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Invariants of the $\mathbb{Z}_2$ orbifolds of the Podleś two spheres

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Abstract. There are two $\mathbb{Z}_2$ orbifolds of the Podleś quantum two-sphere, one being the quantum two-disc D_q and other the quantum two-dimensional real projective space $\mathbb{R}P_q^2$. In this article we calculate the Hochschild and cyclic homology and cohomology groups of these orbifolds and also the corresponding Chern–Connes indices.

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1. Introduction

In noncommutative geometry, Podleś noncommutative two-spheres [11] are noncommutative low-dimensional manifolds. These are $SU_q(2)$-homogeneous spaces. The Podleś quantum 2-spheres are denoted by $S_{q,s}^2$, where

$$q^n \neq 1 \quad \text{for all } n \in \mathbb{N}.$$

and $s \in [0, 1]$. An extensive study of the Dirac operators, spectral triples and corresponding local index formulae for the Podleś quantum 2-spheres can be found in articles [1] and [2]. Detailed quantum isometries of the Podleś quantum 2-spheres can be found [4]. In [7] the authors studied the quantum disc D_q and the two dimensional quantum real projective space $\mathbb{R}P_q^2$ arising from two involutive automorphisms of $S_{q,1}^2$. It is worth noting that for $s > 0$ [15],

$$C\big(S_{q,s}^2\big) \cong C\big(S_{q,1}^2\big).$$

All automorphisms of $S_{q,s}^2$ are known to be diagonal as described in [9].

Let $\mathcal{A}$ denote the C*-algebra $C(S_{q,s}^2)$ and $\rho \in \mathrm{Aut}(\mathcal{A})$. Using a free resolution for $\mathcal{A}$, Masuda, Nakagami, and Watanabe calculated the Hochschild and cyclic homology of the Podleś quantum 2-spheres [10]. In [5] the author used this resolution to calculate the Hochschild homologies $H_\bullet(\mathcal{A}, {}_\rho\mathcal{A})$ associated to an automorphism ρ of the algebra $\mathcal{A}$. These are isomorphic to the twisted Hochschild homologies $HH_\bullet^\rho(\mathcal{A})$ [6].

The Hochschild homology groups $H_n(\mathcal{A},\mathcal{A})$ vanish for $n \geq 2$, but there exist automorphisms ρ with $H_n(\mathcal{A}, {}_\rho\mathcal{A}) \neq 0$ for $n = 0, 1, 2$. These automorphisms are positive powers of the canonical modular automorphism associated with the $SU_q(2)$-invariant linear functional. These are not order two automorphisms and we will not discuss them here, but it is interesting to note this phenomenon which has been studied in detail elsewhere.

Classical orbifolds are geometric objects arising from the linear action of finite groups on manifolds. When dealing with the algebras, orbifold are equivalently defined as the fixed point algebra for the groups action. Since the fixed point algebra for a discrete action on a C*-algebra is Morita equivalent to the associated crossed product algebra, we shall deal with the crossed product algebra to compute the Hochschild and cyclic (co)homology and $K_\bullet$ groups. To my knowledge, the Hochschild and periodic cyclic (co)homology of these two $\mathbb{Z}_2$ orbifolds of the Podleś two spheres are not known in the literature. In this article, we compute the Hochschild and cyclic homology and cohomology groups of the two $S^2_{q,1}$ $\mathbb{Z}_2$-orbifolds D_q and $\mathbb{R}P^2_q$ [7]. We also compute the Chern–Connes indices for each of these orbifolds by pairing the even periodic cocycles with the projections. Similar calculations for other algebras can be found in [13, 14].

2. $\mathbb{Z}_2$ actions on the Podleś quantum sphere

The C*-algebra of the Podleś quantum 2-sphere, $\mathcal{A}$, is the closure of the $*$-algebra generated by A and B satisfying the following relations:

$$A = A^*, \quad BA = q^2AB, \quad B^*B + A^2 = (1-s^2)A + s^2,$$
$$BB^* + q^4A^2 = (1-s^2)q^2A + s^2.$$

It is known that an automorphism of $\mathcal{A}$ acts diagonally on the generators, explicity for $\rho \in \mathrm{Aut}(\mathcal{A})$ and $\lambda \in \mathbb{C}$, ρ acts in one of the following two ways:

$$\sigma_\lambda(B) = \lambda B, \quad \sigma_\lambda(A) = A, \quad \sigma_\lambda(B^*) = \lambda^{-1}B^*,$$
$$\mu_\lambda(B) = \lambda B, \quad \mu_\lambda(A) = -A, \quad \mu_\lambda(B^*) = \lambda^{-1}B^*.$$

For $\rho = \sigma_{-1}$ involution, the algebra $\mathcal{A} \rtimes_{\sigma_{-1}} \mathbb{Z}_2$ is associated to the quantum disc D_q, while for the involution $\rho = \mu_{-1}$; the algebra $\mathcal{A} \rtimes_{\mu_{-1}} \mathbb{Z}_2$ corresponds to the quantum real projective space $\mathbb{R}P^2_q$[7].

3. Strategy of the proof

We use the paracyclic decomposition of the crossed product algebras to decompose the homology groups [3]. We quote a result of [3] to deduce a decomposition of the homology group of the algebra $\mathcal{A} \rtimes \Gamma$.

Theorem 3.1 ([3, Proposition 4.6]). *If* Γ *is finite and* $|\Gamma|$ *is invertible in k, then there is a natural isomorphism of cyclic homology and*

$$HH_\bullet(\mathcal{A} \rtimes \Gamma) = HH_\bullet\big(H_0(\Gamma, (\mathcal{A})^\sharp_\Gamma\big),$$

where $(H_0(\Gamma, (\mathcal{A})^\sharp_\Gamma)$ *is the cyclic module*

$$H_0\big(\Gamma, (\mathcal{A})^\sharp_\Gamma\big)(n) = H_0\big(\Gamma, k[\Gamma] \otimes (\mathcal{A})^{\otimes n+1}\big).$$

Since Γ is abelian, we can conclude that the group homology $H_0(\Gamma, \mathcal{A}^\natural_\Gamma)$ splits the complex into $|\Gamma|$ disjoint parts.

$$H_0\big(\Gamma, \mathcal{A}^\natural_\Gamma\big)(n) = H_0\big(\Gamma, k[\Gamma] \otimes (\mathcal{A})^{\otimes n+1}\big) = \bigoplus_{t\in\Gamma} \big(({}_t\mathcal{A})^{\otimes n+1}\big)^\Gamma$$

For each $t \in \Gamma$, the algebra ${}_t\mathcal{A}$ is set-wise $\mathcal{A}$ with the twisted Hochschild differential ${}_tb$ acting as

$${}_tb(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) + (-)^n \big((t \cdot a_n)a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}\big)$$

on the complex ${}_t\mathcal{A}^{\otimes(\bullet+1)}$. We therefore decompose Hochschild homology $HH_\bullet(\mathcal{A} \rtimes \Gamma)$ as follows:

$$HH_\bullet(\mathcal{A} \rtimes \Gamma) = HH_\bullet\big(H_0(\Gamma, \mathcal{A}^\natural_\Gamma)\big) = \bigoplus_{t\in\Gamma} HH_\bullet\big(({}_t\mathcal{A}^\bullet)^\Gamma\big).$$

It suffcies to calculate $HH_\bullet(({}_t\mathcal{A}^\bullet)^\Gamma)$ for each $t \in \Gamma$. To calculate $HH_\bullet(({}_t\mathcal{A}^\bullet)^\Gamma)$, we recall the lemma below.

Lemma 3.2 ([12]). *Let*

$$J_* := 0 \xleftarrow{d} A \xleftarrow{d} (\mathcal{A}^{\otimes 2}) \xleftarrow{d} (\mathcal{A}^{\otimes 3}) \xleftarrow{d} (\mathcal{A}^{\otimes 4}) \xleftarrow{d} (\mathcal{A}^{\otimes 5}) \xleftarrow{d} \cdots$$

be a chain complex. For a given Γ *action on* $\mathcal{A}$*, consider the following chain complex, with chain map* $d^\Gamma : (\mathcal{A}^{\otimes n})^\Gamma \to (\mathcal{A}^{\otimes n-1})^\Gamma$ *induced from the map* $d : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n-1}$.

$$J^\Gamma_* := 0 \xleftarrow{d^\Gamma} \mathcal{A}^\Gamma \xleftarrow{d^\Gamma} (\mathcal{A}^{\otimes 2})^\Gamma \xleftarrow{d^\Gamma} (\mathcal{A}^{\otimes 3})^\Gamma \xleftarrow{d^\Gamma} (\mathcal{A}^{\otimes 4})^\Gamma \xleftarrow{d^\Gamma} (\mathcal{A}^{\otimes 5})^\Gamma \xleftarrow{d^\Gamma} \cdots$$

With the Γ *action commuting with the differential* d*. We have the following group equality* $H_\bullet(J^\Gamma_*, d^\Gamma) = H_\bullet(J_*, d)^\Gamma$.

Hence using the above lemma we have the following decomposition of the Hochschild homology group $H_\bullet(\mathcal{A} \rtimes_\rho \mathbb{Z}_2)$:

$$H_\bullet(\mathcal{A} \rtimes_\rho \mathbb{Z}_2) = H_\bullet(\mathcal{A}, \mathcal{A})^\rho \oplus H_\bullet(\mathcal{A}, {}_\rho\mathcal{A})^\rho.$$

Where ${}_\rho\mathcal{A}$, set-wise $\mathcal{A}$, is an $\mathcal{A}^e(=\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}})$ bi-module with the following actions:

$$\alpha\cdot a=(-1\cdot\alpha)a \quad \text{and} \quad a\cdot\alpha=a\alpha, \quad \text{for } \alpha\in\mathcal{A} \text{ and } a\in{}_\rho\mathcal{A}.$$

Hence in order to understand $H_\bullet(\mathcal{A}\rtimes_\rho\mathbb{Z}_2)$ we need to understand the ρ invariant subgroups of $H_\bullet(\mathcal{A},\mathcal{A})$ and $H_\bullet(\mathcal{A},{}_\rho\mathcal{A})$. We recall the MNW resolution which we describe below.

In the article [10], the authors presented a resolution of $\mathcal{A}$,

$$\cdots\to\mathcal{M}_{n+1}\to\mathcal{M}_n\cdots\to\mathcal{M}_2\to\mathcal{M}_1\to\mathcal{M}_0\to\mathcal{A}\to 0$$

by free left $\mathcal{A}^e$-modules $\mathcal{M}_n$, with $\operatorname{rank}(\mathcal{M}_0)=1$, $\operatorname{rank}(\mathcal{M}_1)=3$, and $\operatorname{rank}(\mathcal{M}_n)=4$ for $n\geq 2$. Adapting their notations, $\mathcal{M}_1$ has basis $\{e_A,e_B,e_{B^*}\}$, with $d_1\colon\mathcal{M}_1\to\mathcal{M}_0=\mathcal{A}^e$ given by

$$d_1(e_t)=t\otimes 1-1\otimes t^o, \quad t=A,B,B^*.$$

The module $\mathcal{M}_2$ has basis $\{e_a\wedge e_B, e_A\wedge e_B^*, \vartheta_S^{(1)},\vartheta_T^{(1)}\}$, with $d_2\colon\mathcal{M}_2\to\mathcal{M}_1$ given by

$$\begin{aligned}
d_2\big(1_{\mathcal{A}^e}\otimes(e_A\wedge e_{B^*})\big)&=\big(A\otimes 1-1\otimes q^2A^o\big)\otimes e_{B^*}\\
&\qquad-\big(q^2B^*\otimes 1-1\otimes B^{*o}\big)\otimes e_A,\\
d_2\big(1_{\mathcal{A}^e}\otimes(e_A\wedge e_{B^*})\big)&=\big(q^2A\otimes 1-1\otimes A^o\big)\otimes e_B-\big(B\otimes 1-1\otimes q^2B^o\big)\otimes e_A,\\
d_2\big(1_{\mathcal{A}^e}\otimes\vartheta_S^{(1)}\big)&=-q^{-1}\big(B\otimes 1\otimes e_{B^*}+1\otimes B^{*o}\otimes e_B\big)\\
&\qquad-q\big(q^2(A\otimes 1+1\otimes A^o)\big)\otimes e_A,\\
d_2\big(1_{\mathcal{A}^e}\otimes\vartheta_T^{(1)}\big)&=-q^{-1}\big(1\otimes B^o\otimes e_{B^*}+B^*\otimes 1\otimes e_B\big)\\
&\qquad-q^{-1}\big(A\otimes 1+1\otimes A^o\big)\otimes e_A.
\end{aligned}$$

The maps $d_i; i\geq 3$ are not needed in this article. Interested readers can refer to [10] for a complete description.

There exists chain homotopy equivalence between the MNW and the bar resolution of $\mathcal{A}$.

$$\begin{array}{ccccccccc}
\longrightarrow & \mathcal{M}_2 & \xrightarrow{d_2} & \mathcal{M}_1 & \xrightarrow{d_1} & \mathcal{M}_0 & \xrightarrow{d_0} & \mathcal{A} & \longrightarrow 0\\
& {\scriptstyle h_2}\uparrow\downarrow{\scriptstyle f_2} & & {\scriptstyle h_1}\uparrow\downarrow{\scriptstyle f_1} & & {\scriptstyle h_0}\uparrow\downarrow{\scriptstyle f_0} & & \downarrow\cong & \\
\longrightarrow & \mathcal{A}^{\otimes 4} & \xrightarrow{b'} & \mathcal{A}^{\otimes 3} & \xrightarrow{b'} & \mathcal{A}^{\otimes 2} & \xrightarrow{b'} & \mathcal{A} & \longrightarrow 0
\end{array}$$

Let $(\mathcal{M},d)$ denote the MNW resolution of $\mathcal{A}$ and let $(\mathcal{N},b')$ be the bar resolution. Then we have $\{\mathcal{M}\}_i=\mathcal{M}_i$ and $\{\mathcal{N}\}_i=\mathcal{A}^{\otimes(i+2)}$ with the maps $\{f\}_{i\geq 0}\colon\mathcal{M}_i\to\mathcal{A}^{\otimes(i+2)}$ lifting the identity map on $\mathcal{A}$ onto the complex between the resolutions. Similarly, let $\{h\}_{i\geq 0}\colon \mathcal{A}^{\otimes(i+2)} \to \mathcal{M}_i$ be the chain homotopy equivalence map between the resolutions. Explicitly these maps are as follows:

$$f_0(a_1 \otimes a_2^o) = (a_1, a_2), \quad f_1(e_t) = (1, t, 1), \quad \text{for } t = A, B, B^*.$$

The maps f_i, $i \geq 2$ can be found inductively satisfying the relation $f_i \cdot d_{i+1} = b' \cdot f_{i+1}$.

Since a Poincaré–Birkhoff–Witt (PBW) basis for $\mathcal{A}$ consists of the monomials

$$\{A^k B^j\}_{j,k\geq 0}, \{A^k B^{*(j+1)}\}_{j,k\geq 0}$$

we define h_1 on the above PBW basis elements. For $a, b \in \mathcal{A}$, $h_0(a,b) = a \otimes b^o$ and for $t = B, B^*$, we have

$$\begin{aligned} h_1(a, A^n t^m, b) &= (a \otimes b^o)\{(t^m)^o (A^{n-1})^o e_A + (t^m)^o (A^{n-2})^o A e_A + \cdots \\ &\cdots + (t^m)^o A^{n-1} e_A + A^n (t^{m-1})^o e_t + A^n (t^{m-2})^o t e_t + \cdots + A^n t^{m-1} e_t\}. \end{aligned}$$

In order to locate the $\mathbb{Z}_2$ invariant cocycles of $H^\bullet(\mathcal{A}, {}_\rho\mathcal{A})$, we need to use the resolution homotopy maps

$$h_*\colon \mathcal{N}_* \to \mathcal{M}_*$$

and

$$f_*\colon \mathcal{M}_* \to \mathcal{N}_*.$$

We push a cocycles $\mathcal{D}$ into the bar complex and let $\mathbb{Z}_2$ act on it. Then, in the MNW complex, we compare the pullback of this $\mathbb{Z}_2$-acted cocycle with $\mathcal{D}$ to check the $\mathbb{Z}_2$ invariance.

4. The quantum disc

Theorem 4.1 (Hochschild and cyclic homology). *The Hochschild and cyclic homology groups of D_q are as follows:*

$$H_\bullet\big(C(D_q), C(D_q)\big) \cong \begin{cases} \mathbb{C}^{\mathbb{N}}, & \text{for } \bullet = 0, 1, \\ 0, & \text{for } \bullet > 1, \end{cases}$$

$$HP_\bullet\big(C(D_q)\big) \cong \begin{cases} \mathbb{C}^4, & \text{for } \bullet = 2n, \\ 0, & \text{for } \bullet = 2n+1. \end{cases}$$

Proof. Using the paracyclic decomposition of the quantum 2-disc algebra we have the following:

$$H_\bullet\big(C(D_q), C(D_q)\big) = H_\bullet(\mathcal{A}, \mathcal{A})^{\sigma_{-1}} \oplus H_\bullet(\mathcal{A}, {}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}}.$$

From [5] we know that

$$H_0(\mathcal{A}, \mathcal{A}) = \mathbb{C}[1] \oplus \mathbb{C}[A] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^m] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^{*m}].$$

We push each of these cyclces into the bar complex using the map f_0 and let σ_{-1} act on them, thereafter we pull it back to the MNW complex using the map h_0. We see that:

$$H_0(\mathcal{A}, \mathcal{A})^{\sigma_{-1}} = \mathbb{C}[1] \oplus \mathbb{C}[A] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^{2m}] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^{*2m}].$$

Similar calculation reveals that

$$H_0\big(\mathcal{A}, {}_{\sigma_{-1}}\mathcal{A}\big)^{\sigma_{-1}} = \mathbb{C}[1] \oplus \mathbb{C}[A].$$

Hence the homology group $H_0(C(D_q), C(D_q)) \cong \mathbb{C}^{\mathbb{N}}$ has countable generators described above. Similarly we compute $H_1(C(D_q), C(D_q))$, in this case we use maps h_1 and f_1 to locate the invariant cyclic cocycles in the groups $H_\bullet(\mathcal{A}, \mathcal{A})^{\sigma_{-1}}$ and $H_\bullet(\mathcal{A}, {}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}}$. The group $H_1(C(D_q), C(D_q))$ is generated by the cycles

$$1 \otimes {}_{\sigma_{-1}}e_A, \quad 1 \otimes e_A, \quad \big\{B^{2j+1} \otimes e_B\big\}_{j\geq 0} \quad \text{and} \quad \big\{B^{*2j+1} \otimes e_{B^*}\big\}_{j\geq 0}.$$

Here ${}_{\sigma_{-1}}e_A$ denotes the invariant copy of the cycle $e_A \in H_0(\mathcal{A}, {}_{\sigma_{-1}}\mathcal{A})$. For $\rho = \mathrm{id}$ and σ_{-1}, $H_\bullet(\mathcal{A}, {}_\rho\mathcal{A}) = 0\ \forall \bullet > 1$. Therefore we conclude that

$$H_\bullet\big(C(D_q), C(D_q)\big) = 0 \quad \text{for } \bullet > 1.$$

In a similar way we compute $HC_\bullet(C(D_q), C(D_q))$. We observe that the cyclic homology group [5]

$$HC^\rho_{2n}(\mathcal{A}) = \mathbb{C}[1] \oplus \mathbb{C}[A] \quad \text{and} \quad HC^\rho_{2n+1}(\mathcal{A}) = 0 \quad \text{for } \rho \in \{\sigma_{-1}, \mathrm{id}\}.$$

Using the paracyclic decomposition for the cyclic homology $HC_\bullet(C(D_q))$ we have:

$$HC_\bullet\big(C(D_q)\big) = HC_\bullet(\mathcal{A}, \mathcal{A})^{\sigma_{-1}} \oplus HC_\bullet(\mathcal{A}, {}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}}.$$

We now check that all the cycles of $HC_{2n}(\mathcal{A}, \mathcal{A}) \oplus HC_{2n}(\mathcal{A}, {}_{\sigma_{-1}}\mathcal{A})$ are σ_{-1} invariant. □

Corollary 4.2 (Hochschild and cyclic cohomology). *The Hochschild and cyclic homology groups of D_q are as follows:*

$$H^\bullet\big(C(D_q), C(D_q)'\big) \cong \begin{cases} \mathbb{C}^{\mathbb{N}}, & \text{for } \bullet = 0, 1 \\ 0, & \text{for } \bullet > 1, \end{cases}$$

$$HP^\bullet\big(C(D_q)\big) \cong \begin{cases} \mathbb{C}^4, & \text{for } \bullet = 2n, \\ 0, & \text{for } \bullet = 2n+1. \end{cases}$$

Proof. The universal coefficient theorem gives a relation between the Hochschild homology and the cohomology with dual algebra as the coefficient [8],

$$H_\bullet\big(C(D_q), C(D_q)\big)' = H^\bullet\big(C(D_q), C(D_q)'\big).$$

Hence $H^\bullet(D_q, D_q') = 0$ for $\bullet > 1$. And similarly we conclude that for $\bullet = 0, 1$;

Hochschild cohomology groups are countably infinite in dimension. For calculating periodic cyclic cohomology we consider the B, S, I long exact sequence for Hochschild and cyclic cohomology

$$\cdots \to HH^1({}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}} \xrightarrow{B} HC^0({}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}} \xrightarrow{S} HC^2({}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}} \xrightarrow{I} HH^2({}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}} \xrightarrow{B} HC^1({}_{\sigma_{-1}}\mathcal{A})^{\sigma_{-1}} \xrightarrow{S} \cdots .$$

Since $HC^{\rho}_{2n}(\mathcal{A}) = k[1] \oplus k[A]$ for $\rho = \{\mathrm{id}, \sigma_{-1}\}$[5, Prop. 5.2], and the above spectral sequence stabilises, we conclude that the group

$$HP^{\mathrm{even}}\big(C(D_q), C(D_q)\big) \cong \mathbb{C}^4.$$

This group is generated by $[\tau_0]$, $[f_A]$ and $[{}_{\sigma_{-1}}\tau_0]$ and $[{}_{\sigma_{-1}} f_A]$, where for $\tau_0(1) = 1$ and $\tau_0(a) = 0$ for all $a \in \mathcal{A}$, and f_A is the Haar state on $\mathcal{A}(SU_q(2))$ restricted to the Podleś quantum sphere and is given by

$$f_A(A^{r+1}) = (1-q^4)(1-q^{2r+4})^{-1}$$

and for $s > 0$ and it vanishes on the PBW basis elements $A^r B^s$ and $A^r B^{*s}$; [16]

$$f_A(A^r B^s) = 0 = f_A(A^r B^{*s}).$$

Similarly ${}_{\sigma-1}\tau_0$ is a cyclic cocycle with ${}_{\sigma_{-1}}\tau_0(1_{{}_{\sigma_{-1}}\mathcal{A}}) = 1$ and ${}_{\sigma_{-1}}\tau_0(a) = 0$ for all $a \in {}_{\sigma_{-1}}\mathcal{A}$. Likewise we defined the Haar measure ${}_{\sigma_{-1}} f_A$ on the PBW basis elements of ${}_{\sigma_{-1}}\mathcal{A}$. □

5. The quantum real projective space

Theorem 5.1 (Hochschild and cyclic homology). *The Hochschild and cyclic homology groups of* $\mathbb{R}P^2_q$ *are as follows:*

$$H_\bullet\big(C(\mathbb{R}P^2_q), C(\mathbb{R}P^2_q)\big) \cong \begin{cases} \mathbb{C}^{\mathbb{N}}, & \text{for } \bullet = 0, 1, \\ 0, & \text{for } \bullet > 1, \end{cases}$$

$$HC_\bullet\big(C(\mathbb{R}P^2_q), C(\mathbb{R}P^2_q)\big) \cong \begin{cases} \mathbb{C}^2, & \text{for } \bullet = 2n, \\ 0, & \text{for } \bullet = 2n+1. \end{cases}$$

Proof. Similar to the quantum disc case, we have the following decomposition:

$$H_\bullet\big(C(\mathbb{R}P^2_q)\big) = H_\bullet(\mathcal{A}, \mathcal{A})^{\mu_{-1}} \oplus H_\bullet(\mathcal{A}, {}_{\mu_{-1}}\mathcal{A})^{\mu_{-1}}.$$

Since from [5] we know that

$$H_0(\mathcal{A}, \mathcal{A}) = \mathbb{C}[1] \oplus \mathbb{C}[A] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^m] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^{*m}].$$

We observe that:

$$H_0(\mathcal{A},\mathcal{A})^{\mu_{-1}} = \mathbb{C}[1] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^{2m}] \oplus \Sigma^{\oplus}_{m\geq 1}\mathbb{C}[B^{*2m}].$$

Similarly we conclude

$$H_0\big(\mathcal{A}, {}_{\mu_{-1}}\mathcal{A}\big)^{\mu_{-1}} = \mathbb{C}[1].$$

Hence we have

$$H_0\big(C(\mathbb{R}P_q^2), C(\mathbb{R}P_q^2)\big) \cong \mathbb{C}^{\mathbb{N}}.$$

For the first homology $H_1(C(\mathbb{R}P_q^2), C(\mathbb{R}P_q^2))$, the group $H_1(\mathcal{A}, {}_{\mu_{-1}}\mathcal{A})$ vanishes hence it is a countably infinite dimensional group generated by the elements

$$1 \otimes e_A, \quad \big\{B^{2j+1} \otimes e_B\big\}_{j\geq 0} \quad \text{and} \quad \big\{B^{*2j+1} \otimes e_{B^*}\big\}_{j\geq 0}.$$

Higher Hochschild homology groups $HH^{\rho}_{\bullet}({}_{\mu_{-1}}\mathcal{A})$ for ($\bullet > 1$) vanishes [5], thereby we conclude that:

$$H_{\bullet}\big(C(\mathbb{R}P_q^2), C(\mathbb{R}P_q^2)\big) = 0 \quad \text{for } \bullet > 1.$$

To compute $HC_{\bullet}(C(\mathbb{R}P_q^2), C(\mathbb{R}P_q^2))$, we observe that the cyclic homology group $HC_{\bullet}^{\mu_{-1}}(\mathcal{A}) = 0$ for all $\bullet > 0$ [5], therefore

$$HC_{\bullet}\big(C(\mathbb{R}P_q^2)\big) = HC_{\bullet}(\mathcal{A},\mathcal{A})^{\mu_{-1}}.$$

It can easily be checked that both the cycles of $HC_{2n}(\mathcal{A},\mathcal{A})$ are μ_{-1} invariant. □

Corollary 5.2 (Hochschild and cyclic cohomology). *The Hochschild and cyclic homology groups of* $\mathbb{R}P_q^2$ *are as follows:*

$$H^{\bullet}\big(C(\mathbb{R}P_q^2), C(\mathbb{R}P_q^2)\big) \cong \begin{cases} \mathbb{C}^{\mathbb{N}}, & \textit{for } \bullet = 0, 1, \\ 0, & \textit{for } \bullet > 1, \end{cases}$$

$$HP^{\bullet}\big(C(\mathbb{R}P_q^2)\big) \cong \begin{cases} \mathbb{C}, & \textit{for } \bullet = 2n, \\ 0, & \textit{for } \bullet = 2n+1. \end{cases}$$

Proof. As in the case of quantum 2-disc, we conclude that for $\bullet = 0, 1$ the Hochschild cohomology groups are countably infinite in dimension and vanishes for $\bullet > 1$. The periodic cyclic cohomology group

$$HP^{\text{even}}\big(C(\mathbb{R}P_q^2), C(\mathbb{R}P_q^2)\big) \cong \mathbb{C}$$

and is generated by $\mathbb{C}[1]$. This is so because $H_{\bullet}(\mathcal{A}, {}_{\tau_{-1}}) = 0$ for $\bullet > 0$ and of the two cocycles of $HP^{\text{even}}(\mathcal{A})$, only the one dimensional subspace spanned by [1] is τ_{-1} invariant. □

6. Chern–Connes indices

Chern–Connes indices are useful invariants in noncommutative geometry. Explicitly, for a C*-algebra $\mathcal{B}$ over $\mathbb{C}$ we have the following map [8, Section 8]:

$$ch_{0,n}\colon K_0(\mathcal{B}) \to HC_{2n}(\mathcal{B}).$$

defined by

$$[e] \mapsto \mathrm{tr}(c(e)),$$

where $K_0(\mathcal{B})$ is the Grothendieck group of the ring $\mathcal{B}$. Using this map the projections of $\mathcal{B}$ can be paired with the periodic even cyclic cocycles in the following way.

$$K_0(\mathcal{B}) \times HC^{2n}(\mathcal{B}) \xrightarrow{\mathrm{ch}\times\mathrm{id}} HC_{2n}(\mathcal{B}) \times HC^{2n}(\mathcal{B}) \to \mathbb{C}.$$

In this section we calculate the above pairing for the quantum disc D_q and the quantum real projective space $\mathbb{R}P_q^2$. The vanishing of the second Hochschild homology leaves the two orbifolds with fewer periodic cocycles than expected. While the algebra $C(D_q)$ is a Toeplitz algebra and hence $K_0(C(D_q)) \cong \mathbb{Z}$ [18, p. 191]. It is generated by the projection $[1_{D_q}]$. For the noncommutative quantum 2-disc D_q we have the following Chern–Connes index table:

	$S\tau_0$	Sf_A	$S_{\sigma_{-1}}\tau_0$	$S_{\sigma_{-1}}f_A$
$[1_{D_q}]$	1	0	0	0

Similarly we have a description of the group $K_0(\mathbb{R}P_q^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ [7] generated by [1] and $[P]$. The following is the Chern–Connes index table for $\mathbb{R}P_q^2$:

	$S\tau_0$
$[1_{\mathbb{R}P_q^2}]$	1
$[P]$	0

7. Conclusion

We see that the quantum parameter q does not appear in the Chern–Connes indices of both the $\mathbb{Z}_2$ orbifolds. This can be attributed to the vanishing of the second homology of the Podleś quantum 2-spheres. While the Chern–Connes indices for the Podleś quantum 2-spheres was computed by twisting the Hochschild cohomology, this way the “dimension drop” was avoided resulting in a rich invariant set [17]. Here we can not use these twisted quantum 2-sphere invariants as none of the twists are involutive and hence does not appear in the paracyclic decomposition. Whence the vanishing of several projection of the Podleś quantum 2-spheres after $\mathbb{Z}_2$ action leaves few projections on the quotient space.

It is worth noticing that the Chern–Connes index for the noncommutative torus orbifolds had several projections. This can be related to that fact that there is no "dimension drop" in the periodic cyclic cohomology. Though Hochschild cohomology does depend on the parameter θ, but the periodic cyclic cohomolgy is θ invariant [19]. In other words, we can safely say that the noncommutative torus is an ideal noncommutative manifold and the dimension drop for the Podleś quantum sphere indicates that we need to look for other invariants which may characterise the properties of spaces like Podleś quantum 2-spheres, e.g. the quantum $SL(2)$ [6].

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