Lecture 3: The Root Locus Method

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1 The Root Locus method

The Root Locus method, as the name suggests looks at the locii of the roots as some parameter of interest within the system is varied. Since we already know that the position of the roots of the characteristic equation strongly influence the step response of the system, we can find values of the parameter which will position the roots appropriately, using the method of Root Locus. This method involves root locus diagrams. And we will see clever ways of drawing them quickly. We do not *actually* plot them point by point for every value of the parameter.

Definition: The root locus is the locii of the values of s for which $1 + KG^*H^* = 0$ as the real parameter $K$is varied from 0 to ∞.

Consider the characteristic polynomial of the closed loop system

$$1 + G(s)H(s) = 0 \tag{1}$$

where $G(s)H(s)$ is of the form

$$\frac{(s^m + b_1 s^{m-1} + ...b_m)}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n} \tag{2}$$

We are interested in the zeros (or roots) of equation (1). This also means that we are interested in the zeros of the numerator of equation (1). In particular, we are interested in the roots as say, a_2 coefficient is varied. We expand as follows

$$1 + GH = \frac{(s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n) + (s^m + b_1 s^{m-1} + ...b_m)}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n} = 0 \tag{3}$$

Since we are interested in the zeros of the above equation, the denominator is of no consequence. We can ignore it and get

$$num(1 + GH) = (s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n) + (s^m + b_1 s^{m-1} + ...b_m) = 0 \tag{4}$$

The above equation still gives the roots of equation (1). Next,we divide both sides of the above equation by terms not associated with a_2 (to see a familiar form) and write

$$num(1+GH) = 1+\frac{a_2 s^{n-2}}{(s^n + a_1 s^{n-1} + a_3 s^{n-3} + ...a_n) + (s^m + b_1 s^{m-1} + ...b_m)} = 1+a_2 G^*(s)H^*(s) = 0 \tag{5}$$

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Note that a_2 could have been associated with more than one term. This above equation also still gives the same zeros as equation (1). So we can generalize and say that if we are interested in the numerator zeros of $1 + GH$ and their variation with respect to a parameter K embedded in $1 + GH$, we can always rewrite it as $1 + KG^*H^* = 0$ preserving the zeros of $1 + GH$. Now K is varied from 0 to ∞. So we write

$$G^*(s)H^*(s) = -\frac{1}{K}. \tag{6}$$

When K is 0, $G^*(s)H^*(s)$ is at infinity and when K is at ∞, $G^*(s)H^*(s)$ is at 0. Thus, as K is varied, the roots begin at the poles of $G^*(s)H^*(s)$ and end at the zeros of $G^*(s)H^*(s)$. *This is important information.* Since the zeros and poles of $G^*(s)H^*(s)$ are known, we already know the starting and ending points of the root locus diagram.

1.1 Example

Suppose we have

$$1 + GH = 1 + \frac{1}{(s+1)(s+K)}$$

and we are interested in the zero positions of 1+GH as K is varied from 0 to ∞. As in the previous section we write

$$1 + GH = \frac{s^2 + s(K+1) + K + 1}{(s+1)(s+K)}$$

The zeros of the above equation are determined by the numerator alone. You cant do anything to the denominator to take 1+GH to zero. So this is entirely determined by the numerator of 1+GH. And so we rewrite the numerator as

$$s^2 + s + 1 + K(s+1) = 0$$

or

$$1 + \frac{K(s+1)}{s^2 + s + 1} = 0$$

so that

$$G^*H^* = \frac{(s+1)}{s^2 + s + 1} = -\frac{1}{K}$$

At K=0, we begin at the poles of G^*H^* which are at 0.5 ± 0.866. It can be seen by putting K=0 in 1+GH that the zeros of 1+GH also begin here. All this can as well be ascertained for this simple example from 1 + GH itself.

2 Phase of G^*H^*

In equation (6) K being real and positive, the phase of G*H* is always π for all values of K. This is illustrated in figure 1 with respect to the following transfer function.

$$G(s)^*H^*(s) = \frac{s+1}{s(s+5)[(s+2)^2 + 4]}$$

with zero @ $s = -1$
and poles @ $s = 0, -5, -2 \pm j2$

The question we ask is if a point $p = -1 + j2$ is a point on the root locus ? We have to substitute $s = -1 + j2$ in the transfer function and get the phase angle of the complex function G^*H^*, i.e.,

$$\angle G(s)^* H^*(s) = \frac{\angle s + 1}{\angle s \angle (s+5) \angle [(s+2)^2 + 4]},$$

where $s = -1 + j2$. From the above expression for G^*H^* it can be seen that the phase subtended at the point of interest $p = -1 + j2$ by the zeros has to be added and that subtended by the poles in the denominator should be subtracted. This can be done graphically also as in figure 1. At the point $p = -1 + j2$, the zero subtends 90 degrees and the 4 poles subtend 116.6, 0, 76 and 26.6 degrees. Thus the total phase of $G*H*$ at $s = -1 + j2$ is $\phi = 90 - 116.6 - 0 - 76 - 26.6 = -129.2$ degrees which is not $\pi = 180$. Hence, $s = -1 + j2$ is not a point on the root locus.

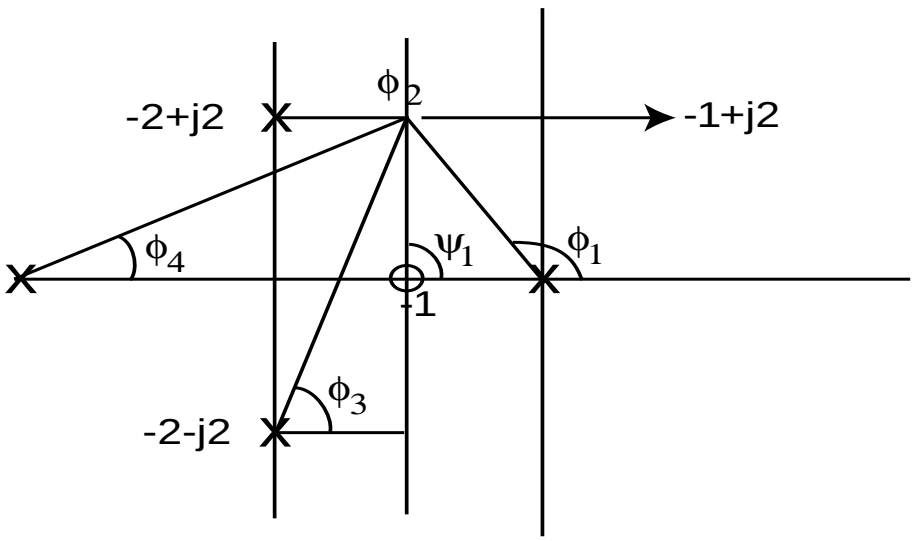

Figure 1: Figure illustrating phase of a root locus.

3 Root Locus example

We take a simple example to demonstrate the features of the Root Locus method. Consider the transfer function

$$G^*H^* = \frac{1}{s(s+1)}$$

zeros are denoted by m. In this case $m = 0$
poles are denoted by n. In this case $n = 2$

We are interested in

$$\begin{array}{lll} Poles & of & \dfrac{G}{1+GH} \\ Zeros & of & 1+GH \\ Zeros & of & 1+KG^*H^* \end{array}$$

$$1 + KG^*H^* = 0$$

$$
\begin{aligned}
orG^*H^* &= -\frac{1}{K} \\
|G^*H^*| &= |\frac{1}{K}|
\end{aligned}
$$

as K goes from zero to ∞.

- Hence the root locus moves from

$$
\begin{aligned}
&poles \;\; of \;\; G^*H^* \;\; to \;\; zeros \;\; of \;\; G^*H^* \\
&poles \;\; of \;\; 1 + KG^*H^* \;\; to \;\; zeros \;\; of \;\; G^*H^*
\end{aligned}
$$

From our example,

$$
1 + KG^*H^* = 1 + \frac{K}{s(s+1)} = s^2 + s + K = 0
$$

$$
s_{12} = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}
$$

1) For $0 \leq K \leq 1/4 \implies$roots are real and between -1 and 0.
2)@ K $= 1/4 \implies$there are two roots @ $-\frac{1}{2}$.
3)ForK $> 1/4 \implies$ the roots become complex, with real part $-\frac{1}{2}$ and imaginary part proportional to $\sqrt{K}$.

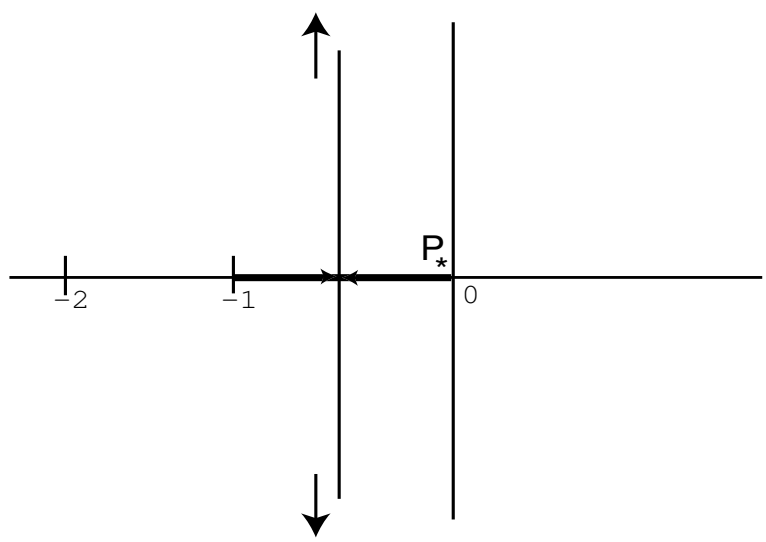

Figure 2: Root locus of $s^2 + s + K$.

- There are two poles ($\mathbf{n = 2}$) of $1 + KG^*H^*$ and hence there will be **2 branches**.
- **Branches begin on the poles** of $1 + KG^*H^*$, here 0 and -1.
- The root locus should end on the zeros of G^*H^*. But here G^*H^* has no finite zeros (**m=0**). Hence **(n-m)=2** branches go off to ∞.

- The loci meet at a point on the real axis and break off at a **break away point**, at specific angles $\pm 90^0$ and **go to infinity following asymptotes and their angles**.
- Hence by proper choice of K we can position the zeros of $1 + GH$ and prescribe dynamics of the system.
- We need more sophisticated techniques when complicated transfer functions are involved.

4 Initial set of root locus rules

- **Root locus on the real axis**: A section of the root locus will exist on the real axis, where to the right of the section there are a total of odd number of (zeros + poles). Like in the example, the section of the root locus between the pole at 0 and 1 has a single pole at 0 at the right end. If we look at the section between -1 and -2, a root locus cannot exist here, because to the right of this section there are two poles (0 and 1) and no zeros, hence a total of even number of poles and zeros.
- **Departure angle from a pole**: Take a point P very close to the pole at 0. As the departing root locus should have π phase, let pole 0 subtend an angle ϕ w.r.t. x axis. Since this point is very close to 0, the angle subtended by the pole at -1 is 0°. Hence the phase of G*H* at point P very close to 0 is

$$\frac{\angle\ from\ zeros}{\angle\ from\ poles}$$

or

$$-0 - \phi = \pi.$$

This results in $\phi = 180^o$. Similarly, for the departure from the pole at -1, if we consider a point P very close to -1. Here, the pole at 0 subtends an angle of 180° and say the pole at -1 subtends χ. Then the phase of G*H* becomes

$$-180 - \chi = 180^o.$$

Thus $\chi = 0^o\ or\ 360^o$.

The root locus departs from pole 0 at an angle 180° along the x axis in the negative direction. And it departs from the pole at -1 at an angle 0° along the positive x axis. Hence these two branches meet at a point on the real axis. That point is called the break away point.

- **Break in and break away point**:

$$G^*H^* = \frac{b}{a} \qquad b = 1$$

$$a = s^2 + s$$

break away point is computed using this equation:

$$b\frac{da}{ds} - a\frac{db}{ds} = 0$$

$$1(2s + 1) = 0$$

$$2s + 1 = 0$$

$$s = -\frac{1}{2}$$

- **Arrival and departure angles at a break in/out point**: Departure from a break away point is simple if you right away accept the rule, but complicated if one were to prove it. In our given example, the break away point was $s = -1/2$. We can check if this is a true break away point by substituting the value of s in the numerator of $1 + kG^*H^* = 0$ and finding if the value of k is real positive. Substituting $s = -1/2$ in

$$s^2 + s + k = 0, \quad gives \;\; k_o = 1/4$$

and so $s = -1/2$ is a break away point. We replace

$$1 + kG^*H^* = 1 + \frac{K^* + k_o}{s(s+1)} = 1 + \frac{K^* + 1/4}{s(s+1)}$$

The numerator of the above equation is

$$s^2 + s + K^* + 1/4 = 0 \;\; rewritten \; as \; 1 + \frac{K^*}{s^2 + s + 1/4} = 1 + \frac{K^*}{(s+1/2)^2}$$

We begin anew, starting with $K^* = 0...\infty$. Since, right at the beginning we have a double pole, the departure angles are

$$-2\phi_l = 180 + 360 * l \quad l = 0, 1$$

This gives $\phi_{0,1} = 90^o, 270^o$.

A simpler way is to 1) find the angles of the arriving locii 2) divide the angle between any 2 branches by 2 3) and rotate the picture by this angle. In our example, the two branches arrive along the real axis and hence the angle between the two is 180°. Divide $180^o/2 = 90$ and rotate the arrival picture by 90°.

It is worth knowing that the root locus when formulated in terms of K^*, the locii arrive at $K^* = 0$ from the negative ∞ and hence we go along 0 to $-\infty$ for K^* or

$$1 - K^*G^*H^* = 0$$

for positive values of K^* and find the departure angles. This gives

$$G^*H^* = \frac{1}{K^*}$$

The departure angle equation for this case is

$$\Sigma\phi_{zero} - \Sigma\phi_{poles} - m\phi_l = 360 * l, \quad l = 0, 1...(m-1)$$

where m is the multiplicity of the pole. For our case since there are no zeros and other poles

$$-2\phi_l = 360 * l, \quad l = 0, 1$$

This gives $\phi_{0,1} = 0, 180^o$.

- **center of asymptotes**: When there are more poles (n) than zeros (m), the remaining (n-m) branches arrive at infinity along definite rays (asymptotes) which appear to emanate from a virtual point on the x axis. This point is required in order to plot the asymptotes. The center of asymptotes α is given by

$$\alpha = \frac{\sum poles - \sum zeros}{n-m} = \frac{-1-0}{2-0} = -\frac{1}{2}.$$

- **Angle of asymptotes**:

$$\begin{aligned} \phi_l &= \frac{180^0 + 360l}{n-m} = \frac{180^0}{2} \quad and \quad \frac{540^0}{2} \qquad l = 0, 1, 2, ..., n-m-1. \\ &= 90^o, 270^o. \end{aligned}$$

5 Root locus for a complicated case

Let us take a slightly more complicated case (figure 3)

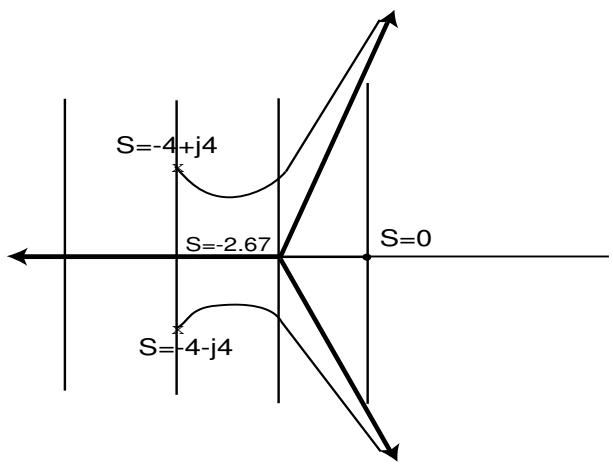

Figure 3: System root locus with three poles and no zeros.

$$KG^*H^* = \frac{K}{s[(s+4)^2+16]} = \frac{K}{s[s-(-4 \pm j4)]}$$

$$|G^*H^*| = \frac{1}{|K|}$$

- number of branches =number of poles = 3
- the branches begin at $K = 0$, $|G^*H^*| \to \infty$, so begin at the poles of G^*H^*
- the branches end at $K = \infty$, $|G^*H^*| \to 0$, end at the zeros of G^*H^*
- $m = 0$, $n = 3$ and so the number of asymptotes is n-m=3

- asymptotes are angles at which they reach ∞.

- Computation of asymptotes:

$$\begin{aligned}\phi_l &= \frac{180^0 + 360l}{n-m} \qquad\qquad l = 0, 1, 2, ..., n-m-1. \\ &= 60^0, 180^0, 300^0.\end{aligned}$$

- Origination of the asymptotes:

$$\alpha = \frac{\sum poles - \sum zeros}{n-m} = \frac{-4-4+0}{3-0} = -2.67.$$

- Departure angles from the three poles: Take a point very close to $-4 + j4$.

$$\begin{aligned}\text{Phase angle from} \quad s = 0 \quad &= 135^0. \\ s = -4 - 4j \quad &= 90^0.\end{aligned}$$

$$\pi = -135^0 - 90^0 - \phi$$

$$\phi = -45^0$$

- Crossing imaginary axis. Routh Criteria

- Break away point

$$G^* H^* = \frac{b}{a}$$

break away point:

$$b\frac{da}{ds} - a\frac{db}{ds} = 0$$

$$b = 1 \longrightarrow \frac{db}{ds} = 0$$

$$a = s^3 + 8s^2 + 32s \longrightarrow \frac{da}{ds} = 3s^2 + 16s + 32$$

We get $3s^2 + 16s + 32 = 0$ the solution to this is $s = -2.67 \pm j1.89$. Now, not all break in points are true break-in points. They can be spurious. If we substitute the s value in GH, then the resultant K should come out to be real. If not, then the s values are not valid break-in points. By this criterion, the above s values are not break-in points.

x

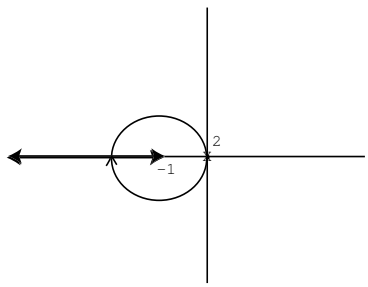

Figure 4:

6 Another root locus

$$KGH = \frac{K(s+1)}{s^2}$$

- <u>2 poles so 2 loci</u>

  One asymptote

$$= \frac{180^0 + l.360^0}{2} \qquad l = 0, 1, 2, ...$$

$$= 180^0$$

  Single asymptote so no origination.

- <u>Departure angle</u>$\longrightarrow$as many departure angles as poles at a point.

$$\begin{aligned} 0 \quad -2\phi_l &= \pi + l.360^0 \\ -2\phi_1 &= \pi = -90^0 \\ -2\phi_2 &= 540^0 = -270^0 = \pm 90^0 \end{aligned}$$

- <u>Break away point</u>

$$b = s + 1 \qquad \frac{db}{ds} = 1$$

$$a = s^2 \qquad \frac{da}{ds} = 2s$$

$$(s+1)2s - s^2 = 0$$

$$s = 0, -2$$