(b)}$.

(iv) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha K^{(a)} \preceq K^{(b)} \preceq \beta K^{(a)}.$$

(v) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha a_{\underline{s}} \leq b_{\underline{s}} \leq \beta a_{\underline{s}}, \quad \underline{s} \in \overrightarrow{\mathbb{N}}_0^r.$$

Proof. The equivalence of (i), (ii), (iii) and (iv) follows easily from Theorem 5.2. Assume that (ii) holds. Then (v) is easily verified by choosing any polynomial p in $\mathcal{P}_{\underline{s}}$ and using

$$\|p\|^2_{\mathcal{H}^{(a)}} = \frac{\|p\|^2_{\mathcal{F}}}{a_{\underline{s}}} \quad \text{and} \quad \|p\|^2_{\mathcal{H}^{(b)}} = \frac{\|p\|^2_{\mathcal{F}}}{b_{\underline{s}}}$$

in (5.12). Also, it is trivial to see that (v) implies (iv). ∎

COROLLARY 5.5: *Let $\nu_1, \nu_2 > \frac{a}{2}(r-1)$. Then the d-tuple of multiplication operators $\boldsymbol{M}^{(\nu_1)}$ on $\mathcal{H}^{(\nu_1)}$ and $\boldsymbol{M}^{(\nu_2)}$ on $\mathcal{H}^{(\nu_2)}$ are similar if and only if $\nu_1 = \nu_2$.*

Proof. Suppose that $\boldsymbol{M}^{(\nu_1)}$ and $\boldsymbol{M}^{(\nu_2)}$ are similar. Then, by Theorem 5.4, there exist constants $\alpha, \beta > 0$ such that $\alpha(\nu_1)_{\underline{s}} \leq (\nu_2)_{\underline{s}} \leq \beta(\nu_1)_{\underline{s}}$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$. Take $\underline{s} = (s_1, 0, \ldots, 0)$, $s_1 \in \mathbb{N}_0$. By the properties of the Gamma function we have

$$\frac{(\nu_1)_{\underline{s}}}{(\nu_2)_{\underline{s}}} = \frac{(\nu_1)_{s_1}}{(\nu_2)_{s_1}} \sim {s_1}^{\nu_1 - \nu_2}.$$

Hence $\nu_1 = \nu_2$. The other implication is trivial. ∎

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