

RECENT DEVELOPMENTS IN LATTICE QCD

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ABSTRACT

I review the current status of several lattice QCD results. I concentrate on new analytical developments and on numerical results relevant to phenomenology.

1. Introduction

Lattice regularisation of field theories provides a framework for their rigorously non-perturbative study from the first principle. Such a formulation is useful for studying both fixed-point field theories (where the lattice regulator is ultimately removed by taking the scaling limit), and effective field theories (where the lattice regulator acts as a fixed cutoff beyond which the theory loses its meaning). After almost a decade in hibernation—testing algorithms, optimising parameters, building computers—lattice studies have now reached a stage where phenomenologically useful results are beginning to be produced with all the systematic errors under control. I review here some of the recent exciting developments, but also remind the reader that there is still a long way to go and many new things to learn.

A convenient way to write down the lattice theory is in the Euclidean path integral framework, where the integration variables are defined on a hypercubic space-time grid and the Lagrangian density is discretised by replacing derivatives by finite differences. Then we can express the expectation values of observables, for instance in QCD [1], as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU] [d\bar{\psi} d\psi] \mathcal{O} \exp[-S_G - S_F] , \quad (1.1)$$

$$Z = \int [dU] [d\bar{\psi} d\psi] \exp[-S_G - S_F] . \quad (1.2)$$

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Here the group matrices U represent the gauge connection between adjacent lattice sites, ψ are the quark fields, S_G and S_F respectively are the discretised gauge and fermion actions. The simplest choice for the gauge action is the plaquette action:

$$S_G = \beta \sum_{x,\mu < \nu} [1 - \text{Tr}(U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\nu}^\dagger)] \quad , \quad \beta \equiv 6/g^2 \quad . \quad (1.3)$$

The two popular lattice fermion schemes are: (a) Wilson fermions corresponding to (the hopping parameter κ controls the quark mass)

$$S_F^W = \sum_x \bar{\psi}_x \psi_x - \kappa \sum_{x,\mu} [\bar{\psi}_x (1 - \gamma_\mu) U_{x,\mu} \psi_{x+\mu} + \bar{\psi}_{x+\mu} (1 + \gamma_\mu) U_{x,\mu}^\dagger \psi_x] \quad , \quad (1.4)$$

and (b) staggered fermions corresponding to (χ, η_μ are the spin-diagonalised forms of ψ, γ_μ)

$$S_F^S = m \sum_x \bar{\chi}_x \chi_x + \frac{1}{2} \sum_{x,\mu} [\bar{\chi}_x \eta_\mu U_{x,\mu} \chi_{x+\mu} - \bar{\chi}_{x+\mu} \eta_\mu U_{x,\mu}^\dagger \chi_x] \quad . \quad (1.5)$$

The lattice theory does not have the same symmetry properties as the continuum field theory. It is anticipated that the explicitly broken symmetries (e.g. rotational and chiral) would be recovered in the continuum limit (as the lattice spacing a is taken to zero holding the physical scale fixed), and the numerical evidence indeed points in this direction.

The Monte Carlo importance sampling method to evaluate the path integral is a brute force statistical analysis. Nonetheless, it is attractive because the calculation does not have any free parameters other than the gauge coupling and the quark masses. The coupling $g = g(a)$ is a function of the lattice cutoff. It is asymptotically free and determines the scaling behaviour as one approaches the continuum limit. The quark masses are adjustable parameters which can be freely varied between the non-relativistic quark model case ($m \rightarrow \infty$) and the chiral limit ($m \rightarrow 0$). Such a freedom to choose parameters is a tremendous advantage in understanding the relativistic effects and sea quark contributions. The lattice theory does not have an inherent scale; all lattice results come out in units of the spacing a . The absolute value of the lattice cutoff a has to be fixed by assigning some dimensionful physical quantity its experimental value, and afterwards the results can be expressed in physical units (say in GeV). In other words, only dimensionless quantities, such as mass ratios, are uniquely predicted in lattice calculations.

Computer simulations can of course deal with only a finite system. The available computer power dictates the number of points that can be simulated on a space-time grid, and then the parameters have to be chosen so as to keep the largest correlation length

well within the finite box. To extract physical information out of such a truncated lattice world, three crucial extrapolations are necessary:

- (1) The thermodynamic limit $L \rightarrow \infty$ removes the infrared cutoff.
- (2) The continuum limit $a \rightarrow 0$ removes the ultraviolet cutoff.
- (3) The chiral limit $m \rightarrow 0$ allows one to reach realistic quark masses.

These extrapolations help us convert results obtained on a finite lattice, with non-zero lattice spacing and not so light quark masses to physical numbers. They are governed by specific scaling rules—the finite volume scaling, the renormalisation group evolution and the chiral perturbative expansion respectively. There are two aspects to applying these rules for removing lattice artifacts: (i) the analytical functional forms governing the limiting behaviour, and (ii) the maximum values of L^{-1} , a and m from where one can safely extrapolate keeping the systematic uncertainties under control. The details of the former have been worked out over the past several years for many quantities of interest, while the latter have to be determined empirically by studying lattice results obtained with different values of L^{-1} , a and m .

The numerical results have greatly improved over the years due to innovations in designing special purpose parallel computers, in finding optimal simulation algorithms and in estimating errors of the inherently statistical results. The available computer power has roughly grown as $CPU \propto e^{year}$ over the last decade. The algorithms for the pure gauge theory have evolved to successfully face the problem of critical slowing down. To date, however, there is no algorithm which can simulate the full theory of QCD (i.e. including light dynamical quarks) at a satisfactorily fast rate using the currently available computers. It is the combination of both analytical and numerical techniques that has brought the subject of lattice QCD to the stage it has reached today, and developments on both these fronts are needed to make the results still more substantive in future.

Table 1 shows the current status of lattice QCD results for various physical observables. In many cases, stable results have been obtained within the quenched approximation. This is actually an uncontrolled approximation, where all the vacuum polarisation quark loops are ignored (or rather absorbed in the renormalisation of the gauge coupling). Its computationally less intensive nature, however, has made it quite popular. A priori one doesn't know how much to trust these results, but in practice they often turn out to be not too far off the real numbers. Simulations of the full theory have become feasible with the rapid development of supercomputers, but all of the calculations so far have been at the qualitative and exploratory level. One knows how to attack various problems, and the

Present Status of Lattice QCD Calculations			
Hadronic Property	Measured Quantity	Quenched	Full Theory
Light Hadron Spectrum	Mesons and Baryons	S	Q
	Glueballs	S	A
Topological Structure	Susceptibility	Q	A
Chiral Symmetry	$\langle \bar{q}q \rangle$	S	Q
Finite Temperature QCD	T_c , Latent Heat, Screening Lengths	S	Q
Decay Constants	f_π, f_K, f_ρ	S	Q
$SU(3)$ Mass Splittings	$\langle h \bar{q}\lambda_8 q h\rangle$	S	Q
Non-singlet Axial Couplings	g_A, F_A, D_A	S	Q
Magnetic Moments	Baryon Octet	Q	-
Form Factors	Electromagnetic	Q	-
Twist-2 Structure Functions	$\bar{q} \overleftrightarrow{D}_\mu \overleftrightarrow{D}_\nu q$	Q	Q
Singlet Scalar Coupling	$\pi - N$ σ -term	S	Q
Singlet Axial Coupling	g_1	A	A
θ_{QCD} Influence	Neutron Electric Dipole Moment	A	-
Final State Interactions	$I = 2$ $\pi - \pi$ Scattering	S	Q
$K^0 - \bar{K}^0$ Mixing	B_K, ϵ	S	A
Direct CP-violation	ϵ'	S	A
$\Delta I = \frac{1}{2}$ Rule	$K \rightarrow \pi\pi$ Decays	Q	-
$D\bar{D}, B\bar{B}$ Mixing	B_D, B_B, f_D, f_B	S	-
D, B decays	Semi-leptonic and Non-leptonic	Q	-
Heavy Quark Spectrum	D, B, ψ, Υ States	S	-

Table 1: Current status of results obtained from lattice QCD Monte Carlo simulations. S=Stable, Q=Qualitative, A=Attempted.

formal machinery has been set up. The future objective is to first verify and refine what we already know about QCD from indirect methods (quark models, perturbation theory, spectral sum rules, large- N_c expansions etc.), and then proceed on to predict unknown parameters and new phenomena.

The topics I have selected for discussion below are only a sample of the many re-

sults obtained in lattice QCD, the selection being based on my own judgement of the bearing the topics have on open questions and phenomenological issues. There are also non-perturbative problems other than QCD, where lattice technology is making important contributions these days. These include the electro-weak sector of the standard model (Higgs and Yukawa theories), random surfaces (quantum gravity), correlated electron systems (high T_c superconductivity) and numerous models of statistical mechanics. The interested reader is referred to the recent proceedings of the annual lattice field theory meetings [2] [3] [4] for further details.

2. Analytical Developments

2.1. Infrared Limit of QCD Strings

Regge phenomenology, for instance linear trajectories in the $m^2 - J$ plane for various hadron multiplets, was the origin of string theory for hadrons. 't Hooft's $1/N$ expansion performed a topological reorganisation of weak coupling perturbation series for $SU(N)$ gauge theories [5], while Wilson's strong coupling expansion provided an explicit realisation of how a lattice gauge theory may resemble a string theory [1]. Forming a connection between asymptotic freedom at short distances and an effective string picture at long distances has remained a tempting idea for many years. Numerical results for pure gauge lattice QCD demonstrate how the static potential between a quark-antiquark pair changes over from the short distance Coulomb form to a long distance linear confining form (see Fig. 3 below), and even provide a profile of the flux tube [6]. The advances in conformal field theory in recent years have brought us to a stage where we can try to put together various pieces of the puzzle and take concrete steps towards understanding the nature of the infrared limit of QCD strings [7] [8].

A string theory of QCD has to be interpreted in an effective field theory language without worrying about renormalisability or other ultraviolet problems. A careful quantisation of the Nambu-Goto string away from its critical dimension gives an effective string action, which involves the induced (rather than Liouville) metric and which can be expanded in terms of higher dimensional operators [9]. This is not a free theory of world sheet fields X^μ . In fact it is known that the gauge theory strings have non-local contact interactions [10]. Lattice strong coupling expansions show that these contact interactions are repulsive in character, leading even to self-avoiding surfaces in some particular cases [11].

The high temperature behaviour of the free energy of the QCD flux tube can be calculated [12]. A comparison of the result with that for the Nambu-Goto string shows that the two do not agree unless more and more excitations become available to the Nambu-Goto string with increasing temperature. The QCD string, like the Nielsen-Olesen vortex which is also a gauge string, is “fat” and the extra degrees of freedom needed may just be the internal shape excitations. It is also possible that a more appropriate description of the QCD string can be given in terms of (a) a rigid string which is asymptotically free but not unitary [13], (b) a string with Dirichlet boundary conditions [14].

Gliozzi has emphasised the restrictions on the effective string picture arising from the underlying gauge theory [8]. The gauge string carries a colour electric flux, so the effective theory must possess a Z_2 automorphism corresponding to $q \leftrightarrow \bar{q}$ or $A_\mu \leftrightarrow -A_\mu$. The presence of the flux also means that the surface is orientable and its two ends are incompatible (a q can annihilate with a \bar{q} but not with another q). This incompatibility implies that the lowest mode propagating along the string world sheet cannot be the ground state with conformal weight $h = 0$; the lowest physical mode must have $h > 0$. The two ends of the colour flux tube can be distinguished by supplementing the theory with an additional conserved quantum number. Gliozzi uses for this purpose the winding number of the free boson compactified on a circle with radius R_f corresponding to the finite thickness of the string. In such a case, a free fermion mode (massive soliton) can exist in the spectrum as an allowed topological excitation. Such a fermion obeys the gauge theory constraints and naturally accounts for the repulsive character of the string self-interactions. Gliozzi’s conjecture is that this fermion is the lowest excitation characterising the infrared limit of QCD string. This situation corresponds to a $c = 1$ conformal field theory where the lowest excitation is a fermion with conformal weight $h = 1/32$. Under this assumption, dimensionless ratios such as $T_c/\sqrt{\sigma}$, $m_G/\sqrt{\sigma}$, $R_f\sqrt{\sigma}$ have been calculated [15]. They turn out to be universal numbers dependent only on the embedding dimension of the string and independent of the gauge group, and not too far off from the numerical lattice results. It also can be argued that at the deconfinement temperature, the effective string theory has only a discrete set of degrees of freedom, i.e. it becomes a topological conformal field theory [8].

2.2. Lattice Gauge Theory in Terms of Dual Variables

Sharatchandra and collaborators have carried out exact duality transformations for lattice gauge theories. Such transformations separate topological degrees of freedom and thus isolate the effects of compactness of the gauge group. For example, the $U(1)$ pure gauge theory in $3 + 1$ dimensions can be written as [16]:

$$Z = \sum_{p_\mu \in \mathbb{Z}} \exp[-g^2 \sum_{x, \mu \neq \nu} (\Delta_\mu p_{x, \nu} - \Delta_\nu p_{x, \mu})^2] , \quad (2.1)$$

where $p_{x, \mu}$ is the integer valued dual vector potential living on the links of the lattice and Δ_μ stands for the discretised derivative. For the more interesting case of non-Abelian gauge theories, the Gauss's law constraint can be solved exactly in the Hamiltonian formulation [17]. This reduces the dynamics to local gauge invariant variables which create and annihilate a unit of colour electric flux. The dynamics then can be recast into an Abelian gauge theory framework [18]. This formulation places on a firm footing 't Hooft's conjecture [19] that topological excitations of the Abelian subgroup of $SU(N)$ determine the confinement mechanism. The formulation is also closely related to the discretised models of random surfaces and quantum gravity. For instance, in the case of $SU(2)$, the lattice can be interpreted as a discretised membrane with half-integral link lengths obeying triangle inequality of angular momentum addition [17]. A generalisation to the $SU(3)$ theory in terms of the integers characterising the various irreducible representations has also been carried out [20]. It is also straightforward to convert the formulation to the Lagrangian framework, e.g. the $SU(2)$ theory on a hypercubic lattice in $2 + 1$ dimensions can be rewritten as a sum over products of $6j$ -symbols resulting from addition of angular momenta [21]. Such explicit transformations with their elegant geometrical interpretations need to be explored further. A combination of these techniques with an appropriately chosen lattice (e.g. as in Ref. [11]), of course assuming universality, might lead to important new results. It would also be interesting to see how the dynamics of the $SU(2)$ gauge theory (characterised by half-integers) differs from that of the $SO(3)$ gauge theory (characterised by integers).

2.3. Rigorous Inequalities

One can derive rigorous inequalities among correlation functions for vector-like gauge theories such as QCD. The basis of such inequalities is the positivity of the measure in the Euclidean path integral:

$$[dA_\mu] [d\bar{\psi} d\psi] \exp[-S_G - S_F] \geq 0 . \quad (2.2)$$

This property has been exploited, both on the lattice [22] and in the continuum [23], to derive inequalities among 2–point correlation functions. Such inequalities yielded the result, for example, that the pion is the lightest hadron.

These arguments can be extended to multi-point correlation functions [24]. Consider the correlation functions of 4–Fermion operators between two pseudoscalar meson states at zero spatial momentum. The Cauchy-Schwarz inequality can be applied to the “off-shell” correlation functions (both the mesons on the same side of the operator) for operators with Dirac tensor structure $\Gamma \otimes \Gamma^\dagger$. The matrix elements are then bounded from below by their values in the Vacuum Saturation Approximation (VSA). This inequality can be extended to the chiral limit, since the “off-shell” threshold amplitude is real and the final state interactions of the psuedo-Goldstone bosons vanish as powers of momenta in the chiral limit. Therefore, for matrix elements which have a smooth chiral limit (e.g. the $\Delta I = 3/2$ and the $\Delta S = 2$ operators fall in this class), the inequality in the chiral limit provides a reasonable indication of the size of the “on-shell” matrix elements at finite quark mass.

The largest matrix elements are obviously the ones with the structure $\gamma_5 \otimes \gamma_5$. In practice, the VSA almost saturates the matrix elements in this case [25]. Furthermore, the numerical results show that the matrix elements for $\gamma_5 \otimes \gamma_5$ are larger by an order of magnitude or more compared to those of other Dirac tensor structures. It follows that when $\gamma_5 \otimes \gamma_5$ occurs as one of the terms amongst the various contractions of the correlation function at tree level, it totally overwhelms all the other terms. (Note that the matrix elements of $\gamma_5 \otimes \gamma_5$ go to a constant in the chiral limit and are not suppressed.) The resulting B –parameter for the full correlation function then is not far from 1, even though the inequality strictly does not hold. This is the case for the electro-penguin operators [26], whose B –parameters turn out to be within 10% of unity.

We also note that the correlation inequality holds for the bare correlation functions without any reference to the cutoff scale, while the matrix elements appearing in it may have non-vanishing anomalous dimensions and be scale dependent. In such a case, the anomalous dimensions must satisfy the constraint

$$\gamma_{\mathcal{O}\mathcal{O}^\dagger} \geq 2\gamma_{\mathcal{O}} \quad , \quad (2.3)$$

so that the correlation inequality holds at any arbitrary scale. In cases where $\mathcal{O}\mathcal{O}^\dagger$ and/or \mathcal{O} are not eigenstates of the anomalous dimension matrix, the result of Eq.(2.3) applies to the largest anomalous dimensions occuring in the decomposition of the operators among various eigenstates.

Similar correlation inequalities can also be applied to a system in a finite volume [24]. Lüscher’s analysis [27] shows how the scattering length can be extracted from finite volume correlation functions measured at two particle threshold. Explicitly, the lowest energy of a two particle state in a sufficiently large cubic box of length L differs from twice the energy of a single particle state by an amount proportional to the scattering length a_0 :

$$\delta E = E_2 - 2E_1 \propto -\frac{a_0}{L^3} . \quad (2.4)$$

The inequality then tells us that the interaction between two mesons in the gluon exchange channel is attractive at threshold, i.e. the scattering length is positive. It also can be deduced that the total threshold interaction between two pseudoscalar mesons is attractive in the flavour antisymmetric representation (e.g. $\mathbf{20}$ of flavor $SU(4)$).

In case of QED (which is a vector gauge theory too) the interaction between neutral atoms is dominated by the massless photon exchange compared to the massive electron exchange, at least at long distances. With some additional assumptions (the atomic propagator in individual QED field configurations is complex in general), it can be shown that the photon exchange polarization interaction is always attractive [24]. This is an essential ingredient in understanding why any gas condenses in to a liquid at low temperatures.

For hadrons containing a single heavy quark, the fact that the heavy quark propagator is just a unitary matrix, can be exploited to derive a lower bound on the $\bar{\Lambda}$ parameters of the heavy quark effective theory [28]. These $\bar{\Lambda}$ parameters are the differences between the hadron masses and the mass parameter for the heavy quark, and they characterise the $1/m$ corrections in the heavy quark effective theory.

2.4. Chiral Singularity in the Quenched Approximation

The neglect of dynamical quark loops leaves the quenched theory non-unitary. One of the consequences of this rather adhoc approximation is that there is an additional unphysical pseudo-Goldstone boson in the chiral limit—the flavour singlet η' . This means that in the surrounding cloud, the quenched hadrons have an extra η' that is absent in the full QCD. The pseudoscalar meson cloud is an important part of the structure of the hadrons which manifests itself at small quark masses in terms of chiral logarithms characterising the infrared chiral singularity. The differences in chiral logarithms between the quenched and the full theory results can be evaluated in chiral perturbation theory by writing the hadron propagators in terms of quark lines and omitting the diagrams with

closed quark loops [29], or by adding supersymmetric scalar ghosts to the QCD Lagrangian such that the unphysical ghost loops cancel the effect of the closed quark loops [30]. It turns out that the unphysical behaviour of the η' gives rise to undesirable singularities in the chiral limit of the quenched theory. For example, the quark condensate diverges and the pseudoscalar meson mass does not obey the GellMann-Oakes-Renner formula. The implications of such a behaviour are not yet clear. The quenched approximation has been of enormous use in many calculations upto date (mainly because at present even the best algorithms and the fastest computers are not good enough to do a reasonable job of simulating the full theory), so one doesn't want to discard it rightaway. The optimistic view is that the effects of the unphysical η' cloud are negligible as long as one works above a certain quark mass.

2.5. Lattice Artifacts, Improved Actions and Operators

The discretised lattice theory with a finite cutoff contains, relative to its continuum analogue, non-universal higher dimensional terms suppressed by powers of the lattice spacing (modulo logarithmic renormalisations). These terms must be either removed or made negligible by a suitable choice of lattice parameters, before scaling the lattice results to the continuum limit. Monte Carlo Renormalisation Group approach has shown that for the simple plaquette action, one at least needs $\beta \geq 6.2$ to get to within 10% of the asymptotic scaling behaviour [31]. This result is also confirmed by looking at the scaling behaviour of physical observables such as string tension, glueball masses and the phase transition temperature. When the quarks are included, a good criterion for judging the scaling behaviour is to look at how fast the masses of particles belonging to the same continuum multiplet but different lattice multiplets come together. Again the results for the quenched theory confirm the above estimate [32].

In principle the lattice artifacts causing departures from the scaling behaviour can be systematically eliminated by improving both the action and the operators. The methodology for a perturbative improvement program was outlined by Symanzik [33], while the Monte Carlo Renormalisation Group approach yields a non-perturbatively improved action as a byproduct. Redundant operators (i.e. the ones that can be eliminated using the classical equations of motion) are a great convenience in simplifying the various terms appearing in an improved action. For the pure gauge theory, the scaling violations are

$O(a^2)$, and both perturbative [34] and non-perturbative [35] approaches find that a negative admixture of 1×2 Wilson loop is required to improve the scaling behaviour of the action.

Improvement for the fermions is more important, since they have stronger scaling violations, at $O(a)$. Moreover, the divergences arising out of explicit breaking of the continuum symmetries on the lattice can make even the leading scaling behaviour non-perturbative. For example, the breaking of chiral symmetries gives rise to linear ultraviolet divergences [36], which when combined with lattice artifacts formally suppressed by powers of a give corrections which are suppressed only as $O(g^2) = O(1/\ln(a))$. These corrections are, therefore, more important than the $O(a)$ corrections for small enough a . When they do not mix the lattice operators of interest with lower dimensional operators, they can be expressed as renormalisation Z -factors relating the continuum and the lattice versions of the operators:

$$\mathcal{O} = Z_{\mathcal{O}}^L \mathcal{O}^L . \quad (2.5)$$

$Z_{\mathcal{O}}^L$ are functions of the lattice spacing a and should ideally be determined non-perturbatively. They have been discussed extensively in the literature [37], and it turns out that they can be estimated reasonably well using the mean field improved perturbation theory described in the following subsection.

Next consider the $O(a)$ terms. The Wilson term in the action used for eliminating the fermion doubling is $O(D^2a)$ and falls in this class. It is found that if the fermion doubling problem is removed using the redundant interaction $\bar{\psi}(\not{D} + m)^2\psi$ instead of the Wilson term $\bar{\psi}D^2\psi$, then these artifacts can be pushed to $O(a^2)$ [38]. In addition to improving the action (or propagators), it is necessary to have an improvement program for the operators [39], to completely get rid of all the $O(a)$ terms. Such terms include logarithmic corrections of type $O(g^2a \ln a)$ arising from renormalisations, which become $O(a)$ corrections after taking in to account the scaling of g^2 . For staggered fermions the scaling violations arising from the quark propagators are already $O(a^2)$; it is solely the lattice representation of the operators that may give rise to $O(a)$ artifacts.

Once the lattice artifacts have been removed, the standard weak coupling perturbation theory can be used, say at 1-loop and at a particular scale, to perform an appropriate matching between the lattice and continuum renormalisation schemes. The exception to this rule occurs when there is an unwanted mixing of lattice operators in to lower dimensional operators. Such mixing has to be gotten rid of non-perturbatively by imposing

a physical constraint. The Wilson fermion approach, due to lack of chiral symmetry, suffers from this problem, e.g. the 4-Fermion operators describing the $\Delta I = \frac{1}{2}$ weak decays get mixed with the quark bilinear operators. Such a handicap is not present in the staggered fermion approach, making it easier to relate the staggered matrix elements to the continuum ones.

2.6. Lattice Perturbation Theory with Tadpole Summation

Weak coupling perturbation theory is essential for proper renormalisation of lattice operators so that they can be matched with their continuum analogues. Assuming that the lattice spacing a is small, the straightforward perturbative expansion is

$$U_\mu \equiv \exp[iagA_\mu] = 1 + iagA_\mu + \dots . \quad (2.6)$$

The quantum corrections to this expansion, however, do not vanish as powers of a . On the lattice, the contractions of the A_μ 's with each other generate ultraviolet divergences which can cancel the additional powers of a . The most troublesome of these contractions are the quadratically divergent tadpole diagrams, which are absent in the continuum but present on the lattice. The quadratic divergence precisely cancels the powers of a accompanying g , leaving behind terms which are suppressed only by powers of g^2 and not of a . Lepage and Mackenzie have proposed a mean field method to suppress these uncomfortably large corrections [40]. In this method the tadpoles are summed up modifying Eq.(2.6) to

$$U_\mu = u_0(1 + iagA_\mu + \dots) , \quad (2.7)$$

with the convention that all tadpole diagrams are to be dropped from the perturbative calculations of renormalisation constants. u_0 here is a non-perturbative parameter to be taken from the numerical simulations; convenient choices are the Landau gauge expectation value of the gauge link, or the fourth root of the expectation value of the plaquette. Rewriting the lattice gauge action so as to make the factors of u_0 explicit, one finds that the effective gauge coupling to be used in perturbative calculations becomes $g_{eff}^2 = g^2/u_0^4$. Similarly the continuum fermion fields are better described by the operators $\sqrt{2u_0}\kappa\psi$, $\sqrt{u_0}\chi$.

This rewriting of perturbation theory has brought the numerical data for $g^2 \approx 1$ in good agreement with scaling [40]; asymptotic scaling in terms of the bare lattice coupling fails miserably for these data.

2.7. Renormalizations in Heavy Quark Effective Theory

In the limit of infinite quark mass, the dynamics of QCD is invariant under spin-flavour symmetries of the heavy quark degrees of freedom. This has been exploited by Isgur and Wise, for hadrons containing a single heavy quark, to relate many form factors and to reduce many matrix elements down to a few unknown functions [41]. Despite the impressive simplifications arising in the limiting case, it is a must to estimate the size of the $O(1/m)$ corrections to the leading terms, in order to attach proper meaning to the results and obtain reliable predictions for physical heavy quark states. The virtue of the formalism of heavy quark effective theory, developed over the past few years, lies in providing a model independent $m \rightarrow \infty$ limit, which can be systematically improved with power series expansions in $1/m$ [42].

In the continuum, the problem is commonly studied using the static (i.e. constant velocity frame) formulation. This language can handle the $O(1/m)$ corrections but not the $O(\alpha_s/m)$ ones, and hence cannot be applied to hadrons containing more than one heavy quark [43]. On the lattice, however, there is no such restriction on the dynamics of the heavy quark. The simplest choice is to just use the standard fermion action with a large mass and study the mass dependence of various matrix elements. In such a case, as explained below, proper normalisation factors have to be included to obtain sensible results.

At finite lattice spacing, even for a free field theory, the mass parameter in the Lagrangian does not agree with the position of the pole in the propagator. This mismatch has to be eliminated when the matrix elements are extracted using the LSZ reduction formula. In the mean field theory approach, a tree level wavefunction renormalisation is sufficient to get rid of the problem [44]. Combining this correction factor with the tadpole summation factor discussed in the previous subsection, the effective fermion masses to be matched with a continuum theory become

$$m_{eff}^W = \ln\left[1 + \frac{1}{2u_0}\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right)\right] , \quad m_{eff}^S = \sinh^{-1}(m/u_0) , \quad (2.8)$$

while the effective lattice fields representing the continuum fermions take the form

$$\psi_{eff} = [1 + 2u_0\kappa - (\kappa/\kappa_c)]^{1/2}\psi , \quad \chi_{eff} = [u_0^2 + m^2]^{1/4}\chi . \quad (2.9)$$

Clearly the effect of the correction factors becomes more and more important as the lattice masses increase. It is easy to see, for example using a hopping parameter expansion, that

one must incorporate these factors to obtain the correct values for the matrix elements in the infinite quark mass limit [45].

An alternative is to use the non-relativistic expansion of the fermion action with the expansion coefficients determined by appropriate matching conditions [46]. The mass term is dropped altogether, and the matrix elements of leading as well as subleading operators are measured in the effective theory. This effective theory, however, is not renormalisable and contains power law divergences. This feature is reflected in non-perturbative contributions to the expansion coefficients. The debate on how well these coefficients can be estimated is not settled as yet.

The mean field prescription is able to get rid of lattice artifacts which are $O(ma)$, but not those which are $O(\Lambda_{QCD}a)$. Thus fractional errors of $O(\Lambda_{QCD}/m)$ are left behind. Maiani *et al.* argue that the only way out is to fix the coefficients using non-perturbative matching conditions [47], in general reducing the predictive power of the theory. Lepage *et al.* argue that the power law divergences do not give rise to large non-perturbative corrections for the couplings typically used in lattice simulations—by restricting m to be of order $1/a$ or smaller and using the mean field improved perturbation theory to estimate the coefficients, the fractional error can be reduced to the level of a few percent [48]. The issue should get resolved when the tests of the mean field improved perturbation theory [40] are pushed to the stage where non-perturbative terms show up.

There is yet another approach to the problem possible, based on the fact that the heavy quark effective theory treats space and time asymmetrically. Perhaps a better description of the continuum physics can be obtained if space and time hopping are treated differently on the lattice too [49]. This is a topic requiring further study.

3. Spectral Results

In Euclidean (imaginary) time, particle propagators evolve as $\exp(-E\tau)$, so the farther they propagate the more they get dominated by the lowest energy eigenstates. It is therefore straightforward to extract the masses of the lowest eigenstates from the asymptotic behaviour of propagators. A few excited states can also be handled, as long as they are stable against strong decay, by applying variational techniques to the wavefunctions representing the creation/annihilation operators. Unstable states (i.e. resonances) are not easy to deal with, and no satisfactory solution to extraction of their properties exists yet.

Some progress has been made though in relating the behaviour of multi-particle states in a finite volume to properties of resonances [50].

Considerable improvement in the signal to noise ratio can be achieved, if the operators are designed so as to maximise their overlap with the desired states and reduce the contamination from excited states. It is here where one's intuition about hadronic wavefunctions (based on phenomenological models) is helpful, and a lot of progress has been made in this direction in recent years. Wavefunctions can be directly determined on the lattice using spatially extended hadron operators. For heavy-light mesons, Coulomb gauge wavefunctions evaluated for a spinless relativistic quark moving in the static $q\bar{q}$ potential (which can be calculated on the same lattices) are excellent approximations to the actual distributions [51]. For light hadrons, Coulomb gauge wavefunctions provide a reasonable description of their internal structure [52], but there is enough room for finding a better prescription. It should be noted that even though such wavefunctions correspond to an uncontrolled truncation in a non-abelian gauge theory, they are a useful starting point for understanding the hadron structure and they certainly improve the signal to noise ratio.

A typical set up in computing matrix elements is to first calculate the Green's functions with the desired interaction operator and the appropriate incoming and outgoing states, let the time separations become large enough so that the lightest states saturate the correlation functions, and then remove the factors corresponding to the external legs following the LSZ reduction technique. Schematically,

$$\begin{aligned} \langle \mathcal{O}_f(\tau_f) | \mathcal{O}_{int} | \mathcal{O}_i(\tau_i) \rangle &= \sum_{j,k} \langle \phi_j | \mathcal{O}_{int} | \phi_k \rangle e^{-E_j \tau_f} e^{E_k \tau_i} \langle \mathcal{O}_f | \phi_j \rangle \langle \phi_k | \mathcal{O}_i \rangle \\ &\xrightarrow[\tau_i \rightarrow -\infty]{\tau_f \rightarrow \infty} \langle h_f | \mathcal{O}_{int} | h_i \rangle e^{-E_f \tau_f} e^{E_i \tau_i} \langle \mathcal{O}_f | h_f \rangle \langle h_i | \mathcal{O}_i \rangle . \end{aligned} \quad (3.1)$$

Here $\phi_j(\phi_k)$ denote all the states consistent with the quantum numbers corresponding to the annihilation (creation) operators $\mathcal{O}_f(\mathcal{O}_i)$, and $h_f(h_i)$ are the lowest ones amongst them.

3.1. Light Hadron Spectrum

The most easily calculable hadron masses on the lattice are those of the pseudoscalar and vector mesons and the spin- $\frac{1}{2}$ baryon. It has become customary to denote them as m_π , m_ρ and m_N (N stands for the nucleon) respectively. These masses are determined as function of the quark mass on the lattice, and a convenient way to represent the results

Figure 1: APE mass ratio plot for the quenched Wilson fermion spectrum.

is in terms of dimensionless mass ratios. One such plot, called the APE plot, depicting the behaviour of m_N/m_ρ vs. $(m_\pi/m_\rho)^2$ is shown in Fig. 1. With increasing quark mass, m_π/m_ρ monotonically varies from 0 to 1, each particular value for m_π/m_ρ representing a theory for which all other dimensionless ratios can be determined.

The data shown in Fig. 1 are for quenched Wilson fermions [53] [54] [55]. The lattice parameters for these simulations are shown in Table 1, together with the parameters of other recent calculations. They indicate, in addition to the size of the state of the art lattice simulations, our present understanding of the cutoff limits a_{max} and L_{min} for reliable extrapolations to the continuum.

The solid curves in the figure, constrained so as to pass through the physical points, represent the qualitative expectations based on phenomenological models. The curve on lower left is the behaviour expected from chiral perturbation theory, while the curve on the upper right is the trend in the heavy quark mass expansion. The important questions are how close the lattice results are to the phenomenological expectations and whether the lattice numbers can be extrapolated to the experimental point, indicated by a “?” in the figure. Before looking at various systematic differences, it should be noted that the quenched theory results do not have to agree with the real world—a priori we do not know what the results should look like. For example, the pion cloud around the hadrons is quite different in the quenched approximation compared to the real world. Also the quenched rho cannot decay and its mass can differ from the real rho mass by an amount comparable to its width [56]. Thus the apparent agreement, within $\approx 10\%$, between the quenched lattice results and phenomenological expectations has to be taken with caution. On one hand, it is a good sign that quenched lattice QCD is not completely off the mark. On the other hand, it may be just a lucky coincidence for these particular variables, which would not hold for some other observables (cf. subsection 2.4 and section 5).

There are several technical issues involved in understanding the lattice data. Contamination from excited states is present in the correlators and has to be carefully eliminated in order to extract the asymptotic mass values. The nucleon, with a small gap between the lowest and the first excited state, is particularly susceptible to this problem. Improved operators help, but even then the truly asymptotic signal may not be easy to get to [57] [58]. This effect has led to a variation amongst the numerical results quoted by various groups. The situation is expected to get clarified soon with the use of sophisticated hadron wavefunctions which couple more strongly to the ground states.

Due to lattice artifacts, we typically have

$$(m_N/m_\rho)_{\text{latt}} = (m_N/m_\rho)_{\text{cont}}[1 + O(a)] \quad . \quad (3.2)$$

The lattice data should thus converge towards a universal curve as $a \rightarrow 0$. Also, the masses are shifted from their infinite volume values, if the lattice size L is too small. The nucleon is physically larger than the mesons and hence more easily distorted by the finite box size. Empirical evidence over the last few years has shown that both the finite lattice spacing and finite volume effects shift the APE curve upwards.

Fig. 1 shows that for the usual Wilson fermions, with $\beta \geq 6.0$, the scaling violations in mass ratios are reasonably under control. More encouraging results have been obtained

Ref.	β	Size	$a(\text{Fermi})$	$\pi/a(\text{GeV})$	$L(\text{Fermi})$
[53]	5.93	$24^3 \times 36$	1/9	5.7	2.7
[54]	6.00	$24^3 \times 32$	1/10	6.3	2.4
[57]	6.00	$24^3 \times 54$	1/10	6.3	2.4
[53]	6.17	$30 \times 32^2 \times 40$	1/12	7.5	2.6
[55]	6.30	$24^3 \times 32$	1/16	10.	1.5
[59]	6.00	$24^3 \times 32$	1/10	6.3	2.4
[60]	6.00	$24^3 \times 40$	1/10	6.3	2.4
[32]	6.20	$32^3 \times 48$	1/13	8.1	2.5
[32]	6.40	$32^3 \times 48$	2/35	11.	1.8

Table 2: Various lattice parameters of recent quenched hadron spectrum calculations. The upper and lower halves of the table correspond to Wilson and staggered fermion calculations respectively.

with an improved Wilson fermion action. Ref. [61] finds that with an $O(a)$ improved action, the scaling violations in mass ratios are substantially reduced between $\beta = 5.7$ and 6.0. There is not much to be gained with the improved action, however, for $\beta \geq 6.2$ [62]. This behaviour is in agreement with the logic that by using a more complicated action one can approach the continuum limit faster. The prospects for staggered fermions are not so bright; the scaling violations, though formally only of $O(a^2)$, are quite large at $\beta = 5.7$ [32].

An example of the finite volume effect on the hadron masses calculated with dynamical quarks is shown in Fig. 2 [63]. Asymptotically, the approach to the thermodynamic limit is dictated by the behaviour of the lightest states in the system—the pions. The corrections can be estimated as pion exchanges between nearest neighbour periodic images and behave as $\exp(-m_\pi L)$ [64]. In the intermediate range, however, the hadron wavefunctions are substantially squeezed. The corrections then are dominated by quark exchanges between all the images, and depend on the sign of the quark boundary conditions. The point particle description has to be softened by form factors and the dominant contribution comes from the zero mode, making the corrections behave like $1/L^3$ [63].

Putting all the evidence together, the numerical results show that in the quenched theory one can safely extrapolate to the continuum for $\beta \geq 6.2$. In the absence of detailed results, it is not yet possible to give such a precise bound for the full theory. The hadron

Figure 2: Finite volume dependence of hadron masses for two flavours of dynamical staggered quarks at $\beta = 5.7$ [63]. The thin lines are the predictions of the asymptotic virtual pion exchange formula, while the thick lines are fits using $1/L^3$ corrections arising from wavefunction squeezing.

masses get to within one or two percent of their infinite volume limits, for quarks heavier than half the strange quark mass, once the lattice size is more than 2.5 Fermi across [63][65].

For lighter quarks, barring the possibility of resonances, extrapolations made using chiral perturbation theory should be reliable. Having brought all the systematic errors within a few percent, it is merely a question of beating down the statistical errors by running a powerful enough computer for a long enough period and obtain quantitative predictions.

3.2. QCD Running Coupling $\alpha(q)$

In principle, it is straightforward to extract the QCD running coupling from lattice calculations. One picks a value for the bare lattice coupling β , calculates on the lattice a physically measurable dimensionful quantity (e.g. f_π), and extracts the value for the lattice cutoff a by comparing the lattice and the physical numbers. This procedure yields the running coupling $g^2(a)$ which can be evolved to other scales as well as converted to other regularisation schemes using perturbation theory.

This straightforward procedure has two important caveats requiring careful treatment:

- (1) Due to lattice artifacts the relation between lattice and continuum results has unphysical and non-universal corrections, e.g.

$$(f_\pi)_{\text{latt}} = (f_\pi)_{\text{cont}} a (1 + O(a)) \quad . \quad (3.3)$$

These scaling violations preclude a naive matching of lattice and continuum numbers.

- (2) The perturbation theory in the bare lattice coupling is not reliable. The problem is exemplified by the following relation between the couplings in the continuum $\overline{\text{MS}}$ and the lattices schemes [66]

$$\frac{1}{\alpha_{\overline{\text{MS}}}(\pi/a)} = \frac{1}{\alpha_L(a)} - 3.880 + O(\alpha) \quad . \quad (3.4)$$

Since $\alpha_L^{-1} \approx 4\pi$, the first order correction is rather large and higher order terms cannot be just ignored.

The first problem can be alleviated to some extent by improving the lattice action and operators as discussed in subsection 2.5 above. Still the remaining unwanted lattice artifacts (terms suppressed by various powers of a and g^2) have to be removed by performing simulations at a variety of bare lattice couplings, confirming that the scaling violations are of the expected nature, and then extrapolating the results to the continuum limit $a \rightarrow 0$.

The second problem makes it mandatory that all the results be expressed in an appropriate renormalisation scheme, where the higher order terms are better behaved. (This is similar in spirit to the shift from the MS to the $\overline{\text{MS}}$ scheme in continuum calculations, and

despite the fact that perturbation theory only provides an asymptotic series.) As discussed in subsection 2.6, the large coefficients originate mainly from the gauge field tadpole diagrams. Lepage and Mackenzie have argued [40] that a continuum-like coupling constant, e.g. $\alpha_{\overline{\text{MS}}}$ or α_{MOM} , with the scale $q \approx \pi/a$ would be a good expansion parameter. Their prescription is to cast all the perturbative expressions in terms of such a coupling constant, which considerably reduces the higher order coefficients, and then determine the coupling constant through a non-perturbatively measured lattice quantity. One simple choice is [40]:

$$\frac{1}{\alpha_{\overline{\text{MS}}}(\pi/a)} = \frac{1}{\alpha_P(a)} - 0.51 + O(\alpha) \quad , \quad \alpha_P = -\frac{3 \ln \langle \text{Tr} U_P \rangle}{4\pi} \quad . \quad (3.5)$$

where U_P is the product of gauge links around an elementary plaquette.

These steps have been followed in Ref. [67] [68] to extract the running coupling of pure gauge QCD, $\alpha_{\overline{\text{MS}}}^{(0)}$, from the static $q\bar{q}$ potential, $V(R)$. Fig. 3 illustrates the accuracy to which $V(R)$ has been determined in lattice simulations, and its remarkable agreement with a simple power series expansion in the $q\bar{q}$ separation R . The running coupling $\alpha_V(R^{-1})$ can be extracted from the short distance Coulomb behaviour of the potential, $V(R) \propto -\alpha_V(R^{-1})/R$. The running coupling $\alpha(\pi/a)$ can also be independently evaluated from the scaling of the string tension (extracted from the long distance part of the potential calculated at a variety of bare lattice couplings β). Both these determinations coincide nicely [67]. Indeed the fact that a single lattice simulation covers both perturbative and non-perturbative aspects of QCD, is an excellent demonstration that the same theory can successfully explain both the high energy jet physics as well as the low energy hadronic properties.

The outstanding drawback of these results for the pure gauge QCD is that there is no direct way to relate the zero quark flavour coupling constant, $\alpha_{\overline{\text{MS}}}^{(0)}$, to the real coupling constant involving a number of light quark flavours, $\alpha_{\overline{\text{MS}}}$. Such a relation necessarily involves non-perturbative physics, and at best one can estimate it using phenomenological models. (For example, assigning the pure gauge theory string tension a phenomenological value, $\sqrt{\sigma} = 0.44$ GeV, is only an educated guess.) The Fermilab group have made such an estimate for the coupling determined using the spin-averaged $1P - 1S$ splitting in the Charmonium system [69]. Charmonium is described well by potential models, so in essence one has to match the potentials of the full and the quenched theory at a distance corresponding to the Charmonium size, $R_{c\bar{c}} \sim 0.5$ Fermi. The two potentials then differ at shorter distances, and the difference can be estimated using the renormalisation group

Figure 3: Static $q\bar{q}$ potential as a function of their separation [67]. The lattice coupling is $\beta = 6.4$, and the short distance lattice artifacts violating rotational symmetry have been removed by subtracting from the lattice results the difference between lattice and continuum 1-gluon exchange potentials.

evolution. Since the quenched β -function is a bit too large, the short distance quenched coupling is a bit too small, i.e. the quenched potential is a bit too shallow. This calculation also has some technical advantages: (a) the orbital splitting is known to be quite insensitive to the value of the quark mass, so no careful tuning of the quark mass is necessary; (b) the quarkonium system is physically smaller than the light hadrons, making it easier to control finite size effects and to bound uncertainties in perturbative renormalisation group evolution.

All the results, converted to the $\overline{\text{MS}}$ scheme and evolved to the energy scale M_Z , are presented in Table 3 for comparison with the experimental numbers [70] [71]. It is seen that the lattice results lie slightly below the experimental ones. Not much should be made of the small discrepancy, which is likely to disappear as the lattice simulations improve and the dominant systematic uncertainty in the conversion from $\alpha_{\overline{\text{MS}}}^{(0)}$ to $\alpha_{\overline{\text{MS}}}$ reduces. Rather

Ref.	Observable	$\alpha_{\overline{\text{MS}}}^{(0)}(M_Z)$	$\alpha_{\overline{\text{MS}}}(M_Z)$
[70]	Experiment	—	0.1134(35)
[71]	Experiment	—	0.123(4)
[69]	$M_{1P} - M_{1S}$	0.0790(9)	0.105(4)
[67]	$q\bar{q}$ Potential	0.0796(3)	—
[68]	$q\bar{q}$ Potential	0.0801(9)	—

Table 3: Comparison of lattice results for $\alpha_{\overline{\text{MS}}}$ with experiment. Errors shown are statistical.

the close agreement between the experimental and lattice results, both in the value and in the size of the error, is a triumph of lattice QCD calculations.

In the calculations described above, a single simulation had to simultaneously obey the requirements for a_{max} and L_{min} . This limits the energy range over which the running coupling can be studied, since the lattice dimensions are constrained by the available computer power. Lüscher and collaborators have proposed an approach based on discrete renormalisation group to extend this energy range [72]. In this approach, computation of a non-perturbatively defined coupling constant on lattices of size L and $2L$, for the same bare coupling, provides an integral of the β -function between scales L and $2L$. Short distance lattice artifacts are eliminated by applying finite size scaling techniques to several pairs of results with sizes L/a and $2L/a$ and then extrapolating to $a \rightarrow 0$. By using multiple steps of this type, the separation between the low energy end (where a non-perturbative input such as the string tension sets the overall scale) and the high energy end (where perturbative scaling can be applied with precision) is extended. In the intervening range, the running coupling is determined non-perturbatively. Encouraging results have been obtained in this manner for the pure gauge $SU(2)$ theory [72].

3.3. $I = 2$ Pion Scattering Length

An understanding of the behaviour of multi-particle states below inelastic threshold can be obtained by turning the limitation of finite lattice volume in to an advantage. When the volume of the system is large enough so that the particle wavefunctions are not badly distorted by the finite box size, the shifts in the energy levels of two-particle states from

their infinite volume limit are related to their scattering phase shifts [73]. In the case of two pions in a finite box, the interaction energy at threshold is

$$\delta E = E_{2\pi} - 2m_\pi = -\frac{4\pi a_0^I}{m_\pi L^3} \left(1 + O\left(\frac{1}{L}\right)\right) , \quad (3.6)$$

where a_0^I is the s -wave scattering length for the two pions in an isospin I state. Numerically, the interaction energy is small in the region where this formula applies, and an easy way to extract δE is from the ratio

$$\frac{G_{2\pi}(0; \tau)}{(G_\pi(0; \tau))^2} \propto e^{-\delta E \tau} . \quad (3.7)$$

At present, the lattice data only allow extraction of the $I = 2$ scattering length, in which case quark annihilation diagrams are absent. (The annihilation diagrams, which are a necessary ingredient for the $I = 0$ scattering interaction, are in general noisy. Moreover, they are likely to suffer from large systematic uncertainty in the quenched approximation.) By extending the flavour symmetry group to $SU(4)$ on the lattice, scattering lengths in all possible flavour representations can be calculated. The calculations have been carried out for both staggered and Wilson fermions [74] [75], and the results compare well with the predictions of lowest order chiral perturbation theory [76], e.g.

$$4\pi a_0^{I=2} f_\pi^2 / m_\pi = -1/4 . \quad (3.8)$$

In fact, somewhat surprisingly, the deviations from the chiral predictions are found to be rather small even for pseudoscalar meson masses of ~ 700 MeV. Comparison of staggered and Wilson fermion results also provides a test of how well the current algebra is restored for Wilson fermions as $a \rightarrow 0$.

3.4. Heavy-Light Pseudoscalar Decay Constant

The decay constant f_P for a heavy-light pseudoscalar meson of mass m_P happens to be a good testing ground for the ideas of heavy quark effective field theory. In case of QCD with N_f light quark flavours, it can be expanded as [77]

$$\phi_P \equiv f_P \sqrt{m_P} \left[\frac{\alpha(m_P)}{\alpha(m_B)} \right]^{6/(33-2N_f)} = \phi_\infty \left(1 + \frac{A}{m_P} + \frac{B}{m_P^2} + \dots \right) . \quad (3.9)$$

Here ϕ_∞ is the limiting value and A, B are constants except for a weak logarithmic dependence on m_P . The question of phenomenological importance is the size of the $1/m_P$ corrections at the physical B and D masses.

Lattice QCD calculations have mapped out the behaviour of ϕ_P vs. $1/m_P$ as illustrated in Fig. 4 [78]. As a function of $1/m_P$, ϕ_P shows a sizeable and negative departure from its asymptotic value—there is about a factor of two variation between the $m = \infty$ limit and the D mass. This feature is consistent with observations in the QCD spectral sum rule approach [79].

As far as the results for $f_B(\text{static}) \equiv \phi_\infty/\sqrt{m_B}$ are concerned, there is a large spread amongst the results obtained by different groups [78][80] [81]. This discrepancy is likely to be due to inadequate isolation of the lightest pseudoscalar meson state, and should shrink with the use of optimised operators by all the groups. The uncertainty for the physical f_B and f_D is, fortunately, much smaller. Some of the systematic errors, such as uncertainties in the lattice scale a and the axial current renormalisation constant Z_A , can be cut down by expressing the results as ratios of pseudoscalar decay constants. The quenched lattice results then become [78][80][81]: f_D/f_B , f_{B_s}/f_B and f_{D_s}/f_D around 1.1 with errors less than 5%; while $f_B/f_\pi = 1.4(1)$.

A precise value for f_B has important phenomenological implications. Constraints from analyses of $K\bar{K}$ and $B\bar{B}$ mixing as well as b -decays, together with the anticipation $m_t \geq 130$ GeV, leave open two possibilities for the CP -violating phase δ in the CKM quark flavour mixing matrix [82]. The possibility of δ in the first(second) quadrant corresponds to large(small) CP -violation in the decay $B \rightarrow K_S J/\psi$. The knowledge of f_B can help distinguish between these two choices, since $f_B/f_\pi = 1.5 - 2$ in the former case while $f_B/f_\pi \approx 1$ in the latter. The lattice results described above, provided that the unknown systematic error due to the quenched approximation is not too large, favour δ being in the first quadrant.

4. 4-Fermion Weak Interaction Matrix Elements

There are many situations in the standard model where perturbation theory in the electro-weak sector is adequate but non-perturbative aspects of QCD substantially alter the amplitudes. The usual machinery of operator product expansion, integration of heavy gauge and matter fields, renormalisation group evolution and operator mixing leaves one

Figure 4: The decay constant for a heavy-light pseudoscalar meson ($\bar{Q}q$), rescaled to take in to account the leading mass dependence and the anomalous dimension, as a function of the pseudoscalar mass inverse [78]. The $M_P^{-1} = 0$ value is from the static approximation. The uncorrected points refer to the results prior to including the normalisation constants of subsection 2.7.

with low energy 4-Fermion effective interactions whose matrix elements have to be non-perturbatively evaluated taking in to account all the QCD corrections. Several of these amplitudes involve the (pseudo-)Goldstone bosons and so chiral symmetry plays an important role in restricting the structure of the amplitudes. Both Wilson and staggered fermion formulations have been applied on the lattice to the study of the problem, and results from the two approaches should agree for a consistent estimate. Due to its residual

chiral symmetry, the staggered fermion approach has an inherent advantage; at present the best results for the weak interaction matrix elements come from this approach.

4.1. $K^0 - \bar{K}^0$ Mixing B -Parameter

This parameter characterises the magnitude of the CP -violation parameter ϵ , and is defined as

$$B_K = \frac{\langle \bar{K}^0 | \bar{s} \gamma_\mu (1 - \gamma_5) d \bar{s} \gamma_\mu (1 - \gamma_5) d | K^0 \rangle}{\frac{8}{3} \langle \bar{K}^0 | \bar{s} \gamma_\mu \gamma_5 d | 0 \rangle \langle 0 | \bar{s} \gamma_\mu \gamma_5 d | K^0 \rangle} . \quad (4.1)$$

It is the ratio of the true matrix element to its value in the Vacuum Saturation Approximation. The lattice results show that the VSA does not work for individual components of the matrix element. Indeed the vector and axial parts separately are logarithmically divergent in the chiral limit. It is only when all the pieces are combined together that the net result is finite [83]. Simulations using staggered fermions have been carried out on large enough lattices and with sufficient statistical precision to expose underlying systematic effects. The values obtained for the renormalisation group invariant combination $B_K g^{-4/9}$ show a systematic decrease with decreasing lattice spacing, and can be extrapolated to around 0.55 – 0.60 for $a \rightarrow 0$ [84].

Results for B_K obtained using Wilson fermions have to undergo a non-perturbative subtraction procedure to eliminate the chiral symmetry violating lattice artifacts. Though the final value for B_K again comes out to be around 0.6, the error on it is at least a factor of 5 larger compared to the staggered fermion case [25]. Simulations using an improved Wilson fermion action considerably eliminate the need for a non-perturbative subtraction and hold a better promise for the future [85].

4.2. B -parameters for Left-Right Operators

The parameter ϵ' characterises direct CP -violation in $K \rightarrow 2\pi$ decay amplitudes. In this case the matrix elements are those of the imaginary part of the weak interaction Hamiltonian. The coefficient functions in front of the effective operators depend on the top quark mass and new physics, and a measurement can decide whether anything beyond the standard model is necessary to understand CP -violation [86]. In the currently allowed range of m_t there are substantial cancellations between various pieces contributing to ϵ' . Matrix elements of the QCD penguin operators are the main ingredients to our understanding of the situation. Isospin violating corrections, from unequal quark masses and electro-penguin operators, also make significant contributions to ϵ' . All the necessary

matrix elements correspond to 4–Fermion operators with left-right chiral structure. The lattice data show that they are not too far off from the VSA guess, i.e. $B_{LR} \approx 1$ [87] [25]—a result that can be understood to some extent on the basis of the rigorous inequalities of subsection 2.3.

5. Sea Quark Content of the Hadrons

Most of the numerical results regarding the hadronic spectral parameters show little difference between the quenched approximation and the full theory (although the bare gauge couplings in the two cases are quite different). Also the phenomenological quark models, which ignore the sea quarks altogether, do a good job in fitting many of the experimental results. All this suggests a welcome phenomenological simplification of the theory (i.e. the dominant effect of the sea quarks is to generate constituent quarks as valence quarks with renormalised properties), but one would like to understand the dynamical reason behind it.

To find unambiguous signatures of the sea quarks (or failures of the quenched approximation), one has to look for instances where the sea quarks amount to more than mere renormalisation of the gauge coupling. Good places to search are then the cases where the quark model expectations are not a good guide. Of particular interest are the strange quark bilinear matrix elements of the proton $\langle p | \bar{s} \Gamma s | p \rangle$. Especially the scalar density ($\Gamma = 1$) and the axial current ($\Gamma = i\gamma_\mu \gamma_5$) bilinears are connected to dynamically broken classical symmetries of QCD and have a significant scope for mixing with the gluonic sector/sea quarks.

The sea quark matrix elements are technically more difficult to calculate than the valence quark matrix elements. The main reason for this is that the correlation of an insertion on the vacuum quark loop with the hadron propagator, occurring via multiple gluon exchanges is statistically noisy. It turns out that, instead of directly computing the 3–point correlation functions, such correlations are easier to extract by making the hadron propagate through a background external field (i.e. creating the sea quark configurations with an extra source term $S_\Gamma = h_\Gamma \sum_x \bar{\psi}(x) \Gamma \psi(x)$ added to the standard fermion action), and then evaluating numerical derivatives with respect to the external field strength h_Γ .

5.1. The $\pi - N$ σ -term

The effect of dynamical quarks is clearly seen in the case of the pion-nucleon sigma term (with $m = m_u = m_d$):

$$\sigma_{\pi N} = m \langle N | (\bar{u}u + \bar{d}d) | N \rangle = m \left(\frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d} \right) m_N \quad , \quad (5.1)$$

where the quark mass derivatives are to be evaluated at fixed gauge coupling. The most recent analysis [88] of experimental data (better data are definitely desirable) gives $\sigma_{\pi N} \approx 45$ MeV. Within the first order flavour $SU(3)$ breaking parametrisation, the valence quark component is only $\sigma_{\pi N}^{\text{val}} \equiv m(3F_S - D_S) \simeq 26$ MeV, leaving ample room for the sea quark component $\sigma_{\pi N}^{\text{sea}} \equiv 2mS_S$.

Our Wilson fermion results [89] for the ratio of the full matrix element to its valence part show that the sea contribution is 1 – 2 times the valence part. When the staggered fermion results of the Columbia group [90] are analysed in a similar manner [91], the contribution to $\sigma_{\pi N}$ from the sea and the valence components is again found to be comparable. This feature is qualitatively in agreement with the experimental data and gives an indication of the importance of insertions on quark loops. The overall magnitude of $\sigma_{\pi N}$ is systematically lower than the experimental value in these simulations, however, an effect likely to be due to not having explored small enough quark masses and weak enough gauge couplings.

The analogous scalar density matrix elements of the ρ and the Δ show similar factors between the valence and the full value, while the magnitude of the matrix elements is roughly proportional to the number of valence quarks. This suggests a model for constituent quarks in which the quarks are dressed strongly, and in a manner which is independent of the state that they are in. Indeed, it can be reasoned [92] that the large sea quark component we see is due to change in the overall scale of the theory; the sea quarks influence the β -function through vacuum polarisation. The really surprising feature of the lattice results then is that even relatively heavy sea quarks (corresponding to the pseudoscalar meson mass up to 1 – 1.2 GeV) give a contribution to scalar density matrix elements that is comparable to the valence component.

5.2. Polarisation of Sea Quarks

The EMC result on polarised muon-proton scattering is difficult to digest without a significant contribution from the sea quarks. Lattice measurement of sea quark and gluon components of the structure function g_1 can be attempted in two ways. The simpler approach is to replace the insertion on the sea quark loops by an effective gluon operator. The measurement of the non-forward matrix element of $\text{Tr}(F\tilde{F})$ and extrapolation of the result to zero momentum transfer then gives an estimate of the sea quark contribution to the axial current matrix elements. Application of this method within the quenched approximation at $\beta = 5.7$ has given an upper bound $|S_A| < 0.08$ [93]. A more sophisticated calculation using four flavours of dynamical staggered fermions on $16^3 \times 24$ lattices at $\beta = 5.35$ and $ma = 0.01$ (corresponding to pseudoscalar mass of about 0.55 GeV) has produced the first tantalising result: $S_A = 0.05(1)$ per flavour [94] [95].

An alternative approach is to directly determine the axial current coupling to the sea quarks in the proton. A hint for a possible signal has been seen in a quenched approximation calculation [96]. We have attempted a full theory simulation with two flavours of dynamical Wilson fermions in a background singlet axial current field [91] [97]. In the first trial run, we used 8^4 lattices at $\beta = 5.3$. The quark mass parameters were $\kappa = 0.165, 0.166, 0.167$ with the corresponding background axial current field strengths $2\kappa h_A = 0.005, 0.004, 0.003$. The pseudoscalar masses are around 0.9, 0.8, 0.6 GeV for these relatively heavy quarks. With these lattice parameters, we have been unable to find any numerical signal for S_A , and can express our results only as 2σ upper bounds: $|S_A| < 0.05, 0.08, 0.11$ per flavour.

Assuming flavour $SU(3)$ symmetry, these numbers are a factor of 2 to 3 below what is required to confirm the EMC result on the lattice. This feature sharply contrasts with the scalar density case, where sea and valence quark components are comparable in magnitude at similar quark masses. It is entirely possible that S_A moves away from zero at comparatively small quark masses only and consequently would be much more difficult to measure.

Lattice results for non-singlet scalar density and axial current matrix elements have turned out to be reasonable [98]. Combining all the observations for scalar density and axial current couplings of the baryon octet with the heavy quark limits and constituent quark model expectations we find: (a) F_S , mS_S , F_A and D_A are finite in the heavy quark limit and are smooth functions of the quark mass, while (b) mD_S and S_A vanish in the

heavy quark limit as m^{-2} and depart significantly from zero only when the quark mass becomes of the order of Λ_{QCD} . This suggests that flavour $SU(3)$ breaking effects are probably small in the former case, though likely to be sizeable in the latter case.

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