Generalized rays in first-order optics: Transformation properties of Gaussian Schell-model fields

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Propagation characteristics of Gaussian Schell-model fields through first-order optical systems and in free space are analyzed by the method of generalized rays. This allows the development of a simple geometrical description of these processes. The invariance of the degree of global coherence is established in full generality. Asymptotic behavior under free propagation and the emergence of a far-zone universal structure are analyzed. New invariants associated with incoherent superpositions of such fields are found.

I. INTRODUCTION

Recently, there has been much interest in the radiation field generated by partially coherent planar (scalar) sources and several useful results have been established, particularly regarding the radiometric properties of such sources. As rightly noted by Wolf,[1] Walther's[2] classic paper of 1968 has acted as the nucleus for most of these developments.[3] In these situations the source is adequately described by the source-plane cross-spectral density. Many model sources have been studied in detail; the Gaussian quasihomogeneous sources[4] and the Gaussian Schell-model sources[5] have received particular attention.

More recently, the notion of generalized light rays has been introduced in statistical wave optics.[6] This notion leads to a ray picture of wave optics which is exact at the level of the two-point correlation function and is applicable equally well to coherent, partially coherent, and incoherent fields. In paraxial situations the generalized rays behave in an extremely simple way both under free propagation and action by optical systems.[7]

In the present paper we use the method of generalized rays to analyze the behavior of Gaussian Schell-model (GSM) fields under the action of first-order systems (FOS). A first-order system is an optical system which changes input ray parameters of location and direction into output parameters by a simple matrix transformation according to Eq. (3.2). It can also be defined via the generalized Huyghens representation.[7] We begin in Sec. II by extending the notion of the GSM field to include a quadratic phase front. When the generalized rays corresponding to these fields are computed, the following fact emerges in a natural and basic way: There exists a one-to-one correspondence between the family of GSM fields and the set of 2×2 symmetric positive-definite matrices whose determinant is bounded above by unity. These matrices are explicitly written in terms of the GSM field parameters.

In Sec. III we undertake the study of the behavior of GSM fields under passage through FOS. The generalized rays map this problem into that of studying the linear transformations of the parameter matrix leading to a representation of the group $SL(2,\mathbb{R})$. This immediately shows that the action of an FOS induces a one-to-one map on the GSM family. The ratio of the transverse coherence length to the intensity width is left invariant in this process; it follows that the GSM family breaks into nonintersecting subfamilies each of which is closed under action by FOS. It is further shown that for every GSM field there exists a one-parameter subgroup of FOS which leaves it invariant. In Sec. IV we develop a graphical representation of GSM fields and FOS based on a three-dimensional Minkowski space and Lorentz transformations, which makes the main results easy to visualize. It helps us answer the following question: For a given FOS, are there any GSM fields left invariant by it? It allows generalizing the Kogelnik "*abcd* law" to partially coherent GSM fields. In Sec. V we specialize our analysis to free propagation and show that the pencils associated with the GSM fields exhibit a universal structure in the far zone. Section VI contains some concluding remarks.

II. GAUSSIAN SCHELL-MODEL FIELDS AND THE ASSOCIATED GENERALIZED RAY DENSITY DISTRIBUTIONS

We will be interested in the action of axially symmetric FOS on time-stationary wave fields. For such fields, different frequency components of the ensemble can be analyzed completely independently.[6] Hence, we present our analysis for a fixed frequency ω which we suppress. Let us choose a Cartesian coordinate system (x,y,z) such that the z axis is along the system axis. We will specify the field through its cross-spectral density in a transverse plane $z \equiv z_0$. If suppressing z_0 and denoting by $\underline{\rho}$ the transverse two-vector part (x,y) of the three vector (x,y,z) the cross spectral density factors in the form

$$\Gamma(\underline{\rho}_1,\underline{\rho}_2)=[I(\underline{\rho}_1)I(\underline{\rho}_2)]^{1/2}g(\underline{\rho}_1-\underline{\rho}_2) \ , \tag{2.1}$$

we then have a Schell-model field.' Clearly, g is the normalized degree of coherence and, from Eq. (2.1), we see that it is translation invariant for Schell-model fields. When both **I,** the intensity distribution, and g are Gaussian

$$I(\underline{\rho})=(A/2\pi\sigma_I^2)\exp(-\underline{\rho}^2/2\sigma_I^2) \ ,$$
$$g(\underline{\rho}_1-\underline{\rho}_2)=\exp(-|\underline{\rho}_1-\underline{\rho}_2|^2/2\sigma_g^2) \ , \tag{2.2}$$

then the field is said to be a Gaussian Schell-model (GSM). Here A is a constant independent of $\underline{\rho}$. By integrating $I(\underline{\rho})$ one finds that A is the total irradiance of the field. It is useful to rewrite the cross-spectral density of the GSM field in the following form:

$$\Gamma(\underline{\rho}+\tfrac{1}{2}\underline{\rho}', \underline{\rho}-\tfrac{1}{2}\underline{\rho}')=\frac{A}{2\pi\sigma_I^2}\exp\left[-\frac{1}{2}\left[\frac{\underline{\rho}^2}{\sigma_I^2}+\frac{(\rho')^2}{\gamma^2}\right]\right] , \tag{2.3}$$

where

$$\frac{1}{\gamma^2}=\frac{1}{\sigma_g^2}+\frac{1}{4\sigma_I^2} \tag{2.4}$$

γ is **an** effective parameter which controls the diffraction properties of this **field.**[9]

Some well-known families of Gaussian fields are special cases of the GSM fields: When $\sigma_g \ll \sigma_I$ we have the Gaussian quasihomogeneous field, and the coherent Gaussian field obtains when $\sigma_g \rightarrow \infty$. Thus, the results of the analysis to follow contain, as special cases, the corresponding results for these limiting families.

When a GSM field is acted on by a lens, it picks up a quadratic phase front. For this and other reasons, it is useful to generalize the GSM field to allow for a phase curvature; by GSM field we will mean, henceforth, one whose cross-spectral density is of the form''

$$\Gamma(\underline{\rho}_1,\underline{\rho}_2)=\frac{A}{2\pi\sigma_1^2}\exp\left[-\frac{1}{4}\left[\frac{\underline{\rho}_1^2+\underline{\rho}_2^2}{\sigma_I^2}\right]-\frac{1}{2}\frac{(\underline{\rho}-\underline{\rho}_2)^2}{\sigma_g^2}\right]\exp\left[-\frac{ik}{2R}(\underline{\rho}_1^2-\underline{\rho}_2^2)\right] . \tag{2.5}$$

When $R>0$ ($R<0$) we have a diverging (converging) phase front.

From the defining equation (2.5) it is clear that the GSM fields form a three-parameter family, σ_I, σ_g, and R or, equivalently, σ_I, γ, and R, being the three parameters. We suppress the parameter A for our interest is in the behavior of GSM fields under the action of systems for which the total irradiance A remains invariant.

Next we compute the generalized rays[6] generated by the GSM field. They are related' to the cross-spectral density through the Wigner-Moyal transform:'[1]

$$W(\underline{\rho}, \underline{S})=(2\pi)^{-2}\int d^2\underline{\rho}' e^{ik\underline{S}\cdot\underline{\rho}'}\Gamma(\underline{\rho}+\tfrac{1}{2}\underline{\rho}', \underline{\rho}-\tfrac{1}{2}\underline{\rho}') \tag{2.6}$$

The Wolf function $W(\underline{\rho},\underline{S})$ represents the intensity of the generalized pencil of rays going in the direction ($\underline{S}$, $S_z=(1-\underline{S}^2)^{1/2}$) through the point $(\underline{\rho},z_0)$. By virtue of Γ being Hermitian, W is real, but it is not pointwise positive definite. Thus, the generalized pencils consist of both shining and dark **rays.**[12] Both types of rays travel along straight lines in free space.

Since Eq. *(2.6)* is invertible it follows that the cross-spectral density in any transverse plane can be reconstructed, in an exact way, from knowledge of the generalized pencils. **Thus,** it becomes clear that the generalized rays offer an exact ray picture of wave optic phenomena involving only the two-point (and no higher-order) correlation function.

The generalized rays corresponding to the GSM field are easily computed owing to the elementary nature of Gaussian integrals. Substitution of Eq. *(2.5)* in Eq. *(2.6)* yields

$$W(\underline{\rho}, \underline{S})=A\left|\frac{\gamma}{2\pi\sigma_I}\right| e^{-\underline{\rho}^2/2\sigma_I^2}e^{-(\underline{S}-\underline{\rho}/R)^2k^2\gamma^2/2} . \tag{2.7}$$

We find that the ray pattern at every point is Gaussian with its peak in the direction of (ρ,R).

As in conventional ray optics, it is useful to treat $\underline{\rho}$, $\underline{S}$ as a column vector

$$\underline{q}=\begin{bmatrix}\underline{\rho}\\ \underline{S}\end{bmatrix} . \tag{2.8}$$

Now Eq. *(2.7)* can be readily rewritten in a compact form

$$W(\underline{\rho},\underline{S})\equiv W(\underline{q})=\frac{A}{\pi^2}\det(\underline{G})\exp(-k\underline{q}^T\underline{G}\underline{q}) \ , \tag{2.9}$$

where $\underline{q}^T$ **is** the transpose of $\underline{q}$ and the GSM field parameter matrix $\underline{G}$ is given by

$$\underline{G}=\frac{1}{2}\begin{bmatrix}\dfrac{1}{k\sigma_1^2}+\dfrac{k\gamma^2}{R^2} & -\dfrac{k\gamma^2}{R}\\ \dfrac{-k\gamma^2}{R} & k\gamma^2\end{bmatrix} . \tag{2.10}$$

It has the following properties:

$$\underline{G}^T=\underline{G} \ , \tag{2.11a}$$

$$\mathrm{tr}\underline{G}>0 \ , \tag{2.11b}$$

$$0<\det\underline{G}\leq 1 \ . \tag{2.11c}$$

That is, G is symmetric and positive definite with its determinant bounded from above by unity. The ratio σ_g/σ_I is known as the degree of global coherence.[13] It is related to G in a simple way:

$$\frac{\sigma_g^2}{\sigma_I^2}=\frac{4\det\underline{G}}{1-\det\underline{G}} . \tag{2.12}$$

We have made use of Eq. (2.4) in obtaining Eqs. (2.11c) and (2.12).

By virtue of Eq. (2.6) for a given fixed k there is a one-to-one correspondence between GSM fields and Wolf functions of the form (2.9), which in turn are in one-to-one correspondence with 2×2 real matrices satisfying Eq. (2.11). We have established the following result: ***There is a one-to-one correspondence between the* GSM *family of a fixed irradiance and the family of* 2x2 *real symmetric positive definite matrices whose determinant is bounded from above by unity.*** Given the GSM field one can immediately construct the parameter matrix G through Eq. (2.10). Conversely, given G one can compute the field parameters through

$$\gamma^2=2G_{22}/k,\quad 1/R=-G_{12}/G_{22} ,$$
$$\sigma_I^2=(G_{22}/2k)\det\underline{G},\quad \sigma_g^2=\frac{2G_{22}}{k(1-\det\underline{G})} . \tag{2.13}$$

III. TRANSFORMATION OF GSM FIELDS BY FIRST-ORDER SYSTEMS

An axially symmetric FOS can be specified through its ray-transfer matrix S:

$$S=\begin{bmatrix} a & b \\ c & d\end{bmatrix},\quad ad-bc=1 \tag{3.1}$$

i.e., $S\in SL(2,\mathbb{R})$. Its action on the Wolf function is to produce the following map:'

$$W_{\rm out}(\underline{q})=W_{\rm in}(S^{-1}\underline{q}) , \tag{3.2}$$

where $W_{\rm in}$ and $W_{\rm out}$ are, respectively, the input and output Wolf functions.

To derive the transformation of GSM fields by FOS we substitute Eq. (2.9) in Eq. (3.2) and obtain

$$W_{\rm out}(\underline{q})=\frac{A_{\rm in}}{\pi^2}\det(\underline{G}_{\rm in})e^{-k\underline{q}^T\underline{G}_{\rm out}\underline{q}} , \tag{3.3}$$

where

$$\underline{G}_{\rm out}=(S^{-1})^T\underline{G}_{\rm in}S^{-1} . \tag{3.4}$$

This is a useful result. The use of generalized rays has mapped the problem of transformation of GSM fields by FOS to one of studying the transformation of symmetric positive-definite 2×2 matrices under $SL(2,\mathbb{R})$ by the rule (3.4), thus circumventing elaborate calculations involving integrals.

Let us examine $\underline{G}_{\rm out}$. Since $\underline{G}_{\rm in}$ is symmetric, so also is $\underline{G}_{\rm out}$. By virtue of S being unimodular,

$$\det\underline{G}_{\rm out}=\det\underline{G}_{\rm in} ,$$
$$0<\det\underline{G}_{\rm out}\leq 1 . \tag{3.5}$$

Further, since $S^{-1}(S^{-1})^T$ and $\underline{G}_{\rm in}$ are both positive definite

$$\mathrm{tr}\underline{G}_{\rm out}=\mathrm{tr}[\underline{G}_{\rm in}S^{-1}(S^{-1})^T]>0 . \tag{3.6}$$

Thus, from Eq. (2.11) we see that $\underline{G}_{\rm out}$ is a ***bonafide*** GSM field parameter matrix and, hence, from Eq. (3.3) and Eq. (2.9), $W_{\rm out}(\underline{q})$ corresponds to a GSM field with irradiance $A_{\rm out}=A_{\rm in}$. Also, since $\det\underline{G}$ is an invariant of this map we see from Eq. (2.12) that the degree of global coherence is preserved by this map.

We have the following result: ***The action of an FOS induces a one-to-one map on the family of GSM fields; the degree of global coherence is an invariant of this map.***

A special case of this result is already known in the work of Collett and Wolf.[4] They studied the behavior of a Gaussian quasihomogeneous field (a special case of GSM field) under free propagation (a special case of FOS) and found that the degree of global coherence was an invariant. Our analysis using generalized rays has led to a twofold generalization of this result.

In the light of our last result it is easily seen that the action of FOS divides the three-parameter GSM family into nonintersecting two-parameter subfamilies, each subfamily being characterized by a fixed value of σ_g/σ_I or, equivalently, of $\det G$. Each subfamily is closed under action by FOS in the strong sense that an FOS transforms it onto itself in a one-to-one fashion. Consequently, a GSM field belonging to one subfamily cannot be transformed into one belonging to a different subfamily by any FOS. In particular, a GSM field which is not quasihomogeneous cannot be transformed into a quasihomogeneous field using FOS alone.

First-order systems form a three-parameter group $SL(2,\mathbb{R})$ [which is the same as $Sp(2,\mathbb{R})$]. But our last result shows that they effect only a two-parameter transformation on the GSM family. The reason for this can be traced to the following fact: ***For every* GSM *field there exists a one-parameter subgroup of FOS which leaves it invariant.***

Proof: Again, let $\underline{G}$ be the parameter matrix of the given GSM field. Write $\det\underline{G}=\gamma^2/4\sigma_I^2=\kappa^2$. By virtue of our last result there exists an FOS S_0 which transforms $\underline{G}$ into the following special form:

$$S_0:\ \underline{G}\to\underline{G}_0=(S_0^{-1})^T\underline{G}S_0^{-1}=\begin{bmatrix}\kappa & 0\\ 0 & \kappa\end{bmatrix} \tag{3.7}$$

We shall call $\underline{G}_0$ the standard form of $\underline{G}$. In fact, S_0 can be explicitly constructed in the following way. Let us denote by $S_l(f)$ and $S_m(\beta)$, respectively, a thin lens of focal length f and a magnifier of linear magnification p. Their ray-transfer matrices are

$$S_l(f)=\begin{vmatrix} 1 & 0\\ 1/f & 1\end{vmatrix},\quad S_m(\beta)=\begin{vmatrix}\beta & 0\\ 0 & \beta^{-1}\end{vmatrix} . \tag{3.8}$$

Choosing $f=R$ and $m=(k\sigma_I\gamma)^{-1/2}$ we have

$$S_l(R):\ \underline{G}\to\underline{G}'=(S_l^{-1})^T\underline{G}\,S_l^{-1}=\begin{bmatrix}\frac{1}{2k\sigma_I^2} & 0\\ 0 & k\gamma^2/2\end{bmatrix},$$
$$S_m((k\sigma_I\gamma)^{-1/2}):\ G'\to G_0=(S_m^{-1})^T\underline{G}'S_m^{-1}=\begin{bmatrix}\kappa & 0\\ 0 & \kappa\end{bmatrix}. \tag{3.9}$$

We have thus found an FOS S_0 which casts $\underline{G}$ into its standard form:

$$S_0=S_m((k\sigma_I\gamma)^{-1/2})S_l(R),$$
$$(S_0^{-1})^T\underline{G}\,S_0^{-1}=\underline{G}_0. \tag{3.10}$$

Now we note that the one-parameter subgroup of FOS SO(2) leaves $\underline{G}_0$ invariant:

$$S_\theta\in SO(2):\ S_\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix},\quad 0\le\theta\le 2\pi,$$
$$S_{\theta_2}S_{\theta_1}=S_{\theta_1}S_{\theta_2}=S_{\theta_1+\theta_2}. \tag{3.11}$$
$$S_\theta:\ \underline{G}_0\to(S_\theta^{-1})^T\underline{G}_0S_\theta^{-1}=\underline{G}_0.$$

From Eqs. (3.10) and (3.11) it follows that the one-parameter subgroup of FOS

$$S'_\theta=S_0^{-1}S_\theta S_0,\quad 0\le\theta\le 2\pi \tag{3.12}$$

leaves $\underline{G}$ invariant. This completes the proof.

For the special case of a coherent Gaussian field with no phase curvature, $\sigma_g=\infty$, $R=\infty$, and the GSM matrix becomes

$$\underline{G}=\begin{bmatrix}\frac{1}{2k\sigma_I^2} & 0\\ 0 & 2k\sigma_I^2\end{bmatrix} \tag{3.13}$$

Our last result specialized to this case shows that

$$S'_\theta=\begin{bmatrix}\cos\theta & (2k\sigma_I^2)^{-1}\sin\theta\\ -2k\sigma_I^2\sin\theta & \cos\theta\end{bmatrix},\quad 0\le\theta\le 2\pi \tag{3.14}$$

is the one-parameter subgroup which leaves an equiphase Gaussian field invariant. Of special interest is a particular element of this subgroup corresponding to $\theta=\pi/2$. This is a scaled Fourier transform operation and we recover the familar result: an equiphase Gaussian function is invariant under an appropriately scaled Fourier transformation.

IV. GEOMETRICAL REPRESENTATION AND ANALYSIS

In the preceding sections it has been shown that there is a one-to-one correspondence between GSM fields and two-dimensional real matrices G with the properties (2.11), such that the effect of an FOS on the former can be expressed by the change (3.4) in G. (Here and in the following the value of the wave number **k** is to be held fixed.) In the present section we develop a transparent geometrical representation of the transformation law (3.4), which makes it very easy to understand the origin of the various results already obtained. In particular, it shows us how to define a complex parameter β for *any* GSM field, which changes according to the well-known Kogelnik *abcd* law under the action of any FOS.

The basic fact to be used is that SL(2,ℝ) is the spinor group corresponding to the group SO(2,1) of proper Lorentz transformations in a three-dimensional "space time."[15] Since G transforms linearly in S according to (3.4), one expects to be able to construct a three-component real column vector out of the elements of $\underline{G}$, such that they undergo a three-dimensional Lorentz transformation determined by S. To realize this, we express $\underline{G}$ in (2.10) as a real linear combination of the unit matrix and the Pauli matrices σ_1,σ_3:

$$\underline{G}=x_0-x_1\sigma_3+x_2\sigma_1=\begin{bmatrix}x_0-x_1 & x_2\\ x_2 & x_0+x_1\end{bmatrix}. \tag{4.1}$$

(Because of the symmetry of $\underline{G}$ the Pauli matrix σ_2 does not appear.) This parametrization of $\underline{G}$ is related to the earlier one by

$$x_0=\frac{1}{4}\left[k\gamma^2\left(1+\frac{1}{R^2}\right)+\frac{1}{k\sigma_I^2}\right],$$
$$x_1=\frac{1}{4}\left[k\gamma^2\left(1-\frac{1}{R^2}\right)-\frac{1}{k\sigma_I^2}\right], \tag{4.2}$$
$$x_2=-\frac{k\gamma^2}{2R}.$$

Evaluating the determinant of (4.1), we see that the degree of global coherence is related to the Minkowski squared length of $\underline{x}$:

$$\det\underline{G}=\kappa^2=x_0^2-x_1^2-x_2^2=\frac{\gamma^2}{4\sigma_I^2}. \tag{4.3}$$

The conditions (2.11a) and (2.11b), characterizing $\underline{G}$, appear as

$$0<\kappa\le 1,\quad x_0>0. \tag{4.4}$$

This leads to the following statement: For a fixed **k**, there is a one-to-one correspondence between the family of all GSM fields and the set of positive timelike vectors in a fictitious three-dimensional space-time, with Lorentz-invariant length κ lying in (0,1]. The appropriateness of this description is seen when (3.4) is stated in terms of **x**: The effect of an FOS corresponding to the matrix S in SL(2,ℝ), Eq. (3.1), is to take x into **x'** according to[16]

$$\underline{G}'=(S^{-1})^T\underline{G}\,S^{-1}:\underline{x}'=\Lambda(S)\underline{x},$$
$$\Lambda(S)=\begin{bmatrix}\frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & -ab-cd\\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & cd-ab\\ -ac-bd & bd-ac & ad+bc\end{bmatrix}. \tag{4.5}$$

Here $\underline{x}$ is a three-component column vector

$$\begin{vmatrix} x_0 \\ x_1 \\ x_2 \end{vmatrix},$$

and similarly for $\underline{x}'$. It is straightforward to derive (4.5) and also to check that $\Lambda(S)$ is a proper Lorentz transformation belonging to the group SO(2,1). Further, for any two **FOS** $S,S'\in SL(2,R)$ acting in succession, it can be seen that

$$\Lambda(S')\Lambda(S)=\Lambda(S'S) . \tag{4.6}$$

This description of the family of all **GSM** fields can be depicted diagramatically as in the figure. The region of interest is enclosed by the (positive) branch of the timelike hyperboloid $\kappa=1$ and the (positive) light cone $\kappa=0$; it includes the former but not the latter. completely coherent **GSM** fields, corresponding to $\sigma_g=\infty$, are represented by vectors $\underline{x}$ lying on the hyperboloid $\kappa=1$. **As** one approaches the quasihomogeneous limit $(\sigma_I/\sigma_g)\rightarrow\infty$, one comes closer and closer to the cone $\kappa=0$. **A** general **GSM** field corresponds to an $\underline{x}$ lying on a general hyperboloid with $0<\kappa\leq 1$; this is shown as an intermediate hyperboloid in the figure. The action of an FOS S is to move an $\underline{x}$ on a hyperboloid with a certain value of κ to another point $\underline{x}'$ *on* the *same* hyperboloid. The basic results of Sec. III become obvious in this representation: (a) Each Lorentz transformation belonging to **SO(2,1),** and representing some FOS, maps the region of $\underline{x}$ space relevant to us onto itself in a one-to-one invertible way; (b) each point of this region represents, in a one-to-one way, some **GSM** field; (c) the mappings $\underline{x}\rightarrow\underline{x}'=\Lambda(S)\underline{x}$ preserve the hyperboloid corresponding to each allowed value of κ; (d) thus the **GSM** fields corresponding to points on each hyperboloid form a two-parameter subfamily with a common degree of global coherence, transforming into each other and not taken into a **GSM** field "belonging" to a distinct hyperboloid, under the action of any FOS. To these may now be added the remark that the matrices S and $-S$ in SL(2,$\mathbb{R}$) must be identified as representing one and the same FOS.

The process of taking a **GSM** field $\underline{G}$ to its standard form $\underline{G}_0$ corresponds to Lorentz transforming a general vector $\underline{x}$ to the "rest frame" value $(\kappa,0,0)$. The FOS denoted by S_θ, $0\leq\theta\leq 2\pi$ in (3.11) are represented by purely "spatial rotations" in the x_1-x_2 plane leaving x_0 unaffected. For a general **GSM** field $\underline{x}$ the FOS S leaving it invariant are the Lorentz transformations in SO(2,1) "with $\underline{x}$ as axis." The converse question can now be answered: If an FOS S is given, are there any **GSM** fields which are invariant under action by S? Since the points $\underline{x}$ which we use are all positive timelike, the answer is as follows: If S is equivalent, by conjugation with a suitable element of **SL(2,$\mathbb{R}$),** to S_θ for some θ, then there exist **GSM** fields invariant under S, otherwise not. In the former case, if S is given we can calculate these **GSM** fields by the converse to the calculations in Sec. III. Examples of FOS which definitely alter *every* **GSM** field are the SL(2,$\mathbb{R}$) matrices

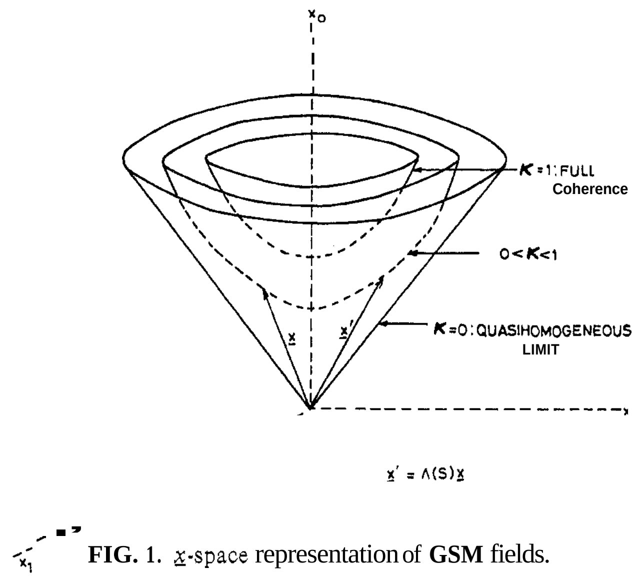

FIG. 1. $\underline{x}$-space representation of **GSM** fields.

$$\begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix}, \quad \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}, \quad \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} . \tag{4.7}$$

for any real α. These are, respectively, the magnifier, "boost," and free propagation one-parameter subgroups. The physical realizations of the boosts using lenses and free propagations, which are quite different for $\alpha>0$ and $\alpha<0$, are described in Sec. VI.

As a final application, we show how to generalize the Kogelnik *abcd* law[17] from the fully coherent case to a general partially coherent **GSM** field. If we define a complex parameter β in terms of x by

$$\beta=-\frac{x_0+x_1}{x_2+i\kappa} , \tag{4.8}$$

then when x is changed to x' by Eq. (4.5), we find that β changes to β' via

$$\beta'=\frac{a\beta+b}{c\beta+d} . \tag{4.9}$$

A more transparent way to express β is to use **(4.2):**

$$\frac{1}{\beta}=\frac{1}{R}-\frac{i}{k\gamma\sigma_I} \tag{4.10}$$

This generalizes the well-known expression" in the coherent case. The point, of course, is that the three-dimensional representation of this section makes it clear that such a generalization must necessarily exist.

V. FREE PROPAGATION: ASYMPTOTIC BEHAVIOR OF THE GENERALIZED PENCILS

Now we specialize our analysis to a special class of first-order systems, namely, free propagation through a distance D whose ray-transfer matrix is

$$S_D=\begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix} . \tag{5.1}$$

Let us denote by $\underline{G}$, $\underline{G}'$ the input and output GSM field parameter matrices and by A the invariant det$\underline{G}$. Then substitution of Eq. (5.1) in (3.4) yields

$$\underline{G}' = \begin{bmatrix} G_{11} & G_{12}-DG_{11} \\ G_{12}-DG_{11} & G_{22}-2G_{12}D+G_{11}D^2 \end{bmatrix} . \qquad (5.2)$$

We find that under free propagation G_{11} is invariant in addition to A. To see the significance of this new invariant, we first note that the transverse plane

$$D = G_{12}/G_{11} \qquad (5.3)$$

is an equiphase plane, i.e., $R' \rightarrow \infty$. We further note from (2.13) and (5.2) that this is the plane where σ_I^2 (and also σ_g^2) as a function of D assumes its minimum value. Thus, we find that the GSM beam has a "waist" at a distance D from the input plane; the waist is to the right (left) of the input plane accordingly as (G_{12}) and, hence, R <0 (>0). Denoting by σ_w the intensity width at the waist we immediately obtain the significance of the invariant G_{11}:

$$G_{11} = \frac{1}{2k\sigma_w^2} . \qquad (5.4)$$

To examine the asymptotic behavior of the ray density function $W_Z(\underline{\rho},\underline{S})$ after propagation through a large distance Z from the waist, it is useful to renormalize the observation plane transverse coordinates in the following way:

$$\underline{\bar{\rho}} = \underline{\rho}/Z . \qquad (5.5)$$

Evidently! $\underline{\bar{\rho}}$ represents the angular position of the observation point with respect to the beam waist. Using (5.5) and (5.2) in (2.9) we have

$$\begin{aligned} W_Z(\underline{\rho},\underline{S}) &\equiv \bar{W}_D(\underline{\bar{\rho}},\underline{S}) \\ &= \frac{A\Delta}{\pi^2}\exp\{-k[G_{11}Z^2(\underline{\bar{\rho}}-\underline{S})^2+(\Delta\underline{S}^2)/G_{11}]\} \\ &\simeq \frac{A}{\pi k Z^2}\frac{\Delta}{G_{11}}\delta^{(2)}(\underline{\bar{\rho}}-\underline{S})\exp\left[-k\frac{(\Delta\underline{\bar{\rho}}^2)}{G_{11}}\right] . \end{aligned} \qquad (5.6)$$

Here we made use of the well-known result

$$\lim_{\Lambda\rightarrow\infty}\frac{\Lambda^2}{\pi}e^{-\Lambda^2\underline{\rho}^2} = \delta^{(2)}(\underline{\rho}) , \qquad (5.7)$$

$\delta^{(2)}(\underline{\rho})$ being the two-dimensional Dirac delta function. The far-zone pencils are radial and have a ***universal structure*** controlled by a single parameter Δ/G_{11}. We deduce that all GSM fields having the same value of A/G $_{11}$ will result in the same far-zone pencil structure. This equivalence statement is about the far-zone pencils and, hence, it is more general than the paraxial version of the Wolf-Collett[18] equivalence theorem which is for the far-zone intensity distribution. The latter obtains when one integrates (5.6) over $\underline{S}$,

$$\begin{aligned} I_Z(\underline{\rho}) &\equiv \int d^2(k\underline{S})W_Z(\underline{\rho},\underline{S}) \\ &= \frac{A}{\pi Z^2}\frac{k\Delta}{G_{11}}\exp\left[-\frac{k\Delta}{G_{11}}\left[\frac{\underline{\rho}}{Z}\right]^2\right] , \end{aligned} \qquad (5.8)$$

and makes use of the fact that

$$\Delta/G_{11} = k\gamma_w^2/2 , \qquad (5.9)$$

where γ_w is the value of γ at the waist. Even Eq. (5.8) is a generalization of the Wolf-Collett theorem for the following reason: Whereas their formulation assumes the "source plane" to be an equiphase surface, our treatment does not place any such requirement and, in fact, explicitly allows for a phase curvature in the "source plane."

Finally, we note that the approximation leading to Eq. (5.7) conserves the total irradiance as can be seen by integrating (5.8) over $\underline{\rho}$.

VI. CONCLUDING REMARKS

We have analyzed the passage of GSM fields through FOS using the method of generalized rays. This method, while staying exact within wave optics, reduces the problem of otherwise dealing with complicated integrals into one of multiplying 2 x 2 matrices. This aspect, combined with the geometrical picture of viewing this process as Lorentz transformation in 2 + 1 Minkowski space, helps one find a complete answer to any question related to this class of problems, much more easily than will be possible using the conventional wave optic methods. Thus, we found that the GSM family is closed under action by FOS. We have further shown that, given any GSM field, there always exists a one-parameter subgroup of **FOS** which leaves it invariant.

In Eq. (4.7) we identified three subgroups of FOS which definitely modified every GSM field. While the magnifier and free propagation subgroups are well known, the boost subgroup is not as commonly known in the context of first-order optics. It turns out that they, and in fact any **FOS,** can be synthesized using thin lenses separated by free propagation sections. Let us denote by S_D and S_f, respectively, the ray-transfer matrices for free propagation through a distance D and action by a thin lens of focal length f. Then the "antiboosts" $(\alpha>0)$ can be synthesized even in the simple configuration $S_{D_2}S_fS_{D_1}$ involving one concave lens with

$$D_1 = D_2 = (\cosh\alpha - 1)(\sinh\alpha)^{-1} ,$$

$$f = -(\sinh\alpha)^{-1} .$$

But the boosts $(\alpha<0)$ as FOS are qualitatively different and cannot be synthesized even in any configuration involving two lenses. They can, however, be realized in the three-lens configuration.[19]

$$S_{f_2}S_{D_2}S_fS_DS_{f_1}S_{D_1}$$

with

$$D = D_1 = 2x > 0 , \quad f_1 = x ,$$

$$f = x\sinh\alpha(\sinh\alpha - x\cosh\alpha - x)^{-1}, \quad D_2 = -\sinh\alpha,$$

$$f_2 = -\sinh\alpha(\cosh\alpha+1)^{-1} .$$

Our analysis can be simply extended to fields which are incoherent superpositions (convex combinations) of GSM **fields.**[20] For such fields, it is clear from the geometrical picture presented in Sec. IV that there will exist, in addition to the invariant norm of each three-vector representing the individual GSM fields, new invariants corresponding to the Lorentz inner products of these vectors. For instance, if the input field is an incoherent superposition of two GSM fields with parameters σ_I, γ and $\sigma_I', \sigma_I', \gamma'$ the input plane being the equiphase plane (the waist plane) for either field, then the additional invariant corresponding to the inner product is, from Eq. (4.2),

$$\frac{\gamma^2}{(\sigma_I')^2}+\frac{(\gamma')^2}{\sigma_I^2} \, .$$

Thus, if we are dealing with convex combination of n GSM fields then there will be $n(n+1)/2$ invariants. It should be emphasized that the derivation of such invariants will at best be quite tedious if one uses the traditional methods.

There already exists rich literature on the behavior of coherent Gaussian beams under action by FOS and the associated *abcd* law. Our geometrical picture in Sec. IV and the *abcd* law brings both coherent and partially coherent Gaussian fields under the same fold, rendering this literature applicable to all GSM fields.

For simplicity, the analysis in this paper was restricted to axially symmetric GSM fields and axially symmetric FOS. We hope to analyze the behavior of anisotropic GSM fields in arbitrary FOS elsewhere.

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