

**BOSONIC MEAN FIELD THEORY OF THE SPIRAL PHASES  
OF HEISENBERG ANTIFERROMAGNETS ON A CHAIN**

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**ABSTRACT**

We develop a novel bosonic mean field theory to describe the spiral phases of a Heisenberg antiferromagnet on a one-dimensional chain, in terms of three bosons at each site. The ground state is disordered and for large values of the spin  $S$ , two different and exponentially small energy gaps are found. The spin-spin correlation function is computed and is shown to decay exponentially at large distances. Our mean field theory is also shown to be exact in a large- $N$  generalization.

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The study of the ground state and excitation spectrum of two dimensional quantum antiferromagnets has aroused considerable interest [1–14], particularly since their relevance to high  $T_c$  superconductivity was realized [1,2]. A wide variety of approaches have been used - field theory methods [3–6], linear spin-wave theories[7], analogy with neutral superfluidity [8], bosonic and fermionic mean field theories [9–13], and numerical methods [14] - to study the problem.

The first question that was tackled was the ground state and excitation spectrum of unfrustrated Heisenberg antiferromagnets. Field theory methods showed that in one dimension, the ground state and its excitations had completely different properties depending upon whether the spin was an integer or half-integer. For integer spins, the ground state was exponentially disordered at long distances and excitations had a gap, whereas for half-integer spins, the ground state was only algebraically disordered and had massless excitations [3,4]. This difference had its origin in the topological term that was induced in the long wave-length effective field theory. But this difference disappeared in two dimensions, where no topological term was found to be induced [5]. Moreover, numerical evidence [15] favoured a Neel ordered ground state in two dimensions.

The question of the ground state and excitation spectrum of a frustrated model is far more complex and is, as yet, not completely understood. The mapping to conformal field theories [4] has led to the expectation that the phase diagram for half-integer spins has a region where the model has algebraic disorder and massless excitations. Outside this region, the system is expected to be dimerized, except at specific points. For integer spins, the region of massless excitations is replaced by massive excitations. However, there exist few explicit results. For two dimensional models, the phase diagram is even more uncertain, because the various different approaches lead to different answers. In particular, the existence of a spin-liquid state (predicted by fermionic mean field theories) is still unconfirmed. (For a recent summary of existing results, see Ref. [16].)

In this paper, we introduce a new (bosonic) mean field theory (MFT) involving a representation of spins by three bosons in the adjoint representation of the  $SO(3)$  group of spins, precisely to study this question. In particular, we address the following specific issue. We study the frustrated Heisenberg antiferromagnet (AFM) on a chain using our three boson

representation (3BR), with the aim of obtaining explicit results. This method (like all mean field methods) is insensitive to the presence of topological terms. However, since the ultimate aim is to study spirals in two dimensions, where topological terms are not expected, this is not a serious handicap. This 3BR works well for spiral phases, reproducing the zero modes at  $q = 0$  and  $q = \theta$  (where  $\theta$  is the spiral angle) as expected from symmetry considerations. The same representation also works for the spiral phases of the triangular AFM in two dimensions and is expected to work for the helicoidal phases of the frustrated model on a two dimensional square lattice. One advantage of the 3BR is that no rotation of all the spins to a ferromagnetic configuration is needed. This makes an analysis of helicoidal phases in two dimensions technically much simpler [17].

In the 3BR employed here, we represent the spins at every site by a triplet of bosons. To enforce the spin nature of the operators, two constraints are required at each site which is in contrast to the more commonly used representation of spins in terms of two Schwinger bosons (2BR) where only one constraint per site is required. These two constraints are imposed on an average. Following the method of Sarker et al[12], we perform a Hartree-Fock-Mean-Field (HF-MF) averaging to obtain the ground state energy in terms of six variational parameters, whose values, in turn, are obtained by extremizing the energy. In the  $S \rightarrow \infty$  limit, the spin-wave spectrum is reproduced. For  $S$  large but finite, our solution yields two exponentially small energy gaps. We can compute the two spin correlation function and show that beyond some length scale, the correlation function falls off exponentially. Finally, we show that our HF-MF treatment becomes exact in the large- $N$  limit by generalizing the 3BR (forming the triplet representation of  $SO(3)$ ) to an  $N$ BR (forming the  $N$ -plet representation of  $SO(N)$ ).

Let us start with the frustrated Heisenberg AFM on a chain described by the Hamiltonian

$$H = \sum_n ( \mathbf{S}_n \cdot \mathbf{S}_{n+1} + \delta \mathbf{S}_n \cdot \mathbf{S}_{n+2} ) \quad (1)$$

where we have normalized the exchange constant  $J = 1$ . For  $\delta < 1/4$ , the classical ground state is Neel ordered, whereas for  $\delta > 1/4$ , the classical ground state is a spiral where all the spins lie in a plane and the relative angle  $\theta$  between any two spins is given by

$$\cos \theta = -1/4\delta \quad (2)$$

so that  $\pi/2 < \theta < \pi$ . The classical ground state energy per unit spin is given by

$$\frac{E_0}{N} = - \left( \delta + \frac{1}{8\delta} \right) S^2 \quad (3)$$

to leading order in  $S$ , where  $N$  is the total number of spins. For  $\delta = 1/2$ , (a special case called the Majumdar-Ghosh model [18]), this model was recently studied by us [19]. In the long distance, large- $S$  limit, we mapped the model to an  $SO(3)$ -valued field theory and using the  $\beta$ -functions of the theory, we showed that the ground state was exponentially disordered and exhibited a gap. However, we were unable to generalize that method to arbitrary  $\delta$ . (See also Ref. [20]).

We first perform a spin-wave analysis of this Hamiltonian using Villain's action-angle variables [21]. For the general spiral case - *i.e.*,  $\delta > 1/4$ , - the spin-wave spectrum [22] (valid for large- $S$ ) is given by

$$\omega_k = 2S \left[ \left( -\cos\theta - \delta \cos 2\theta + \cos ka + \delta \cos 2ka \right) \left( -\cos\theta - \delta \cos 2\theta + \cos\theta \cos ka + \delta \cos 2\theta \cos 2ka \right) \right]^{1/2}. \quad (4)$$

(The lattice spacing is  $a$ ). Within the first Brillouin zone,  $\omega_k$  vanishes at  $ka = 0, +\theta$  and  $-\theta$  with a linear dispersion, and the spin-wave velocities are given by

$$\begin{aligned} c_0 &= Sa (1 + 4\delta) (1 - 1/16\delta^2)^{1/2} \\ \text{and } c_\theta &= c_0 (1 - 1/2\delta + 1/8\delta^2)^{1/2} \end{aligned} \quad (5)$$

at  $ka = 0$  and  $ka = \pm\theta$  respectively. Thus, the mode at  $k = 0$  has a higher velocity than the two modes at  $ka = \pm\theta$ . We shall see that this spectrum is reproduced by our bosonic MFT in the large- $S$  limit.

We now set up the bosonic MFT by expressing the spin in terms of bosons. The 3BR expresses the components of a spin  $\mathbf{S}$  in terms of three bosons as

$$S_\alpha = -i \epsilon_{\alpha\beta\gamma} a_\beta^\dagger a_\gamma \quad (6)$$

where  $\alpha, \beta$  and  $\gamma$  run from 1 to 3,  $\epsilon_{\alpha\beta\gamma}$  is completely antisymmetric and repeated indices are always summed over. Using  $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$ , we can check that the spin algebra is satisfied. But to also have  $S_\alpha S_\alpha = S^2 = S(S+1)$ , we need to impose the constraints

$$a_\alpha^\dagger a_\alpha = S \quad \text{and} \quad a_\alpha^\dagger a_\alpha^\dagger a_\beta a_\beta = 0 \quad (7)$$

on any physical state. Notice that the first equation in Eq. (7) implies that the 3BR works only for integer spins. By enforcing these constraints, we can check that for any  $S$ , the total number of orthonormal states is  $2S + 1$  as expected. The second constraint in Eq. (6) can equivalently be rephrased as the two constraints

$$a_\alpha^\dagger a_\alpha^\dagger = 0 \quad \text{and} \quad a_\beta a_\beta = 0 \quad (8)$$

in the sense of matrix elements between any two physical states. This is the form in which it is employed later.

To understand the connection between spin order parameters and appropriate expectation values of the bosons, consider a spin operator lying in the  $\hat{x}$ - $\hat{y}$  plane, in the  $S \rightarrow \infty$  limit, - *i.e.*, in the classical limit, - with expectation value

$$\langle S_\alpha \rangle = S (\cos\phi, \sin\phi, 0). \quad (9)$$

Then Eqs. (5) and (6) imply that

$$\langle a_1 \rangle = i \sqrt{\frac{S}{2}} \sin\phi, \quad \langle a_2 \rangle = -i \sqrt{\frac{S}{2}} \cos\phi, \quad \langle a_3 \rangle = \sqrt{\frac{S}{2}} \quad (10)$$

upto an arbitrary phase multiplying all the bosons. So if  $\phi_{mn}$  is the angle between the  $m^{th}$  and  $n^{th}$  spins, we have

$$\langle \mathbf{S}_m \cdot \mathbf{S}_n \rangle \simeq \langle \mathbf{S}_m \rangle \cdot \langle \mathbf{S}_n \rangle = S^2 \cos\phi_{mn} \quad (11)$$

which in turn implies that

$$\begin{aligned} \langle a_{m\alpha}^\dagger a_{n\alpha} \rangle &\simeq \langle a_{m\alpha}^\dagger \rangle \langle a_{n\alpha} \rangle = S \cos^2(\phi_{mn}/2) \\ \langle a_{m\alpha} a_{n\alpha} \rangle &\simeq \langle a_{m\alpha} \rangle \langle a_{n\alpha} \rangle = S \sin^2(\phi_{mn}/2), \end{aligned} \quad (12)$$

again upto arbitrary overall phases. Thus, spiral ordering of spins implies non-zero expectation values of bosonic bilinears. In fact,  $\langle a_{m\alpha}^\dagger a_{n\alpha} \rangle$  is the Ferromagnetic (FM) order parameter and  $\langle a_{m\alpha} a_{n\alpha} \rangle$  is the Antiferromagnetic (AFM) order parameter.

Next, we observe that the product  $\mathbf{S}_m \cdot \mathbf{S}_n$  can be written as

$$\mathbf{S}_m \cdot \mathbf{S}_n = : Y_{m,n}^\dagger Y_{m,n} : - X_{m,n}^\dagger X_{m,n} \quad (13)$$

where

$$Y_{m,n} = a_{m\alpha}^\dagger a_{n\alpha} \quad \text{and} \quad X_{m,n} = a_{m\alpha} a_{n\alpha}. \quad (14)$$

In terms of the bilinears  $X_{m,n}$  and  $Y_{m,n}$ , the Hamiltonian in Eq. (1) can be rewritten as

$$H = \sum_n [ : Y_{n,n+1}^\dagger Y_{n,n+1} : - X_{n,n+1}^\dagger X_{n,n+1} + \delta : Y_{n,n+2}^\dagger Y_{n,n+2} : - \delta X_{n,n+2}^\dagger X_{n,n+2} \\ + \lambda_n (a_{n\alpha}^\dagger a_{n\alpha} - S) - \rho_n (a_{n\alpha} a_{n\alpha}) - \rho_n^* a_{n\alpha}^\dagger a_{n\alpha}^\dagger ], \quad (15)$$

with  $\lambda_n, \rho_n$  and  $\rho_n^*$  being the Lagrange multiplier fields introduced to enforce the constraints in Eqs. (7) and (8) at each site. We now make a HF decomposition by writing

$$A^\dagger A = \langle A^\dagger \rangle A + A^\dagger \langle A \rangle - \langle A^\dagger \rangle \langle A \rangle, \quad (16)$$

where  $A = X_{n,n+1}, Y_{n,n+1}, X_{n,n+2}$  and  $Y_{n,n+2}$  in turn. Such a decomposition (in contrast to the Peierls variational decomposition, which allows for all possible decouplings where each four boson term is written as products of pairs in three different ways) is justified later by a large- $N$  argument. Further, we make the MF ansatz that the parameters  $\lambda_n = \lambda$ ,  $\rho_n = \rho_n^* = \rho$ ,  $\langle X_{n,n+1} \rangle = X_1$ ,  $\langle X_{n,n+2} \rangle = X_2$ ,  $\langle Y_{n,n+1} \rangle = Y_1$  and  $\langle Y_{n,n+2} \rangle = Y_2$  are all independent of  $n$  and real. Notice that this ansatz - in particular, the reality of  $\rho$ ,  $X_i$  and  $Y_i$  - breaks the local gauge invariance  $a_{n\alpha} \rightarrow e^{i\theta_n} a_{n\alpha}$  of the Hamiltonian. However, physical quantities such as spin-spin correlations remain gauge-invariant [12]. Also note that non-zero values for  $X_1$ ,  $X_2$  and  $Y_1$ ,  $Y_2$  imply the existence of short-range AFM and FM orderings respectively.

We now diagonalize the MF Hamiltonian by a Bogoliubov transformation to obtain

$$\frac{H_{MF}}{N} = - \lambda S + X_1^2 + \delta X_2^2 - Y_1^2 - \delta Y_2^2 \\ + a \int_0^{\pi/a} \frac{dk}{\pi} ( \omega_k b_{k\alpha}^\dagger b_{ka} + \frac{3}{2} \omega_k - \frac{3}{2} \mu_k ) \quad (17)$$

where

$$\omega_k = ( \mu_k^2 - \nu_k^2 )^{1/2} \\ \mu_k = \lambda + 2Y_1 \cos ka + 2\delta Y_2 \cos 2ka \quad (18) \\ \text{and } \nu_k = \rho + 2X_1 \cos ka + 2\delta X_2 \cos 2ka,$$

and the bosons  $b_{k\alpha}$  are related to the bosons  $a_{k\alpha}$  by the standard Bogoliubov transformation given by

$$\begin{aligned} a_{k\alpha} &= \cosh\theta_k b_{k\alpha} + \sinh\theta_k b_{-k\alpha}^\dagger \\ a_{k\alpha}^\dagger &= \sinh\theta_k b_{-k\alpha} + \cosh\theta_k b_{k\alpha}^\dagger. \end{aligned} \quad (19)$$

The factors of three in Eq. (17) arise because the constraints in Eq. (7) have only been imposed on the average, resulting in the decoupling of the three bosons. (This tripling of number of branches of the spin-wave spectrum is an unfortunate feature of this MFT. A similar doubling of number of branches occurred in the MFT based on the 2BR [11][12].) Thus, we obtain the MF ground state energy as

$$\frac{E_{MF}}{N} = - (S + 3/2) \lambda + X_1^2 + \delta X_2^2 - Y_1^2 - \delta Y_2^2 + \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \omega_k. \quad (20)$$

The equations for the six variational parameters  $\lambda$ ,  $\rho$ ,  $X_i$  and  $Y_i$  are obtained by extremizing the energy and are given by

$$\begin{aligned} S + \frac{3}{2} &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\mu_k}{\omega_k} \\ 0 &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\nu_k}{\omega_k} \\ Y_1 &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\mu_k}{\omega_k} \cos ka \\ X_1 &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\nu_k}{\omega_k} \cos ka \\ Y_2 &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\mu_k}{\omega_k} \cos 2ka \\ \text{and } X_2 &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\nu_k}{\omega_k} \cos 2ka. \end{aligned} \quad (21)$$

These equations look rather intractable. But, in fact, it is possible to obtain the solutions to leading order in  $S$ . We know that as  $S \rightarrow \infty$ , the solution should approach the classical spiral ground state configuration asymptotically. Hence, to leading order in  $S$ , we must have

$$\begin{aligned} Y_1 &= S \cos^2 \theta/2, & X_1 &= S \sin^2 \theta/2 \\ Y_2 &= S \cos^2 \theta, & X_2 &= S \sin^2 \theta \\ \lambda &= -2S (\cos\theta + \delta \cos 2\theta) & \text{and } \rho &= 0, \end{aligned} \quad (22)$$

where  $\theta$  is defined in Eq. (2). (One can check that these are the correct asymptotic values of the parameters by comparing them with the values given in Eq. (12) as well as by comparing the dispersions given in Eqs. (18) and (4). Note also that by substituting this solution in Eq. (20), the MF energy agrees with the classical energy to leading order in  $S$ .) However, with these asymptotic solutions, notice that  $\omega_k \rightarrow 0$  at  $k = 0$  and  $k = \theta/a$ . Hence, the right hand sides (R.H.S.) of Eq. (21) are log divergent and can only equal the left hand sides (L.H.S.), which are large but finite, if we allow for a small mass generation. Let us assume that near  $k \sim 0$  and  $k \sim \theta$ , the dispersions of  $\omega_k$  are given by

$$\begin{aligned} \omega_k &\simeq \sqrt{\Delta_0^2 + c_0^2 k^2}, \quad k \sim 0 \\ \text{and } \omega_k &\simeq \sqrt{\Delta_\theta^2 + c_\theta^2 (k - \theta/a)^2}, \quad k \sim \theta/a, \end{aligned} \quad (23)$$

where  $c_0$  and  $c_\theta$  are the spin-wave velocities given in Eq. (5), and  $\Delta_0$  and  $\Delta_\theta$  are the two small masses generated. In fact, we will show that the  $\Delta_i$  are exponentially small - *i.e.*, of  $O(e^{-S})$ . Notice also that the non-singular regions on the R.H.S. of Eq. (21) are of  $O(1)$  (not of  $O(S)$ ) and do not contribute to establishing the equality of the L.H.S. and R.H.S to  $O(S)$ . Hence, to leading order in  $S$ , we can simply assume that the integrals on the R.H.S. are dominated by their values at  $k = 0$  and  $k = \theta/a$ . In this limit, Eqs. (21) reduce to

$$\begin{aligned} S + 3/2 &= \mu_0 I_0 + \mu_{\theta/a} I_{\theta/a} \\ 0 &= \nu_0 I_0 + \nu_{\theta/a} I_{\theta/a} \\ Y_1 &= \mu_0 I_0 + \mu_{\theta/a} \cos\theta I_{\theta/a} \\ X_1 &= \nu_0 I_0 + \nu_{\theta/a} \cos\theta I_{\theta/a} \\ Y_2 &= \mu_0 I_0 + \mu_{\theta/a} \cos 2\theta I_{\theta/a} \\ \text{and } X_2 &= \nu_0 I_0 + \nu_{\theta/a} \cos 2\theta I_{\theta/a}, \end{aligned} \quad (24)$$

with

$$\begin{aligned} I_0 &= \frac{3a}{2} \int_0^\epsilon \frac{dk}{\pi} \frac{1}{\omega_k} \simeq \frac{3a}{2\pi c_0} \ln \frac{\epsilon c_0}{\Delta_0} \\ \text{and } I_{\theta/a} &= \frac{3a}{2} \int_{\theta/a - \epsilon/2}^{\theta/a + \epsilon/2} \frac{dk}{\pi} \frac{1}{\omega_k} \simeq \frac{3a}{2\pi c_\theta} \ln \frac{\epsilon c_\theta}{\Delta_\theta}. \end{aligned} \quad (25)$$

where  $\Delta/c \ll \epsilon \ll \pi/a$ . We can now explicitly check that the  $O(S)$  terms on both sides

of Eq. (24) are satisfied when

$$\begin{aligned} \Delta_0 &\sim \frac{c_0}{a} \exp \left[ -\frac{2\pi S}{3} \sqrt{\frac{4\delta - 1}{4\delta + 1}} \right] \\ \text{and } \Delta_\theta &\sim \frac{c_\theta}{a} \exp \left[ -\frac{\pi S}{3} \sqrt{\frac{4\delta - 1}{(4\delta + 1)(1 - 1/2\delta + 1/8\delta^2)}} \right]. \end{aligned} \quad (26)$$

To summarize, the solutions to the Eqs. (21) are given in Eqs. (22) and they lead to two exponentially small mass gaps in the theory given in Eqs. (26).

It is interesting to compare the values for the gaps in Eqs. (26) for  $\delta = 1/2$  with the values obtained for the same  $\delta$  using the field theory approach [19]. The one-loop  $\beta$ -function of the field theory of the Majumdar-Ghosh model led to the single mass gap  $\Delta \sim \exp(-1.8S)$ , whereas here, the bosonic MF treatment yields  $\Delta_0 \sim \exp(-1.2S)$  and  $\Delta_\theta \sim \exp(-.86S)$ , which agree, at least upto the order of magnitude of coefficient of  $S$ . The two mass gaps obtained here (like the two spin-waves) appear to reflect the fact that the spiral ordering of the ground state picks a particular plane. Thus, it is not surprising that fluctuations, and hence, onset of disorder, within the plane and perpendicular to the plane, have different mass scales. The field theory method, presumably, was not sensitive enough to see this feature.

Let us now calculate the spin-spin correlation function within the bosonic MFT. From Eqs. (13) and (14), we see that the product of any two spins can be written as a product of bilinears, so that the spin-spin correlation function is given by

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_n \rangle = \langle : Y_{0,n}^\dagger Y_{0,n} : \rangle - \langle X_{0,n}^\dagger X_{0,n} \rangle. \quad (27)$$

Using only the Wick contractions allowed by the HF decomposition in Eq. (16), we find that the spin-spin correlation can be written as

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_n \rangle = |\langle Y_{0,n} \rangle|^2 - |\langle X_{0,n} \rangle|^2, \quad (28)$$

where  $\langle Y_{0,n} \rangle$  and  $\langle X_{0,n} \rangle$  are obtained by using the Bogoliubov transformation in Eq. (19) as

$$\begin{aligned} \langle Y_{0,n} \rangle &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \left( \frac{\mu_k}{\omega_k} - 1 \right) \cos nka \\ \text{and } \langle X_{0,n} \rangle &= \frac{3a}{2} \int_0^{\pi/a} \frac{dk}{\pi} \frac{\nu_k}{\omega_k} \cos nka. \end{aligned} \quad (29)$$

Once again, the integrals are dominated by the regions near  $k \sim 0$  and  $k \sim \theta$ . We now explicitly compute the correlation function in two limiting cases. When  $na$ , the distance between the two spins measured in terms of the lattice spacing  $a$ , is small, - *i.e.*,  $na \ll \Delta_0^{-1}, \Delta_\theta^{-1}$ , we find that  $Y_{0,n} \sim S \cos^2 (n\theta/2)$  and  $X_{0,n} \sim \sin^2 (n\theta/2)$  so that

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_n \rangle = S^2 \cos n\theta, \quad \text{for } na \ll \Delta_0^{-1}, \Delta_\theta^{-1}. \quad (30)$$

This is not surprising, because at short distances, we expect the system to be ordered. But at long distances, - *i.e.*, when  $na \gg \Delta_0^{-1}, \Delta_\theta^{-1}$ , - we find that

$$\begin{aligned} \langle Y_{0,n} \rangle + \langle X_{0,n} \rangle &\sim S \int_0^\epsilon \frac{dk}{\pi} \frac{\cos nka}{\sqrt{\mu_k - \nu_k}} \sim S e^{-na\Delta_0/c_0}, \text{ and} \\ \langle Y_{0,n} \rangle - \langle X_{0,n} \rangle &\sim S \int_{\theta/a-\epsilon/2}^{\theta/a+\epsilon/2} \frac{dk}{\pi} \frac{\cos nka}{\sqrt{\mu_k + \nu_k}} \sim S \cos n\theta e^{-na\Delta_\theta/c_\theta} \end{aligned} \quad (31)$$

Hence, for large enough distances,

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_n \rangle \sim S^2 \cos n\theta \exp \left[ -na \left( \frac{\Delta_0}{c_0} + \frac{\Delta_\theta}{c_\theta} \right) \right], \quad \text{for } na \gg \Delta_0^{-1}, \Delta_\theta^{-1}, \quad (32)$$

- *i.e.*, the correlation function falls off exponentially.

Let us now justify the HF decompositions or Wick contractions used in Eqs. (16) and (27). Naively, the four boson terms that appear in the product of two spins can be decomposed (or contracted) in three different ways. However, in our MF treatment, we have only allowed one possible decomposition. For example,  $X_{m,n}^\dagger X_{m,n} = a_{m\alpha}^\dagger a_{n\alpha}^\dagger a_{m\beta} a_{n\beta}$  is only decomposed as  $\langle a_{m\alpha}^\dagger a_{n\alpha}^\dagger \rangle a_{m\beta} a_{n\beta} + a_{m\alpha}^\dagger a_{n\alpha}^\dagger \langle a_{m\beta} a_{n\beta} \rangle$ . The other possible contractions  $\langle a_{m\alpha}^\dagger a_{n\beta} \rangle$  and  $\langle a_{m\alpha}^\dagger a_{m\beta} \rangle$  which are down by factors of  $1/3$ , because the  $SO(3)$  indices  $\alpha$  and  $\beta$  are not summed over, have been ignored. The justification for this treatment comes from the large- $N$  generalization of the model. The 3BR is generalized to an NBR ( $N$  boson representation) of  $SO(N)$  ‘spins’. In the  $N \rightarrow \infty$  limit, the other contractions which are down by a factor of  $1/N$  can certainly be ignored.

Our large- $N$  generalization is similar in spirit to the generalization of the 2BR of  $SU(2)$  spins to the  $N$  boson representation of  $SU(N)$  spins discussed in Ref. [11]. But just for completeness, we mention some details of our large- $N$  model. We write the components of an  $SO(N)$  spin  $S_\alpha$  as

$$S_\alpha = -i g_{\beta\gamma}^\alpha a_\beta^\dagger a_\gamma \quad (33)$$

with  $\alpha = 1, \dots, N(N-1)/2$  and  $\beta, \gamma = 1, \dots, N$ . Furthermore, we have the relations

$$g_{\beta\gamma}^\alpha = -g_{\gamma\beta}^\alpha \quad \text{and} \quad g_{\beta\gamma}^\alpha g_{\delta\epsilon}^\alpha = \delta_{\beta\delta} \delta_{\gamma\epsilon} - \delta_{\beta\epsilon} \delta_{\gamma\delta}. \quad (34)$$

To reproduce the spin algebra and the correct number of states, the constraints in Eq. (18) are now replaced by

$$a_\alpha^\dagger a_\alpha = NS/3 \quad \text{and} \quad a_\alpha^\dagger a_\alpha^\dagger a_\beta a_\beta = 0. \quad (35)$$

Clearly, this representation works only if  $NS/3$  is an integer. The Hamiltonian for general  $N$  given by

$$H = \frac{3}{N} \sum_n ( \mathbf{S}_n \cdot \mathbf{S}_{n+1} + \delta \mathbf{S}_n \cdot \mathbf{S}_{n+2} ) \quad (36)$$

can be written in terms of the bosons in Eq. (33) along with the Lagrange multiplier fields to enforce the constraints just as was done earlier. In fact, the entire analysis can be reproduced. Here, however, our aim in introducing the large- $N$  formalism was only to justify the HF decoupling procedure that we used.

To conclude, a notable feature of our analysis is that we do not need to rotate the spins of the ground state of interest in order to make it look FM and then proceed with the analysis. For the 2BR, such a rotation is usually performed [11]-[13]. We are currently using the 3BR to study frustrated spin models in two dimensions [17].

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