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Non-linear Field Theory of a Frustrated Heisenberg Spin Chain

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Abstract

We derive a continuum field theory for the Majumdar-Ghosh model in the large-S limit, where the field takes values in the manifold of the $SO(3)$ group. No topological term is induced in the action and the cases for integer spin and half-integer spin appear to be indistinguishable. A one-loop $\beta-$function calculation indicates that the theory flows towards a strong coupling (disordered) phase at long distances. This is verified in the large-N limit, where all excitations are shown to be massive.

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1. Introduction

The study of quantum spin models - quantum antiferromagnets in particular - has attracted a great deal of interest [1 - 4] , ever since it was recognised that the ground state properties of the Heisenberg antiferromagnet (AFM) and its generalisations in two dimensions could be relevant to high T_c superconductors [5]. One widely used method of analysis which is applicable in the long-wavelength limit has been the continuum field theory approach, wherein an approximate field theory is derived for the quantum AFM in the large-S limit and the properties of the ground state are obtained using standard field theory techniques.

For one-dimensional quantum spin chains, the field theory approach led to some interesting and unexpected results. By mapping the Heisenberg AFM to a non-linear sigma model defined on the manifold of S^2 , Haldane [1, 2] argued that an integer spin system would have a ground state with massive excitations and exponential disorder, but a half-integer spin system would have massless excitations and algebraic order. The distinction between integer spins and half-integer spins was shown to be caused by the presence of a topological term in the action. In two dimensions, however, for the Heisenberg AFM on bipartite lattices, the current understanding [3 , 4] is that no topological term is induced in the effective long-distance action and the system always exhibits long-range Néel order (at zero temperature) with massless excitations.

However, for frustrated spin systems in both one and two dimensions, the ground states have not yet been conclusively determined. It has been plausibly argued that the frustrated $J - J'$ model (with both nearest-neighbour and next-nearest-neighbour couplings) on bipartite two-dimensional lattices would have a flux-phase ground state [6]. Another example of a two-dimensional frustrated model - the Heisenberg AFM on a triangular lattice - has been studied [7] in the field theory limit. It was shown there that no topological term is induced and no distinction between integer and half-integer spins is observed.

In one dimension, the Majumdar-Ghosh model [8], which has both nearest-neighbour and next-nearest-neighbour couplings with a particular ratio of their strengths, is a specific example of a frustrated spin system. This model has been solved for $S = 1/2$ and has been shown to have doubly degenerate valence bond ground states with massive excitations [8]. However, frustrated spin models in one dimension have not yet been studied from a field theoretic point of view. In this paper, we shall address ourselves to a study of the Majumdar-Ghosh model in the large-S, long-wavelength limit. We shall derive an appropriate field theory for the model and analyse it. In Sec. 2, we briefly review the results from the spin wave, field theory and large-N approaches to the unfrustrated Heisenberg AFM. In Sec. 3, we study the Majumdar-Ghosh model in detail. We derive a spin wave theory about the classical ground state using Villain's approach [9] and show that the model has three spin wave modes, with two of them having the same velocity and the third having a higher one. We derive a continuum field theory for the model in the large-S, long-wavelength limit and discuss its symmetries. No topological term is induced in the action. Hence the long-wavelength theory appears to be indifferent to the distinction between integer and half-integer spins. We then perform a one-loop $\beta-$function analysis of the field theory which indicates that the system flows towards a disordered phase at long distances. Finally, we generalise the field theory to an appropriate large-N field theory and show that a saddle-point approximation indicates that all the excitations are massive. In Sec. 4, we summarise the evidence offered in the earlier sections and present our conclusions.

2. The Heisenberg Antiferromagnet

In this section, we shall briefly review [1 , 2] the study of the simplest spin chain - the nearest-neighbour Heisenberg AFM - given by the Hamiltonian

$$H = J \sum_{<i,j> \in n.n} \mathbf{S}_i \cdot \mathbf{S}_j \tag{2.1}$$

with $J > 0$. The classical ground state (Néel state) of this model has neighbouring spins antiparallel. The standard spin-wave analysis can be carried out about this ground state using the Holstein-Primakoff transformation [10]

$$\begin{aligned} S_i^z &= S - a_i^\dagger a_i \\ S_i^- &= a_i^\dagger \ (2S - a_i^\dagger a_i)^{1/2} \approx \sqrt{2S} \ a_i^\dagger \\ S_i^+ &= (2S - a_i^\dagger a_i)^{1/2} \ a_i \approx \sqrt{2S} \ a_i \end{aligned} \tag{2.2}$$

where the second approximate equality is true in the large-S limit. Two spin-wave modes are found at $k = 0$ and $k = \pi$ satisfying the relativistic dispersion $E = ck$ with $c = 2JSa$. However, it is well-known [11] that there exists no long-range order in one dimension, due to infra-red divergences in the theory. Hence, a small fluctuation analysis about an ordered state is plagued with infinities and breaks down in the disordered phase.

The next approach is the continuum field theory approach initiated by Haldane [1], which led to the startling result that integer spins have massive modes and exponential disorder, whereas half-integer spins have massless modes and algebraic order. Here, however, we shall follow Affleck's [2] method to obtain the field theory. Continuum fields $\vec{\phi}$ and $\mathbf{l}$ are defined as

$$\begin{aligned} \mathbf{l}_{2i+1/2} \ &= \ \frac{(\mathbf{S}_{2i} + \mathbf{S}_{2i+1})}{2a} \ \equiv \ \mathbf{l}(x) \\ \text{and} \quad \vec{\phi}_{2i+1/2} \ &= \ \frac{(\mathbf{S}_{2i} - \mathbf{S}_{2i+1})}{2S} \ \equiv \ \vec{\phi}(x) \end{aligned} \tag{2.3}$$

where a is the lattice spacing. These variables satisfy the identities

$$\begin{aligned} &\mathbf{l} \cdot \vec{\phi} = 0 \\ \text{and} \quad &\vec{\phi}^2 = 1 + \frac{1}{S} \ - \ \frac{a^2 \mathbf{l}^2}{S^2} \end{aligned} \tag{2.4}$$

so that in the large-S limit, $\vec{\phi}^2 = 1$, - *i.e.*, $\vec{\phi}$ takes values on the manifold of the sphere S^2. The commutation relations for the spins imply that the fields satisfy the commutation relations given by

$$\begin{aligned} [\ l_\alpha,\ l_\beta\ ] &= i\epsilon_{\alpha\beta\gamma}\ l_\gamma\ \delta(x-y) \\ [\ l_\alpha,\ \phi_\beta\ ] &= i\epsilon_{\alpha\beta\gamma}\ \phi_\gamma\ \delta(x-y) \\ [\ \phi_\alpha,\ \phi_\beta\ ] &= 0 \end{aligned} \tag{2.5}$$

which indicates that $\mathbf{l}(x)$ can be identified as the angular momentum of the field $\vec{\phi}$ constructed as $\mathbf{l} = \vec{\phi} \times \vec{\pi}$, (where $\vec{\pi}$ is the canonically conjugate momentum to $\vec{\phi}$). The Hamiltonian in Eq. (2.1) can be rewritten in terms of these fields and expanded in a derivative expansion. In the long-wavelength limit, we only need to keep terms upto two space-time derivatives. In fact, for $\mathbf{l}$, since it is already proportional to $\vec{\pi}$, we only need to keep terms upto $\mathbf{l}^2$, $\dot{\mathbf{l}}$ and $\mathbf{l}'$, where the dot and prime denote time and space derivatives respectively. In this limit, the Hamiltonian can be reexpressed as

$$H = \int dx\ [\ \frac{cg^2}{2}\ (\ \mathbf{l} + \frac{S}{2}\ \vec{\phi}\ '\ )^2\ +\ \frac{c}{2g^2}\ \vec{\phi}\ '^{\ 2}\ ] \tag{2.6}$$

where $c = 2JSa$ is the spin-wave velocity and the coupling constant $g^2 = 2/S$, so that large-S corresponds to weak coupling.

This Hamiltonian in Eq. (2.6) follows from the Lagrangian given by

$$\mathcal{L} = \frac{\dot{\vec{\phi}}^{\,2}}{2cg^2} - \frac{c\vec{\phi}\ '^{\ 2}}{2g^2} + \frac{S}{2}\vec{\phi} \cdot \vec{\phi}\ ' \times \dot{\vec{\phi}} \tag{2.7}$$

where the field $\vec{\phi}$ is subject to the constraint $\vec{\phi}^2 = 1$. Setting the spin-wave velocity $c = 1$, the action can be written in a relativistically invariant way as

$$\mathcal{S} = \int d^2x\ [\ \frac{1}{2g^2}\ \partial_\mu\vec{\phi} \cdot \partial^\mu\vec{\phi}\ +\ \frac{\theta}{8\pi}\epsilon^{\mu\nu}\vec{\phi} \cdot \partial_\mu\vec{\phi} \times \partial_\nu\vec{\phi}\ ] \tag{2.8}$$

with $\theta = 2\pi S$. The θ term can actually be shown to be a total derivative, which affects neither the classical equations of motion nor the perturbative Feynman rules. However, it does have a topological significance. On compactified Euclidean space, both x_μ and $\vec{\phi}$ can be represented as points on a sphere S^2. The functional

$$Q(\vec{\phi}) = \frac{1}{8\pi}\int d^2x\ \epsilon^{\mu\nu}8\ [\vec{\phi} \cdot \partial_\mu\vec{\phi} \times \partial_\nu\vec{\phi}]\ \equiv\ Q \tag{2.9}$$

measures the winding of the sphere $S^2(\vec{\phi})$ onto the sphere $S^2(x_\mu)$ and is always quantised as an integer for any field configuration, since $\Pi_2(S^2) = Z$. Thus, for integer values of S, the contribution of the topological term to the path integral is given by $e^{-2\pi i S Q} = 1$ - *i.e.*, it is irrelevant. The action in Eq. (2.8) without the topological term is well-known to describe a theory which flows to a strong coupling, massive phase with masses of the order $e^{-\pi S}$. But for half-integer spins, the topological term $e^{-2\pi i S Q} = e^{-i\pi Q}$ is either $+1$ or -1 depending on the value of Q and is certainly relevant. In that case, the topological term cannot be ignored. The current understanding [2] is that the action in Eq. (2.8) with a topological term is actually massless and describes a conformally invariant field theory with algebraic order - *i.e.*, with a power-law fall-off of correlation functions.

The result that the field theory in Eq. (2.8) describes a massive phase can also be confirmed by generalising the S^2 manifold of the $\vec{\phi}$ field to an S^N manifold and then studying the large-N limit of the field theory. The appropriate Lagrangian is given by

$$\mathcal{L} = \frac{N}{2g^2} \left[\, \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \, + \, i\lambda(\vec{\phi}^2 - 1) \, \right] \tag{2.10}$$

where the constraint has been enforced using an explicit Lagrange multiplier field. We can now integrate out $\vec{\phi}$ and obtain

$$\mathcal{S}_{\text{eff}}(\lambda) = \frac{N}{2} \left[\, - \int d^2x \, \frac{i\lambda}{g^2} \, + \, \text{tr} \, \ln \, (-\partial^2 + i\lambda) \, \right] . \tag{2.11}$$

In the large-N limit, we can ignore fluctuations of λ and evaluate $\mathcal{S}_{\text{eff}}(\lambda)$ at the saddle-point defined by $\partial \mathcal{S}_{\text{eff}} / \partial \lambda = 0$. We find that the saddle-point equation gives

$$\frac{1}{g^2} = \frac{1}{2\pi} \ln \, \frac{\Lambda}{m} \tag{2.12}$$

where Λ is an ultra-violet cut-off. This determines the mass parameter m in terms of the cut-off and the bare parameter g as

$$m = \Lambda e^{-2\pi/g^2} = \Lambda e^{-\pi S} \tag{2.13}$$

which agrees with the expectation from the one-loop renormalisation group flow.

Thus, the current understanding of the Heisenberg AFM spin chain is that both for integer and half-integer spins, there exists short-range order, but for integer spins, at large distances, long-wavelength fluctuations destroy the order. For half-integer spins, even in the long-distance regime, excitations are massless and there exists algebraic order.

3. The Majumdar-Ghosh Model

The Hamiltonian for the Majumdar-Ghosh model is given by

$$H = J \sum_i \mathbf{S}_i \cdot [\, \mathbf{S}_{i+1} + \frac{1}{2} \mathbf{S}_{i+2} \,] \tag{3.1}$$

The classical ($S \to \infty$) ground state must have any three neighbouring spins adding upto zero. This can be seen by recasting the Hamiltonian in Eq. (3.1) as

$$H = \frac{J}{4} \sum_i (\mathbf{S}_i + \mathbf{S}_{i+1} + \mathbf{S}_{i+2})^2 + \text{constant} \tag{3.2}$$

For this to hold all along the chain, the spins must lie on a plane and must successively twist by $2\pi/3$ as shown in Fig. 1.

For $S = 1/2$, this model has dimerised ground states which are just products of local singlet pairs. Since each spin can form a singlet by pairing with its neighbour to the left or with its neighbour to the right, it is clear that two such ground states are possible. These states were proven to be exact ground states by Majumdar and Ghosh. Later, it was shown [12] that these two states are the only ground states and that there exists a gap in the spectrum.

However, the model remains unsolved for all higher spins. Here we shall obtain the ground state properties of the model in the large-S, long-wavelength limit. Even though naive spin-wave analysis is inadequate because of infra-red divergences in one-dimensional systems, we perform a spin wave analysis to obtain the spin wave modes and their velocities. The purpose of this calculation is to have a check on the field theory formulation. This is done in Sec. (A). In Sec. (B), we derive the field theory, discuss its symmetries and show that it reproduces the spin-wave calculations in the appropriate limit. In Sec. (C), we study the renormalisation group flows of the coupling constants. The global symmetries of the field theory allow for four independent coupling constants. We study the flow of these couplings by deriving the one-loop $\beta-$functions and integrating them numerically. We obtain the scale at which a transition to the strong coupling (disordered) phase occurs. Finally, in Sec. (D), we show that in the large-N limit, all excitations become massive.

This confirms the fact that the theory is in the disordered phase in the long-wavelength limit.

Sec. (A): Spin-wave Analysis

Since the classical ground state of the Majumdar-Ghosh Hamiltonian is planar, it is more convenient to obtain the spin-wave theory by parametrising [13] the spin operators in terms of Villain's variables [9], rather than the usual Holstein-Primakoff variables. The spin variables at each site are expressed in terms of two conjugate variables S_i^z and φ_i as

$$\begin{aligned} S_i^+ &= e^{i\varphi_i} \left[(S + \frac{1}{2})^2 - (S_i^z + \frac{1}{2})^2 \right]^{1/2} \\ S_i^- &= \left[(S + \frac{1}{2})^2 - (S_i^z + \frac{1}{2})^2 \right]^{1/2} e^{-i\varphi_i} \\ S_i^z &= \frac{1}{i} \frac{\partial}{\partial \varphi_i} \end{aligned} \tag{3.3}$$

The periodic operator φ_i ($\varphi_i = \varphi_i + 2\pi$) satisfies the commutation relation

$$[\varphi_i , \frac{S_j^z}{S}] = \frac{i}{S} \delta_{ij} . \tag{3.4}$$

In the large-S limit, the discreteness of S_i^z /S and the periodicity of φ_i can be neglected and S_i^z and φ_i can be thought of as canonically conjugate, continuous momentum and position operators. In the classical ($S \to \infty$) limit, in fact, $S_i^z /S \to 0$ and φ_i is fixed at its classical value $\bar{\varphi}_i$, which is the angle made by the classical spin relative to the x-axis. The large-S approximation involves a systematic $1/S$ expansion around this classical ground state.

For the Majumdar-Ghosh Hamiltonian, from Fig. 1, it is clear that $\bar{\varphi}_i - \bar{\varphi}_j$ for any two neighbouring spins is always $2\pi/3$. Let us now compute the spin wave spectrum for the Hamiltonian by expanding the square roots in Eq. (3.3) and keeping terms only upto order $1/S$, and by expanding the angles φ_i to quadratic order about $\bar{\varphi}_i$ - *i.e.* we have

$$\begin{aligned} S_i^+ &\simeq e^{i\bar{\varphi}_i} (S + \frac{1}{2}) (1 + i\theta_i - \frac{\theta_i^2}{2}) \left(1 - \frac{1}{2} \frac{(S_i^z + \frac{1}{2})^2}{(S + \frac{1}{2})^2} \right) \\ S_i^- &\simeq e^{-i\bar{\varphi}_i} (S + \frac{1}{2}) (1 - i\theta_i - \frac{\theta_i^2}{2}) \left(1 - \frac{1}{2} \frac{(S_i^z + \frac{1}{2})^2}{(S + \frac{1}{2})^2} \right) \end{aligned} \tag{3.5}$$

where θ_i describes in-plane fluctuations (fluctuations about the angle $\bar{\varphi}_i$) and S_j^z/S describes out-of-plane fluctuations. Substituting these operators in the Hamiltonian in Eq. (3.1), we find that

$$\begin{aligned} H = J \sum_i [\, S_i^z\, S_{i+1}^z \, - \, S_i^{z2} \cos\Theta \, - \, \frac{S^2}{2} (\theta_i \, - \theta_j\,)^2 \cos\Theta\,] \\ + \frac{J}{2} \sum_i [\, S_i^z\, S_{i+2}^z \, - \, S_i^{z2} \cos 2\Theta \, - \, \frac{S^2}{2} (\theta_i \, - \theta_j\,)^2 \cos 2\Theta\,] \end{aligned} \tag{3.6}$$

where $\Theta = \bar{\varphi}_i - \bar{\varphi}_j = 2\pi/3$. Upon Fourier transforming, we can write the Hamiltonian in the standard oscillator form as

$$H = J\, [\, \sum_q \frac{S_q^z\, S_{-q}^z}{2\, m_q} \, + \, \frac{1}{2}\, k_q\, \theta_q\, \theta_{-q}\,] \tag{3.7}$$

where

$$\omega(\, q\,) \; = \; \sqrt{\frac{k_q}{m_q}} \; = \; -\, J\, S \cos\Theta \sqrt{\, (3-\gamma_q)\, (3-\gamma_q/\cos\Theta\,)} \tag{3.8}$$

with

$$\gamma_q = 2\, \cos q \; + \; \cos 2q. \tag{3.9}$$

Here, $-\pi < q < \pi$. Also note that $\omega(q) = \omega(-q)$. The spin-wave spectrum is now obtained by expanding $\omega(q)$ about its zeroes -*i.e.*, about any q_0 where $\omega(q_0) = 0$ - and identifying its velocity as $\partial\omega/\partial q \;|_{q=q_0}$. It is clear from Eq. (3.8) that $\omega(q)$ has three zeroes at $q = 0,\ 2\pi/3$ and $-2\pi/3$ within the first Brillouin zone. At $q = 0$, the spin-wave velocity is given by

$$c\; (q \; = \; 0) \; = \; J\; S\; a \sqrt{\frac{27}{4}} \tag{3.10}$$

and at both $q = 2\pi/3$ and $q = -2\pi/3$, it is given by

$$c\; (q \; = \; \pm\, 2\pi/3\,) \; = \; J\; S\; a \sqrt{\frac{27}{8}}. \tag{3.11}$$

Hence, we have three spin-wave modes, two with the same velocity and one with a higher velocity.

As was mentioned in Sec. 2, spin-wave theory about an ordered state is not appropriate in one dimension, because infra-red divergences in the theory drive any ordered state towards disorder. However, the spin-wave velocities computed here will serve as a consistency check on the field theory of the model, which will be derived in the next subsection.

Sec. (B): The Continuum Field Theory

The first task in the derivation of a continuum field theory for the Majumdar-Ghosh model is the identification of 'small' and 'large' variables from a local set of spins, analogous to the $\mathbf{l}$ and $\vec{\phi}$ fields in Eq. (2.3). Since, the classical ground state is a three sub-lattice N$\acute{e}$el state, we define field variables involving three local spins as

$$\begin{aligned} \mathbf{l}_{3i} &= \frac{(\ \mathbf{S}_{3i-1} \ + \ \mathbf{S}_{3i} \ + \ \mathbf{S}_{3i+1}\ )}{3\ a} \\ (\vec{\phi}_1)_{3i} &= \frac{(\ \mathbf{S}_{3i-1} \ - \ \mathbf{S}_{3i+1}\ )}{\sqrt{3}\ S} \\ (\vec{\phi}_2)_{3i} &= \frac{(\ \mathbf{S}_{3i-1} \ + \ \mathbf{S}_{3i+1} \ - \ 2\ \mathbf{S}_{3i}\ )}{3\ S} \end{aligned} \tag{3.12}$$

where a is the lattice spacing. Note that we have grouped the spins as $(3i-1, 3i, 3i+1)$. The two other possible groupings are $(3i-2, 3i-1, 3i)$ and $(3i, 3i+1, 3i+2)$. All three groupings lead to the same field theory. (We shall verify this later and see how these three possibilities are related to symmetries in the field theory.)

For the classical configuration in Fig. 1, we see that $\mathbf{l} \ = \ 0$ and the fields $\vec{\phi}_1$ and $\vec{\phi}_2$ satisfy the conditions given by

$$\vec{\phi}_1^2 \ = \ \vec{\phi}_2^2 \ = \ 1 \quad \text{and} \quad \vec{\phi}_1 \cdot \vec{\phi}_2 \ = \ 0. \tag{3.13}$$

We may therefore expect $\mathbf{l}$ to be a 'small' variable (proportional to a space-time derivative in the long-distance limit) and $\vec{\phi}_1$ and $\vec{\phi}_2$ to be the two 'large' and non-linear variables, describing three degrees of freedom (since $\vec{\phi}_1 \cdot \vec{\phi}_2 \ = \ 0$). The actual identities satisfied by these fields are given by

$$\begin{aligned} \vec{\phi}_1^2 \ + \ \vec{\phi}_2^2 \ &= \ 2 \ + \ \frac{2}{S} \ - \ \frac{2a^2\mathbf{l}^2}{S^2} \\ \vec{\phi}_1^2 \ - \ \vec{\phi}_2^2 \ &= \ - \ \frac{4a}{S} \ \vec{\phi}_2 \cdot \mathbf{l} \\ \vec{\phi}_1 \cdot \vec{\phi}_2 \ &= \ - \ \frac{2a}{S} \vec{\phi}_1 \cdot \mathbf{l}, \end{aligned} \tag{3.14}$$

which reduces to Eq. (3.13) in the long-distance (where $\mid a\mathbf{l} \mid << \mid \vec{\phi}_1 \mid, \mid \vec{\phi}_2 \mid$) and large-S limit. Moreover, the $\mathbf{l}$, $\vec{\phi}_1$ and $\vec{\phi}_2$ fields satisfy the commutation relations given by

$$\begin{aligned} [\ l_\alpha \ (x), \ l_\beta \ (y) \ ] \ &= \ i \ \epsilon_{\alpha\beta\gamma} \ l_\gamma(x) \ \delta(x-y) \\ [\ l_\alpha \ (x), \phi_{a\beta} \ (y) \ ] \ &= \ i \ \epsilon_{\alpha\beta\gamma} \ \phi_{a\gamma}(x) \ \delta(x-y) \\ [\ \phi_{a\alpha} \ (x), \phi_{b\beta} \ (y) \ ] \ &= \ 0 \end{aligned} \tag{3.15}$$

where a, b take the values 1 and 2. These relations can be easily derived from the commutation relations of the spin operators by replacing the lattice δ-functions by continuum ones - *i.e.*, $\delta_{i,j}/3a \ \longrightarrow \ \delta(x-y)$. Thus, $\mathbf{l}$ can be identified as the angular momentum operators of the fields $\vec{\phi}_1$ and $\vec{\phi}_2$.

We can now derive the continuum Hamiltonian by rewriting Eq. (3.1) in terms of the fields $\mathbf{l}$, $\vec{\phi}_1$ and $\vec{\phi}_2$ and then Taylor expanding the fields upto second derivatives of $\vec{\phi}_1$ and $\vec{\phi}_2$ and first derivatives of $\mathbf{l}$, and by replacing the sum over sites by an integral over space. Thus, in terms of the fields, we obtain

$$H = \int dx \left[\frac{cg^2}{2} \ (\ \mathbf{l} \ + \ \frac{S\vec{\phi}_1'}{\sqrt{3}} \ )^2 \ + \ \frac{c}{2g^2} \ (\ \vec{\phi}_1'^{\ 2} \ + \ \vec{\phi}_2'^{\ 2} \ ) \right]$$

where

$$\begin{aligned} c = JSa \ (\ \frac{27}{8} \ )^{1/2} \\ \text{and} \quad g^2 = \sqrt{6}/S \ . \end{aligned} \tag{3.16}$$

As before, we see that large-S corresponds to the weak coupling, perturbative regime. Since analysis of the field theory is simpler from a Lagrangian formulation, let us deduce a Lagrangian for this field theory. We define a third unit vector

$$\vec{\phi}_3 = \vec{\phi}_1 \times \vec{\phi}_2$$

with

$$\vec{\phi}_1 \cdot \vec{\phi}_3 = \vec{\phi}_2 \cdot \vec{\phi}_3 = 0 \quad , \quad \vec{\phi}_3^{\,2} = 1 \ . \tag{3.17}$$

This field satisfies the commutation relations expressed in Eq. (3.15) for $a,\ b\ =\ 3$ as well. Let us now define an orthogonal matrix $\underline{R}$ whose entries are

$$\underline{R} = \begin{pmatrix} \phi_{11} & \phi_{21} & \phi_{31} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix} \ . \tag{3.18}$$

Thus, the three columns of $\underline{R}$ are given by the vectors $\vec{\phi}_1$, $\vec{\phi}_2$ and $\vec{\phi}_3$ respectively. Notice that since $\underline{R}$ is orthogonal ($\underline{R}^T\ \underline{R} = \underline{R}\ \underline{R}^T\ = I$), it only has three independent degrees of freedom. If we now introduce a diagonal matrix

$$I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{3.19}$$

it is clear that the second term in the Hamiltonian in Eq. (3.16) is proportional to tr $(\underline{R}'^T \underline{R}' I_2)$. But the problem is to figure out a time derivative term in the Lagrangian involving the $\underline{R}$ fields that would lead to the first term in the same Hamiltonian.

To figure this out, we note that our problem involves an $SO(3)$-valued field at each space-time point. Hence, we need the method of quantisation on a group manifold which is derived in Ref. [14]. Following their work, we can show that the required Lagrangian is given by

$$\mathcal{L} = \frac{1}{4cg^2}\ \mathrm{tr}\ (\dot{\underline{R}}^T\ \dot{\underline{R}})\ -\ \frac{c}{2\ g^2}\mathrm{tr}\ (\underline{R}'^T\ \underline{R}'\ I_2)\ +\ \frac{S}{\sqrt{3}}\ \vec{\phi}_1 \cdot \vec{\phi}_1' \times \dot{\vec{\phi}}_1 . \tag{3.20}$$

Let us verify that the Hamiltonian in Eq. (3.16) can be derived from this Lagrangian [14]. $\underline{R}$ may be parametrised by three numbers ξ_α, which are local coordinates on the manifold of $SO(3)$. Thus, the derivatives of $\underline{R}$ are

$$\partial_\mu \underline{R}\ =\ \partial_\mu \xi_\alpha\ \partial_\alpha \underline{R} \tag{3.21}$$

where $\partial_\alpha\ =\ \partial / \partial_\alpha$. The momenta π_α canonical to the ξ_α can be found from the Lagrangian as

$$\pi_\alpha\ =\ \frac{\partial\ \mathcal{L}}{\partial\ \dot{\xi}_\alpha}\ =\ \frac{1}{4cg^2}\ \mathrm{tr}\ [\ (\partial_\alpha\ \underline{R}^T)\ \dot{\underline{R}}\ +\ \dot{\underline{R}}^T\ (\partial_\alpha\ \underline{R})\ ]\ +\ \frac{S}{\sqrt{3}}\ (\vec{\phi}_1 \times \vec{\phi}_1')\ \cdot \partial_\alpha \vec{\phi}_1 \tag{3.22}$$

leading to the Hamiltonian

$$H \;=\; \int dx\, [\frac{1}{4cg^2}\ \mathrm{tr}\ (\dot{\underline{R}}^T\ \dot{\underline{R}})\ -\ \frac{c}{2g^2}\mathrm{tr}\ (\underline{R}'^T\ \underline{R}'\ I_2)\ ]. \tag{3.23}$$

Now, the π_α are locally defined momenta conjugate to the locally defined coordinates ξ_α . The globally defined momenta (conjugate to the globally defined coordinate variables $\underline{R}$) are given by

$$l_\alpha \;=\; \pi_\beta\ N_{\beta\alpha} \tag{3.24}$$

where $(\partial_\beta\ \underline{R}\ )\ N_{\beta\alpha} \;=\; i\ T_\alpha\ \underline{R}$ and T_α are the generators of $SO(3)$, so that $(T_\alpha)_{\beta\gamma} \;=\; i\ \epsilon_{\alpha\beta\gamma}$. Hence, we find that

$$l_\alpha \;=\; \frac{i}{2cg^2}\mathrm{tr}\ (\dot{\underline{R}}^T\ T_\alpha\ \underline{R})\ +\ \frac{iS}{\sqrt{3}}\ (\vec{\phi}_1 \times \vec{\phi}_1{}')\cdot(T_\alpha\ \vec{\phi}_1). \tag{3.25}$$

Here, we have used the fact that the first column of the matrix $T_\alpha\ \underline{R}$ is given by $T_\alpha\ \vec{\phi}_1$. The construction of l_α guarantees that

$$\begin{aligned} [\ l_\alpha\ (x),\ l_\beta\ (y)\ ] \;&=\; i\ \epsilon_{\alpha\beta\gamma}\ l_\gamma(x)\ \delta(x-y) \\ [\ l_\alpha\ (x),\ \underline{R}\ (y)\ ] \;&=\; T_\alpha\ \underline{R}\ (x)\ \delta(x-y) \end{aligned} \tag{3.26}$$

which is identical (in component notation) to Eq. (3.15). Moreover, by substituting for $\dot{\underline{R}}$ in Eq. (3.23) in terms of l_α using Eq. (3.25), we can verify that the Hamiltonian in Eq. (3.23) is identical to that in Eq. (3.16).

Let us now study the Lagrangian given in Eq. (3.20), which we have just proved is the continuum field theory of the Majumdar-Ghosh model in the large-S limit. Notice that the last term in Eq. (3.20) is a total derivative, and therefore, has no effect perturbatively. In fact, we shall now argue that this topological term actually vanishes for all smooth field configurations in Euclidean space, and has no non-perturbative effect either. To demonstrate this, we observe that the Euclidean action (setting $c = 1$) is given by

$$S_E \;=\; \int\ d^2x\ \{\frac{1}{4g^2}\ (\dot{\vec{\phi}}_1^{\,2}\ +\ \dot{\vec{\phi}}_2^{\,2}\ +\ \dot{\vec{\phi}}_3^{\,2}\ )\ +\ \frac{1}{2g^2}\ (\vec{\phi}_1{}'^{\,2}\ +\ \vec{\phi}_2{}'^{\,2}\ )\}\ -\ \frac{4\pi i S}{\sqrt{3}}\ Q\ [\vec{\phi}_1], \tag{3.27}$$

where $Q\ [\vec{\phi}_1]$ is defined as $Q\ [\vec{\phi}]$ was defined in Eq. (2.9). Finiteness of S_E implies that $\vec{\phi}_1$, $\vec{\phi}_2$ and $\vec{\phi}_3$ must go to constants at infinity, so that space-time is compactified to S^2 . The functional $Q\ [\vec{\phi}_1]$ must still be an integer since $\vec{\phi}_1$ also lies on S^2 ($\vec{\phi}^2 = 1$). However, since $\vec{\phi}_1$ is actually part of an $SO(3)$ matrix $\underline{R}$ and $\Pi_2(SO(3)) = 0$, $Q[\vec{\phi}_1]$ must actually vanish. Let us demonstrate this explicitly. We define a vector

$$\vec{\phi}\ (\theta) \ = \ \vec{\phi}_1 \cos\theta \ + \ \vec{\phi}_2 \sin\theta. \tag{3.28}$$

Clearly, $Q\ [\vec{\phi}(\theta)]$ is also an integer, since $\vec{\phi}(\theta)$ also takes values on S^2 . However, since $\vec{\phi}(\pi) = -\vec{\phi}(0)$, $Q[\vec{\phi}(\pi)] = -Q[\vec{\phi}(0)]$. Since $Q[\vec{\phi}]$ is restricted to integer values, it cannot change when θ is continuously varied. Hence, the only consistent possibility is that $Q[\vec{\phi}]$ must vanish for all θ.

Thus, we have the important result that there is no topological term in the field theory of the Majumdar-Ghosh model in 1 + 1 dimensions. Notice that the proof of the vanishing of the topological term needed the global existence of two orthonormal vectors ϕ_1 and ϕ_2 , whereas in Sec. 2, the field theory of the Heisenberg AFM involved only one globally defined field ϕ . Hence, the field theory defined in Sec. 2 did possess a topological term which distinguished between integer and half-integer spins. Here, however, we can work with the simpler $SO(3)$-valued field theoretic action defined by

$$S_E = \int d^2x\ [\ \frac{1}{4g^2}\ \mathrm{tr}\ (\dot{\underline{R}}^T \underline{R}) \ + \ \frac{1}{2g^2}\ \mathrm{tr}\ (\underline{R}^{\prime T}\ \underline{R}^\prime\ I_2)\ ]\ . \tag{3.29}$$

This action is invariant under the global symmetry $\underline{R} \rightarrow A\underline{R}B$, where A is an $SO(3)$ matrix and B is a block diagonal matrix of the form

$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.30}$$

Thus, the theory has an $SO(3)_L \times SO(2)_R$ symmetry, where the 'L' index mixes the rows and the 'R' index mixes the columns. The $SO(3)_L$ denotes the original spin symmetry of the Hamiltonian generated by the angular momentum $\mathbf{l}(x)$. Equivalently, we may think of this symmetry as the symmetry of the 'top' $\underline{R}$ about a 'space-fixed' set of axes that affects the three vectors $\vec{\phi}_1$, $\vec{\phi}_2$ and $\vec{\phi}_3$ in the same way without mixing them. On the other hand, the $SO(2)_R$ is a rotation about the 'body-fixed' axis $\vec{\phi}_3$ which mixes $\vec{\phi}_1$ and $\vec{\phi}_2$.

This symmetry is not present in the original spin Hamiltonian in Eq. (3.1) but only arises in the long-wavelength field theory developed around the ground state of Fig.1. We now recall the statement following Eq. (3.12). To define the fields, the spins can be grouped in threes in three different ways. These three ways are related to each other by rotations in the $(\vec{\phi}_1, \vec{\phi}_2)$ plane by the angles $0, 2\pi/3$ and $4\pi/3$. So we should certainly have expected a cyclic Z_3 symmetry in the field theory. However, the vanishing of the topological term in the action appears to have allowed an enlargement of the symmetry to a full continuous $SO(2)$ symmetry.

Finally, let us derive the spin-wave spectrum from the Lagrangian and check whether we get the right spin-wave velocities. The classical ground state is clearly given by the configuration $\mathbf{S}_{3i} = S\ (0,1,0)$, $\mathbf{S}_{3i-1} = S\ (-\sqrt{3}/2, -1/2, 0)$ and $\mathbf{S}_{3i+1} = S\ (\sqrt{3}/2, -1/2, 0)$, which can be expressed in terms of the matrix $\underline{R}$ by the identity matrix i.e. $\underline{R} = I$. Small fluctuations about this ordered state can be parametrised by

$$\underline{R} = \begin{pmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{pmatrix} \tag{3.31}$$

where α, β and γ are small fluctuation fields. In terms of these fields, the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2cg^2}\ (\dot{\alpha}^2\ +\ \dot{\beta}^2\ +\ \dot{\gamma}^2)\ -\ \frac{c}{2g^2}\ (\alpha'^2\ +\ \beta'^2\ +\ 2\gamma'^2\ )\ . \tag{3.32}$$

Thus, the α and β modes have the velocity c and the γ-mode has the velocity $\sqrt{2}c$ in agreement with the spin-wave velocities in Eq. (3.10) and (3.11) since $c = JSa\ (27/8)^{1/2}$ from Eq. (3.16). From the matrix $\underline{R}$ in Eq. (3.31), it is clear that the γ-mode describes fluctuations within the plane of the classical vectors $\vec{\phi}_1$ and $\vec{\phi}_2$ - *i.e.*, within the plane of the classical spins. The α and β modes describe out-of-plane fluctuations.

Thus, we have proved that the field theory of the Majumdar-Ghosh Hamiltonian reproduces the standard spin-wave results in the weak coupling limit.

Sec. (C): One-loop β-function and the flow of coupling constants

In the previous section, we derived an effective low-energy field theory for the Majumdar-Ghosh model and showed that it reproduced the standard spin-wave scenario by expanding about the classical, ordered ground state. However, more generally, we can study the long-distance properties of such a field theory by applying the renormalisation group. The $SO(3)_L \times SO(2)_R$ global symmetry of the action in Eq. (3.29) implies that at any scale, the effective (Euclidean) Lagrangian can have four different coupling constants and can be written as

$$\begin{aligned}\mathcal{L}_E = \left(\frac{1}{2g_1^2} - \frac{1}{4g_2^2} \right) \mathrm{tr}\, (\dot{\underline{R}}^T \dot{\underline{R}}) + \left(\frac{1}{2g_2^2} - \frac{1}{2g_1^2} \right) \mathrm{tr}\, (\dot{\underline{R}}^T \dot{\underline{R}}\, I_2) \\ + \left(\frac{1}{2g_3^2} - \frac{1}{4g_4^2} \right) \mathrm{tr}\, (\underline{R}'^T \underline{R}') + \left(\frac{1}{2g_4^2} - \frac{1}{2g_3^2} \right) \mathrm{tr}\, (\underline{R}'^T \underline{R}'\, I_2) \,.\end{aligned} \tag{3.33}$$

At microscopic distances of the order of the lattice spacing a, where we derived the field theory, we have

$$g_1^2 = g_2^2 = g_3^2 = 2g_4^2 = g^2$$

with

$$g^2 = \sqrt{6}/S \,. \tag{3.34}$$

(Compare Eq. (3.29) and Eq. (3.33)). But these values change as we move to larger distance scales in accordance with the renormalisation group equations. (Note that an $SO(3)_L \times SO(3)_R$ symmetric Lagrangian would have had $g_1 = g_2$ and $g_3 = g_4$). The unusual parametrisation in Eq. (3.33) has been chosen for convenience in studying the evolution of the small fluctuations α, β and γ, so that the spin-wave (Minkowski) Lagrangian takes the simple form given by

$$\mathcal{L} = \frac{1}{2g_1^2} (\dot{\alpha}^2 + \dot{\beta}^2) - \frac{1}{2g_3^2} (\alpha'^2 + \beta'^2) + \frac{1}{2g_2^2} \dot{\gamma}^2 - \frac{1}{2g_4^2} \gamma'^2 \,. \tag{3.35}$$

From this Lagrangian, we read off the velocity of the α and β modes to be g_1/g_3 and that of the γ-mode to be g_2/g_4 .

To study the flow of the coupling constants to the long distance regime, we calculate the four β-functions

$$\beta(g_i^2) = \frac{d}{dy}\, g_i^2(y) \tag{3.36}$$

where $y = \ln(L/a\ )$ is a measure of the distance scale L. This is done using the background field formalism [15]. We expand the field $\underline{R}(x)$ as

$$\underline{R}(x) = \ \underline{R}_0(x)\ e^{i\eta_\alpha(x) T_\alpha} \tag{3.37}$$

where $\underline{R}_0(x)$ is a slowly varying field, which we take to be a solution of the Euler-Lagrange equations of motion of the Lagrangian in Eq. (3.33), and $\eta_\alpha(x)$ are rapidly varying fields. We then integrate over the η_α fields to obtain an effective action for $\underline{R}_0$, from which the β-functions are obtained. (The field $\underline{R}(x)$ has to be expanded as $\underline{R}(x) = \underline{R}_0(x)\ \exp(i\eta_\alpha(x)\ T_\alpha)$ rather than $\underline{R}(x) = \ \exp(i\eta_\alpha(x)\ T_\alpha)\ \underline{R}_0(x)$ in order to maintain the $SO(3)_L\ \times SO(2)_R$ symmetry for the effective action for $\underline{R}_0$).

On expanding the right hand side of Eq. (3.33) to second order in $\eta_\alpha(x)$ (the first order terms vanish because $\underline{R}_0$ extremises the action), we find that the Euclidean propagator $M_{\alpha\beta}(\mathbf{k}) = \langle \eta_\alpha(\mathbf{k}) \eta_\beta(-\mathbf{k}) \rangle$ in momentum space $\mathbf{k} = (k_0\ , k_1\ )$ has a diagonal form with

$$\begin{aligned} M_{11} = M_{22} &= \frac{g_1^2\ g_3^2}{g_3^2\ k_0^2\ +\ g_1^2\ k_1^2} \\ \text{and} \quad M_{33} &= \frac{g_2^2\ g_4^2}{g_4^2\ k_0^2\ +\ g_2^2\ k_1^2} \end{aligned}\ . \tag{3.38}$$

In addition, there are two kinds of vertices between the $\underline{R}_0$-fields (denoted by dotted lines) and the η-fields (denoted by solid lines) as shown in Figs. 2 (a) and 2 (b). The vertex in Fig. 2 (a) has one derivative of $\underline{R}_0$ and one derivative of η. (Since the momentum flowing along the $\underline{R}_0$ line is negligible compared to the momentum flowing along the η lines, we have shown the η-momenta to be $\mathbf{k}$ and $-\mathbf{k}$ respectively). The Feynman rule for this vertex is given by

$$\begin{aligned} \Gamma_{\alpha\beta} = \ -ik_0 \Big[& (\frac{1}{g_1^2}\ -\ \frac{1}{2g_2^2}\ )\ \mathrm{tr}\ \dot{\underline{R}}_0^T\ \underline{R}_0\ (\ T_\alpha\ T_\beta\ -\ T_\beta\ T_\alpha\ ) \\ & +\ (\frac{1}{g_2^2}\ -\ \frac{1}{g_1^2}\ )\ \mathrm{tr}\ \dot{\underline{R}}_0^T\ \underline{R}_0\ (\ T_\alpha\ I_2\ T_\beta\ -\ T_\beta\ I_2\ T_\alpha\ ) \Big] \\ -ik_1 \Big[& (\frac{1}{g_3^2}\ -\ \frac{1}{2g_4^2}\ )\ \mathrm{tr}\ \underline{R}_0'^T\ \underline{R}_0\ (\ T_\alpha\ T_\beta\ -\ T_\beta\ T_\alpha\ ) \\ & +\ (\frac{1}{g_4^2}\ -\ \frac{1}{g_3^2}\ )\ \mathrm{tr}\ \underline{R}_0'^T\ \underline{R}_0\ (\ T_\alpha\ I_2\ T_\beta\ -\ T_\beta\ I_2\ T_\alpha\ ) \Big]\ . \end{aligned} \tag{3.39}$$

The vertex in Fig. 2(b) has two derivatives of $\underline{R}_0$ and no derivatives of η, and its Feynman rule is given by

$$\begin{aligned}\tilde{\Gamma}_{\alpha\beta} \;=\; & -\; (\frac{1}{g_2^2} \;-\; \frac{1}{g_1^2}\;)\ \mathrm{tr}\ \{\dot{\underline{R}}_0^T\ \dot{\underline{R}}_0\ (T_\alpha\ I_2\ T_\beta \;-\; \frac{1}{2}\ T_\alpha\ T_\beta\ I_2 \;-\; \frac{1}{2} I_2\ T_\alpha\ T_\beta\ )\} \\ & -\; (\frac{1}{g_4^2} \;-\; \frac{1}{g_3^2}\;)\ \mathrm{tr}\ \{\underline{R}'^T_0\ \underline{R}'_0\ (T_\alpha\ I_2\ T_\beta \;-\; \frac{1}{2}\ T_\alpha\ T_\beta\ I_2 \;-\; \frac{1}{2} I_2\ T_\alpha\ T_\beta\ )\}\ .\end{aligned} \tag{3.40}$$

The one-loop contribution to the effective action is therefore obtained by evaluating the diagrams in Figs. 3 (a) and 3 (b), which arise from the vertices in Figs. 2 (a) and 2 (b) respectively. Both the diagrams in Fig. 3 have symmetry factors of 1/2 . The momentum integrals are cut off at short distances by the lattice spacing a and at long distances by the length scale L at which we wish to evaluate the effective action - *i.e.*, $L^{-1} \;\leq\; k_0, k_1 \;\leq\; a^{-1}$. It is also convenient to use identities like

$$(\ \dot{\vec{\phi}}_1 \cdot \dot{\vec{\phi}}_2\ )^2 \;=\; \frac{1}{2}\ (\ \dot{\vec{\phi}}_1^{\,2} \;+\; \dot{\vec{\phi}}_2^{\,2} \;-\; \dot{\vec{\phi}}_3^{\,2}\ ) \tag{3.41}$$

which follow from the constraints in Eq. (3.13) and Eq. (3.17). We then obtain the β-functions given by

$$\begin{aligned}\beta(g_1^2\ ) \;&=\; \frac{g_1^4}{2\pi}\ [\ \frac{g_1^2\ g_3\ g_4}{g_2^2}\ \frac{2}{(g_1\ g_4 \;+\; g_2\ g_3\ )} \;+\; g_1\ g_3\ (\frac{1}{g_1^2} \;-\; \frac{1}{g_2^2}\ )\ ] \\ \beta(g_2^2\ ) \;&=\; \frac{g_2^4}{2\pi}\ [\ g_1^3\ g_3\ (\frac{2}{g_1^2} \;-\; \frac{1}{g_2^2}\ )^2 \;+\; 2\ g_1\ g_3\ (\frac{1}{g_2^2} \;-\; \frac{1}{g_1^2}\ )\ ] \\ \beta(g_3^2\ ) \;&=\; \frac{g_3^4}{2\pi}\ [\ \frac{g_3^2\ g_1\ g_2}{g_4^2}\ \frac{2}{(g_1\ g_4 \;+\; g_2\ g_3\ )} \;+\; g_1\ g_3\ (\frac{1}{g_3^2} \;-\; \frac{1}{g_4^2}\ )\ ] \\ \beta(g_4^2\ ) \;&=\; \frac{g_4^4}{2\pi}\ [\ g_3^3\ g_1\ (\frac{2}{g_3^2} \;-\; \frac{1}{g_4^2}\ )^2 \;+\; 2\ g_1\ g_3\ (\frac{1}{g_4^2} \;-\; \frac{1}{g_3^2}\ )\ ].\end{aligned} \tag{3.42}$$

Let us now integrate the β-functions numerically starting from their values at $y \;=\; 0$ given in Eq. (3.34). The one-loop β-functions have the property that they are invariant under the rescaling

$$y \;\longrightarrow\; \lambda\ y \quad \text{and} \quad g_i^2 \;\longrightarrow g_i^2/\lambda \tag{3.43}$$

which implies that the β-functions for different values of g_i^2 (or equivalently $1/S$) are related by a scaling law. This also means that if one or more of the coupling constants go to infinity at any length scale $y = y_0$, then the product $y_0 g^2$ (where $g^2 = \sqrt{6}/S$), must be a number independent of the value of g^2.

From our numerical study of the renormalisation group flows, we find that the coupling constants run in such a way that the velocity of the γ-mode increases and the velocity of the α and β modes decreases. (This behaviour was verified over a large range of initial values starting from $g^2 = 0.001$ through $g^2 = 1.0$). Although all four couplings increase monotonically, the coupling g_2 goes to infinity first, and this happens at a length scale given by

$$y_0 g^2 = 4.5 \tag{3.44}$$

(At that point, the other three couplings have the values $g_1^2 (y_0)/g^2 = 3.6$, $g_3^2 (y_0)/g^2 = 28$ and $g_4^2 (y_0)/g^2 = 12$). The blowing up of one or more of the couplings is normally interpreted as a sign that the system becomes disordered at that length scale - i.e. at

$$L_0 \sim a\, e^{y_0} \sim a\, e^{4.5/g^2} \sim a\, e^{1.8S} . \tag{3.45}$$

Beyond this length scale, the action in Eq. (3.29) is no longer valid and a new action (presumably describing massive excitations and, perhaps, some massless excitations) must become applicable. Although our calculation cannot be used to obtain any information other than the scale L_0 at which the perturbative calculations break down, it is interesting to speculate on the possible scenarios at longer distances. We can think of three possible scenarios.

(a) All three modes α, β and γ become massive at L_0, so that there is a gap in the spectrum $\Delta \sim 1/L_0$.

(b) Only the γ-mode becomes massive at L_0. The α and β modes remain massless until a longer distance L_1 after which they too become massive.

(c) Only the α and β modes become massive at L_0, but the γ-mode becomes massive only at a longer distance L_1.

Clearly, much more analysis is needed to decide between the three possibilities [16].

Sec. (D): A Large-N Approximation

A non-perturbative method of studying the mass generation in any field theory is to go to the large-N limit of an appropriate generalisation of the model. For the S^2 model studied in Sec. 2, the generalisation to an S^N model was obvious. Here, however, the obvious generalisation of $\underline{R} \in SO(3)$ to $\underline{R} \in SO(N)$ runs into problems. To study the $SO(N)$-valued field theory , it is necessary to impose the constraints using Lagrange multiplier fields. But this increases the number of degrees of freedom from $N(N - 1)/2$ to N^2 , which does not match asymptotically in the large-N limit.

To find a suitable generalisation, let us write the Euclidean $SO(3)$ -valued field theory as

$$\mathcal{L}_E = \frac{1}{2g^2} [\dot{\vec{\phi}}_1^{\,2} + \dot{\vec{\phi}}_2^{\,2} - (\vec{\phi}_1 \cdot \dot{\vec{\phi}}_2)^2 + \vec{\phi}_1'^{\,2} + \vec{\phi}_2'^{\,2}] \tag{3.46}$$

where we have eliminated $\dot{\vec{\phi}}_3^2 = \dot{\vec{\phi}}_1^2 + \dot{\vec{\phi}}_2^2 - 2(\vec{\phi}_1 \cdot \dot{\vec{\phi}}_2)^2$, but we still have to impose the constraints $\vec{\phi}_1^{\,2} = \vec{\phi}_2^{\,2} = 1$ and $\vec{\phi}_1 \cdot \vec{\phi}_2 = 0$. Note that $\mathcal{L}_E$ still exhibits the $SO(3) \times SO(2)$ symmetry. We can now generalise the three dimensional vector fields $\vec{\phi}_1$ and $\vec{\phi}_2$ to N-dimensional vector fields and write the large-N Lagrangian as

$$\mathcal{L}_E = \frac{N}{2g^2} [\dot{\vec{\phi}}_1^2 + \dot{\vec{\phi}}_2^2 - (\vec{\phi}_1 \cdot \dot{\vec{\phi}}_2)^2 + \vec{\phi}_1'^{\,2} + \vec{\phi}_2'^{\,2}] \, . \tag{3.47}$$

This Lagrangian has an $SO(N)\times SO(2)$ symmetry and since it still satisfies the same three constraints, it has $2N - 3$ degrees of freedom. We introduce three Lagrange multiplier fields to impose the constraints and a fourth field to reduce the quartic term in (3.47) to a quadratic form. We then obtain

$$\begin{aligned} \mathcal{L}_E = \frac{N}{2g^2} [\dot{\vec{\phi}}_1^2 + \dot{\vec{\phi}}_2^2 + \vec{\phi}_1'^2 + \vec{\phi}_2'^2 + i\lambda_1 (\vec{\phi}_1^2 - 1) + i\lambda_2 (\vec{\phi}_2^2 - 1) \\ + 2i\lambda_3 \vec{\phi}_1 \cdot \vec{\phi}_2 + \lambda_4^2 + \lambda_4 (\vec{\phi}_1 \cdot \dot{\vec{\phi}}_1 - \vec{\phi}_2 \cdot \dot{\vec{\phi}}_1)] \, . \end{aligned} \tag{3.48}$$

(By integrating over the λ_i fields in the path integral, we get back the Lagrangian in Eq. (3.47)). Now, by integrating out the $\vec{\phi}_i$ fields, we compute the effective action as a function of λ_i as

$$
\begin{aligned}
e^{-S(\lambda_i\ )} &= \int \mathcal{D}\vec{\phi}_1 \mathcal{D}\vec{\phi}_2 \ \exp\,[-\int \ d^2x\ \mathcal{L}_E(\vec{\phi}_1, \vec{\phi}_2, \lambda_i\ )\ ] \\
&= \int \prod_{i=1}^{N} \mathcal{D}\phi_{1i}\mathcal{D}\phi_{2i}\ \exp\,[-\sum_{i=1}^{N} \int \frac{d^2k}{(2\pi)^2}\ M_i\ ]
\end{aligned}
\tag{3.49}
$$

where

$$
M_i\ = (\phi_{1i}\ (-k)\ \phi_{2i}\ (k)\ )\ \begin{pmatrix} k^2\ +\ i\lambda_1 & i\lambda_3\ +\ i\lambda_4\ k_0 \\ i\lambda_4\ k_0 & k^2\ +\ i\lambda_2 \end{pmatrix} \begin{pmatrix} \phi_{1i}(k) \\ \phi_{2i}(-k) \end{pmatrix}
\tag{3.50}
$$

so that

$$
\begin{aligned}
S(\lambda_i\ ) = \ & \frac{N}{2}\,[\,-\,i\,\frac{\lambda_1}{g^2}\ -\ i\,\frac{\lambda_2}{g^2}\ ] \\
+\frac{N}{2}\int \frac{d^2k}{(2\pi)^2} \Big[& \ln\{\ k^2\ +\ \frac{i}{2}\,(\lambda_1+\lambda_2\ )\ +\ i\,\sqrt{\frac{(\lambda_1-\lambda_2\ )^2}{4}\ +\ (\lambda_3+\lambda_4\ k_0\ )^2}\ \} \\
& + \ln\{\ k^2\ +\ \frac{i}{2}\,(\lambda_1+\lambda_2\ )\ -\ i\,\sqrt{\frac{(\lambda_1-\lambda_2\ )^2}{4}\ +\ (\lambda_3+\lambda_4\ k_0\ )^2}\ \}\Big]\,.
\end{aligned}
\tag{3.51}
$$

Because of the factor of N in $S(\lambda_i\ )$, in the large-N limit, the integral over λ_i in the path integral is dominated by the saddle-point of $S(\lambda_i\ )$ given by $\partial S/\partial\lambda_i\ = 0$. This is found to be at $\lambda_1\ = \lambda_2\ = -i\ m^2$ and $\lambda_3\ = \lambda_4\ = 0$ where the saddle-point equation gives

$$
\frac{1}{g^2}\ = \int \frac{d^2k}{(2\pi)^2}\ \frac{1}{(k^2\ +\ m^2\ )^2}\quad .
\tag{3.52}
$$

Hence

$$
m = \Lambda\ \exp\,(-\ 2\pi/g^2\ ) = \Lambda\ \exp\,(-2.57\ S)
\tag{3.53}
$$

where Λ is the ultra-violet cut-off a^{-1} . We can verify that this solution is actually a minimum by computing $D_{ij}\ = \partial^2 S/\partial\lambda_i\ \partial\lambda_j$ at the saddle-point and showing that all its eigenvalues are positive. In fact,

$$
D_{ij}\ = \begin{pmatrix} 1/4\pi m^2 & 0 & 0 & 0 \\ 0 & 1/4\pi m^2 & 0 & 0 \\ 0 & 0 & 1/2\pi m^2 & 0 \\ 0 & 0 & 0 & 1/g^2 \end{pmatrix}
\tag{3.54}
$$

at the saddle-point. Substituting the values of λ_i at the saddle-point back in Eq. (3.48), we find that

$$\mathcal{L}_E \;=\; \frac{N}{4g^2} \, [\, \dot{\vec{\phi}}_1^{\,2} \;+\; \dot{\vec{\phi}}_2^{\,2} \;+\; \vec{\phi}_1^{\,\prime\, 2} \;+\; \vec{\phi}_2^{\,\prime\, 2} \;+\; m^2 \, (\vec{\phi}_1^{\,2} \;+\; \vec{\phi}_2^{\,2} \,) \;-\; 2m^2 \,], \qquad (3.55)$$

so that all $2N$ fields have become massive.

Thus, the large-N approximation confirms our earlier result that the field theory of the large-S limit of the Majumdar-Ghosh model actually describes a disordered phase with a mass gap $m \sim e^{-KS}$, where K is some number of order one.

4. Discussion and Outlook

Our results in the previous section clearly indicate that the Majumdar-Ghosh model is exponentially disordered beyond some distance scale and has a mass gap for all large spins S. Thus, for all half-integer spins, by applying the Lieb-Schultz-Mattis theorem [17], we see that the ground state has to be doubly degenerate, with each state breaking parity (defined as reflection about a site). For $S = 1/2$, these ground states are just the dimerised ground states mentioned at the beginning of Sec. 3. For higher half-integer spins, the ground states are not easy to visualise, although the theorem is known to hold. However, for integer spins, the degeneracy or non-degeneracy of the ground state is still an open question.

Can the procedure described in Secs. 2 and 3 be used to study general spiral phases? The derivation of the field theory presented in Secs. 2 and 3 crucially depended upon grouping together all the spins in one period and then identifying 'large' and 'small' variables. These became the non-linear fields and their canonical momenta in the long-distance field theory. This procedure clearly fails for the general spiral phase for which the classical ground state is aperiodic.

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Figure Captions

1. Classical ground state of the Majumdar-Ghosh model.
2. Interaction vertices between $\underline{R}_0$ and η. The dotted lines denote the $\underline{R}_0$ fields and the solid lines denote the η fields.
3. One-loop effective action for $\underline{R}_0$ obtained by integrating over the η fields.