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# ERRATUM TO "NON-UNIFORMLY FLAT AFFINE ALGEBRAIC HYPERSURFACES"

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**Abstract.** In this erratum, we correct an erroneous result in [PV2] and prove that the affine algebraic hypersurfaces $xy^2=1$ and $z=xy^2$ are not interpolating with respect to the Gaussian weight.

## §1. Introduction

Let $(X,\omega)$ be a Stein Kähler manifold of complex dimension $n$, equipped with a holomorphic line bundle $L \to X$ with smooth Hermitian metric $e^{-\varphi}$, and let $Z \subset X$ be a complex analytic subvariety of pure dimension $d$. To these data, assign the Hilbert spaces

$$\mathscr{B}_n(X,\varphi) := \left\{ F \in H^0(X,\mathcal{O}_X(L)) \ ; \ ||F||^2_X := \int_X |F|^2 e^{-\varphi}\frac{\omega^n}{n!} < +\infty \right\}$$

and

$$\mathfrak{B}_d(Z,\varphi) := \left\{ f \in H^0(Z,\mathcal{O}_Z(L)) \ ; \ ||f||^2_Z := \int_{Z_{\mathrm{reg}}} |f|^2 e^{-\varphi}\frac{\omega^d}{d!} < +\infty \right\}.$$

Such Hilbert spaces are called *(generalized) Bergman spaces*. When the underlying manifold is $\mathbb{C}^n$ and the weight $\varphi$ is a Bargmann–Fock weight, the spaces are called *(generalized) Bargmann–Fock spaces*.

We say that $Z$ is interpolating if the restriction map

$$\mathscr{R}_Z : H^0(X,\mathcal{O}_X(L)) \to H^0(Z,\mathcal{O}_Z(L))$$

induces a surjective map on Hilbert spaces. If the induced map

$$\mathscr{R}_Z : \mathscr{B}_n(X,\varphi) \to \mathfrak{B}_d(Z,\varphi)$$

is surjective, then one says that $Z$ is an *interpolation subvariety*, or simply *interpolating* with respect to $\varphi$. It can be easily shown that if $Z$ is interpolating, the map above is *bounded*.

In [PV2], Pingali and Varolin claimed that (Theorems 2 and 3) the (nonuniformly flat) curve $C_2 = \{(x,y) \in \mathbb{C}^2 \mid xy^2 = 1\}$ and the surface $S = \{(x,y,z) \in \mathbb{C}^3 \mid z = xy^2\}$ are interpolating with respect to a smooth weight $\varphi$ satisfying $m\omega_0 \le \sqrt{-1}\partial\bar{\partial}\phi \le M\omega_0$, where $\omega_0$ is the Euclidean metric and $m, M > 0$ are positive constants. The purported proof of the claim rested heavily on Lemma 3.2, which aimed to generalize the QuimBo trick [BOC]. Unfortunately, Lemma 3.2 is false. (However, for Theorems 1 and 4, we do not need

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Lemma 3.2. Instead, Lemma 6 in [L] in conjunction with elliptic regularity is enough.) In this erratum, we in fact prove that the negations of Theorems 2 and 3 in [PV2] are true.

THEOREM 1. *The curve* $C_2$ *is not interpolating with respect to the Gaussian weight* $|x|^2+|y|^2$.

Using Theorem 6.1 in [PV2], we can easily see that the following result holds.

THEOREM 2. *The surface* $S$ *is not interpolating with respect to the Gaussian weight* $|x|^2+|y|^2+|z|^2$.

These results lead us to suspect that perhaps uniform flatness might be equivalent to being interpolating (with respect to the Gaussian weight) for smooth affine *algebraic* hypersurfaces. For smooth affine *analytic* hypersurfaces, this expectation is false as shown in [PV1].

## §2. Proof of Theorem 1

Let $f_n(x,y)=y^{-(2n+1)}$, then $f_n\in\mathcal{O}(C_2)$.
Now,

$$\begin{aligned}||f_n||^2&=\int_{C_2}|f_n(x,y)|^2e^{-(|x|^2+|y|^2)}dA\\&=\int_{\mathbb{C}^*}|y^{-(2n+1)}|^2e^{-(|y|^{-4}+|y|^2)}\left(1+4|y|^{-6}\right)dV(y)\\&=\pi\int_{r=0}^{\infty}r^{-(2n+1)}e^{-(r+r^{-2})}\left(1+4r^{-3}\right)dr.\end{aligned}\tag{1}$$

For $\frac{1}{2}<s<\frac{3}{2}$ and $\frac{1}{2}<t<\frac{3}{2}$, let us consider the following integral:

$$\begin{aligned}\int_0^{\infty}e^{-(sr+tr^{-2})}4r^{-3}dr&=\left[e^{-sr}\int e^{-tr^{-2}}4r^{-3}dr\right]_0^{\infty}-\int_0^{\infty}-se^{-sr}\left(\int e^{-tr^{-2}}4r^{-3}dr\right)dr\\&=\left[e^{-sr}\frac{2}{t}e^{-tr^{-2}}\right]_0^{\infty}+\int_0^{\infty}se^{-sr}\frac{2}{t}e^{-tr^{-2}}dr\\&=\frac{2s}{t}\int_0^{\infty}e^{-(sr+tr^{-2})}dr.\end{aligned}$$

Therefore, we have

$$\int_0^{\infty}e^{-(sr+tr^{-2})}\left(1+4r^{-3}\right)dr=\left(1+\frac{2s}{t}\right)\int_0^{\infty}e^{-(sr+tr^{-2})}dr.\tag{2}$$

Differentiating (2) with respect to $s$, we arrive at the following:

$$\int_0^{\infty}-re^{-(sr+tr^{-2})}\left(1+4r^{-3}\right)dr=\left(1+\frac{2s}{t}\right)\int_0^{\infty}-re^{-(sr+tr^{-2})}dr+\frac{2}{t}\int_0^{\infty}e^{-(sr+tr^{-2})}dr.\tag{3}$$

Setting $s=1$ in (3), we have

$$\int_0^{\infty}re^{-(r+tr^{-2})}\left(1+4r^{-3}\right)dr=\int_0^{\infty}re^{-(r+tr^{-2})}dr+\frac{2}{t}\int_0^{\infty}(r-1)e^{-(r+tr^{-2})}dr.\tag{4}$$

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Differentiating (4) $(n+1)$ times with respect to $t$, we see that

$$\begin{aligned}
&\int_0^\infty r\left(-r^{-2}\right)^{n+1} e^{-(r+tr^{-2})}\left(1+4r^{-3}\right)dr\\
&=\int_0^\infty r\left(-r^{-2}\right)^{n+1} e^{-(r+tr^{-2})}dr+2\int_0^\infty (r-1)e^{-r}\frac{d^{n+1}}{dt^{n+1}}\left(\frac{e^{-tr^{-2}}}{t}\right)dr\\
&=(-1)^{n+1}\int_0^\infty r^{-2n-1}e^{-(r+tr^{-2})}dr+2(-1)^{n+1}\int_0^\infty (r-1)e^{-r}\sum_{k=0}^{n+1}\frac{(n+1)!}{(n+1-k)!}\frac{r^{-2(n+1-k)}}{t^{k+1}}e^{-tr^{-2}}dr\\
&=(-1)^{n+1}\int_0^\infty r^{-2n-1}e^{-(r+tr^{-2})}dr+2(-1)^{n+1}(n+1)!\int_0^\infty (r-1)e^{-(r+tr^{-2})}\sum_{k=0}^{n+1}\frac{r^{-2(n+1-k)}}{(n+1-k)!}\frac{1}{t^{k+1}}dr.
\end{aligned} \tag{5}$$

Substituting $t=1$ in (5), we get

$$\begin{aligned}
\int_0^\infty r^{-(2n+1)}e^{-(r+r^{-2})}\left(1+4r^{-3}\right)dr &= \int_0^\infty r^{-2n-1}e^{-(r+r^{-2})}dr\\
&\quad +2(n+1)!\int_0^\infty (r-1)e^{-(r+r^{-2})}\sum_{k=0}^{n+1}\frac{r^{-2k}}{k!}dr.
\end{aligned} \tag{6}$$

Now,

$$\begin{aligned}
&\int_0^\infty r^{-2n-1}e^{-(r+r^{-2})}dr\\
&=\left[e^{-r}\int r^{-2(n-1)}e^{-r^{-2}}r^{-3}dr\right]_0^\infty-\int_0^\infty -e^{-r}\left(\int r^{-2(n-1)}e^{-r^{-2}}r^{-3}dr\right)dr\\
&=\frac{(-1)^{n-1}}{2}\left[e^{-r}\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(n-1)!}{k!}(-r^{-2})^k e^{-r^{-2}}\right]_0^\infty\\
&\quad+\frac{(-1)^{n-1}}{2}\int_0^\infty e^{-r}\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(n-1)!}{k!}(-r^{-2})^k e^{-r^{-2}}dr\\
&=\frac{(n-1)!}{2}\int_0^\infty e^{-(r+r^{-2})}\sum_{k=0}^{n-1}\frac{r^{-2k}}{k!}dr\\
&\le\frac{(n-1)!}{2}\int_0^\infty e^{-(r+r^{-2})}e^{r^{-2}}dr.\\
&\le(n-1)!
\end{aligned} \tag{7}$$

Using (1), (6), and (7), we can see that the following holds:

$$||f_n||^2\le \pi(n-1)!+2\pi(n+1)!\int_0^\infty (r-1)e^{-(r+r^{-2})}\sum_{k=0}^{n+1}\frac{r^{-2k}}{k!}dr<\infty. \tag{8}$$

Suppose $C_2$ is interpolating. Then, there exist $F_n\in\mathscr{B}_2((|x|^2+|y|^2))$ and $C>0$ such that $F_n|_{C_2}=f_n$ and

$$||F_n||\le C||f_n||,\forall n\in\mathbb{N}. \tag{9}$$

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Let

$$F_n(x,y) = \sum_{i,j\geq 0} c_{ij} x^i y^j.$$

Then, we have

$$\begin{aligned} y^{-(2n+1)} &= \sum_{i,j\geq 0} c_{ij} y^{-2i} y^j \\ &= \sum_{i,j\geq 0} c_{ij} y^{-(2i-j)} \\ &= \sum_{2i-j=2n+1} c_{ij} y^{-(2i-j)}. \end{aligned} \tag{10}$$

This equation implies that

$$\sum_{k=1}^{\infty} c_{k+n,2k-1} = 1. \tag{11}$$

Equation (11) implies that there exists an $m \in \mathbb{N}$ such that $|c_{m+n,2m-1}| \geq 2^{-(m+1)}$. Therefore,

$$\begin{aligned} ||F_n||^2 &\geq \sum_{k=1}^{\infty} |c_{k+n,2k-1}|^2 (k+n)!(2k-1)! \\ &\geq |c_{m+n,2m-1}|^2 (m+n)!(2m-1)! \\ &\geq (2^{-(m+1)})^2 (1+n)! 2^{2m-2} \\ &\geq \frac{(n+1)!}{2^4}. \end{aligned} \tag{12}$$

From (8), (9), and (12), we conclude that

$$\frac{(n+1)!}{2^4} \leq C\left(\pi(n-1)! + 2\pi(n+1)! \int_0^{\infty} (r-1) e^{-(r+r^{-2})} \sum_{k=0}^{n+1} \frac{r^{-2k}}{k!} dr\right).$$

This inequality implies that

$$\frac{1}{2^4} \leq \pi C\left(\frac{1}{n(n+1)} + 2\int_0^{\infty} (r-1) e^{-(r+r^{-2})} \sum_{k=0}^{n+1} \frac{r^{-2k}}{k!} dr\right).$$

We are led to a contradiction because $\left(\frac{1}{n(n+1)} + 2\int_0^{\infty} (r-1) e^{-(r+r^{-2})} \sum_{k=0}^{n+1} \frac{r^{-2k}}{k!} dr\right) \to 0$, as $n \to \infty$. □

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