

Vertex transitive graphs G with $\chi_D(G) > \chi(G)$ and small automorphism group*

Niranjan Balachandran

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai, India

Sajith Padinhatteeri †

Department of ECE, Indian Institute of Science, Bangalore, India

Pablo Spiga

Dipartimento Di Matematica E Applicazioni, University of Milano-Bicocca, Milano, Italy

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Abstract

For a graph G and a positive integer k , a vertex labelling $f: V(G) \rightarrow \{1, 2, \dots, k\}$ is said to be k -distinguishing if no non-trivial automorphism of G preserves the sets $f^{-1}(i)$ for each $i \in \{1, \dots, k\}$. The distinguishing chromatic number of a graph G , denoted $\chi_D(G)$, is defined as the minimum k such that there is a k -distinguishing labelling of $V(G)$ which is also a proper coloring of the vertices of G . In this paper, we prove the following theorem: Given $k \in \mathbb{N}$, there exists an infinite sequence of vertex-transitive graphs $G_i = (V_i, E_i)$ such that

1. $\chi_D(G_i) > \chi(G_i) > k$,
2. $|\text{Aut}(G_i)| < 2k|V_i|$, where $\text{Aut}(G_i)$ denotes the full automorphism group of G_i .

In particular, this answers a question posed by the first and second authors of this paper.

Keywords: Distinguishing chromatic number, vertex transitive graphs, Cayley graphs.

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E-mail addresses: niranj@math.iitb.ac.in (Niranjan Balachandran), sajithp@iisc.ac.in (Sajith Padinhatteeri), pablo.spiga@unimib.it (Pablo Spiga)

1 Introduction

Let G be a graph. An automorphism of G is a permutation φ of the vertex set $V(G)$ of G such that, for any $x, y \in V(G)$, $\varphi(x), \varphi(y)$ are adjacent if and only if x, y are adjacent. The automorphism group of a graph G , denoted by $\text{Aut}(G)$, is the group of all automorphisms of G . A graph G is said to be vertex transitive if, for any $u, v \in V(G)$, there exists $\varphi \in \text{Aut}(G)$ such that $\varphi(u) = v$.

Given a positive integer r , an r -coloring of G is a map $f: V(G) \rightarrow \{1, 2, \dots, r\}$ and the sets $f^{-1}(i)$, for $i \in \{1, 2, \dots, r\}$, are the color classes of f . An automorphism $\varphi \in \text{Aut}(G)$ is said to fix a color class C of f if $\varphi(C) = C$, where $\varphi(C) = \{\varphi(v) : v \in C\}$. A coloring of G , with the property that no non-trivial automorphism of G fixes every color class, is called a distinguishing coloring of G .

Collins and Trenk in [5] introduced the notion of the distinguishing chromatic number of a graph G , which is defined as the minimum number of colors needed to color the vertices of G so that the coloring is both proper and distinguishing. Thus, the distinguishing chromatic number of G is the least integer r such that the vertex set can be partitioned into sets V_1, V_2, \dots, V_r such that each V_i is independent in G , and for every non-trivial $\varphi \in \text{Aut}(G)$ there exists some color class V_i with $\varphi(V_i) \neq V_i$. The distinguishing chromatic number of a graph G , denoted by $\chi_D(G)$, has been the topic of considerable interest recently (see, for instance, [1, 2, 3, 4]).

One of the many questions of interest regarding the distinguishing chromatic number concerns the contrast between $\chi_D(G)$ and the cardinality of $\text{Aut}(G)$. For instance, the Kneser graphs $K(n, r)$ have very large automorphism groups and yet, $\chi_D(K(n, r)) = \chi(K(n, r))$ for $n \geq 2r + 1$, and $r \geq 3$ (see [2]). The converse question is compelling: Are there infinitely many graphs G_n with ‘small’ automorphism groups and satisfying $\chi_D(G_n) > \chi(G_n)$?

The question as posed above is not actually interesting for two reasons. First, for all even n , $\chi_D(C_n) > \chi(C_n) = 2$ and $|\text{Aut}(C_n)| = 2n$, where C_n is the cycle of length n . Second, if one stipulates that G also has arbitrarily large chromatic number, then here is a construction for such a graph. Start with a rigid graph G with a leaf vertex x and having large chromatic number (one can obtain this by minor modifications to a random graph, for instance); then, blow up the leaf vertex x to a new disjoint set X whose neighbor in the new graph \tilde{G} is the same as the neighbor of x in G . In fact one can arrange for $\chi_D(\tilde{G}) - \chi(\tilde{G})$ to be as large as one desires. Furthermore, since $|\text{Aut}(\tilde{G})| = |X|!$, this provides examples of graphs for which the automorphism groups are relatively ‘small’ in terms of the order of the graph.

In the example above, the fact that $\chi_D(G)$ is larger than $\chi(G)$ is accounted for by a ‘local’ reason, and that is what makes the problem stated above not very interesting. However, if one further stipulates that the graph is vertex-transitive, then the same question is highly non-trivial. In [1], the first and second authors constructed families of vertex-transitive graphs with $\chi_D(G) > \chi(G) > k$ and $|\text{Aut}(G)| = O(|V(G)|^{3/2})$, for any given k . In this paper, we improve upon that result:

Theorem 1.1. *Given $k \in \mathbb{N}$, there exists an infinite family of graphs $G_n = (V_n, E_n)$ satisfying:*

1. $\chi_D(G_n) > \chi(G_n) > k$,
2. G_n is vertex transitive and $|\text{Aut}(G_n)| < 2k|V_n|$.

Our family of graphs consists of Cayley graphs. To recall the definition, let A be a group and let S be an inverse-closed subset of A , i.e., $S = S^{-1}$, where $S^{-1} := \{s^{-1} : s \in S\}$. The Cayley graph $\text{Cay}(A, S)$ is the graph with vertex set A and the vertices u and v are adjacent in $\text{Cay}(A, S)$ if and only if $uv^{-1} \in S$.

We start with a brief description of the graphs of our construction. For q , an odd prime, let \mathbb{F}_q^n denote the n -dimensional vector space over \mathbb{F}_q . Our graphs shall be Cayley graphs $\text{Cay}(\mathbb{F}_q^n, S)$ for some suitable inverse-closed set $S \subset \mathbb{F}_q^n$ which is obtained by taking a union of a certain collection of lines in \mathbb{F}_q^n and then deleting the zero element of \mathbb{F}_q^n . More precisely, let $\mathcal{H}_0 := \{(x_1, x_2, \dots, x_{n-1}, 0) : x_i \in \mathbb{F}_q, 1 \leq i \leq n-1\}$ and let $\mathbf{0}$ denote the element $(0, \dots, 0) \in \mathbb{F}_q^n$. For each line (1-dimensional subspace of \mathbb{F}_q^n) $\ell \subset \mathbb{F}_q^n$ satisfying $\ell \cap \mathcal{H}_0 = \{\mathbf{0}\}$, pick ℓ independently with probability $1/2$ to form the random set \tilde{S} . Our connection set S for the Cayley graph $\text{Cay}(\mathbb{F}_q^n, S)$ is defined by $S := \{v \in \mathbb{F}_q^n : v \in \ell \text{ for some } \ell \in \tilde{S}\} \setminus \{\mathbf{0}\}$. Our main theorem states that with high probability, $G_{n,S} := \text{Cay}(\mathbb{F}_q^n, S)$ satisfies the conditions of Theorem 1.1.

To show that these graphs have ‘small’ automorphism groups, we prove a stronger version of Theorem 4.3 of [6] in this particular context, which is also a result of independent interest.

Theorem 1.2. *Let q be a prime power, let n be a positive integer with $n \geq 2$ and let G be the additive group of the n -dimensional vector space \mathbb{F}_q^n over the finite field \mathbb{F}_q of cardinality q , and let $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{\mathbf{0}\}$ be the multiplicative group of the field \mathbb{F}_q with its natural group action on G by scalar multiplication, and write $K := \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$. If S is an inverse-closed subset of G with $K \leq \text{Aut}(\text{Cay}(G, S))$, then either*

- (i) $\text{Aut}(\text{Cay}(G, S)) = K$, or
- (ii) there exists $\varphi \in \text{Aut}(\text{Cay}(G, S)) \setminus K$ with φ normalizing G .

Remark 1.3. Theorem 1.2 is valid even though the connection set S is not inverse-closed. Since we deal with Cayley graphs the phrase inverse-closed subset is used in the statement of the theorem.

The rest of the paper is organized as follows. We start with some preliminaries in Section 2 and then include the proofs of Theorems 1.1 and 1.2 in the next section. We conclude with some remarks and some open questions.

2 Preliminaries

We begin with a few definitions from finite geometry. For more details, one may see [13, 14]. By $\text{PG}(n, q)$ we mean the Desarguesian projective space obtained from the affine space $\text{AG}(n+1, q)$.

Definition 2.1. A cone with vertex $A \subset \text{PG}(k, q)$ and base $B \subset \text{PG}(n-k-1, q)$, where $\text{PG}(k, q) \cap \text{PG}(n-k-1, q) = \emptyset$, is the set of points lying on the lines connecting points of A and B .

Definition 2.2. Let V be an $(n+1)$ -dimensional vector space over a finite field \mathbb{F} . A subset S of $\text{PG}(V)$ is called an \mathbb{F}_q -linear set if there exists a subset U of V that forms an \mathbb{F}_q -vector space, for some $\mathbb{F}_q \subset \mathbb{F}$, such that $S = \mathcal{B}(U)$, where

$$\mathcal{B}(U) := \{\langle u \rangle_{\mathbb{F}} : u \in U \setminus \{\mathbf{0}\}\}$$

and where $\langle u \rangle_{\mathbb{F}}$ denotes the projective point of $\text{PG}(V)$, corresponding to the vector u of $U \subset V$.

Further details about \mathbb{F}_q -linear sets can be found in [14], for instance.

The projective space $\text{PG}(n, q)$ can be partitioned into an affine space $\text{AG}(n, q)$ and a hyperplane at infinity, denoted by H_∞ .

Definition 2.3. Following [13], we say that a set of points $U \subset \text{AG}(n, q)$ determines the direction $d \in H_\infty$, if there is an affine line through d meeting U in at least two points.

We now state the main theorem of [13] which will be relevant in our setting.

Theorem 2.4. Let $U \subset \text{AG}(n, \mathbb{F}_q)$, $n \geq 3$, $|U| = q^k$. Suppose that U determines at most $\frac{q+3}{2}q^{k-1} + q^{k-2} + \dots + q^2 + q$ directions and suppose that U is an \mathbb{F}_p -linear set of points, where $q = p^h$, $p > 3$ prime. If $n - 1 \geq (n - k)h$, then U is a cone with an $(n - 1 - h(n - k))$ -dimensional vertex at H_∞ and with base a \mathbb{F}_q -linear point set $U_{(n-k)h}$ of size $q^{(n-k)(h-1)}$, contained in some affine $(n - k)h$ -dimensional subspace of $\text{AG}(n, q)$.

We end this section by recalling another result that appears in [6] as Theorem 4.2.

Theorem 2.5. Let G be a permutation group on Ω with a proper self-normalizing abelian regular subgroup. Then $|\Omega|$ is not a prime power.

3 Proofs of the Theorems

In this section we prove Theorems 1.1 and 1.2 starting with the proof of Theorem 1.2. We believe that this result is only the tip of an iceberg: its current statement has been tailored to the context of our setting, and uses some ideas that appear in [6, Section 3] and [9].

Proof of Theorem 1.2. We suppose that (i) does not hold, that is, K is a proper subgroup of $\text{Aut}(\text{Cay}(G, S))$; we show that (ii) holds. Write $\Gamma := \text{Cay}(G, S)$.

Let B be a subgroup of $\text{Aut}(\Gamma)$ with $K < B$ and with K maximal in B . Suppose that $K \triangleleft B$. As G is characteristic in K , we get $G \triangleleft B$. In particular, every element φ in $B \setminus K$ satisfies (ii).

Suppose then that K is not normal in B . Since K is maximal in B and $G \triangleleft K$, we have $\mathbf{N}_B(G) = K$. Suppose that there exists $b \in B \setminus K$ such that $L := \langle G, G^b \rangle$ (the smallest subgroup of B containing G and G^b) satisfies $L \cap K = G$. We claim that we are now in the position to apply Theorem 2.5 (and implicitly some ideas from [9]). Indeed, as $\mathbf{N}_L(G) = \mathbf{N}_B(G) \cap L = K \cap L = G$, L is a transitive permutation group on the vertices of Γ with a proper regular self-normalizing abelian subgroup G . (Observe that G is a proper subgroup of L because $b \notin \mathbf{N}_B(G) = K$.) By Theorem 2.5, $|G|$ is not a prime power, which is a contradiction because $|G| = q^n$. This proves that, for every $b \in B \setminus K$, we have $\langle G, G^b \rangle \cap K > G$.

Fix $b \in B \setminus K$. Now, G and G^b are abelian and hence $G \cap G^b$ is centralized by $\langle G, G^b \rangle$. From the preceding paragraph, there exists $k \in \langle G, G^b \rangle \cap K$ with $k \notin G$. Observe now that $K = \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ is a Frobenius group with kernel $G = \mathbb{F}_q^n$ and complement \mathbb{F}_q^* . Therefore, k acts by conjugation fixed-point-freely on $G \setminus \{0\}$. As k centralizes $G \cap G^b$, we deduce $|G \cap G^b| = 1$.

Let $C := \bigcap_{x \in B} K^x$ be the core of K in B . As $G \cap G^b = 1$ for all $b \in B \setminus K$, $K \cap K^b$ has no non-identity q -elements. Therefore $C \cap G = 1$. As $C \triangleleft B$ and $C \leq K$, C is

a normal subgroup of the Frobenius group K intersecting its kernel on the identity. This yields $C = 1$.

Let Ω be the set of right cosets of K in B . From the paragraph above, B acts faithfully on Ω . Moreover, as K is maximal in B , the action of B on Ω is primitive. Therefore B is a finite primitive group with a solvable point stabilizer K . In [11], Li and Zhang have explicitly determined such primitive groups: these are classified in [11, Theorem 1.1] and [11, Tables I–VII]. Now, using the terminology in [11], a careful (but not very difficult) case-by-case analysis on the tables in [11] shows that B is a primitive group of affine type, that is, B contains an elementary abelian normal r -subgroup V , for some prime r . For this analysis it is important to keep in mind that the stabilizer K is a Frobenius group with kernel the elementary abelian group $G \cong \mathbb{F}_q^n$ and $n \geq 2$.

Let $|V| = r^t$. Now, the action of B on Ω is permutation equivalent to the natural action of $B = V \rtimes K$ on V , with V acting via its regular representation and with K acting by conjugation. Observe that $q \neq r$, because K acts faithfully and irreducibly as a linear group on V and hence K contains no non-identity normal r -subgroups. Observe further that $|B| = |V||K| = r^t \cdot q^n \cdot (q - 1)$.

We are finally ready to reach a contradiction and to do so, we go back studying the action of B on the vertices of Γ . Observe that B is solvable because V is solvable and so is $B/V \cong K$. We write B_0 for the stabilizer in B of the vertex $\mathbf{0}$ of Γ . As G acts regularly on the vertices of Γ , we obtain $B = B_0G$ and $B_0 \cap G = 1$. In particular, $|B_0| = r^t \cdot (q - 1)$. Observe that B_0 is a Hall Π -subgroup of the solvable group B , where Π is the set of all the prime divisors of $q - 1$ together with the prime r . As V is a Π -subgroup, from the theory of Hall subgroups (see for instance [7], Theorem 3.3), V has a conjugate contained in B_0 . Since $V \triangleleft B$, we have $V \leq B_0$. This is clearly a contradiction because V is normal in B , but B_0 is core-free in B , being the stabilizer of a point in a transitive permutation group. □

For the next lemma, recall that

$$\mathcal{H}_0 := \{(x_1, x_2, \dots, x_{n-1}, 0) : x_i \in \mathbb{F}_q, 1 \leq i \leq n - 1\}.$$

In what follows, $G_{n,S}$ will denote the Cayley graph $\text{Cay}(\mathbb{F}_q^n, S)$ and $S = \tilde{S} \setminus \{\mathbf{0}\}$ for some set $\tilde{S} = \bigcup_{\ell \in \mathcal{L}} \ell$, where \mathcal{L} is a collection of lines in \mathbb{F}_q^n with each $\ell \in \mathcal{L}$ satisfying $\ell \cap \mathcal{H}_0 = \{\mathbf{0}\}$.

Lemma 3.1. *If $\mathcal{L} \neq \emptyset$, then $\chi(G_{n,S}) = q$.*

Proof. Observe that each line that belongs to the set S gives rise to a clique of size q in the graph $G_{n,S}$. Therefore $\chi(G_{n,S}) \geq q$. On the other hand, for a fixed $v \in S$, the partition $(C_\lambda)_{\lambda \in \mathbb{F}_q}$, where $C_\lambda := \{w + \lambda v : w \in \mathcal{H}_0\}$, of the vertex set \mathbb{F}_q^n is a proper coloring of the graph $G_{n,S}$. Indeed, for any distinct $x = w_1 + \lambda v, y = w_2 + \lambda v$ in C_λ , we have $x - y = w_1 - w_2 \notin S$ because $w_1 - w_2 \in \mathcal{H}_0$ and $S \cap \mathcal{H}_0 = \emptyset$. Therefore the sets C_λ are independent in $G_{n,S}$ for each $\lambda \in \mathbb{F}_q$. □

Lemma 3.2. *Assume that q is prime. Let \tilde{S} be the random set corresponding to a union of lines ℓ in \mathbb{F}_q^n with $\ell \cap \mathcal{H}_0 = \{\mathbf{0}\}$ and where each $\ell \in \mathbb{F}_q^n$ is chosen independently with probability $\frac{1}{2}$; and let $S = \tilde{S} \setminus \{\mathbf{0}\}$. Then*

$$\mathbb{P}(\chi_D(G_{n,S}) > q) \geq 1 - \exp\left(-\frac{q^{n-3}}{4}\right).$$

Proof. First, note that $\mathbb{E}(|S|) = \frac{q^{n-1}}{2}$, so taking $\delta = \frac{1}{q}$ and $\mu = \mathbb{E}(|S|)$ in the Chernoff bound (see (2.6) on page 26 of [10]) we obtain

$$\mathbb{P}\left(|S| < \frac{q^{n-1} - q^{n-2}}{2}\right) \leq \exp\left(-\frac{q^{n-3}}{4}\right).$$

In particular, with probability at least $1 - \exp(-q^{n-3}/4)$, we have $|S| > \frac{q^{n-1} - q^{n-2}}{2}$. We may thus assume $|S| > \frac{q^{n-1} - q^{n-2}}{2}$ in what follows.

We claim that every color class in a proper q -coloring of $G_{n,S}$ is an affine hyperplane of \mathbb{F}_q^n . To see why, let C_1, \dots, C_q be independent sets in $G_{n,S}$ witnessing a proper q -coloring of $G_{n,S}$. Fix $v \in S$ and consider the line $\ell_v := \{\lambda v : \lambda \in \mathbb{F}_q\}$ along with its translates $\ell_v + w := \{\lambda v + w : \lambda \in \mathbb{F}_q\}$, for $w \in \mathcal{H}_0$. Each set $\ell_v + w$ is a clique of size q in $G_{n,S}$, and these cliques partition the vertex set of $G_{n,S}$, so in particular each C_i contains at most one vertex from each of these translates $\ell_v + w$. Consequently, $|C_i| \leq q^{n-1}$ for all $i \in \{1, \dots, q\}$. By size considerations, it follows that $|C_i| = q^{n-1}$ for each $i \in \{1, \dots, q\}$.

Consider a color class C . Suppose C determines at least $\frac{q+3}{2}q^{n-2} + q^{n-3} + \dots + q^2 + q + 1$ directions. Then if $\langle C \rangle$ denotes the set of all vertices in the affine lines intersecting at least two points in C , we have $|\langle C \rangle| + |S| > 1 + q + \dots + q^{n-1}$, so $\langle C \rangle \cap S \neq \emptyset$. However, this contradicts the assumption that C is an independent set in $G_{n,S}$. Therefore C determines at most $\frac{q+3}{2}q^{n-2} + q^{n-3} + \dots + q^2 + q$ directions. Since q is prime, by Corollary 10 in [13], it follows that C is an \mathbb{F}_q -linear set. Hence, by Theorem 2.4, the color class C is a cone with an $n - 2$ (projective) dimensional vertex \mathcal{V} at H_∞ and an affine point u_1 as base. In particular, the affine plane corresponding to the \mathbb{F}_q -subspace spanned by \mathcal{V} passing through the affine point u_1 is contained in C . Since $|C| = q^{n-1}$, it follows that C is this affine hyperplane, and this proves the claim.

To complete the proof, observe that for each $\lambda \in \mathbb{F}_q^* \setminus \{1\}$, the map $\varphi_\lambda(x) = \lambda x$, $x \in \mathbb{F}_q^n$ fixes each color class. Moreover, φ_λ fixes the set S and $\varphi_\lambda(u) - \varphi_\lambda(v) = \varphi_\lambda(u - v)$, so φ_λ is a non-trivial automorphism which fixes each color class. Therefore $\chi_D(G_{n,S}) > q$. \square

Lemma 3.3. *If $n \geq 6$ and $q \geq 5$ is prime, then $\text{Aut}(G_{n,S}) \cong \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ with probability at least*

$$1 - 2^{-\frac{q^{n-1}}{3}}.$$

Proof. Since $G_{n,S}$ is a Cayley graph on the additive group $G = \mathbb{F}_q^n$, by Theorem 1.2, either $\text{Aut}(G_{n,S}) = K \cong \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ or there exists $\varphi \in \text{Aut}(G_{n,S}) \setminus K$ with φ normalizing $G = \mathbb{F}_q^n$. We show that with probability at least $1 - 2^{-\frac{q^{n-1}}{3}}$, there is no φ satisfying the latter condition.

Suppose $\varphi \in \text{Aut}(G_{n,S})$ normalizes \mathbb{F}_q^n . If $a = \varphi(\mathbf{0})$ and $\lambda_a : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ is the right translation via a , then $\lambda_a^{-1}\varphi$ is an automorphism of $G_{n,S}$ normalizing \mathbb{F}_q^n and with $(\lambda_a^{-1}\varphi)(\mathbf{0}) = (\lambda_a^{-1})(\varphi(\mathbf{0})) = (\lambda_a^{-1})(a) = a - a = \mathbf{0}$. Therefore, without loss of generality, we may assume that $\varphi(\mathbf{0}) = \mathbf{0}$. Since S is the neighbourhood of $\mathbf{0}$ in $G_{n,S}$, we get $\varphi(S) = S$. Moreover, since φ acts as a group automorphism on \mathbb{F}_q^n , we have $\varphi \in \text{GL}_n(q)$.

Now, for $\varphi \in \text{GL}_n(q)$, let E_φ denote the event $\varphi(S) = S$. Let \mathcal{L} denote the set of all lines ℓ with $\ell \cap \mathcal{H}_0 = \emptyset$. Also, let $\text{Orb}_\varphi(\ell) = \{\ell, \varphi(\ell), \varphi^2(\ell), \dots, \varphi^k(\ell)\}$ where $\varphi^{k+1}(\ell) = \ell$. Then

$$\mathbb{P}(E_\varphi) \leq \prod_{i=1}^{N_\varphi} 2^{1 - |\text{Orb}_\varphi(\ell_i)|} = 2^{N_\varphi - |\mathcal{L}|},$$

where N_φ denotes the number of distinct orbits of φ in \mathcal{L} . Setting $\mathcal{G} = \text{GL}(n, q) \setminus \{\lambda I : \lambda \in \mathbb{F}_q^*\}$, we have

$$\mathbb{P}\left(\bigcup_{\varphi \in \mathcal{G}} E_\varphi\right) \leq \sum_{\varphi \in \mathcal{G}} \mathbb{P}(E_\varphi) \leq 2^{-|\mathcal{L}|} \sum_{\varphi \in \mathcal{G}} 2^{N_\varphi}. \tag{3.1}$$

Let $F_\varphi := |\{\ell \in \mathcal{L} : \varphi(\ell) = \ell\}|$ and $F := \max_{\varphi \in \mathcal{G}} F_\varphi$. Now $N_\varphi \leq F + \frac{|\mathcal{L}| - F}{2} = \frac{F + |\mathcal{L}|}{2}$. Thus, it suffices to give a suitable upper bound for F . Towards that end, we note that, if $F_\varphi = F$ for $\varphi \in \mathcal{G}$, then every line ℓ fixed by φ corresponds to an eigenvector of φ . If $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$ denote the eigenspaces of φ for some distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then

$$F_\varphi \leq \sum_{i=1}^k \binom{\dim \mathcal{E}_i}{1}_q - \binom{\dim(\mathcal{E}_i \cap \mathcal{H}_0)}{1}_q \leq q^{n-2} + 1.$$

Similarly, we have $|\mathcal{L}| = \binom{n}{1}_q - \binom{n-1}{1}_q = q^{n-1}$, and so by (3.1), we have

$$\mathbb{P}\left(\bigcup_{\varphi \in \mathcal{G}} E_\varphi\right) \leq |\mathcal{G}| 2^{\frac{F - |\mathcal{L}|}{2}} < q^{n^2} 2^{-\frac{q^{n-1} - q^{n-2} - 1}{2}} < 2^{-\frac{q^{n-1}}{3}},$$

for $q \geq 5, n \geq 6$. □

Computations and estimates similar to the ones presented in the proof of Lemma 3.3 have been proved useful in a variety of problems, see for instance [1, 8] and [12, Section 6.4].

Proof of Theorem 1.1. Given $k \in \mathbb{N}$ with $k \geq 4$, pick a prime number q with $k < q < 2k$. For $n \geq 6$, consider the random graph $G_{n,S}$ of the group \mathbb{F}_q^n as constructed above. By Lemmas 3.1, 3.2 and 3.3, with positive probability, the graph $G_{n,S}$ satisfies the statements of the lemmas, and hence satisfies the conclusions of Theorem 1.1. □

4 Concluding remarks

- We observe that, for S chosen randomly as in the proof of our result, the distinguishing chromatic number of $G_{n,S}$ is $q + 1$ with high probability. Indeed, consider the q -coloring C described in Lemma 3.1. Re-color the vertex $\mathbf{0}$ using an additional color. Then the coloring described by the partition $C' = C \cup \{\mathbf{0}\}$ is a proper, distinguishing coloring of $G_{n,S}$ with $q + 1$ colors. In fact, C' is clearly proper, and to show that it is distinguishing, consider $\varphi \in \text{Aut}(G_{n,S}) = \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ (by Lemma 3.3) that fixes every color class. Write $\varphi(x) = \lambda x + b$ with $\lambda \in \mathbb{F}_q^*, b \in \mathbb{F}_q^n$. Since φ fixes the color class containing $\mathbf{0}$, we have $b = \mathbf{0}$. Also, x and λx cannot be in same color class unless $\lambda = 1$. Therefore φ is the identity automorphism.

It is interesting to determine if one can obtain families of vertex-transitive graphs with $\chi_D(G) > \chi(G) + 1$, with ‘small’ automorphism groups and with $\chi(G)$ being arbitrarily large. In fact, for $k \in \mathbb{N}$, there is no known family of vertex-transitive graphs for which $\chi_D(G) > \chi(G) + 1 > k$ and $|\text{Aut}(G)| = O(|V(G)|^{O(1)})$. It is plausible that Cayley graphs over certain groups may provide the correct constructions.

- Theorem 1.1 establishes, for any fixed k , the existence of vertex-transitive graphs $G_n = (V_n, E_n)$ with $\chi_D(G_n) > \chi(G_n) > k$ and with $|\text{Aut}(G_n)| < 2k|V_n|$. It would be interesting to obtain a similar family of graphs that satisfy with $\chi_D(G_n) > \chi(G_n) > k$ and with $|\text{Aut}(G_n)| \leq C|V_n|$, for some absolute constant C .

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