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# DILATION OF CONTRACTIVE TUPLES: A SURVEY

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## 0. Introduction

In this survey article we start with the unitary dilation of a single contraction due to Sz.-Nagy and Foias [45]. Ando gave a generalization to a pair of commuting contractions [2]. He proved that such a pair has a simultaneous commuting dilation. Then came the startling revealation from Varopoulos [46], Parrott [36] and Crabb-Davie [18] that this phenomenon can not be generalized any further. They produced examples of triples of commuting contractions which fail to have any commuting isometric dilation. The next stage of developments saw the successful attempt of dilating a special class of tuples, viz., the contractive ones. Drury [27], in connection to his generalization of von Neumann's inequality, and then Arveson [6] proved the *standard commuting* dilation for commuting contractive tuples. Several authors pursued the idea of dilating any contractive tuple (commuting or not) to isometries with orthogonal ranges. Some ideas along this direction can already be seen in an early paper of Davis [21]. In more concrete form this dilation appeared in the papers of Bunce [17] and Frazho [28]. A real extensive study of this notion has been carried out by Popescu in a series of papers, see [40] - [44] and also [3], [4] with Arias. He has neat generalizations of many results from one variable situation. This dilation is called the *standard non-commuting* dilation. Davidson, Kribs and Shpigel [20] derive more information about this dilation for finite rank tuples. Then of course arose the natural question that if one starts with a commuting contractive tuple, then what is the relation between the two dilations that it posseses. A recent article by Bhat, Bhattacharyya and Dey [16] show that the standard commuting dilation is the maximal commuting dilation sitting inside the standard non-commuting dilation.

Section 1 is about unitary dilation of a contraction and von Neumann's inequality. Simultaneous commuting unitary dilation of a pair of commuting contractions and the von Neumann inequality for such pairs is taken up in Section 2. In this section, we also show that in general a triple of commuting contractions does not have a commuting unitary dilation. In Section 3, the contractive tuples are introduced and for a commuting contractive tuple, the standard commuting dilation is constructed. We also study two special contractive tuples. Section 4 is about the standard non-commuting dilation of a general contractive tuple. In Section 5, we outline the proof that the standard commuting dilation is the maximal commuting piece of the standard non-commuting dilation.

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## 1. Dilation of a Single Contraction

All the Hilbert spaces in this note are over the complex field and are separable. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the notations $\mathcal{K} \supset \mathcal{H}$ and $\mathcal{H} \subset \mathcal{K}$ will mean that $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ or that $\mathcal{H}$ is isometrically embedded into $\mathcal{K}$, i.e., there is a linear isometry $A$ mapping $\mathcal{H}$ into $\mathcal{K}$. In the latter case, we shall identify $\mathcal{H}$ with the closed subspace $A\mathcal{H}$ of $\mathcal{K}$. Any bounded operator $T$ on $\mathcal{H}$ is then identified with the bounded operator $ATA^*$ on $A\mathcal{H}$.

Definition 1.1. *Let $\mathcal{H} \subset \mathcal{K}$ be two Hilbert spaces. Suppose $T$ and $V$ are bounded operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then $V$ is called a dilation of $T$ if*

$$T^n h = P_{\mathcal{H}} V^n h$$

*for all $h \in \mathcal{H}$ and all non-negative integers $n$ where $P_{\mathcal{H}}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$. A dilation $V$ of $T$ is called minimal if $\overline{span}\{V^n h : h \in \mathcal{H}, n = 0, 1, 2, \ldots\} = \mathcal{K}$. An isometric (respectively unitary) dilation of $T$ is a dilation $V$ which is an isometry (respectively unitary).*

Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$ and let $\|\cdot\|$ denote the operator norm on $\mathcal{B}(\mathcal{H})$. An element $T$ of $\mathcal{B}(\mathcal{H})$ is called a contraction if $\|T\| \leq 1$. Given any pair $(\mathcal{H}, T)$ of a Hilbert space and a contraction acting on it, the following classical theorem of Sz.-Nagy and Foias constructs a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an isometric dilation $V$ of $T$ on $\mathcal{K}$. Two dilations $V_1$ and $V_2$ on the Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, of the same operator $T$ on $\mathcal{H}$ are called *unitarily equivalent* if there is a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $UV_1U^* = V_2$.

Theorem 1.2. *For every contraction $T$ on a Hilbert space $\mathcal{H}$, there is a minimal isometric dilation which is unique up to unitary equivalence.*

Proof: Let $D_T = (\mathbf{1}_{\mathcal{H}} - T^*T)^{1/2}$ be the unique positive square root of the positive operator $\mathbf{1}_{\mathcal{H}} - T^*T$. This is called the *defect operator* of $T$. Let $\mathcal{D}_T = \overline{\text{Range}} D_T$. Then the dilation space is constructed by setting

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \cdots .$$

The space $\mathcal{H}$ is identified with the subspace of $\mathcal{K}$ consisiting of elements of the form $(h, 0, 0, \ldots)$ where $h \in \mathcal{H}$. This is the way $\mathcal{H}$ is isometrically embedded into $\mathcal{K}$. Define the operator $V$ on $\mathcal{K}$ by

$$V(h_0, h_1, h_2, \ldots) = (Th_0, D_T h_0, h_1, h_2, \ldots).$$

For every $h \in \mathcal{H}$, we have $\|Th\|^2 + \|D_T h\|^2 = \|h\|^2$, so that the operator $V$ defined above is an isometry.

Now for $h_0 \in \mathcal{H}$ and $(k_0, k_1, k_2, \ldots) \in \mathcal{K}$, we have

$$\begin{aligned} < V^* h_0, (k_0, k_1, k_2, \ldots) > &= < h_0, V(k_0, k_1, k_2, \ldots) > \\ &= < (h_0, 0, 0, \ldots), (Tk_0, D_T k_0, k_1, k_2, \ldots) > \\ &= < h_0, Tk_0 > \\ &= < T^* h_0, k_0 > = < T^* h_0, (k_0, k_1, k_2, \ldots) > . \end{aligned}$$

Thus $V^* h = T^* h$ for all $h \in \mathcal{H}$. The operator $V$ on $\mathcal{K}$ has the property that $\mathcal{H}$ is is left invariant by $V^*$. When this happens, $\mathcal{H}$ is called a co-invariant subspace of $V$. In such a case, it is easy to see that $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$ for $n = 1, 2, \ldots$ and $T$ is called a piece of $V$. So $V$ is an isometric dilation of $T$. Minimality becomes clear by observing that

$$V^n(h_0, h_1, h_2, \ldots) = (T^n h_0, D_T T^{n-1} h_0, \ldots, D_T h_0, h_1, h_2, \ldots)$$

from which it follows that

$$V^n(h_0,0,0,\dots) = (T^n h_0, D_T T^{n-1} h_0, \dots, D_T h_0, 0, 0, \dots). \tag{1.1}$$

This gives another proof of $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$ for $n = 1,2,\dots$. Moreover, it follows from (1.1) that

$$\overline{span}\{V^n(h,0,0,\dots) : h \in \mathcal{H}, n = 0,1,2,\dots\} = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \cdots = \mathcal{K}.$$

It just remains to show that the minimal dilation is unique up to unitary equivalence. To that end, first note that given any isometric dilation $V$ of $T$ and two elements $h$ and $h'$ of $\mathcal{H}$, we have

$$\langle V^n h, V^m h'\rangle = \begin{cases} \langle V^{n-m}h, h'\rangle = \langle T^{n-m}h, h'\rangle \text{ if } n \geq m \geq 0, \\ \langle h, V^{m-n}h'\rangle = \langle h, T^{m-n}h'\rangle \text{ if } m \geq n \geq 0. \end{cases}$$

Thus $\langle V^n h, V^m h'\rangle$ (and hence the inner product of two finite sums of the form $\sum_{n=0}^{N} V^n h_n$ and $\sum_{n=0}^{N'} V^n h_n$) does not depend on a particular choice of minimal dilation $V$. Now take two minimal dilations, say $V_1$ and $V_2$ on the spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively and define $U$ on span$\{V_2^n h : h \in \mathcal{H}, n = 0,1,2,\dots\}$ by

$$U(\sum_{n=0}^{N} V_2^n h_n) = \sum_{n=0}^{N} V_1^n h_n.$$

This is a well-defined and isometric linear transformation from a dense subspace of $\mathcal{K}_2$ onto a dense subspace of $\mathcal{K}_1$. It is well-known that such an $U$ extends to a unitary operator from $\mathcal{K}_2$ to $\mathcal{K}_1$. Moreover, note that for $h \in \mathcal{H}$, we have $Uh = U(V_2^0)h = V_1^0 h = h$, thus the isometric embedding of $\mathcal{H}$ into $\mathcal{K}_2$ and $\mathcal{K}_1$ are left intact by $U$. Of course, $U$ has been so costructed that $UV_2U^* = V_1$. ■

The basic ingredient for the construction above is the operator called *unilateral shift.*

Definition 1.3. *An isometry $S$ on a Hilbert space $\mathcal{M}$ is called a unilateral shift if there is a subspace $\mathcal{L}$ of $\mathcal{M}$ satisfying*
*(i) $S^n\mathcal{L} \perp \mathcal{L}$ for all $n = 1,2,\dots$ and*
*(ii) $\mathcal{L} \oplus S\mathcal{L} \oplus S^2\mathcal{L} \cdots = \mathcal{M}$.*
*The subspace $\mathcal{L}$ is called the generating subspace for $S$ and* $\dim \mathcal{L}$ *is called the multiplicity of $S$.*

A unilateral shift has a unique generating subspace and is determined up to *unitary equivalence* by its multiplicity, i.e., if $S_1$ and $S_2$ are two unilateral shifts on Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively with the same multiplicity, then there is a unitary $U : \mathcal{L}_1 \to \mathcal{L}_2$ such that $US_1U^* = S_2$. The proofs and other facts about unilateral shifts can be found in, for example [45].

Let $S$ be the unilateral shift defined on the space $\mathcal{D}_T \oplus \mathcal{D}_T \cdots$ by

$$S(h_1, h_2, \dots) = (0, h_1, h_2, \dots), \text{ where } h_1, h_2, \dots \in \mathcal{D}_T.$$

Then $S$ has multiplicity equal to rank$D_T$ and the block operator matrix of the dilation $V$ is

$$\left(\begin{array}{c|ccc} T & 0 & 0 & 0 \\ \hline D_T & & & \\ 0 & & S & \\ 0 & & & \end{array}\right).$$

The next step is to obtain a unitary dilation.

DEFINITION 1.4. *Let $\mathcal{H} \subset \mathcal{K}$ be two Hilbert spaces. Suppose $V$ and $U$ are bounded operators on $\mathcal{H}$ and $\mathcal{K}$ respectively such that*

$$U^n h = V^n h \text{ for all } h \in \mathcal{H}.$$

*Then $U$ is called an extension of $V$. A unitary extension is an extension which is also a unitary operator.*

An extension $U$ of a bounded operator $V$ is also a dilation of $V$ because $P_{\mathcal{H}} U^n h = P_{\mathcal{H}} V^n h = V^n h$ for any $h \in \mathcal{H}$ and $n \geq 1$. It moreover has the property that $\mathcal{H}$ is an invariant subspace for $U$. It is this second property that makes it clear that in general, contractions could not have isometric extensions.

REMARK 1.5. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose $V$ is a dilation of $T$ on a Hilbert space $\mathcal{K}_1 \supset \mathcal{H}$ and $U$ is an extension of $V$ on a Hilbert space $\mathcal{K}_2 \supset \mathcal{K}_1$. Then $U$ is a dilation of $T$. Indeed,*

$$\begin{aligned} P_{\mathcal{H}} U^n h &= P_{\mathcal{H}} V^n h \text{ because } U \text{ is an extension of } V \text{ and } h \in \mathcal{H} \subset \mathcal{K}_1 \\ &= T^n h \text{ because } V \text{ is a dilation of } T. \end{aligned}$$

DEFINITION 1.6. *A unitary operator $U$ on a Hilbert space $\mathcal{M}$ is called a bilateral shift if there is a subspace $\mathcal{L}$ of $\mathcal{M}$ satisfying*
*(i) $U^n \mathcal{L} \perp \mathcal{L}$ for all integers $n \neq 0$ and*
*(ii) $\oplus_{n=-\infty}^{\infty} U^n \mathcal{L} = \mathcal{M}$.*
*The subspace $\mathcal{L}$ is called a generating subspace for $U$ and* $\dim \mathcal{L}$ *is called the multiplicity of $U$.*

LEMMA 1.7. *A unilateral shift $V$ on $\mathcal{M}$ always has an extension to a bilateral shift. Moreover, the extension preserves multiplicity.*

PROOF: The generating subspace of $V$ is $\mathcal{L} = \mathcal{M} \ominus V\mathcal{M}$. Define $\mathcal{K} = \oplus_{-\infty}^{\infty} \mathcal{L}_n$ where each $\mathcal{L}_n$ is the same as $\mathcal{L}$. For an element $(\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots)$ of $\mathcal{K}$ with $l_n \in \mathcal{L}_n$ for every $n \in \mathbb{Z}$, define

$$U(\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots) = (\ldots, l'_{-2}, l'_{-1}, l'_0, l'_1, l'_2, \ldots),$$

where now $l'_n \in \mathcal{L}_n$ and $l'_n = l_{n-1}$ for all $n \in \mathbb{Z}$. Clearly, $U$ is unitary and $\{(\ldots, 0, 0, l_0, 0, 0, \ldots) : l_0 \in \mathcal{L}_0\}$ is a generating subspace for $U$. This subspace has the same dimension as that of $\mathcal{L}$. An element $\sum_0^{\infty} V^n l_n$ of $\mathcal{H}$ is identified with the element $(\ldots, 0, 0, l_0, l_1, l_2, \ldots)$ of $\mathcal{K}$. This is an isometric embedding.

Now for $h = \sum_0^{\infty} V^n l_n$, we have

$$Uh = U(\ldots, 0, 0, l_0, l_1, l_2, \ldots) = (\ldots, 0, 0, 0, l_0, l_1, \ldots) = \sum_0^{\infty} V^n l_{n-1} = Vh.$$

That completes the proof. ∎

An immediate corollary is the following.

COROLLARY 1.8. *An isometry $V$ on $\mathcal{H}$ always has an extension to a unitary.*

PROOF: By Wold decomposition ([45], page 3), we have $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are reducing subspaces of $V$ and $V = V_0 \oplus V_1$ where $V_0 = V|_{\mathcal{H}_0}$ is a unitary and $V_1 = V|_{\mathcal{H}_1}$ is a unilateral shift.

By the lemma above, $V_1$ can be extended to a bilateral shift, say $U_1$. Now $V_0 \oplus U_1$ is a unitary extension of $V$. ■

THEOREM 1.9. *For every contraction $T$ on a Hilbert space $\mathcal{H}$, there is a minimal unitary dilation which is unique up to unitary equivalence.*

PROOF: Obtaining a unitary dilation is mmediate from the above discussions. We take an isometric dilation and then its unitary extension, say $U_0$. This is a unitary dilation, although may not be minimal. Let

$$\mathcal{K} = \overline{span}\{U_0^n h : h \in \mathcal{H} \text{ and } n = 0, 1, 2, \ldots\}.$$

This is a reducing subspace for $U_0$ and the restriction $U$ of $U_0$ to $\mathcal{K}$ is a minimal unitary dilation.

The uniqueness (up to unitary equivalence) proof is exactly on the same lines as the proof of uniqueness of isometric dilation. ■

The unitary dilation of a contraction gives a quick proof of von Neumann's inequality. In its original proof [35], von Neumann first proved it for Mobius functions and then used the fact that the space of absolutely convergenet sums of finite Blashke products is isometrically isomorphic to the disk algebra, the algebra of all functions which are analytic in the interior and continuous on the closure of the unit disk. See Drury [26] or Pisier [39] for the details of this proof. The following proof using the dilation is due to Halmos [29].

THEOREM 1.10. *(**von Neumann's inequality**): For every polynomial $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, let*

$$\|p\| = \sup\{|p(z)| : |z| \leq 1\}.$$

*If $T$ is a contraction and $p$ is a polynomial, then*

$$\|p(T)\| \leq \|p\|.$$

PROOF: First note that by spetcral theory, if $U$ is a unitary operator and $p$ is a polynomial, then $\sigma(p(U)) = \{p(z) : z \in \sigma(U)\}$.

Since $U$ is unitary, $\sigma(U) \subset \mathbb{T}$ where $\mathbb{T}$ is the unit circle. Thus $\|p(U)\| = \sup\{|p(z)| : z \in \sigma(U)\} \leq \sup\{|p(z)| : z \in \mathbb{T}\} = \|p\|$. Now by the unitary dilation theorem, $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$ which gives

$$\|p(T)\| = \|P_{\mathcal{H}} p(U)|_{\mathcal{H}}\| \leq \|p(U)\| \leq \|p\|.$$

■

The survey article by Drury [26] is an excellent source for more discussions on von Neumann's inequality.

## 2. TUPLES OF COMMUTING CONTRACTIONS

Ando gave a beautiful generalization of Sz.-Nagy and Foias's theorem for two commuting contractions. The concept of dilation for a tuple of operators is similar to Definition 1.1.

DEFINITION 2.1. *Let $\mathcal{H} \subset \mathcal{K}$ be two Hilbert spaces. Suppose $\underline{T} = (T_1, T_2, \ldots, T_n)$ and $\underline{V} = (V_1, V_2, \ldots, V_n)$ are tuples of bounded operators acting on $\mathcal{H}$ and $\mathcal{K}$ respectively, i.e., $T_i \in \mathcal{B}(\mathcal{H})$ and $V_i \in \mathcal{B}(\mathcal{K})$. The operator tuple $\underline{V}$ is called a dilation of the operator tuple $\underline{T}$ if*

$$T_{i_1} T_{i_2} \ldots T_{i_k} h = P_{\mathcal{H}} V_{i_1} V_{i_2} \ldots V_{i_k} h \text{ for all } h \in \mathcal{H},\ \ k \geq 1 \text{ and all } 1 \leq i_1, i_2, \ldots, i_k \leq n.$$

*If $V_i$ are isometries with orthogonal ranges, i.e., $V_i^* V_j = \delta_{ij}$ for $1 \le i, j \le n$, then $\underline{V}$ is called an isometric dilation. A dilation $\underline{V}$ of $T$ is called minimal if $\overline{span}\{V_{i_1} V_{i_2} \dots V_{i_k} h : h \in H, k \ge 0$ and $1 \le i_1, i_2, \dots, i_k \le n\} = \mathcal{K}$.*

THEOREM 2.2. *For a pair $\underline{T} = (T_1, T_2)$ of commuting contractions on a Hilbert space $\mathcal{H}$, there is a commuting isometric dilation $\underline{V} = (V_1, V_2)$.*

PROOF: Let $\mathcal{H}_+ = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ be the direct sum of infinitely many copies of $\mathcal{H}$. Define two isometries $W_1$ and $W_2$ on $\mathcal{H}_+$ as follows. For $h = (h_0, h_1, h_2, \dots) \in \mathcal{H}_+$, set

$$W_i h = (T_i h_0, D_{T_i} h_0, 0, h_1, h_2, \dots), \text{ for } i = 1, 2.$$

Here $D_{T_i}$ are the defect operators as defined in the proof of Theorem 1.2. Clearly, $W_1$ and $W_2$ are isometries. However, they need not commute. We shall modify them to get commuting isometries.

Let $\mathcal{H}_4 = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ and let $v$ be a unitary operator on $\mathcal{H}_4$. We shall specify $v$ later. Identify $\mathcal{H}_+$ and $\mathcal{H} \oplus \mathcal{H}_4 \oplus \mathcal{H}_4 \oplus \cdots$ by the following identification:

$$h = (h_0, h_1, h_2, \dots) \to (h_0, \{h_1, h_2, h_3, h_4\}, \{h_5, h_6, h_7, h_8\}, \dots).$$

Now define a unitary operator $W : \mathcal{H}_+ \to \mathcal{H}_+$ by

$$Wh = (h_0, v(h_1, h_2, h_3, h_4), v(h_5, h_6, h_7, h_8), \dots),$$

where $h = (h_0, h_1, h_2, \dots)$. The unitarity of $W$ is clear since $v$ is a unitary and

$$W^* h = W^{-1} h = (h_0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \dots).$$

We define $V_1 = W W_1$ and $V_2 = W_2 W^{-1}$. These are isometries because they are products of isometries. These act on $\mathcal{H}_+$ and

$$V_i^*(h_0, 0, \dots) = (T_i^* h_0, 0, \dots), \text{ for } i = 1, 2.$$

Now we shall see that $v$ can be chosen so that $V_1$ and $V_2$ commute.

To choose such a $v$, we first compute $V_1 V_2$ and $V_2 V_1$.

$$\begin{aligned} V_1 V_2 (h_0, h_1, \dots) &= W W_1 W_2 v^{-1} (h_0, h_1, \dots) \\ &= W W_1 W_2 ((h_0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \dots) \\ &= W W_1 (T_2 h_0, D_{T_2} h_0, 0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \dots) \\ &= W (T_1 T_2 h_0, D_{T_1} T_2 h_0, 0, D_{T_2} h_0, 0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \dots) \\ &= (T_1 T_2 h_0, v(D_{T_1} T_2 h_0, 0, D_{T_2} h_0, 0), (h_1, h_2, h_3, h_4), (h_5, h_6, h_7, h_8), \dots) \end{aligned}$$

and

$$\begin{aligned} V_2 V_1 (h_0, h_1, \dots) &= W_2 W_1 (h_0, h_1, \dots) \\ &= W_2 (T_1 h_0, D_{T_1} h_0, 0, h_1, h_2, \dots) \\ &= (T_2 T_1 h_0, D_{T_2} T_1 h_0, 0, D_{T_1} h_0, 0, h_1, h_2, \dots). \end{aligned}$$

Since $T_1$ and $T_2$ commute, $V_1 V_2$ will be equal to $V_2 V_1$ if

$$v(D_{T_1} T_2 h, 0, D_{T_2} h, 0) = (D_{T_2} T_1 h, 0, D_{T_1} h, 0) \tag{2.1}$$

for all $h \in \mathcal{H}$. Now a simple calculation shows that

$$\|(D_{T_1} T_2 h, 0, D_{T_2} h, 0)\| = \|(D_{T_2} T_1 h, 0, D_{T_1} h, 0)\|, \text{ for all } h \in \mathcal{H}.$$

Hence one can define an isometry $v$ by (2.1) from $\mathcal{L}_1 \stackrel{\text{def}}{=} \overline{\text{span}}\{(D_{T_1}T_2h, 0, D_{T_2}h, 0) : h \in \mathcal{H}\}$ onto $\mathcal{L}_2 \stackrel{\text{def}}{=} \overline{\text{span}}\{(D_{T_2}T_1h, 0, D_{T_1}h, 0) : h \in \mathcal{H}\}$. To extend $v$ to the whole of $\mathcal{H}_4$ as a unitary operator, i.e., an isometry of $\mathcal{H}_4$ onto itself, one just needs to check that $\mathcal{H}_4 \ominus \mathcal{L}_1$ and $\mathcal{H}_4 \ominus \mathcal{L}_2$ have the same dimension. If $\mathcal{H}$ is finite dimensional, this is obvious because $\mathcal{L}_1$ and $\mathcal{L}_2$ are isometric. When $\mathcal{H}$ is infinite dimensional, note that $\mathcal{L}_1^\perp$ and $\mathcal{L}_2^\perp$ have dimension at least as large as $\dim\mathcal{H}$ because each of $\mathcal{L}_1^\perp$ and $\mathcal{L}_2^\perp$ contain a subspace isomorphic to $\mathcal{H}$, for example the subspace $\{(0, h, 0, 0) : h \in \mathcal{H}\}$. Thus

$$\dim \mathcal{H}_4 \geq \dim(\mathcal{H}_4 \ominus \mathcal{L}_i) \geq \dim \mathcal{H} = \dim \mathcal{H}_4 \text{ for } i = 1, 2.$$

So they have the same dimension. This completes the proof that $v$ can be so defined that $V_1$ and $V_2$ commute. ■

THEOREM 2.3. *Let $V_1$ and $V_2$ be two commuting isometries on a Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ and two commuting unitaries $U_1$ and $U_2$ on $\mathcal{K}$ such that*

$$U_1h = V_1h \textit{ and } U_2h = V_2h \textit{ for all } h \in \mathcal{H}.$$

*In other words, two commuting isometries can be extended to two commuting unitaries.*

We remark here that simple modifications of the proof of this theorem yield that the same is true for any number (finite or infinite) of commuting isometries.
PROOF: Using Corollary 1.8, we first find a unitary extension $U_1$ on a Hilbert space $\tilde{\mathcal{H}}$ of the isometry $V_1$. Without loss of generality we may assume that this extension is *minimal*, i.e.,

$$\tilde{\mathcal{H}} = \overline{\text{span}}\{U_1^n h : n \in \mathbb{Z} \text{ and } h \in \mathcal{H}\}.$$

We want to define an isometric extension $\tilde{V}_2$ on $\tilde{\mathcal{H}}$ of $V_2$ which

(1) would commute with $U_1$,
(2) would be a unitary on $\tilde{\mathcal{H}}$ if $V_2$ already is a unitary.

Assume for a moment that this has been accomplished, i.e., we have found a $\tilde{V}_2$ satisfying the above conditions. Then if $V_2$ happens to be unitary, we are done. If not, then just repeat the construction by applying Corollary 1.8 again, this time to the isometry $\tilde{V}_2$ instead of $V_1$. Since $U_1$ is already a unitary, the resulting extensions that we shall get will be commuting unitaries.

Now we get down to finding a $\tilde{V}_2$ satisfying (1) and (2). To that end, note that

$$\begin{aligned}
\|\sum_{n=-\infty}^{\infty} U_1^n V_2 h_n\|^2 &= \sum_{n,m} \langle U_1^n V_2 h_n, U_1^m V_2 h_m\rangle \\
&= \sum_{n\geq m} \langle U_1^{n-m} V_2 h_n, V_2 h_m\rangle + \sum_{n<m} \langle V_2 h_n, U_1^{m-n} V_2 h_m\rangle \\
&= \sum_{n\geq m} \langle V_1^{n-m} V_2 h_n, V_2 h_m\rangle + \sum_{n<m} \langle V_2 h_n, V_1^{m-n} V_2 h_m\rangle \text{ since } U_1 \text{ extends } V_1 \\
&= \sum_{n\geq m} \langle V_2 V_1^{n-m} h_n, V_2 h_m\rangle + \sum_{n<m} \langle V_2 h_n, V_2 V_1^{m-n} h_m\rangle \text{ since } V_2 \text{ and } V_1 \text{ commute} \\
&= \sum_{n\geq m} \langle V_1^{n-m} h_n, h_m\rangle + \sum_{n<m} \langle h_n, V_1^{m-n} h_m\rangle \text{ since } V_2 \text{ is an isometry} \\
&= \|\sum_{n=-\infty}^{\infty} U_1^n h_n\|^2 \text{ tracing the steps back with } V_2 = \mathbf{1}_{\mathcal{H}}.
\end{aligned}$$

Thus on the dense subspace span$\{U_1^n h : n \in \mathbb{Z}, h \in \mathcal{H}\}$, one can unambiguously define an isometry $\tilde{V}_2$ by

$$\tilde{V}_2 \left( \sum_{n=-\infty}^{\infty} U_1^n h_n \right) = \sum_{n=-\infty}^{\infty} U_1^n V_2 h_n.$$

Of course, $\tilde{V}_2$ extends $V_2$ and commutes with $U_1$. Finally if $V_2$ is a unitary, i.e., a surjective isometry, then it follows from the definition of $\tilde{V}_2$ that it has a dense range. An isometry with a dense range is surjective and hence unitary. We have finished the proof. ■

This theorem of course immediately produces the unitary dilation theorem for two commuting contractions:

THEOREM 2.4. *Given two commuting contractions $T_1$ and $T_2$ on a Hilbert space $\mathcal{H}$, there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and two commuting unitaries $U_1$ and $U_2$ on $\mathcal{K}$ such that*

$$T_1^m T_2^n h = P_{\mathcal{H}} U_1^m U_2^n h, \quad \text{for all } h \in \mathcal{H} \text{ and } m, n = 0, 1, 2, \ldots. \tag{2.2}$$

PROOF First get a commuting isometric dilation $\underline{V} = (V_1, V_2)$ on a space $\mathcal{H}_+$ by Theorem 2.2. Then get a commuting unitary extension $\underline{U} = (U_1, U_2)$ of $\underline{V}$ by Theorem 2.4. Now $\underline{U}$ is the required dilation because $U_1$ and $U_2$ commute and for any $h \in \mathcal{H}$ and $m, n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
P_{\mathcal{H}} U_1^m U_2^n h &= P_{\mathcal{H}} V_1^m V_2^n h \text{ because } \underline{U} \text{ extends } \underline{V} \text{ and } h \in \mathcal{H} \subset \mathcal{H}_+ \\
&= T_1^m T_2^n h \text{ because } \underline{V} \text{ dilates } \underline{T}.
\end{aligned}$$

■

The simultaneous unitary dilation theorem immediately produces a von Neumann's inequality.

COROLLARY 2.5. (**von Neumann's inequality**)*: Let $T_1$ and $T_2$ be two commuting contractions acting on a Hilbert space $\mathcal{H}$. Suppose $p(z_1, z_2)$ is any polynomial in two variables. Then*

$$\|p(T_1, T_2)\| \leq \sup\{|p(z_1, z_2)| : |z_1| \leq 1, |z_2| \leq 1\}.$$

PROOF Let $\mathcal{K} \supset \mathcal{H}$ be a unitary dilation space for $\underline{T}$ and $U_1$ and $U_2$ be two commuting unitaries on $\mathcal{K}$ as obtained from Theorem 2.4. Since $U_1$ and $U_2$ are commuting unitaries, all four of $U_1, U_2, U_1^*$

and $U_2^*$ commmute. Thus the $C^*$-algebra $\mathcal{C}$ generated by $U_1$ and $U_2$ is commutative. So by Gelfand theory ([19], Chapter I), $\mathcal{C}$ is isometrically $*$-isomorphic to $C(\mathcal{M}_\mathcal{C})$ where $\mathcal{M}_\mathcal{C}$ is the set of all multiplicative linear functionals $\chi$ on $\mathcal{C}$. Such functionals satisfy $\|\chi\| = \chi(1) = 1$.

Thus $|\chi(U_i)|^2 = \chi(U_i)\chi(U_i^*) = \chi(U_iU_i^*) = \chi(\mathbf{1}_\mathcal{K}) = 1$. So for any polynomial $p(z_1, z_2)$, we have

$$\begin{aligned} \|p(U_1,U_2)\| &= \sup_{\chi\in\mathcal{M}_\mathcal{C}} |\chi(p(U_1,U_2)| \\ &= \sup_{\chi\in\mathcal{M}_\mathcal{C}} |p(\chi(U_1),\chi(U_2))| \text{ because } \chi \text{ is multiplicative and linear} \\ &\leq \sup_{|z_1|=1,|z_2|=1} |p(z_1,z_2)|. \end{aligned}$$

Hence $\|p(T_1, T_2)\| = \|P_\mathcal{H} p(U_1, U_2)|_\mathcal{H}\| \leq \|p(U_1, U_2)\| \leq \sup_{|z_1|=1,|z_2|=1} |p(z_1, z_2)|$. ■

The simultaneous unitary dilation of a pair of contractions is due to Ando [2] and the proofs given here are essentially the same as his original ones. We shall end this section with the rather striking fact that Ando's theorem does not generalize to more than two commuting contractions. The unitary extension theorem of isometries holds good, as remarked above, for any number of commuting isometries. It is the isometric dilation of contractions which fails for more than a pair of contractions.

Perhaps the easiest way to see it is to construct a triple of commuting contractions which do not have a commuting unitary dilation. To that end, let $\mathcal{L}$ be a Hilbert space and let $A_1, A_2, A_3$ be three unitary operators on $\mathcal{L}$ such that

$$A_1A_2^{-1}A_3 \neq A_3A_2^{-1}A_1.$$

(For example, $A_2 = \mathbf{1}$ and $A_1, A_3$ two non-commuting unitaries will do.) Let $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$ and let $T_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, 3$ be defined as

$$T_i(h_1, h_2) = (0, A_ih_1) \text{ where } h_1, h_2 \in \mathcal{L}.$$

Clearly $\|T_i\| = \|A_i\| = 1$ for $i = 1, 2, 3$ and $T_iT_j = T_jT_i = 0$ for $i, j = 1, 2, 3$. SO $(T_1, T_2, T_3)$ is a commuting triple of contractions on $\mathcal{H}$. Suppose there exist commuting unitary operators $U_1, U_2, U_3$ on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $T_i = P_\mathcal{H} U_i|_\mathcal{H}$ for $i = 1, 2, 3$. Then

$$P_\mathcal{H} U_i(h, 0) = T_i(h, 0) = (0, A_ih), h \in \mathcal{H} \;\; i = 1, 2, 3.$$

Note that $\|U_i(h, 0)\| = \|h\|$ and $\|(0, A_ih)\| = \|A_ih\| = \|h\|$. So $U_i(h, 0) = (0, A_ih)$. Hence $U_kU_j^{-1}U_i(h, 0) = U_kU_j^{-1}(0, A_ih) = U_kU_j^{-1}(0, A_j(A_j^{-1}A_ih) = U_kU_j^{-1}U_j(0, A_j^{-1}A_ih) = U_k(0, A_j^{-1}A_ih) = (0, A_kA_j^{-1}A_ih)$. Since the $U_i$ commute, $U_kU_j^{-1}U_i = U_iU_j^{-1}U_k$ for all $i, j = 1, 2, 3$. So $A_kA_j^{-1}A_i = A_iA_j^{-1}A_k$ for all $i, j = 1, 2, 3$. That is a contradiction. So there is no commuting dilation.

Note that von Nemann's inequality is an immediate corollary of unitary dilation. So one way to show non-existence of dilation is to show that von Neumann's inequality is violated. This is what Crabb and Davie did with a triple of operators acting on an eight-dimensional space [18]. The literature over the years is full of a lot of discussions and considerations of many aspects of the issue originating from this spectacular failure of von Neumann's inequality. The survey article of Drury [26] is very insightful, so is the monograph by Pisier [39]. The reader is referred to Varopoulos [46] for his probabilistic arguments and establishing connection with Grothedieck's inequality, and Parrott [36] who gave an example which satisfies von Neumann's inequality but does not have a unitary dilation. Of more recent interest are the articles by Bagchi and Misra and the author [31], [12].

## 3. Standard Commuting Dilation

This section will show that a commuting contractive tuple has a commuting dilation which is canonical in a sense to be described below. We call it the standard commuting dilation. We shall begin with the relevant definitions along with a little bit of discussion on the free Fock space creation operators which will be used in the next section. Since an $n$-tuple of commuting cotractions does not in general admit a commuting dilation and does not satisfy von Neumann's inequality, one was led to consider a particular class of commuting contractive tuples. Throughout the rest of the paper, $n$ is a fixed positive integer larger than 1.

Definition 3.1. *Let $\mathcal{H}$ be a Hilbert space and let $\underline{T} = (T_1, \ldots, T_n)$ be a tuple of bounded operators acting on $\mathcal{H}$. Then $\underline{T}$ is called a contractive tuple if $\sum_{i=1}^n T_iT_i^* \leq \mathbf{1}_{\mathcal{H}}$. The tuple is called commuting if $T_iT_j = T_jT_i$ for all $i, j = 1, 2, \ldots, n$. The positive operator $(\mathbf{1}_{\mathcal{H}} - \sum_{i=1}^n T_iT_i^*)^{1/2}$ and the closure of its range are respectively called the* Defect operator *of $\underline{T}$ and the* Defect space *of $\underline{T}$ and are denoted by $D_{\underline{T}}$ and $\mathcal{D}_{\underline{T}}$.*

Contractivity of a tuple is equivalent to demanding that for all $h_1, h_2, \ldots, h_n \in \mathcal{H}$,

$$\|T_1h_1 + T_2h_2 + \cdots + T_nh_n\|^2 \leq \|h_1\|^2 + \|h_2\|^2 \cdots + \|h_n\|^2.$$

A prototype of a commuting contractive tuple is the so-called $n$-shift which we shall simpy call the *shift* since $n$ is fixed. This operator tuple will play a central role in the theory of dilation of a commuting contractive tuple, much like that of the unilateral shift in the case of a single contraction. Before we formally define it, we shall briefly sketch here the route that we are going to follow for finding the dilation and how we are going to use this special tuple for our purpose. Suppose we have a single linear contraction $T$ on a Hilbert space. Consider the usual Toeplitz algebra $\mathcal{T}$ (See [6]), i.e., the unital $C^*$-algebra generated by the unilateral shift $S$. Then there is a unique unital completely positive map $\varphi$ on $\mathcal{T}$ which maps $S$ to $T$ and moreover any 'sesqui-polynomial' $\sum a_{k,l}S^k(S^*)^l$ to $\sum a_{k,l}T^k(T^*)^l$ (Keeping powers of $S^*$, $T^*$ only on the right is important). Actually this is a way of looking at Sz. Nagy dilation of contractions. Indeed if we consider the minimal Stinespring representation $\pi$ of $\varphi$, we see that $\pi(S)$ is nothing but the minimal isometric dilation of $T$. We shall imitate this path for finding dilation for a commuting contractive tuple. See Paulsen [37] for discussions on completely positive maps, Stinespring dilation and related material.

Given a Hilbert space $\mathcal{L}$ and $k = 0, 1, 2, \ldots$, we write $\mathcal{L}^{\circledS k}$ for the symmetric tensor product of $k$ copies of $\mathcal{L}$. The space $\mathcal{L}^{\circledS 0}$ is defined as the one dimensional vector space $\mathbb{C}$ with its usual inner product. For $k \geq 2$, $\mathcal{L}^{\circledS k}$ is the subspace of the full tensor product $\mathcal{L}^{\otimes k}$ consisting of all vectors fixed under the natural representation of the permutation group $\sigma_k$,

$$\mathcal{L}^{\circledS k} = \{\xi \in \mathcal{L}^{\otimes k} : U_\pi \xi = \xi, \quad \pi \in \sigma_k\},$$

$U_\pi$ denoting the isomorphism of $\mathcal{L}^{\otimes k}$ defined on elementary tensors by

$$U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\pi^{-1}(1)} \otimes x_{\pi^{-1}(2)} \otimes \cdots \otimes x_{\pi^{-1}(n)}, \qquad x_i \in \mathcal{L}. \tag{3.1}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Then an orthonormal basis for the full tensor product space $\mathcal{L}^{\otimes k}$ is $\{e_{i_1} \otimes \cdots \otimes e_{i_k} : 1 \leq i_1, \ldots, i_k \leq n\}$. The full Fock space over $\mathcal{L}$ and the symmetric Fock space over $\mathcal{L}$ are respectively

$$\Gamma(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \cdots \oplus \mathcal{L}^{\otimes k} \oplus \cdots$$

and

$$\Gamma_s(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus \mathcal{L}^{\circledS 2} \oplus \cdots \oplus \mathcal{L}^{\circledS k} \oplus \cdots .$$

In both the Fock spaces, the one dimensional subspace $\mathbb{C} \oplus \{0\} \oplus \{0\} \oplus \cdots$ is called the vacuum space. The unit norm element $(1, 0, 0, \ldots)$ in this space is called the vacuum vector and is denoted by $\omega$. The projection on to the vaccum space is denote by $E_0$. Define the creation operator tuple $\underline{V} = (V_1, V_2, \ldots, V_n)$ on $\Gamma(\mathbb{C}^n)$ by

$$V_i\xi = e_i \otimes \xi \text{ for } i = 1, 2, \ldots, n \text{ and } \xi \in \Gamma(\mathbb{C}^n).$$

It is easy to see that the $V_i$ are isometries with orthogonal ranges. Denoting by $P_+$ the orthogonal projection onto the subspace $\Gamma_s(\mathcal{L})$ of $\Gamma(\mathcal{L})$, define the tuple of bounded operators $\underline{S} = (S_1, S_2, \ldots, S_n)$ on $\Gamma_s(\mathcal{L})$ by

$$S_i\xi = P_+(e_i \otimes \xi) \text{ for } i = 1, 2, \ldots, n \text{ and } \xi \in \Gamma(\mathbb{C}^n).$$

Since $V_i$ are isometries, the $S_i$ are contractions. The projection $P_+$ acts on the full tensor product space $\mathcal{L}^{\otimes k}$ by the following action on the orthonormal basis:

$$P_+(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = \frac{1}{k!} \sum e_{\pi(i_1)} \otimes e_{\pi(i_2)} \otimes \cdots \otimes e_{\pi(i_k)}$$

where $\pi$ varies over the permutation group $\sigma_k$. Using this it is easy to see that $\underline{S}$ forms a commuting tuple. *The operator tuple $\underline{S}$ is called the commuting $n$-shift.* For contractivity of $\underline{S}$, we start with the following lemma.

Lemma 3.2. *$\mathbf{1}_{\Gamma(\mathbb{C}^n)} - \sum V_i V_i^*$ is the one-dimensional projection onto the vacuum space.*

Proof: We compute the action of $V_i^*$ on the orthonormal basis elements:

$$\begin{aligned} \langle V_i^*(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}), \xi\rangle &= \langle e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, V_i\xi\rangle \\ &= \langle e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, e_i \otimes \xi\rangle \\ &= \begin{cases} \langle e_{i_2} \otimes \cdots \otimes e_{i_n}, \xi\rangle \text{ if } i_1 = i \\ 0 \text{ if } i_1 \neq i. \end{cases} \end{aligned}$$

Thus

$$V_i^*(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = \begin{cases} e_{i_2} \otimes \cdots \otimes e_{i_k} \text{ if } i_1 = i \\ 0 \text{ if } i_1 \neq i. \end{cases} \tag{3.2}$$

Hence it follows that $\sum V_i V_i^*(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ for any $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$.

Now

$$\langle V_i^*\omega, \xi\rangle = \langle \omega, e_i \otimes \xi\rangle = 0 \text{ for any } \xi \in \Gamma(\mathbb{C}^n) \text{ and any } i.$$

Thus $\sum V_i V_i^*(\omega) = 0$ and hence $\mathbf{1}_{\Gamma(\mathbb{C}^n)} - \sum V_i V_i^*$ is the one-dimensional projection onto the vacuum space. ∎

This lemma immediately gives the contractivity property for $\underline{V}$ and $\underline{S}$:

Corollary 3.3. *$\underline{V}$ and $\underline{S}$ are contractive tuples.*

Proof: We have seen in the lemma above that $\mathbf{1}_{\Gamma(\mathbb{C}^n)} - \sum V_i V_i^*$ is a projection and hence a positive operator. Thus $\sum V_i V_i^* \leq \mathbf{1}_{\Gamma(\mathbb{C}^n)}$.

A moment's thought shows that $\Gamma_s(\mathbb{C}^n)$ is a co-invariant subspace for the operator tuple $\underline{V}$, i.e., $V_i^*$ leaves $\Gamma_s(\mathbb{C}^n)$ invariant for each $i = 1, 2, \ldots, n$. Thus $\sum S_i S_i^* = \sum (P_+ \sum V_i P_+)(P_+ V_i^* P_+) = P_+ \sum V_i V_i^* P_+ \leq \mathbf{1}_{\Gamma_s(\mathbb{C}^n)}$. ∎

Before we proceed further, it will be helpful to list some properties of the shift which will be needed later.

LEMMA 3.4. *$\sum_{i=1}^n S_i^* S_i$ is an invertible operator on $\Gamma_s(\mathbb{C}^n)$.*

PROOF:

We shall show that $\sum S_i^* S_i$ is a digonal operator in a natural basis for $\Gamma_s(\mathbb{C}^n)$. Any ordered $n$-tuple of non-negative integers $\underline{k} = (k_1, k_2, \ldots, k_n)$ will be called a multi-index. Let $|\underline{k}| = k_1 + k_2 + \cdots + k_n$. Given such a multi-index $\underline{k}$ with $|\underline{k}| \geq 1$, we shall, for brevity, write

$$\underline{e}^{\underline{k}} = U_\pi (e_1^{\otimes k_1} \otimes e_2^{\otimes k_2} \otimes \cdots \otimes e_n^{\otimes k_n}).$$

Clearly, the set $\{\omega\} \cup \{\underline{e}^{\underline{k}} : \underline{k}$ is a multi-index with $|\underline{k}| \geq 1\}$ forms a basis for the space $\Gamma_s(\mathbb{C}^n)$. For any multi-index $\underline{k}$ with $|\underline{k}| \geq 1$, we have

$$S_i^* \underline{e}^{\underline{k}} = V_i^* \underline{e}^{\underline{k}} = \begin{cases} 0 \text{ if } k_i = 0 \\ \frac{k_i}{|\underline{k}|} \underline{e}^{\underline{k}'} \text{ otherwise,} \end{cases}$$

the last step following from (3.1) and (3.2), here $\underline{k}'$ is the multi-index $(k_1, \ldots k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n)$. Thus we have a formula for the action of $S_i^*$ on the basis. (This immediately gives another proof of the fact that $\sum S_i S_i^* = \mathbf{1}_{\Gamma_s(\mathbb{C}^n)} - E_0$ because $\sum S_i S_i^*(\omega) = 0$ and for any $\underline{k}$ with $|\underline{k}| \geq 1$, $\sum S_i S_i^*(\underline{e}^{\underline{k}}) = (|\underline{k}|)^{-1} \sum k_i (\underline{e}^{\underline{k}}) = \underline{e}^{\underline{k}}$.) Now $\sum S_i^* S_i \omega = n\omega$ and for $|\underline{k}| \geq 1$,

$$\begin{aligned} \sum S_i^* S_i (\underline{e}^{\underline{k}}) &= \sum S_i^* (\underline{e}^{\underline{k}''}) \text{ where } \underline{k}'' \text{ is the multi-index } (k_1, \ldots k_{i-1}, k_i + 1, k_{i+1}, \ldots k_n) \\ &= \sum \frac{k_i + 1}{|\underline{k}| + 1} \underline{e}^{\underline{k}} \\ &= \frac{|\underline{k}| + n}{|\underline{k}| + 1} \underline{e}^{\underline{k}}. \end{aligned}$$

Thus $\sum S_i^* S_i$ is a diagonal operator in the basis mentioned above. Since none of the diagonal coefficients is zero and the sequence of diagonal coeffcients tends to $n$ as $|\underline{k}| \to \infty$, the operator is invertible. ∎

The $C^*$-subalgebra of $\mathcal{B}(\Gamma_s(\mathbb{C}^n))$ generated by $S_1, \ldots, S_n$ will be denoted by $\mathcal{T}_n$ and called the Toeplitz $C^*$-algebra. The Toeplitz $C^*$-algebra is unital. We do not have to, a priori, include $\mathbf{1}_{\Gamma_s(\mathbb{C}^n)}$ in the $C^*$-algebra $\mathcal{T}_n$ because the operator $\sum S_i^* S_i$ is invertible in $\mathcal{B}(\Gamma_s(\mathbb{C}^n))$. Since $C^*$-algebras are inverse closed, $(\sum S_i^* S_i)^{-1}$ is in the $C^*$-algebra generated by $S_1, \ldots, S_n$ and hence $\mathcal{T}_n$ is unital. The subalgebra of $\mathcal{T}_n$ consisting of polynomials in $S_1, \ldots, S_n$ and $\mathbf{1}_{\Gamma_s(\mathbb{C}^n)}$ will be denoted by $\mathcal{A}$. The following two lemmas give more information about the $C^*$-algebra $\mathcal{T}_n$.

LEMMA 3.5. *$\mathcal{T}_n = \overline{span} \mathcal{A}\mathcal{A}^*$.*

PROOF A direct computation yields

$$S_i^* S_j \underline{e}^{\underline{k}} = \frac{|\underline{k}|}{|\underline{k}| + 1} S_j S_i^* \underline{e}^{\underline{k}} \quad \text{for } 1 \leq i, j \leq n$$

and hence the result follows. ∎

LEMMA 3.6. *All compact operators are in $\mathcal{T}_n$..*

PROOF Since $\sum S_i S_i^* = \mathbf{1}_{\Gamma_s(\mathbb{C}^n)} - E_0$, the one-dimensional projection $E_0$ onto the vacuum space is in span$\mathcal{A}\mathcal{A}^*$. Now given any two multi-indices $\underline{k}$ and $\underline{l}$, the operator $\underline{S}^{\underline{k}} E_0 \underline{S}^{*\underline{l}}$ is in span$\mathcal{A}\mathcal{A}^*$. But note that this operator is nothing but the rank-one operator

$$\xi \to \langle \xi, \underline{e}^{\underline{l}} \rangle \underline{e}^{\underline{k}}.$$

As the set $\{\omega, \underline{e}^{\underline{k}} : \underline{k} \text{ is any multi-index }\}$ is a basis for $\Gamma_s(\mathbb{C}^n)$, all rank one operators are in $\overline{\text{span}}\mathcal{A}\mathcal{A}^*$. Thus $\overline{\text{span}}\mathcal{A}\mathcal{A}^*$ contains all finite rank operators and hence all compact operators. ■

We now proceed towards developing the model and dilation for a given commuting contractive tuple $\underline{T}$ on a Hilbert space $\mathcal{H}$. By an operator space we shall mean a vector subspace of $\mathcal{B}(\mathcal{L})$ where $\mathcal{L}$ is a Hilbert space. Given an operator space $\mathcal{E}$ and an algebra $\mathcal{A} \subseteq \mathcal{E}$, a completely positive map $\varphi$ from $\mathcal{E}$ to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is called an $\mathcal{A}$-morphism if

$$\varphi(AX) = \varphi(A)\varphi(X), \text{ for any } A \in \mathcal{A} \text{ and } X, AX \in \mathcal{E}.$$

This is related to the hereditary isomorphisms of Agler [1].

Every unital $\mathcal{A}$-morphism $\varphi : \mathcal{T}_n \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ gives rise to a commuting contractive tuple $(T_1, \dots, T_n)$ on $\mathcal{H}$ by way of $T_i = \varphi(S_i)$, $i = 1, \dots, n$. Indeed, $\sum T_i T_i^* = \varphi(\sum S_i S_i^*) \le \varphi(\mathbf{1}_{\Gamma_s(\mathbb{C}^n)}) = \mathbf{1}_{\mathcal{H}}$ and $T_i T_j = \varphi(S_i)\varphi(S_j) = \varphi(S_i S_j) = \varphi(S_j S_i) = \varphi(S_j)\varphi(S_i) = T_j T_i$ for all $1 \le i, j \le n$. Given any commuting contractive tuple $\underline{T}$ acting on $\mathcal{H}$, our aim is to produce an $\mathcal{A}$-morphism from the $C^*$-algebra $\mathcal{T}_n$ (with its subalgebra $\mathcal{A}$ as defined above) to $\mathcal{B}(\mathcal{H})$. This will be achieved by the help of the following crucial thorem. The $\mathcal{A}$-morphism is the key element in finding dilation and proving von Nemann inequaltiy. In fact, it could be thought of as the model for $\underline{T}$. We shall associate a completely positive map $P_{\underline{T}}$ with $\underline{T}$ which acts on $\mathcal{B}(\mathcal{H})$ by

$$P_{\underline{T}}(X) = \sum_{i=1}^{n} T_i X T_i^*.$$

Since $\underline{T}$ is a contractive tuple, the completely positive map $P_{\underline{T}}$ is contractive and hence $\mathbf{1}_{\mathcal{H}} \ge P_{\underline{T}}(\mathbf{1}_{\mathcal{H}}) \ge P_{\underline{T}}^2(\mathbf{1}_{\mathcal{H}}) \ge \cdots$. This decreasing sequence of positive contractions converges strongly and $A_\infty$ will denote the positive contraction which is the strong limit:

$$A_\infty = \lim_{m \to \infty} P_{\underline{T}}^m(\mathbf{1}_{\mathcal{H}}).$$

The commuting contractive tuple $\underline{T}$ will be called pure if $A_\infty = 0$.

THEOREM 3.7. *Let $(T_1, \dots, T_n)$ be a commuting contractive tuple of operators on a Hilbert space $\mathcal{H}$. Then there is a unique bounded operator $L : \Gamma_s(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}} \to \mathcal{H}$ satisfying $L(\omega \otimes \xi) = D_{\underline{T}}\xi$ and*

$$L(\underline{e}^{\underline{k}} \otimes \xi) = \underline{T}^{\underline{k}} D_{\underline{T}} \xi \tag{3.3}$$

*for every multi-index $\underline{k}$ with $|\underline{k}| = 1, 2, \dots$. In general $\|L\| \le 1$, and if $(T_1, \dots, T_n)$ is a pure tuple, then $L$ is a coisometry: $LL^* = \mathbf{1}_{\mathcal{H}}$.*

PROOF A bounded operator $L$ satisfying (3.3) is obviously unique because $\Gamma_s(\mathbb{C}^n)$ is spanned by the set of vectors

$$\{\omega, \underline{e}^{\underline{k}} : \underline{k} \text{ is a multi-index with } |\underline{k}| = 1, 2, \dots\}.$$

We define $L$ by exhibiting its adjoint, i.e., we shall exhibit a contraction $A : \mathcal{H} \to \Gamma_s(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}}$, define $L = A^*$ and then show that $L$ has the required properties.

For every $h \in \mathcal{H}$, let $Ah$ be the sequence of vectors $(\zeta_0, \zeta_1, \zeta_2, \dots)$ where $\zeta_k \in (\mathbb{C}^n)^{\otimes k} \otimes \mathcal{D}_{\underline{T}}$ is defined by

$$\zeta_k = \sum_{i_1,\dots,i_k=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes D_{\underline{T}} T_{i_k}^* \dots T_{i_1}^* h$$

for $k \geq 1$ and $\zeta_0 = \omega \otimes D_{\underline{T}} h$. Notice that since $T_1^*, \dots, T_n^*$ commute, $\zeta_k$ actually belongs to the symmetric subspace $(\mathbb{C}^n)^{\circledS k} \otimes \mathcal{D}_{\underline{T}}$ so that in fact $A$ maps into $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}}$. To show that $A$ is a contraction we claim that

$$\sum_{k=0}^{\infty} \|\zeta_k\|^2 \leq \|h\|^2.$$

Indeed,

$$\|\zeta_k\|^2 = \sum_{i_1,\dots,i_k=1}^{n} \|D_{\underline{T}} T_{i_k}^* \dots T_{i_1}^* h\|^2 = \sum_{i_1,\dots,i_k=1}^{n} \langle T_{i_1} \dots T_{i_k} D_{\underline{T}}^2 T_{i_k}^* \dots T_{i_1}^* h, h\rangle.$$

Noting that $D_{\underline{T}}^2 = \mathbf{1}_{\mathcal{H}} - P(\mathbf{1}_{\mathcal{H}})$ we find that

$$\sum_{i_1,\dots,i_k=1}^{n} T_{i_1} \dots T_{i_k} D_{\underline{T}}^2 T_{i_k}^* \dots T_{i_1}^* = P^k(\mathbf{1}_{\mathcal{H}} - P(\mathbf{1}_{\mathcal{H}})) = P^k(1_{\mathcal{H}}) - P^{k+1}(\mathbf{1}_{\mathcal{H}}),$$

and hence

$$\|\zeta_k\|^2 = \langle P^k(\mathbf{1}_{\mathcal{H}})h, h\rangle - \langle P^{k+1}(\mathbf{1}_{\mathcal{H}})h, h\rangle.$$

The series $\|\zeta_0\|^2 + \|\zeta_1\|^2 + \dots$ therefore telescopes and we are left with

$$\sum_{k=0}^{\infty} \|\zeta_k\|^2 = \|h\|^2 - \langle A_\infty h, h\rangle \leq \|h\|^2.$$

Now let $\zeta = \underline{e}^{\underline{k}} \otimes \xi$ for some multi-index $\underline{k}$ with $|\underline{k}| = m \geq 1$ and $\xi \in \mathcal{D}_{\underline{T}}$. Then for any $h \in \mathcal{H}$,

$$\begin{aligned}
\langle L(\zeta), h\rangle &= \langle \underline{e}^{\underline{k}} \otimes \xi, Ah\rangle \\
&= \sum_{l=1}^{\infty} \sum_{i_1,\dots,i_l=1}^{n} \langle \underline{e}^{\underline{k}} \otimes \xi, e_{i_1} \otimes \cdots \otimes e_{i_l} \otimes D_{\underline{T}} T_{i_l}^* \dots T_{i_1}^* h\rangle \\
&= \sum_{i_1,\dots,i_m=1}^{n} \langle U_\pi(e_1^{\otimes k_1} \otimes \cdots \otimes e_n^{\otimes k_n}, e_{i_1} \otimes \cdots \otimes e_{i_m}\rangle \langle \xi, D_{\underline{T}} T_{i_m}^* \dots T_{i_1}^* h\rangle \\
&= \langle \xi, D_{\underline{T}}(\underline{T}^{\underline{k}})^* h\rangle = \langle \underline{T}^{\underline{k}} D_{\underline{T}} \xi, h\rangle.
\end{aligned}$$

For $\zeta = \omega \otimes \xi$ with $\xi \in \mathcal{D}_{\underline{T}}$ we have

$$\langle \omega \otimes \xi, Ah\rangle = \langle \omega \otimes \xi, \omega \otimes D_{\underline{T}} h\rangle = \langle \xi, D_{\underline{T}} h\rangle = \langle D_{\underline{T}} \xi, h\rangle,$$

as required. If $(T_1, \dots, T_n)$ is pure, then $A_\infty = 0$. Thus $A$ is an isometry and hence $L$ is a coisometry. ∎

Given a commuting contractive tuple, the following theorem constructs an $\mathcal{A}$-morphism from the Toeplitz $C^*$-algebra into the unital $C^*$-algebra generated by $T_1, T_2, \dots, T_n$. Note that (3.3) implies that $L(\underline{S}^{\underline{k}} \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}}) = \underline{T}^{\underline{k}} L$.

THEOREM 3.8. *For every commuting contractive tuple $(T_1,\dots,T_n)$ acting on a Hilbert space $\mathcal{H}$ there is a unique unital $\mathcal{A}$-morphism*

$$\varphi : \mathcal{T}_n \to \mathcal{B}(\mathcal{H})$$

*such that $\varphi(S_i) = T_i$, $i = 1,\dots,n$.*

PROOF: The uniqueness assertion is immediate since an $\mathcal{A}$-morphism is uniquely determined on the closed linear span of the set of products $\{AB^* : A, B \in \mathcal{A}\}$.

For existence, first assume that the commuting contractive tuple $\underline{T} = (T_1, T_2 \dots, T_n)$ is pure. Recall that this means $P^k(\mathbf{1}_{\mathcal{H}})$ converges strongly to 0 as $k$ tends to $\infty$. We first construct an $\mathcal{A}$-morphism for such a tuple. The lemma above asserts that there is a unique bounded operator $L : \Gamma_s(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}} \to \mathcal{H}$ satisfying $L(\omega \otimes \xi) = D_{\underline{T}}\xi$ for $\xi \in \mathcal{D}_{\underline{T}}$ and

$$L(\underline{e}^{\underline{k}} \otimes \xi) = \underline{T}^{\underline{k}} D_{\underline{T}}\xi,$$

for $|\underline{k}| = 1, 2, \dots$, $\xi \in \mathcal{D}_{\underline{T}}$; moreover, since $(T_1,\dots,T_n)$ is a pure contractive tuple, $L$ is a coisometry.

Let $\varphi : \mathcal{T}_n \to \mathcal{B}(\mathcal{H})$ be the completely positive map

$$\varphi(X) = L(X \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})L^*, \ X \in \mathcal{T}_n.$$

The map $\varphi$ is unital because $L$ is a co-isometry. (3.3) implies that for every $X \in \mathcal{T}_n$ we have $\varphi(\underline{S}^{\underline{k}}X) = L(\underline{S}^{\underline{k}}X \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})L^* = L(\underline{S}^{\underline{k}} \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})(X \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})L^* = \underline{T}^{\underline{k}} L(X \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})L^* = \underline{T}^{\underline{k}}\varphi(X)$. So for any polynomial $f$,

$$\varphi(f(\underline{S})X) = f(\underline{T})\varphi(X)$$

hence $\varphi$ is an $\mathcal{A}$-morphism having the required properties.

The general case is deduced from this by a classical device. For any commuting contractive tuple $\underline{T} = (T_1,\dots,T_n)$ define $\underline{T}_r$ for $0 < r < 1$ to be

$$\bar{T}_r = (rT_1,\dots,rT_n).$$

It is clear that $\underline{T}_r$ is a pure commuting contractive tuple. Thus there is an $\mathcal{A}$-morphism $\varphi_r : \mathcal{T}_n \to \mathcal{B}(\mathcal{H})$ satisfying

$$\varphi_r(S_i) = rT_i, \qquad i = 1,\dots,d.$$

We have

$$\varphi_r((S_{i_1},\dots,S_{i_k})(S_{j_1}^*,\dots,S_{j_l}^*)) = f((rT_{i_1},\dots,rT_{i_k})(rT_{j_1}^*,\dots,rT_{j_l}^*))$$

for $k, l \geq 1$ and $1 \leq i_1,\dots,i_k, j_1,\dots,j_l \leq n$. Since operators of the form $(S_{i_1},\dots,S_{i_k})(S_{j_1}^*,\dots,S_{j_l}^*)$ span $\mathcal{T}_n$ and since the family of maps $\varphi_r$, $0 < r < 1$ is uniformly bounded, it follows that $\varphi_r$ converges point-norm to an $\mathcal{A}$-morphism $\varphi$ as $r \uparrow 1$, and $\varphi(S_i) = T_i$ for all $i$. ∎

COROLLARY 3.9. **von Neumann's inequality:** *Let $\underline{T} = (T_1,\dots,T_n)$ be any commuting contractive tuple acting on a Hilbert space $\mathcal{H}$ and $\underline{S} = (S_1,\dots,S_n)$ be the shift. Then for any polynomial $f$ in $n$-variables,*

$$\|f(T_1,\dots,T_n)\| \leq \|f(S_1,\dots,S_n)\|.$$

PROOF: Making use of the unital completely positive map $\varphi$ of the last theorem which maps $f(S_1,\dots,S_n)$ to $f(T_1,\dots,T_n)$, we have

$$\|f(T_1,\dots T_n)\| = \|\varphi(f(S_1,\dots,S_n))\| \leq \|\varphi\|\|f(S_1,\dots,S_n)\| = \|f(S_1,\dots,S_n)\|.$$

∎

The above theorems lead us to the following dilation theorem for any commuting contractive tuple $\underline{T}$ acting on some Hilbert space $\mathcal{H}$. We need some notation. If $m$ is a positive integer or $\infty$ and $\mathcal{M}$ is a Hilbert space of dimension $m$, we shall mean by $m \cdot \underline{S}$, the operator tuple $(S_1 \otimes \mathbf{1}_{\mathcal{M}}, \dots, S_n \otimes \mathbf{1}_{\mathcal{M}})$ acting on $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}$. In the next theorem, we are going to express $\underline{T}$ as a compression of a direct sum one of whose components might be absent. To assimilate this in a single notation, we make the convention that $m \cdot \underline{S}$ is absent if $m = 0$.

Given a Hilbert space $\mathcal{N}$, and a representation $\beta$ of $\mathcal{T}_n$ on $\mathcal{N}$, the operator tuple

$$\underline{A} \stackrel{\text{def}}{=} m \cdot \underline{S} \oplus \beta(\underline{S})$$

is clearly a commuting contractive tuple on $\hat{\mathcal{H}} \stackrel{\text{def}}{=} (\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}) \oplus \mathcal{N}$. Let $\mathcal{H}$ be a subspace of $\hat{\mathcal{H}}$ such that $A_i^* \mathcal{H} \subseteq \mathcal{H}$ for all $i = 1, \dots, n$. Reacall that such subspaces are called *co-invariant* with respect to the tuple $\underline{A}$. Consider the compression $\underline{T}$ of $\underline{A}$ to $\mathcal{H}$ as follows.

$$T_i \stackrel{\text{def}}{=} P_{\mathcal{K}} A_i|_{\mathcal{H}}.$$

This $\underline{T}$ is clearly a commuting contractive tuple on $\mathcal{H}$ and moreover, for any polynomial $f(z_1, \dots, z_n)$, $f(\underline{T})$ is the compression of $f(\underline{A})$ due to the co-invariance of $\mathcal{H}$ with respect to $\underline{A}$. We prove that every commuting contractive tuple has such a realization with $\beta$ sending all compact operators to zero.

THEOREM 3.10. **Dilation:** *Let $\underline{T}$ be any commuting contractive tuple acting on a separable Hilbert space $\mathcal{H}$ and rank$D_{\underline{T}} = m$ (which is a non-negative integer or $\infty$). Then there is a separable Hilbert space $\mathcal{M}$ of dimension $m$, another separable Hilbert space $\mathcal{N}$ with a commuting tuple of operators $\underline{Z} = (Z_1, \dots, Z_n)$ acting on it, satisfying $Z_1 Z_1^* + \cdots + Z_n Z_n^* = \mathbf{1}_{\mathcal{N}}$ such that:*

*(a) $\mathcal{H}$ is contained in $\hat{\mathcal{H}} \stackrel{\text{def}}{=} (\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}) \oplus \mathcal{N}$ as a subspace and it is co-invariant under $\underline{A} \stackrel{\text{def}}{=} m \cdot \underline{S} \oplus \underline{Z}$.*

*(b) $\underline{T}$ is the compression of $\underline{A}$ to $\mathcal{H}$, that is, $\underline{T}^{\underline{k}} = P_{\mathcal{H}} \underline{A}^{\underline{k}}|_{\mathcal{H}}$ for every multi-index $\underline{k}$.*

*(c) $\hat{\mathcal{H}} = \overline{span}\{\underline{A}^{\underline{k}} h : h \in \mathcal{H}$ and $\underline{k}$ is any multi-index$\}$.*

*Thus any commuting contractive tuple has a minimal commuting dilation. Moreover, this dilation is unique up to unitary equivalence.*

PROOF We consider the minimal Stinespring dilation of $\varphi$. Thus we get a Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$ and a representation $\pi$ of $\mathcal{T}_n$ on $\hat{\mathcal{H}}$ such that

$$\varphi(X) = P_{\mathcal{H}} \pi(X) P_{\mathcal{H}} \text{ for } X \in \mathcal{T}_n,$$

where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$ (We are identifying any operator $Z \in \mathcal{B}(\mathcal{H})$ with $P_{\mathcal{H}} Z P_{\mathcal{H}} \in \mathcal{B}(\hat{\mathcal{H}})$). So

$$\hat{\mathcal{H}} = \overline{\text{span}}\, \{\pi(X)u : X \in \mathcal{T}_n \text{ and } u \in \mathcal{H}\}.$$

The $C^*$-algebra $\mathcal{T}_n$ is separable and hence the Hilbert space $\hat{\mathcal{H}}$ is also separable. The tuple $(\pi(S_1), \dots, \pi(S_n))$ is a dilation of $(T_1, \dots, T_n)$ in the sense that for any polynomial $f$,

$$f(T_1, \dots, T_n) = P_{\mathcal{H}} f(\pi(S_1), \dots, \pi(S_n))|_{\mathcal{H}},$$

and $\mathcal{H}$ is a co-invariant subspace for $(\pi(S_1), \dots, \pi(S_n))$.

Let us denote the set of all compact operators on $\Gamma_s(\mathbb{C}^n)$ by $\mathcal{B}_0(\Gamma_s(\mathbb{C}^n))$ (or just $\mathcal{B}_0$ when there is no chance of confusion). Since $\mathcal{T}_n$ contains $\mathcal{B}_0$, by standard theory of representations of $C^*$-algebras (see [19], Chapter I for example), the representation $\pi$ decomposes as $\pi = \pi_0 \oplus \pi_1$, where $\pi_i : \mathcal{T}_n \to \mathcal{B}(\hat{\mathcal{H}}_i)$ with $\pi_0$ being a non-degenerate representation of $\mathcal{B}_0$ on $\hat{\mathcal{H}}_0$, $\pi_1$ being 0 on $\mathcal{B}_0$ and $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ (one of $\pi_0$ and $\pi_1$ could be absent too). Since the only non-degenerate representation of the $C^*$-algebra of compact operators is the identity representation with some multiplicity and since a represenation which is non-degenerate on an ideal, extends uniquely to the entire $C^*$-algebra, it follows that $\pi_0$ is just the identity representation with some multiplicity i.e., up to unitary isomorphism, $\hat{\mathcal{H}}_0 = \Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}$ and $\pi_0(X) = X \otimes \mathbf{1}_{\mathcal{M}}$ for some Hilbert space $\mathcal{M}$. So if we take $\mathcal{N} = \hat{\mathcal{H}}_1$, and $\pi_1(S_i) = Z_i$ then $(Z_1, \dots, Z_n)$ is a commuting contractive tuple and *(a), (b)* are satisfied. Moreover $\sum Z_i Z_i^* = \mathbf{1}_{\mathcal{N}}$ as $\pi_1$ kills compact operators and $\mathbf{1}_{\Gamma_s(\mathbb{C}^n)} - \sum S_i S_i^*$ is compact.

It remains to prove that the multiplicity i.e., dim $(\mathcal{M})$ is just the rank of $D_{\underline{T}}$. For this, note that dim $\mathcal{M} = \dim($ range $\pi_0(E))$ where $E$ is any one-dimensional projection in $\mathcal{T}_n$. Taking $E = E_0$, the projection onto the vacuum space, and making use of minimality of Stinespring representation, we have

$$\text{range } \pi(E_0) = \{\pi(E_0)\xi : \xi \in \hat{\mathcal{H}}\} = \overline{\text{span}}\{\pi(E_0)\pi(X)u : X \in \mathcal{T}_n, u \in \mathcal{H}\}.$$

Then by Lemma 3.6 and its proof,

$$\begin{aligned}
\text{range } \pi(E_0) &= \overline{\text{span}}\{\pi(E_0)\pi(E_0 X)u : X \in \mathcal{T}_n, u \in \mathcal{H}\} \\
&= \overline{\text{span}}\{\pi(E_0)\pi(X)u : X \in \mathcal{B}_0, u \in \mathcal{H}\} \\
&= \overline{\text{span}}\{\pi(E_0)\pi(\underline{S}^{\underline{k}} E_0 (\underline{S}^{\underline{l}})^*)u : \text{all multi-indices } \underline{k}, \underline{l}, \text{ and } u \in \mathcal{H}\} \\
&= \overline{\text{span}}\{\pi(E_0)\pi((\underline{S}^{\underline{l}})^*)u : \text{all multi-indices } \underline{k}, \underline{l}, \text{ and } u \in \mathcal{H}\}.
\end{aligned}$$

Now we define a unitary $U$ : range $\pi(E_0) \to \overline{\text{range}} D_{\underline{T}}$ by setting

$$U\pi(E_0)\pi((\underline{S}^{\underline{l}})^*)u = D_{\underline{T}}(\underline{T}^{\underline{l}})^* u$$

and extending linearly. Then $U$ is isometric because for $u, v \in \mathcal{H}$ and all $\underline{k}$ and $\underline{l}$,

$$\begin{aligned}
\langle \pi(E_0)\pi((\underline{S}^{\underline{k}})^*)u, \pi(E_0)\pi((\underline{S}^{\underline{l}})^*)v \rangle &= \langle u, \pi(\underline{S}^{\underline{k}})\pi(E_0)\pi((\underline{S}^{\underline{l}})^*)v \rangle \\
&= \langle u, \underline{T}^{\underline{k}} D_{\underline{T}}^2 (\underline{T}^{\underline{l}})^* v \rangle \\
&= \langle D_{\underline{T}}(\underline{T}^{\underline{k}})^* u, D_{\underline{T}}(\underline{T}^{\underline{l}})^* v \rangle.
\end{aligned}$$

Taking $\underline{l} = 0$, it is clear that $U$ is onto. This proves that $\overline{\text{range}} D_{\underline{T}}$ and $\mathcal{M}$ have the same dimensions. Now the uniqueness of the dilation follows from that of Stinespring dilation of a completely positive map. ■

As remarked before in the direct sum for $\hat{\mathcal{H}}$ and $\underline{A}$ appearing in this theorem one of the summands could be absent. $\mathcal{M}$ and $n \cdot \underline{S}$ are absent iff $n = 0$, that is, iff $\sum T_i T_i^* = \mathbf{1}_{\mathcal{H}}$. It can be shown that $\mathcal{N}$ and $\underline{Z}$ is absent if and only if $P_{\underline{T}}^m(\mathbf{1}_{\mathcal{H}})$ converges to zero strongly as $m$ tends to infinity where $P_{\underline{T}}$ is the completely positive map associated with $\underline{T}$.

It is known by the counter-examples of Parrott and Varopoulos that the dilation operator $\underline{A}$ could not be as nice as an isometry. Nevertheless, it is a commuting tuple and the final section will show us that it is the *best* that can be done retaining commutativity.

The ideas of this section are from Arveson [6], Drury [27] and Arias-Popescu [3], [4], [40], [41], [43], although we closely follow the methods of Arveson. In fact, the standard commuting dilation was looked at by Drury [27] in his study of von Neumann inequality for tuples and similar ideas have been explored by Agler [1], Athavale [11] and others for different classes of operators using various reproducing kernels. Popescu concentrated mainly on the non-commutative case more of which we shall see in the next section, many commutative results fall out as special cases from his theory. The crucial operator $L$ of Theorem 3.7 is actually the adjoint of the Poisson transform defined by Popescu. The proof of the dilation theorem here is due to B. V. R. Bhat and the author [15].

## 4. Standard Non-commuting Dilation

In this section, we concentrate on general contractive tuples, which are not necessarily commuting. The dilation will consist of isometries with orthogonal ranges. As in Section 3, we consider a model $n$-tuple $\underline{V} = (V_1, V_2, \dots, V_n)$. Recall from last section that $\underline{V}$ is the tuple of creation operators on the full Fock space over $\mathbb{C}^n$. Let $C^*(\underline{V})$ denote the $C^*$-subalgebra of $\mathcal{B}(\Gamma(\mathbb{C}^n))$ generated by $V_1, V_2, \dots, V_n$. We first note the elemantary observation:

Lemma 4.1. *$V_1, V_2, \dots, V_n$ are isometries with orthogonal ranges.*

Proof For any $\xi_1, \xi_2 \in \Gamma(\mathbb{C}^n)$, and any $i, j \in \{1, 2, \dots, n\}$, we have $\langle V_i\xi_1, V_j\xi_2\rangle = \langle e_i \otimes \xi, e_j \otimes \xi_2\rangle = \langle e_i, e_j\rangle\langle \xi_1, \xi_2\rangle = \delta_{ij}\langle \xi_1, \xi_2\rangle$, where $\delta_{ij}$ is Kronecker delta. ∎

Hence the $C^*$-algebra $C^*(\underline{V})$ is automatically unital. Let $\mathcal{A}$ be the algebra generated by $\mathbf{1}_{\Gamma(\mathbb{C}^n)}$ and $V_1, V_2, \dots, V_n$. This is called the *non-commutative disk algebra*, see Popescu [43]. This is the set of all operators of the form $f(\underline{V})$ where $f$ varies over all polynomials in $n$ non-commuting variables. There is a non-commutative Poisson transform and the $\mathcal{A}$-morphism which will be used for getting dilation. In this case too, the $C^*$-algebra is the same as the operator space $\overline{\text{span}}\mathcal{A}\mathcal{A}^*$ because $V_i^* V_j = \delta_{ij}$.

Lemma 4.2. *All compact operators on $\Gamma(\mathbb{C}^n)$ are in $C^*(\underline{V})$.*

Proof First $E_0 = \mathbf{1}_{\Gamma(\mathbb{C}^n)} - \sum V_i V_i^*$ is in $C^*(\underline{V})$. Now for any $k \geq 1, l \geq 1$, $1 \leq i_1, i_2, \dots, i_k \leq n$ and $1 \leq j_1, j_2, \dots, j_l \leq n$, we see that the operator $V_{i_1}V_{i_2}\dots V_{i_n}E_0(V_{j_1}V_{j_2}\dots V_{j_l})^*$ is the same as

$$\xi \to \langle \xi, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}\rangle e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_l}.$$

Since $\{\omega, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} : k \geq 1 \text{ and } 1 \leq i_1, i_2, \dots, i_k \leq n\}$ forms a basis for $\Gamma(\mathbb{C}^n)$, all rank one operators are in $C^*(\underline{V})$ and hence all compact operators are in $C^*(\underline{V})$. ∎

In the following we are going to describe the Poisson kernel and the Poisson transform associated with a contractive tuple following Popescu [43]. So take a contractive tuple $\underline{T} = (T_1, T_2, \dots, T_n)$ of operators acting on $\mathcal{H}$. The *Poisson kernel* $\{K_r(\underline{T})\}_{0 \leq r \leq 1}$ associated with $\underline{T}$ is the family of bounded operators $K_r(\underline{T}) : \mathcal{H} \to \Gamma(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}}$ defined by $K_r(\underline{T})h = (\zeta_0, \zeta_1, \zeta_2, \dots)$ where $\zeta_0 = \omega \otimes D_{\underline{T}}h$ and for $k \geq 1$,

$$\zeta_k = r^k \sum_{i_1, \dots, i_k = 1}^{n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes D_r(\underline{T})(T_{i_1}T_{i_2}\dots T_{i_k})^* h,$$

where $D_r(\underline{T})$ is the defect operator $(\mathbf{1}_{\mathcal{H}} - \sum r^2 T_i T_i^*)^{1/2}$. Since $(rT_1, \dots, rT_n)$ is pure for $0 < r < 1$, it is easy to check that $K_r(\underline{T})$ is an isometry for $0 < r < 1$ and $K(\underline{T}) \stackrel{\text{def}}{=} K_1(\underline{T})$ is a contraction with $K(\underline{T})^* K(\underline{T}) = \mathbf{1}_{\mathcal{H}} - A_\infty$ where $A_\infty$ is the operator defined in Section 3. Thus $K(\underline{T})$ is an isometry if and only if $\underline{T}$ is pure.

The *Poisson transform* associated with the contractive tuple $\underline{T}$ is the completely positive map $\alpha : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\alpha(X) = \lim_{r \to 1} K_r(\underline{T})^* (X \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}}) K_r(\underline{T}),$$

where the limit is in the norm topology of $\mathcal{B}(\mathcal{H})$. For each $0 < r < 1$, the $K_r(\underline{T})$ satisfies

$$K_r(\underline{T})^* (V_{i_1}V_{i_2}\dots V_{i_k} \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}}) = T_{i_1}T_{i_2}\dots T_{i_k} K_r(\underline{T})^*$$

for $k \geq 1$ and $1 \leq i_1, i_2, \dots, i_k \leq n$. This implies that the Poisson transform is a unital completely contractive $\mathcal{A}$-morphism. Depending on the property of the tuple $\underline{T}$, the image of $K(\underline{T})$ can be contained in a subspace, say $\mathcal{L}_{\underline{T}} \otimes \mathcal{D}_{\underline{T}}$ where $\mathcal{L}_{\underline{T}}$ is a subspace of $\Gamma(\mathbb{C}^n)$. We state the discussion above as a theorem.

THEOREM 4.3. *If $\underline{T} = (T_1, \ldots, T_n)$ is a contractive tuple acting on a Hilbert space $\mathcal{H}$, then there is a unique unital $\mathcal{A}$-morphism $\alpha : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H})$ such that $\alpha(V_j) = T_j$.*

PROOF The only point which needs a proof is uniqueness which is clear too. An $\mathcal{A}$-morphism which sends $V_j$ to $T_j$ for every $j$ has to be unique because $C^*(\underline{V}) = \overline{\text{span}}\mathcal{A}\mathcal{A}^*$. ■

COROLLARY 4.4. **von Neumann's inequality:** *Let $\underline{T} = (T_1, \ldots, T_n)$ be any contractive tuple acting on a Hilbert space $\mathcal{H}$ and $\underline{V} = (V_1, \ldots, V_n)$ be the tuple of creation operators on $\Gamma(\mathbb{C}^n)$. Then for any polynomial $f$ in $n$ non-commuting indeterminates,*

$$\|f(T_1, \ldots, T_n)\| \leq \|f(V_1, \ldots, V_n)\|.$$

PROOF The proof is obvious using the $\mathcal{A}$-morphism $\alpha$ obtained above. ■

Thus for a commuting contractive tuple $\underline{T}$, there are at least two von Neumann inequalities given by Corollary 3.9 and Corollary 4.4. Arias and Popescu noted that given any pure contractive tuple $\underline{T}$, the best von Nemann inequality arises from the smallest co-invariant (with respect to $\underline{V}$) subspace, say $\mathcal{L}_{\underline{T}}$, of $\Gamma(\mathbb{C}^n)$ such that $K(\underline{T})$ takes its values in $\mathcal{L}_{\underline{T}} \otimes \mathcal{D}_{\underline{T}}$. If $B_i = P_{\mathcal{L}_{\underline{T}}} V_i|_{\mathcal{L}_{\underline{T}}}$ for $i = 1, 2, \ldots, n$, then

$$\|f(\underline{T})\| \leq \|f(\underline{B})\| \text{ for all polynomials } f \text{ in } n \text{ non-commuting indeterminates.}$$

See [3]. They also give a formaula for $\mathcal{L}_{\underline{T}}$ in terms of $\underline{T}$.

Theorem 4.3 gives the following dilation theorem for any contractive tuple $\underline{T}$ acting on some Hilbert space $\mathcal{H}$. We are going to use the notation employed in the Dilation theorem of Section 3. Given a Hilbert space $\mathcal{N}$, and a representation $\beta$ of $C^*(\underline{V})$ on $\mathcal{N}$, the operator tuple

$$\underline{A} \stackrel{\text{def}}{=} n \cdot \underline{V} \oplus \beta(\underline{V})$$

is clearly a contractive tuple on $\hat{\mathcal{H}} \stackrel{\text{def}}{=} (\mathcal{H} \otimes \mathcal{M}) \oplus \mathcal{N}$. Let $\mathcal{H}$ be a co-invariant subspace of $\hat{\mathcal{H}}$ under the tuple $\underline{A}$. Then the compression $\underline{T}$ of $\underline{A}$ to $\mathcal{H}$

$$T_i \stackrel{\text{def}}{=} P_{\mathcal{H}} A_i|_{\mathcal{H}}$$

is a contractive tuple on $\mathcal{H}$ and moreover, for any polynomial $f$ in $n$ non-commuting indeterminates, $f(\underline{T})$ is the compression of $f(\underline{A})$ due to the co-invariance of $\mathcal{H}$ with respect to $\underline{A}$. The following theorem proves a converse.

THEOREM 4.5. **Dilation:** *Let $\underline{T}$ be any contractive tuple acting on a separable Hilbert space $\mathcal{H}$ and rank$D_{\underline{T}} = m$ (which is a non-negative integer or $\infty$). Then there is a separable Hilbert space $\mathcal{M}$ of dimension $m$, another separable Hilbert space $\mathcal{N}$ with a tuple of operators $\underline{Z} = (Z_1, \ldots, Z_n)$ acting on it, satisfying $Z_i^* Z_j = \delta_{ij}$ for $1 \leq i, j \leq n$ and $Z_1 Z_1^* + \cdots + Z_n Z_n^* = \mathbf{1}_{\mathcal{N}}$ such that:*

*(a) $\mathcal{H}$ is contained in $\hat{\mathcal{H}} \stackrel{\text{def}}{=} (\mathcal{H} \otimes \mathcal{M}) \oplus \mathcal{N}$ as a subspace and it is co-invariant under $\underline{A} \stackrel{\text{def}}{=} m \cdot \underline{V} \oplus \underline{Z}$.*

*(b) $\underline{T}$ is the compression of $\underline{A}$ to $\mathcal{H}$, i.e., $T_{i_1} T_{i_2} \ldots T_{i_k} h = P_{\mathcal{H}} A_{i_1} A_{i_2} \ldots A_{i_k} h$ for every $h \in \mathcal{H}, k \geq 1$ and $1 \leq i_1, i_2, \ldots, i_k \leq n$.*

*(c)$\hat{\mathcal{H}} = \overline{\text{span}}\{A_{i_1} A_{i_2} \ldots A_{i_k} h$ where $h \in \mathcal{H}, k \geq 1$ and $1 \leq i_1, i_2, \ldots, i_k \leq n\}$.*

*In other words, every contractive tuple $\underline{T}$ has a minimal isometric dilation in the sense of Definition 2.1.*

*Any two such minimal isometric dilations are unitarily equivalent.*

PROOF We consider a Stinespring dilation of $\alpha$. Thus we get a Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$ and a representation $\pi$ of $C^*(\underline{V})$ on $\hat{\mathcal{H}}$ such that

$$\varphi(X) = P_{\mathcal{H}}\pi(X)P_{\mathcal{H}} \text{ for } X \in C^*(\underline{V}),$$

where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$ (We are identifying any operator $Z \in \mathcal{B}(\mathcal{H})$ with $P_{\mathcal{H}}ZP_{\mathcal{H}} \in \mathcal{B}(\hat{\mathcal{H}})$). Now

$$\begin{aligned}
\alpha(V_i)\alpha(V_i^*) &= \alpha(V_iV_i^*) \\
&= P_{\mathcal{H}}\pi(V_iV_i^*)P_{\mathcal{H}} \\
&= P_{\mathcal{H}}\pi(V_i)\pi(V_i^*)P_{\mathcal{H}} \\
&= P_{\mathcal{H}}\pi(V_i)(P_{\mathcal{H}} + P_{\mathcal{H}}^{\perp})(P_{\mathcal{H}} + P_{\mathcal{H}}^{\perp})\pi(V_i^*)P_{\mathcal{H}} \\
&= (P_{\mathcal{H}}\pi(V_i)P_{\mathcal{H}} + P_{\mathcal{H}}\pi(V_i)P_{\mathcal{H}}^{\perp})(P_{\mathcal{H}}\pi(V_i^*)P_{\mathcal{H}} + P_{\mathcal{H}}^{\perp}\pi(V_i^*)P_{\mathcal{H}}) \\
&= \alpha(V_i)\alpha(V_i^*) + (P_{\mathcal{H}}\pi(V_i)P_{\mathcal{H}}^{\perp})(P_{\mathcal{H}}\pi(V_i)P_{\mathcal{H}}^{\perp})^*.
\end{aligned}$$

Thus

$$P_{\mathcal{H}}\pi(V_i)P_{\mathcal{H}}^{\perp} = 0. \tag{4.1}$$

Note that we may and do assume $\pi$ to be a *minimal* Stinespring dilation. So

$$\hat{\mathcal{H}} = \overline{\text{span}}\ \{\pi(X)u : X \in C^*(\underline{V}) \text{ and } u \in \mathcal{H}\}.$$

The $C^*$-algebra $C^*(\underline{V})$ is separable and hence the Hilbert space $\hat{\mathcal{H}}$ is also separable. The tuple $(\pi(V_1), \dots, \pi(V_n))$ is a dilation of $(T_1, \dots, T_n)$ in the sense that for any polynomial $f$ (in $n$ non-commuting variables),

$$f(T_1, \dots, T_n) = P_{\mathcal{H}}f(\pi(V_1), \dots, \pi(V_n))|_{\mathcal{H}},$$

and $\mathcal{H}$ is a co-invariant subspace for $(\pi(V_1), \dots, \pi(V_n))$ in view of (4.1). The proof for obtaining the spaces $\mathcal{M}$ and $\mathcal{N}$ and the operator tuples $\underline{Z}$ and $\underline{A}$ are exactly the same as in Theorem 3.10 and so we omit that. We give proof of the multiplicity i.e., dim $(\mathcal{M})$ = rank of $D_{\underline{T}}$ which involves now non-commuting polynomials. We have

$$\begin{aligned}
& \text{range } \pi(E_0) \\
= \ & \{\pi(E_0)\xi : \xi \in \hat{\mathcal{H}}\} \\
= \ & \overline{\text{span}}\{\pi(E_0)\pi(X)u : X \in C^*(\underline{V}), u \in \mathcal{H}\}\text{range } \pi(E_0) \\
= \ & \overline{\text{span}}\{\pi(E_0)\pi(E_0X)u : X \in C^*(\underline{V}), u \in \mathcal{H}\} \\
= \ & \overline{\text{span}}\{\pi(E_0)\pi(X)u : X \in \mathcal{B}_0(\Gamma(\mathbb{C}^n)), u \in \mathcal{H}\} \\
= \ & \overline{\text{span}}\{\pi(E_0)\pi(V_{i_1}\dots V_{i_k}E_0(V_{j_1}\dots V_{j_l})^*)u : k, l \geq 1; 1 \leq i_1, \dots, i_k, j_1, \dots, j_l \leq n \text{ and } u \in \mathcal{H}\} \\
= \ & \overline{\text{span}}\{\pi(E_0)\pi((V_{j_1}\dots V_{j_l})^*)u : l \geq 1; 1 \leq j_1, \dots, j_l \leq n \text{ and } u \in \mathcal{H}\}.
\end{aligned}$$

Now a unitary $U$ : range $\pi(E_0) \to \overline{\text{range}}D_{\underline{T}}$ can be defined by setting

$$U\pi(E_0)\pi((V_{j_1}\dots V_{j_l})^*)u = D_{\underline{T}}(T_{j_1}\dots T_{j_l})^*u$$

and extending linearly. Then $U$ is isometric because for $u, v \in \mathcal{H}$ and $1 \leq i_1, \dots, i_k, j_1, \dots, j_l \leq n$,

$$\begin{aligned}
\langle \pi(E_0)\pi((V_{i_1}\dots V_{i_k})^*)u, \pi(E_0)\pi((V_{j_1}\dots V_{j_l})^*)v\rangle &= \langle u, \pi(V_{i_1}\dots V_{i_k})\pi(E_0)\pi((V_{j_1}\dots V_{j_l})^*)v\rangle \\
&= \langle u, T_{i_1}\dots T_{i_k}D_{\underline{T}}^2(T_{j_1}\dots T_{j_l})^*v\rangle \\
&= \langle D_{\underline{T}}(T_{i_1}\dots T_{i_k})^*u, D_{\underline{T}}(T_{j_1}\dots T_{j_l})^*v\rangle.
\end{aligned}$$

It is clear that $U$ is onto. This proves that $\overline{\text{range}}D_{\underline{T}}$ and $\mathcal{M}$ have the same dimensions.

The final contention of course follows from the fact that minimal Stinespring dilation is unique up to unitary equivalence. ■

Thus any contractive tuple can be dilated to a tuple of isometries with orthogonal ranges. The dilation obtained in the theorem above is called the *standard non-commuting dilation.* The decomposition of $\underline{A}$ into the direct sum of $m \cdot \underline{V}$ and $\underline{Z}$ can actually be thought of as the Wold decomposition, see Popescu [40]. Clearly, the $m \cdot \underline{V}$ part (the non-commutative shift) is absent if and only if $\underline{T}$ satisfies $\sum T_i T_i^* = \mathbf{1}_{\mathcal{H}}$ and the $\underline{Z}$ part (the anaogue of the unitary part when $n = 1$) is absent if and only if $\underline{T}$ is pure. Thus the dilation tuple $\underline{A}$ satisfies $\sum A_i A_i^* = \mathbf{1}_{\hat{\mathcal{H}}}$ if and only if $\sum T_i T_i^* = \mathbf{1}_{\mathcal{H}}$.

Bunce [17] and Frazho [28] had explored the ideas of isometeric dilation with orgonal ranges. Popescu was the first to study this and other related notions from one variable operator theory systematically and obtained generalizations of such notions as Wold decomposition, characteristic function and functional calculus. He extensively developed the model theory. Popescu's original arguments for the dilation was a bit different. He also treated infinite sequences of operators. Most of the results here will generalize to infinite sequences of commuting or non-commuting operators, we do not go into the infinite case here because notationally it is more difficult to present.

An alternative proof of the dilation theorem also follows from the discrete time case of Bhat's dilation of quantum dynamical semigroups [13].

## 5. Maximality of The Standard Commuting Dilation

In this section, we would answer a natural question. If $\underline{T}$ is a commuting tuple of bounded operators acting on a Hilbert space $\mathcal{H}$, then it possesses two dilations. On one hand, there is the standard commuting one which retains commutativity but the constituent operators are only a direct sum of the commutative shift and a spherical isometry. On the other hand, one has the standard non-commuting one which gives the dilation as isometries with orthogonal ranges, but loses commutativity. So the natural question is what is the best dilation retaining commutativity. The recent result of Bhat, Bhattacharyya and Dey shows that the standard commuting dilation is the *maximal commuting tuple contained* in the minimal isometric dilation. We shall briefly outline the ideas here referring the reader to [16] for the details of proofs.

Since we shall deal with both the standard commuting dilation and the standard non-commuting dilation of a commuting tuple, let us fix some notations. The standard commuting dilation of a commuting contractive tuple will be denoted by $\underline{\tilde{S}}$ and the standard non-commuting dilation of a contractive tuple will be denoted by $\underline{\tilde{V}}$. If $\underline{T}$ is a commuting contractive tuple acting on $\mathcal{H}$, then we have seen that both the dilations are of such a type that $\mathcal{H}$ is a co-invariant subspace. Recall that in such a case, $\underline{T}$ is called a piece of the dilation tuple. For an arbitrary tuple of operators $\underline{R} = (R_1, R_2, \ldots, R_n)$ acting on a Hilbert space $\mathcal{L}$ and having a co-invariant subspace $\mathcal{H}$, the tuple $(P_{\mathcal{H}} R_1|_{\mathcal{H}}, P_{\mathcal{H}} R_2|_{\mathcal{H}}, \ldots, P_{\mathcal{H}} R_n|_{\mathcal{H}})$ is called a piece of $\underline{R}$. It is the same as saying that $(R_1^*|_{\mathcal{H}}, R_2^*|_{\mathcal{H}}, \ldots, R_n^*|_{\mathcal{H}})$ is a part of $\underline{R}^*$ in the sense of Halmos [30]. Let $\underline{R}$ be an $n$-tuple of bounded operators on a Hilbert space $\mathcal{L}$. Consider

$$\begin{aligned} \mathcal{C}(\underline{R}) = \{\mathcal{M} \quad &: \quad \mathcal{M} \text{ is a co-invariant subspace for each } R_i, \\ \text{and} \quad &\quad R_i^* R_j^* h = R_j^* R_i^* h, \text{ for all } h \in \mathcal{M}, \text{ and } i, j = 1, \ldots, n\}. \end{aligned}$$

So $\mathcal{C}(\underline{R})$ consists of all co-invariant subspaces of an $n$-tuple of operators $\underline{R}$ such that the compressions of the tuple $\underline{R}$ to the co-invariant subspace form a commuting tuple. It is a complete lattice, in the sense that arbitrary intersections and closures of spans of arbitrary unions of such spaces are again in this collection. Therefore it has a maximal element. We denote it by $\mathcal{L}^c(\underline{R})$ (or by $\mathcal{L}^c$ when the tuple under consideration is clear).

DEFINITION 5.1. *Suppose $\underline{R}$ is an $n$-tuple of operators on a Hilbert space $\mathcal{L}$. Then the* maximal commuting piece *of $\underline{R}$ is defined as the commuting piece $\underline{R}^c = (R_1^c, \dots, R_n^c)$ obtained by compressing $\underline{R}$ to the maximal element $\mathcal{L}^c(\underline{R})$ of $\mathcal{C}(\underline{R})$. The maximal commuting piece is said to be* trivial *if the space $\mathcal{L}^c(\underline{R})$ is just the zero space.*

The following result is quite useful in determining the maximal commuting piece. Before going into it, we introduce another notation which will be very useful in this section. As we have seen, very often we need to consider products of the form $T_{i_1}T_{i_2}\cdots\ T_{i_k}$, where $1 \le i_1, i_2, \dots, i_k \le n$ and $\underline{T} = (T_1, T_2, \dots, T_n)$ is a tuple of operators acting on $\mathcal{H}$. It will be convernient to have a notation for this. Let $\Lambda$ denote the set $\{1, 2, \dots, n\}$ and $\Lambda^k$ denote the $k$-fold cartesian product of $\Lambda$ with itself for $k \ge 1$. Given $\mathbf{i} = (i_1, \dots, i_k)$ in $\Lambda^k$, $\underline{T}^{\mathbf{i}}$ will mean the operator $T_{i_1}T_{i_2}\cdots T_{i_k}$. Let $\tilde{\Lambda}$ denote $\cup_{n=0}^{\infty}\Lambda^n$, where $\Lambda^0$ is just the set $\{0\}$ by convention and by $\underline{T}^0$ we would mean $\mathbf{1}_{\mathcal{H}}$. In a similar fashion for $\mathbf{i} \in \tilde{\Lambda}$, $e^{\mathbf{i}}$ will denote the vector $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ in the full Fock space $\Gamma(\mathbb{C}^n)$ and $e^0$ is the vacuum $\omega$.

LEMMA 5.2. *Let $\underline{R}$ be an $n$-tuple of bounded operators on a Hilbert space $\mathcal{L}$. Let $\mathcal{K}_{ij} = \overline{span}\{\underline{R}^{\mathbf{i}}(R_iR_j - R_jR_i)h : h \in \mathcal{L}, \mathbf{i} \in \tilde{\Lambda}\}$ for all $1 \le i, j \le n$, and $\mathcal{K} = \overline{span}\{\cup_{i,j=1}^n \mathcal{K}_{ij}\}$. Then $\mathcal{L}^c(\underline{R}) = \mathcal{K}^{\perp}$. In other words, $\mathcal{L}^c(\underline{R}) = \{h \in \mathcal{L} : (R_i^*R_j^* - R_j^*R_i^*)(\underline{R}^{\mathbf{i}})^*h = 0,$ for all $i, j = 1, 2, \dots, n$ and $\mathbf{i} \in \tilde{\Lambda}\}$.*

PROOF: Firstly $\mathcal{K}^{\perp}$ is a co-invariant subspace of $\underline{R}$. This is obvious as each $R_i$ leaves $\mathcal{K}$ invariant. Now for $i, j \in \{1, 2, \dots, n\}$, $h_1 \in \mathcal{L}^c$, and $h_2 \in \mathcal{L}$,

$$\langle (R_i^*R_j^* - R_j^*R_i^*)h_1, h_2\rangle = \langle h_1, (R_jR_i - R_iR_j)h_2\rangle = 0.$$

So we get $(R_i^*R_j^* - R_j^*R_i^*)h_1 = 0$. So $\mathcal{K}^{\perp}$ is in $\mathcal{C}(\underline{R})$. Now if $\mathcal{M}$ is an element of $\mathcal{C}(\underline{R})$, take $i, j \in \{1, \dots, n\}, \mathbf{i} \in \tilde{\Lambda}, h_1 \in \mathcal{M}$, and $h \in \mathcal{L}$. We have

$$\langle h_1, \underline{R}^{\mathbf{i}}(R_iR_j - R_jR_i)h\rangle = \langle (R_j^*R_i^* - R_i^*R_j^*)(\underline{R}^{\mathbf{i}})^*h_1, h\rangle = 0$$

as $(\underline{R}^{\mathbf{i}})^*h_1 \in \mathcal{M}$ and $R_i^*$ commutes with $R_j^*$ on $\mathcal{M}$. Hence $\mathcal{M}$ is contained in $\mathcal{K}^{\perp}$ Now the last statement is easy to see. ∎

COROLLARY 5.3. *Suppose $\underline{R}$, $\underline{T}$ are $n$-tuples of operators on two Hilbert spaces $\mathcal{L}, \mathcal{M}$. Then the maximal commuting piece of $(R_1 \oplus T_1, \dots, R_n \oplus T_n)$ acting on $\mathcal{L} \oplus \mathcal{M}$ is $(R_1^c \oplus T_1^c, \dots, R_n^c \oplus T_n^c)$ acting on $\mathcal{L}^c \oplus \mathcal{M}^c$. The maximal commuting piece of $(R_1 \otimes \mathbf{1}_{\mathcal{M}}, \dots, R_n \otimes \mathbf{1}_{\mathcal{M}})$ acting on $\mathcal{L} \otimes \mathcal{M}$ is $(R_1^c \otimes \mathbf{1}_{\mathcal{M}}, \dots, R_n^c \otimes \mathbf{1}_{\mathcal{M}})$ acting on $\mathcal{L}^c \otimes \mathcal{M}$.*

PROOF: As the product $(\underline{R} \oplus \underline{T})^{\mathbf{i}}$ breaks into the direct sum $\underline{R}^{\mathbf{i}} \oplus \underline{T}^{\mathbf{i}}$ and the product $(\underline{R} \otimes \mathbf{1}_{\mathcal{M}})^{\mathbf{i}}$ breaks into the tensor product $\underline{R}^{\mathbf{i}} \otimes \mathbf{1}_{\mathcal{M}}$, so the proof is clear from the Lemma above. ∎

LEMMA 5.4. *Let $\underline{V} = (V_1, \dots, V_n)$ and $\underline{S} = (S_1, \dots S_n)$ be standard contractive tuples on full Fock space $\Gamma(\mathbb{C}^n)$ and the symmetric Fock space $\Gamma_s(\mathbb{C}^n)$ respectively. Then the maximal commuting piece of $\underline{V}$ is $\underline{S}$.*

PROOF: It is clear that $\underline{S}$ is a commuting piece of $\underline{V}$. To show maximality suppose $x \in \Gamma(\mathbb{C}^n)$ and $\langle x, \underline{V}^{\mathbf{i}}(V_jV_j - V_jV_i)y\rangle = 0$ for all $\mathbf{i} \in \tilde{\Lambda}, 1 \le i, j \le n$ and $y \in \Gamma(\mathbb{C}^n)$. We wish to show that $x \in \Gamma_s(\mathbb{C}^n)$. Suppose $x = \oplus_{m \ge 0} x_m$ with $x_m \in (\mathbb{C}^n)^{\otimes^m}$ for $m \ge 0$. For $m \ge 2$ and any permutation

$\sigma$ of $\{1, 2, \dots, m\}$ we need to show that the unitary $U_\sigma : (\mathbb{C}^n)^{\otimes^m} \to (\mathbb{C}^n)^{\otimes^m}$, defined by

$$U_\sigma(u_1 \otimes \cdots \otimes u_m) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(m)},$$

leaves $x_m$ fixed. Since the group of permutations of $\{1, 2, \dots, m\}$ is generated by permutations $\{(1,2), \dots, (m-1, m)\}$ it is enough to verify $U_\sigma(x_m) = x_m$ for permutations $\sigma$ of the form $(i, i+1)$. So fix $m$ and $i$ with $m \geq 2$ and $1 \leq i \leq (m-1)$. We have

$$\langle \oplus_p x_p, \underline{V}^{\mathbf{i}}(V_k V_l - V_l V_k) y \rangle = 0,$$

for every $y \in \Gamma(\mathbb{C}^n)$, $1 \leq k, l \leq n$. As $\mathbf{i}$ is arbitrary, this means that

$$\langle x_m, z \otimes (e_k \otimes e_l - e_l \otimes e_k) \otimes w \rangle = 0$$

for any $z \in (\mathbb{C}^n)^{\otimes^{(i-1)}}, w \in (\mathbb{C}^n)^{\otimes^{(m-i-1)}}$. This clearly implies $U_\sigma(x_m) = x_m$, for $\sigma = (i, i+1)$. ∎

Here is an important lemma which will be used in the proof of the main theorem.

LEMMA 5.5. *Suppose $\underline{T}, \underline{R}$ are $n$-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$, with $\mathcal{H} \subseteq \mathcal{L}$, such that $\underline{R}$ is a dilation of $\underline{T}$. Then $\mathcal{H}^c(\underline{T}) = \mathcal{L}^c(\underline{R}) \bigcap \mathcal{H}$ and $\underline{R}^c$ is a dilation of $\underline{T}^c$.*

PROOF: We have $R_i^* h = T_i^* h$, for $h \in \mathcal{H}$. Therefore, $(R_i^* R_j^* - R_j^* R_i^*)(\underline{R}^{\mathbf{i}})^* h = (T_i^* T_j^* - T_j^* T_i^*)(\underline{T}^{\mathbf{i}})^* h$ for $h \in \mathcal{H}$, $1 \leq i, j \leq n$, and $\mathbf{i} \in \tilde{\Lambda}$. Now the first part of the result is clear from Lemma 5.2. Further for $h \in \mathcal{L}^c(\underline{R})$, $R_i^* h = (R_i^c)^* h$ and so for $h \in \mathcal{H}^c(\underline{T}) = \mathcal{L}^c(\underline{R}) \bigcap \mathcal{H}$, $(R_i^c)^* h = R_i^* h = T_i^* h = (T_i^c)^* h$. This proves the claim. ∎

Suppose $\sum T_i T_i^* = \mathbf{1}_{\mathcal{H}}$, then it is easy to see that $P_{\underline{T}}^m(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{H}}$ for all $m$ and there is no-way this sequence can converge to zero. So in the pure case the defect operator and the defect spaces are non-trivial. First we restrict our attention to pure tuples (not necessarily commuting).

THEOREM 5.6. *Let $\underline{T}$ be a pure contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal commuting piece $\underline{\tilde{V}}^c$ of the minimal isometric dilation $\underline{\tilde{V}}$ of $\underline{T}$ is a realization of the standard commuting dilation of $\underline{T}^c$ if and only if $\overline{D_{\underline{T}}(\mathcal{H})} = \overline{D_{\underline{T}^c}(\mathcal{H}^c(\underline{T}))}$. In such a case rank $(D_{\underline{T}})$ = rank $(D_{\underline{T}^c})$ = rank $(D_{\underline{\tilde{V}}})$ = rank $(D_{\underline{\tilde{V}}^c})$.*

PROOF: Note that one of the realizations of the standard non-commuting dilation of $\underline{T}$ consists of the space $\Gamma(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}}$ and the operator tuple $\tilde{V} = (V_1 \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}}, \dots, V_n \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})$. The isometric embedding of $\mathcal{H}$ into the dilation space $\Gamma(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}}$ is given by the Poisson transform $K(\underline{T})$. Since $\underline{T}$ is purely contractive it is obvious that $\underline{T}^c$ is a pure contractive tuple. We know from Lemma 5.5 that $\underline{\tilde{V}}^c = \underline{S} \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}}$ on $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{D}_{\underline{T}}$ is a commuting dilation of $\underline{T}^c$. On the other hand, the standard commuting dilation of $\underline{T}^c$ consists of the space $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}$ where $\mathcal{M} = \overline{D_{\underline{T}^c}(\mathcal{H}^c(\underline{T}))}$ and the operator tuple $\underline{S} \otimes \mathbf{1}_{\mathcal{M}}$. Since the standard commuting dilation is unique up to unitary equivalence, these two dilations are same if and only if $\mathcal{D}_{\underline{T}} = \overline{D_{\underline{T}^c}(\mathcal{H}^c(\underline{T}))}$.

The last contention can be seen as follows. When $\underline{\tilde{V}}^c$ is a realization of the standard commuting dilation of $\underline{T}^c$, then rank $(D_{\underline{T}^c})$ = rank $(D_{\underline{\tilde{V}}^c})$. Also as $\underline{\tilde{V}}$ is the minimal isometric dilation of $\underline{T}$, rank $(D_{\underline{T}})$ = rank $(D_{\underline{\tilde{V}}})$. Finally as $\underline{\tilde{V}}^c = (\underline{S} \otimes \mathbf{1}_{\mathcal{D}_{\underline{T}}})$, rank $(D_{\underline{\tilde{V}}^c})$ = rank $(D_{\underline{T}})$. ∎

The main theorem is the following.

THEOREM 5.7. *Suppose $\underline{T}$ is a commuting contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal commuting piece of the minimal isometric dilation of $\underline{T}$ is a realization of the standard commuting dilation of $\underline{T}$.*

Our approach to prove this theorem is as follows. First we consider the standard commuting dilation of $\underline{T}$ on a Hilbert space $\mathcal{H}_1$ as described above. Now the standard tuple $\underline{S}$ is also a contractive tuple. So we have a unique unital completely positive map $\eta : C^*(\underline{V}) \to C^*(\underline{S})$, satisfying

$$\eta(\underline{V}^{\mathbf{i}}(\underline{V}^{\mathbf{j}})^*) = \underline{S}^{\mathbf{i}}(\underline{S}^{\mathbf{j}})^* \ \mathbf{i},\mathbf{j} \in \tilde{\Lambda}.$$

Now clearly $\psi = \varphi \circ \eta$. Consider the minimal Stinespring dilation of the composed map $\pi_1 \circ \eta : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_1)$. Here we obtain a Hilbert space $\mathcal{H}_2$ containing $\mathcal{H}_1$ and a unital $*$-homomorphism $\pi_2 : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_2)$, such that

$$\pi_1 \circ \eta(X) = P_{\mathcal{H}_1}\pi_2(X)|_{\mathcal{H}_1}, \quad \text{for all } X \in C^*(\underline{V}),$$

and $\overline{\text{span}}\ \{\pi_2(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_1\} = \mathcal{H}_2$.

Now we have a commuting diagram as follows

$$\begin{tikzcd}
 & & \mathcal{B}(\mathcal{H}_2) \arrow[d] \\
 & & \mathcal{B}(\mathcal{H}_1) \arrow[d] \\
C^*(\underline{V}) \arrow[r, "\eta"'] \arrow[uurr, "\pi_2"] & C^*(\underline{S}) \arrow[r, "\varphi"'] \arrow[ur, "\pi_1"] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

where all the down arrows are compression maps, horizontal arrows are unital completely positive maps and diagonal arrows are unital $*$-homomorphisms.

Taking $\hat{\underline{V}} = (\hat{V}_1, \ldots, \hat{V}_n) = (\pi_2(V_1), \ldots, \pi_2(V_n))$, we need to show (i) $\hat{\underline{V}}$ is the minimal isometric dilation of $\underline{T}$ and (ii) $\tilde{\underline{S}} = (\pi_1(S_1), \ldots, \pi_1(S_n))$ is the maximal commuting piece of $\hat{\underline{V}}$. Due to uniqueness up to unitary equivalence of minimal Stinespring dilation, we have (i) if we can show that $\pi_2$ is a minimal dilation of $\psi = \varphi \circ \eta$. For proving this we actually make use of (ii). At first the assertion (ii) is proved in a very special case.

DEFINITION 5.8. A $n$-tuple $\underline{T} = (T_1, \ldots, T_n)$ of operators on a Hilbert space $\mathcal{H}$ is called a spherical unitary if it is commuting, each $T_i$ is normal, and $T_1T_1^* + \cdots + T_nT_n^* = \mathbf{1}_{\mathcal{H}}$.

Given an $n$-tuple $\underline{T} = (T_1, \ldots, T_n)$ of operators satisfying $T_1T_1^* + \cdots + T_nT_n^* = \mathbf{1}_{\mathcal{H}}$, it follows by Theorem 3.10 that its standard commuting dilation is a tuple of normal operators, hence each $T_i^*$ is subnormal (or see [10] for this result). If moreover the space $\mathcal{H}$, where the operators $T_i$ act, is a finite dimensional Hilbert space then applying the result that all finite dimensional subnormal operators are normal (see [30]), we get that normality of $T_i$ is automatic. Thus any $n$-tuple $\underline{T} = (T_1, \ldots, T_n)$ of operators satisfying $T_1T_1^* + \cdots + T_nT_n^* = \mathbf{1}_{\mathcal{H}}$ on a finite-dimensional Hilbert space is a spherical unitary.

Note that if $\underline{T}$ is a spherical unitary we have $\varphi(\underline{S}^{\mathbf{i}}(\mathbf{1}_{\Gamma_s(\mathbb{C}^n)} - \sum S_iS_i^*)(\underline{S}^{\mathbf{j}})^*) = \underline{T}^{\mathbf{i}}(\mathbf{1}_{\mathcal{H}} - \sum T_iT_i^*)(\underline{T}^{\mathbf{j}})^*$ $= 0$ for any $\mathbf{i},\mathbf{j} \in \tilde{\Lambda}$. This forces that $\varphi(X) = 0$ for any compact operator $X$ in $C^*(\underline{S})$. Now as the commutators $[S_i^*, S_j]$ are all compact we see that $\varphi$ is a unital $*$-homomorphism. So the minimal Stinespring dilation of $\varphi$ is itself. Hence the standard commuting dilation of a spherical unitary is itself. So the following result would yield Theorem 5.7 for spherical unitaries.

THEOREM 5.9. *Let $\underline{T}$ be a spherical unitary on a Hilbert space $\mathcal{H}$. Then the maximal commuting piece of the minimal isometric dilation of $\underline{T}$ is $\underline{T}$.*

The proof of this theorem involves lengthy computations and we refer the reader to [16] for its proof, omitting the proof here. But assuming this, we prove the main Theorem.

PROOF OF THEOREM 5.7 : As $C^*(\underline{S})$ contains the ideal of all compact operators by standard $C^*$-algebra theory we have a direct sum decomposition of $\pi_1$ as follows. Take $\mathcal{H}_1 = \mathcal{H}_{1C} \oplus \mathcal{H}_{1N}$ where $\mathcal{H}_{1C} = \overline{\text{span}}\{\pi(X)h : h \in \mathcal{H}, X \in C^*(\underline{S})$ and $X$ is compact$\}$ and $\mathcal{H}_{1N} = \mathcal{H}_1 \ominus \mathcal{H}_{1C}$, Clearly $\mathcal{H}_{1C}$ is a reducing subspace for $\pi_1$. Therefore

$$\pi_1(X) = \begin{pmatrix} \pi_{1C}(X) & \\ & \pi_{1N}(X) \end{pmatrix}$$

that is, $\pi_1 = \pi_{1C} \oplus \pi_{1N}$ where $\pi_{1C}(X) = P_{\mathcal{H}_{1C}}\pi_1(X)P_{\mathcal{H}_{1C}}$, $\pi_{1N}(X) = P_{\mathcal{H}_{1N}}\pi_1(X)P_{\mathcal{H}_{1N}}$. As observed by Arveson [6], $\pi_{1C}(X)$ is just the identity representation with some multiplicity. More precisely, $\mathcal{H}_{1C}$ can be factored as $\mathcal{H}_{1C} = \Gamma_s(\mathbb{C}^n) \otimes \overline{D_{\underline{T}}(\mathcal{H})}$, such that $\pi_{1C}(X) = X \otimes \mathbf{1}$, in particular $\pi_{1C}(S_i) = S_i \otimes \mathbf{1}$ where $\mathbf{1}$ here is the identity operator on $\overline{D_{\underline{T}}(\mathcal{H})}$. Also $\pi_{1N}(X) = 0$ for compact $X$. Therefore, taking $Z_i = \pi_{1N}(S_i)$, $\underline{Z} = (Z_1, \ldots, Z_n)$ is a spherical unitary.

Now as $\pi_1 \circ \eta = (\pi_{1C} \circ \eta) \oplus (\pi_{1N} \circ \eta)$ and the minimal Stinespring dilation of a direct sum of two completely positive maps is the direct sum of minimal Stinespring dilations so $\mathcal{H}_2$ decomposes as $\mathcal{H}_2 = \mathcal{H}_{2C} \oplus \mathcal{H}_{2N}$, where $\mathcal{H}_{2C}, \mathcal{H}_{2N}$ are orthogonal reducing subspaces of $\pi_2$, such that $\pi_2$ also decomposes, say $\pi_2 = \pi_{2C} \oplus \pi_{2N}$, with

$$\pi_{1C} \circ \eta(X) = P_{\mathcal{H}_{1C}}\pi_{2C}(X)|_{\mathcal{H}_{1C}}, \ \pi_{1N} \circ \eta(X) = P_{\mathcal{H}_{1N}}\pi_{2N}(X)|_{\mathcal{H}_{1N}},$$

for $X \in C^*(\underline{V})$ with $\mathcal{H}_{2C} = \overline{\text{span}}\ \{\pi_{2C}(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_{1C}\}$ and $\mathcal{H}_{2N} = \overline{\text{span}}\ \{\pi_{2N}(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_{1N}\}$. It is also not difficult to see that $\mathcal{H}_{2C} = \overline{\text{span}}\ \{\pi_{2C}(X)h : X \in C^*(\underline{V}), X$ compact, $h \in \mathcal{H}_{1C}\}$ and hence $\mathcal{H}_{2C}$ factors as $\mathcal{H}_{2C} = \Gamma(\mathbb{C}^n) \otimes \overline{D_{\underline{T}}(\mathcal{H})}$ with $\pi_{2C}(V_i) = V_i \otimes \mathbf{1}$. Also $(\pi_{2N}(V_1), \ldots, \pi_{2N}(V_n))$ is a minimal isometric dilation of spherical isometry $(Z_1, \ldots, Z_n)$. Now we get that $(\pi_1(S_1), \ldots, \pi_1(S_n))$ acting on $\mathcal{H}_1$ is the maximal commuting piece of $(\pi_2(V_1), \ldots, \pi_2(V_n))$.

All that remains to show is that $\pi_2$ is the minimal Stinespring dilation of $\varphi \circ \eta$. Suppose this is not the case. Then we get a reducing subspace $\mathcal{H}_{20}$ for $\pi_2$ by taking $\mathcal{H}_{20} = \overline{\text{span}}\ \{\pi_2(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}\}$. Take $\mathcal{H}_{21} = \mathcal{H}_2 \ominus \mathcal{H}_{20}$ and correspondingly decompose $\pi_2$ as $\pi_2 = \pi_{20} \oplus \pi_{21}$,

$$\pi_2(X) = \begin{pmatrix} \pi_{20}(X) & \\ & \pi_{21}(X) \end{pmatrix}$$

Note that we already have $\mathcal{H} \subseteq \mathcal{H}_{20}$. We claim that $\mathcal{H}_2 \subseteq \mathcal{H}_{20}$. Firstly, as $\mathcal{H}_1$ is the space where the maximal commuting piece of $(\pi_2(V_1), \ldots, \pi_2(V_n)) = (\pi_{20}(V_1) \oplus \pi_{21}(V_1), \ldots, \pi_{20}(V_n) \oplus \pi_{21}(V_n))$ acts, by the first part of Corollary 5, $\mathcal{H}_1$ decomposes as $\mathcal{H}_1 = \mathcal{H}_{10} \oplus \mathcal{H}_{11}$ for some subspaces $\mathcal{H}_{10} \subseteq \mathcal{H}_{20}$, and $\mathcal{H}_{11} \subseteq \mathcal{H}_{21}$. So for $X \in C^*(\underline{V})$, $P_{\mathcal{H}_1}\pi_2(X)P_{\mathcal{H}_1}$, has the form (see the diagram)

$$P_{\mathcal{H}_1}\pi_2(X)P_{\mathcal{H}_1} = \begin{pmatrix} \pi_{10} \circ \eta(X) & 0 & & \\ 0 & 0 & & \\ & & \pi_{11} \circ \eta(X) & 0 \\ & & 0 & 0 \end{pmatrix}$$

where $\pi_{10}, \pi_{11}$ are compressions of $\pi_1$ to $\mathcal{H}_{10}$, $\mathcal{H}_{11}$ respectively. As the mapping $\eta$ from $C^*(\underline{V})$ to $C^*(\underline{S})$ is clearly surjective, it follows that $\mathcal{H}_{10}, \mathcal{H}_{11}$ are reducing subspaces for $\pi_1$. Now as $\mathcal{H}$ is contained in $\mathcal{H}_{20}$, in view of minimality of $\pi_1$ as a Stinespring dilation, $\mathcal{H}_1 \subseteq \mathcal{H}_{20}$. But then the minimality of $\pi_2$ shows that $\mathcal{H}_2 \subseteq \mathcal{H}_{20}$. Therefore, $\mathcal{H}_2 = \mathcal{H}_{20}$. ∎

This theorem puts in perspective the two standard dilations of a commuting tuple, but that is not its only merit. As an application, the article [16] goes on to find a neat classification of all representations of Cuntz algebra $\mathcal{O}_n$ coming from dilations of commuting contractive tuples. Another recent article by Dey [22] shows that if $\underline{T}$ is a tuple of $q$-commuting operators, then the standard $q$-commuting dilation is the maximal $q$-commuting part of its standard non-commuting dilation.

We finish with the comment that the material presented here are merely the basics and starting from here one can pursue any of a number of directions. Popescu has generalized much of the Sz. Nagy - Foias theory to the case of an infinite set of contractions satisfying the contractivity property, see [40] - [44] and his papers with Arias [3], [4]. The dilation theory over the years has inspired many a branch. Some of the prominent upshots are the study of Hilbert modules which have been systematically studied by Douglas, Misra, Paulsen and Varughese [23], [24], [25] and the theory of curvature of contractive modules by Arveson [9]. The dilations of quantum dynamical semigroups by Arveson [7], [8] and Bhat [13], [14] draw their original motivation from Sz-Nagy Foias dilation. In the linked notion of dilating homomorphisms, there arise natural questions of which contractive homomorphisms are completely contractive. For a study of this question on various domains in $\mathbb{C}^n$, see the works of Misra, Pati, Paulsen and Sastry [32], [38], [33], [34]. For various generalizations of the von Neumann inequality to Banach space setting, we refer to the monograph of Pisier [39] and the references therein.

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