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CORRIGENDUM

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Heisenberg uniqueness pairs for the hyperbola

(Bull. Lond. Math. Soc. 53 (2021), no. 1, 16–25)

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Abstract

Recently we found that the sufficient condition in the main result of our paper entitled "Heisenberg uniqueness pairs for the hyperbola, Bull. Lond. Math. Soc. 53 (2021), no. 1, 16–25" is incorrect. The purpose of this corrigendum is to point out the gap in the proof given in the above-mentioned paper and supply a correct sufficient condition with proof. We prove the following result: let Γ be the hyperbola $xy = 1$ in $\mathbb{R}^2$, and $\Lambda_{\beta,\theta}$ be the lattice-cross defined by $\Lambda_{\beta,\theta} = ((\mathbb{Z} + \{\theta\}) \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\theta \in \mathbb{R}$, and $\beta > 0$. Then $(\Gamma, \Lambda_{\beta,\theta})$ is a Heisenberg uniqueness pair for $\beta \leqslant 1$.

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1 | INTRODUCTION

The aim of this paper is to correct the statement of [2, Theorem 1.4] which should be as follows.

Theorem 1.1. *Let* $\Gamma = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ *be the hyperbola and* Λ_β^θ *be the lattice-cross* $\Lambda_\beta^\theta = ((\mathbb{Z} + \{\theta\}) \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$*, where* β *is a positive real and* $\theta = 1/p$*, for some* $p \in \mathbb{N}$*. Then* $(\Gamma, \Lambda_\beta^\theta)$ *is a Heisenberg uniqueness pair whenever* $\beta \leqslant 1$*. Conversely, if* $(\Gamma, \Lambda_\beta^\theta)$ *is a Heisenberg uniqueness pair (HUP), then* $\beta \leqslant p$*.*

The necessity of the condition $\beta \leqslant p$ was proved in [2] and this part is correct. However, we also stated the converse which is unfortunately not true. The aim of this note is to give a counterexample to show that the converse stated in [2, Theorem 1.4] is not correct and we also indicate where the mistake lies in the argument given in [2].

For $p \geqslant 2$ and $\beta = p$, we can construct a counterexample to show that $(\Gamma, \Lambda_\beta^\theta)$ is not an HUP. To see this, following [1, p. 3, Equation 1.2] we can choose a non-zero function $H_{pn_0} \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$, $n_0 \in \mathbb{N}$ such that for $n \in \mathbb{Z}$, $m \in \mathbb{Z} \setminus \{0\}$

$$\int_{\mathbb{R}} e^{\pi i(n+1/p)x} H_{pn_0}(x/p)dx = \delta_{np+1,pn_0} = 0 \text{ and } \int_{\mathbb{R}} e^{\pi imp/x} H_{pn_0}(x/p)dx = 0$$

with $\int_{\mathbb{R}} H_{pn_0}(x/p)dx = \delta_{0,pn_0} = 0$. Therefore, for $\beta = p$ it is immediate that if we consider the measure $d\mu(x) = H_{pn_0}(x/p)dx$, then $\hat{\mu}|_{\Lambda_\beta^\theta} = 0$ but μ is non-zero.

As in [2], let $L_p^\infty(\mathbb{R})$ denote the space of all functions $f \in L^\infty(\mathbb{R})$ such that the map $x \longmapsto e^{-\pi ix/p} f(x)$ is 2-periodic. Then the weak-star closure in $L^\infty(\mathbb{R})$ of the linear span of functions $\{e_n^p(x) = e^{\pi i(n+1/p)x};\ n \in \mathbb{Z}\}$ is $L_p^\infty(\mathbb{R})$. In [2], we erroneously had assumed that the space $L_p^\infty(\mathbb{R})$ equals the space of all $2p$-periodic functions in $L^\infty(\mathbb{R})$. Therefore, our assumption in the proof of [[2], Theorem 2.4] that "The proof is similar to the proof of [12, Lemma 5.2] " does not work for $0 < \beta \leqslant p$. Next, we have the following result.

Theorem 1.2. *Let Γ be the hyperbola $xy = 1$ in $\mathbb{R}^2$, and $\Lambda_{\beta,\theta}$ be the lattice-cross defined by*

$$\Lambda_{\beta,\theta} = ((\mathbb{Z} + \{\theta\}) \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}), \tag{1.1}$$

where $\theta \in \mathbb{R}$, and $\beta > 0$. Then $(\Gamma, \Lambda_{\beta,\theta})$ is an HUP for $\beta \leqslant 1$.

Proof of Theorem 1.2 follows on similar lines as the proof of [4, Theorem 1.6.1] together with [4, Proposition 3.13.1(b)] for the case $0 < \beta < 1$ and [4, Proposition 3.13.3] for the case $\beta = 1$, respectively. However, for the sake of completeness, we briefly summarize the proof of Theorem 1.2. Suppose that there exists $f \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} e^{i\pi(m+\theta)t} f(t)dt = \int_{\mathbb{R}} e^{i\pi nt} f\left(\frac{\beta}{t}\right)\frac{dt}{t^2} = 0,\ m, n \in \mathbb{Z}. \tag{1.2}$$

Let $\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$ and $0 < \beta < 1$. Observe that for any $h \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} e^{i\pi mt} h(t)dt = 0 \text{ for all } m \in \mathbb{Z} \text{ if and only if } \sum_{j\in\mathbb{Z}} h(t + 2j) = 0 \text{ a.e. on } \mathbb{R}. \tag{1.3}$$

In particular, consider $h(t) = e^{\pi i\theta t} f(t)$ in (1.3) and $h(t) = \frac{1}{t^2} f(\frac{\beta}{t})$ in (1.3), respectively. Thus from (1.2) we get that

$$|f(t)| \leqslant \sum_{j\in\mathbb{Z}^\times} |f(t + 2j)| \text{ and } \frac{1}{t^2} f\left(\frac{\beta}{t}\right) + \sum_{j\in\mathbb{Z}^\times} \frac{1}{(t + 2j)^2} f\left(\frac{\beta}{t + 2j}\right) = 0 \text{ a.e. on } \mathbb{R}. \tag{1.4}$$

Combining both the conditions in (1.4), after invoking the change of variables $t \mapsto \beta/t$ in the second identity of (1.4), we get that

$$|f(t)| \leqslant \sum_{j,k\in\mathbb{Z}^\times} \frac{\beta^2}{[2j(t + 2k) + \beta]^2} \left| f\left(\frac{\beta(t + 2k)}{2j(t + 2k) + \beta}\right)\right| \text{ a.e. on } \mathbb{R}. \tag{1.5}$$

A simple calculation shows that if we restrict f to the interval $I := (-1, 1]$, then the identity (1.5) reduced to

$$|f(t)| \leqslant \mathrm{T}_\beta^2 |f|(t) \text{ a.e. } t \in I. \tag{1.6}$$

where the operator $\mathrm{T}_\beta : L^1(I) \longrightarrow L^1(I)$ is given by $\mathrm{T}_\beta f(x) = \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} f(-\frac{\beta}{x+2j})$. For the details about the map T_β, see [4]. Since T_β preserves the cone of positive functions and from (1.6) we have $\mathrm{T}_\beta^2|f| - |f| \geqslant 0$, it follows that $\mathrm{T}_\beta^4|f| \geqslant \mathrm{T}_\beta^2|f|$. By repeating the same argument, for $N \in \mathbb{N}$, we get that $\mathrm{T}_\beta^{2N}|f| \geqslant |f|$ a.e. on I. Now, for $0 < \beta < 1$, in view of [4, Proposition 3.13.1(b)], it implies that $\mathrm{T}_\beta^{2N}|f| \longrightarrow 0$ in $L^1(I)$ as $N \longrightarrow \infty$, and hence, (1.6) implies that $f = 0$ a.e. on I. Then from the second identity in (1.4) we get that $f = 0$ a.e. on $\mathbb{R} \setminus I$. Thus $f = 0$. Finally, for $\beta = 1$ it follows from (1.6) and [4, Proposition 3.13.3] that $f = 0$ a.e. on I. Now, from the second identity in (1.4) we get that $f = 0$ a.e. on $\mathbb{R} \setminus I$. Thus $f = 0$.

However, all the proofs of [2, Theorem 1.4] remain correct if we replace Λ_β^θ, where $\theta = 1/p$ in [2, Theorem 1.4] by the new lattice-cross $\tilde{\Lambda}_{\beta,p} = (\frac{1}{p}\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$ with the following modifications.

(a) Replace $(n + 1/p)$ by n/p, where $n \in \mathbb{Z}$.

(b) Let $\mathcal{L}_p^\infty(\mathbb{R})$ denote the space of all $2p$-periodic functions in $L^\infty(\mathbb{R})$. Then the weak-star closure in $L^\infty(\mathbb{R})$ of the linear span of the functions $\{e_n^p(x) = e^{\pi i n x/p};\ n \in \mathbb{Z}\}$ equals to $\mathcal{L}_p^\infty(\mathbb{R})$.

(c) Let $\mathcal{L}_\beta^\infty(\mathbb{R})$ denote the space of all functions $f \in L^\infty(\mathbb{R})$ such that the map $x \longmapsto f(\frac{p\beta}{x})$ is $2p$-periodic. Then the weak-star closure in $L^\infty(\mathbb{R})$ of the linear span of the functions $\{e_n^\beta(x) = e^{\pi i n\beta/x};\ n \in \mathbb{Z}\}$ equals to $\mathcal{L}_\beta^\infty(\mathbb{R})$.

(d) Replace the sum space $L_p^\infty(\mathbb{R}) + L_\beta^\infty(\mathbb{R})$ by the new sum space $\mathcal{L}_p^\infty(\mathbb{R}) + \mathcal{L}_\beta^\infty(\mathbb{R})$.

(e) $I_\beta(x) = -\frac{p\beta}{x}$.

Remark 1.3. In the article [2] with the above modifications, we have proved that $(\Gamma, \tilde{\Lambda}_{\beta,p})$ is an HUP if and only if $0 < \beta \leqslant p$, where Γ is the hyperbola in the plane. Although this result is immediate from [3], but the modified result in [2] provides an alternate proof without using invariance properties for HUPs.

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