NBI-HE-93-2
January 1993

Observing 4d baby universes in quantum gravity

J. Ambjørn

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

S. Jain

Centre for Theoretical Studies, Indian Institute of Science
Bangalore 560012, India

J. Jurkiewicz[1]

Inst. of Phys., Jagellonian University.,
ul. Reymonta 4, PL-30 059, Kraków 16, Poland

C.F. Kristjansen

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

We measure the fractal structure of four dimensional simplicial quantum gravity by identifying so-called baby universes. This allows an easy determination of the critical exponent γ connected to the entropy of four-dimensional manifolds.

[1]Supported by the KBN grant no. 2 0053 91 01

1 Introduction

A simple way to characterize the fractal structure of 2d quantum gravity was recently introduced in ref. [1]. Using very few assumptions it was possible to prove that the functional behaviour of the number of two-dimensional surfaces with area N_2 and coupled to matter with central charge $c < 1$

$$\mathcal{N}(N_2) \sim N_2^{\gamma(c)-3}\, e^{\mu_0 N_2} \tag{1}$$

is directly related to the fractal structure of 2d quantum gravity. In particular one can determine $\gamma(c)$ by simply counting the number of so-called "minimum bottleneck universes", abbreviated "minbu's", living on a typical surface in the ensemble of surfaces of fixed total area. This number is given by

$$\mathcal{N}_{N_2}(V_2) \sim N_2 V_2^{\gamma(c)-2}, \tag{2}$$

where N_2 is the total area of the surface and $V_2 \ll N_2$ is the area of the minbu's being counted. A "baby universe" of area V_2 is any simply connected region of the surface of area V_2 and boundary length l such that $V_2 \gg l^2$, and a minbu is a baby universe whose l is the smallest possible length consistent with the ultraviolet cutoff. This boundary, along which the minbu is connected to the remainder of the (parent) surface is called a minimal bottleneck. The only assumption going into the derivation of (2) is the distribution (1) and the approach is well suited for dynamically triangulated surfaces [2, 3, 4, 5]. Here the area of a region can be identified with the number of triangles in the region and a minimal bottleneck is a simple closed path consisting of three links which partitions the surface into two parts - the minbu containing $V_2 \gg 1$ triangles, and the parent containing $N_2 - V_2$ triangles.

Until now the measurement of γ by numerical simulations has been a rather painful procedure since one had to perform a whole sequence of simulations (or a so-called grand canonical simulation where the number of triangles changes, which is also an unpleasant task) for different areas N_2 and in the end fit the measured quantities to a formula related to (1). Following the suggestion in [1] one can now extract γ from a single simulation using (2). This is neater also because one does not have to subtract out the leading $e^{\mu_0 N_2}$ piece while making the fit; (2) contains only the universal power law piece. This method also works in practise. Indeed, in a recent numerical simulation it was possible to extract γ for pure 2d gravity with high precision (less than 1%) [9].

2 The model

A model of 4d quantum gravity which generalizes the discretized 2d approach was suggested a year ago [7, 8]. The partition function is given by

$$Z(\kappa_2, \kappa_4) = \sum_{T \in \mathcal{T}} e^{-\kappa_4 N_4 + \kappa_2 N_2} \tag{3}$$

where the sum is over triangulations T in a suitable class of triangulations $\mathcal{T}$. The quantity N_4 denotes the number of 4-simplexes in the triangulation and N_2 the number of triangles. The coupling constant κ_2 is inversely proportional to the bare gravitational coupling constant, while κ_4 is related to the bare cosmological constant. The most important restriction to be imposed on $\mathcal{T}$ is that of a fixed topology. If we allow an unrestricted summation over all topologies in (3) the partition function is divergent [8]. In the following we will always restrict ourselves to considering only manifolds with the topology of S^4.

$Z(\kappa_2, \kappa_4)$ is the grand canonical partition function. It is defined in a region $\kappa_4 \geq \kappa_4^c(\kappa_2)$ in the (κ_2, κ_4) coupling constant plane. The only way in which we can hope to obtain a continuum limit is by letting κ_4 approach $\kappa_4^c(\kappa_2)$ from above. This tentative continuum limit depends only on one coupling constant κ_2. We can write (3) as

$$Z(\kappa_2, \kappa_4) = \sum_{N_4} Z(\kappa_2, N_4) e^{-\kappa_4 N_4}. \tag{4}$$

$Z(\kappa_2, N_4)$ is the canonical partition function where N_4 is kept fixed. Then we have in practise only one coupling constant, κ_2, and the aspects of gravity which do not involve the fluctuation of the total volume of the universe can be addressed in the limit of large N_4. However it is actually possible to extract the critical exponent for volume fluctuations from $Z(\kappa_2, N_4)$. Let us assume that the canonical partition function has the form:

$$Z(\kappa_2, N_4) = N_4^{\gamma(\kappa_2) - 3}\, e^{\kappa_4^c(\kappa_2) N_4} \cdot (1 + \mathcal{O}(1/N_4)) \tag{5}$$

When κ_4 is close to $\kappa_4^c(\kappa_2)$ we can approximate (3) with

$$Z(\kappa_2, \kappa_4) \sim \text{analytic} + \frac{C(\kappa_2)}{(\kappa_4 - \kappa_4^c(\kappa_2))^{\gamma(\kappa_2) - 2}} \tag{6}$$

where "analytic" means possible analytic terms of the form $(\kappa_4 - \kappa_4^c)^n$, $0 \leq n < 2 - \gamma(\kappa_2)$. The entropy exponent of the number of 4d simplicial manifolds, γ, determines the volume fluctuations since we have

$$\left\langle N_4^2 \right\rangle - \langle N_4 \rangle^2 = \frac{d^2 \ln Z(\kappa_2, \kappa_4)}{d\kappa_4^2} \;\sim\; \text{analytic} + \frac{C(\kappa_2)}{(\kappa_4 - \kappa_4^c(\kappa_2))^{\gamma(\kappa_2)}} \tag{7}$$

if we assume $\gamma(\kappa_2) < 2$.

We will try to extract the critical exponent $\gamma(\kappa_2)$ by numerical simulations. At this point it is important to note that it is by no means obvious that the exponent γ exists. Equation (5) is an *ansatz*, inspired from 2d quantum gravity where we know that a similar formula holds when the central charge of matter $c < 1$. In the case of $c = 1$ there are logarithmic corrections, and the form is not known for $c > 1$. In 3d simplicial gravity it was shown [15, 17, 18] that a more likely form of $Z(\kappa_1, N_3)$ (the 3d analogue of (5)) is

$$Z(\kappa_1, N_3) \sim \exp\left[\kappa_3^c(\kappa_1) N_3 \left(1 - \frac{C(\kappa_1)}{N_3^{\alpha(\kappa_1)}}\right)\right] \tag{8}$$

and a possible power law correction would be subleading. However, 3d simplicial gravity seems to differ from 4d simplicial gravity in many respects ([7, 8, 16]), and it was clear from the fine tuning process $\kappa_4 \to \kappa_4^c(\kappa_2)$ in the Monte Carlo simulations that the corrections to the leading term $\exp(\kappa_4^c N_4)$ were less severe in 4d than in 3d. It is therefore indeed possible that the finite size corrections in 4d are power-like, as in 2d simplicial gravity, rather than of the exponential form (8) encountered in 3d.

Let us now define the minimal bottleneck baby universes, which we, following [1], will denote "minbu's". On a 2d triangulated manifold one can check whether we have a minbu in the following simple way: pick a link and two neighbouring links, attached to the two vertices of the first link. If the two links have a vertex in common the three links form a closed loop. In general this loop will be trivial in the sense that it will just be the boundary of one of the two triangles in the surface which contains our starting link. However, it might turn out that there is no triangle in the surface which has the closed loop as its boundary. If we cut the surface along such a closed loop we will have separated it into two disconnected open surfaces each one having the closed loop as its boundary, provided the topology of the surface is that of a sphere (which we will assume). Both components will have spherical topology when closed by adding the triangle which has the loop as boundary. The smallest of the spheres will be a minbu, the largest the "mother universe".

This construction can immediately be generalized to 4d triangulated manifolds. We check for minbu's as follows: pick a 3-simplex (a tetrahedron) and in this the four 2-simplexes (triangles) which constitute its boundary. Identify for each triangle all the 3-simplexes in the manifold which have this triangle as their boundary. We now have four groups of 3-simplexes and all 3-simplexes have one vertex which does not belong to the triangle from which the 3-simplex was constructed. Pick now a 3-simplex from each of the 4 groups and check whether their 4 free vertices coincide.

If that is the case the four 3-simplexes together with the original 3-simplex can be thought of as the (closed) boundary of a 4-simplex. In general the 4-simplex found by this procedure will just be one of the two 4-simplexes which shared the original 3-simplex. However, if that is not the case we will, if the topology of our original 4-manifold is that of a sphere (which we assume as usual), separate it into two non-trivial components if we cut along the closed piecewise linear 3d manifold built out of the five 3-simplexes. Both components have this closed 3-manifold as their boundary and both components will have the topology of S^4 if we close them by adding to each of them a 4-simplex in such a way that their boundary is identified with the boundary of the 4-simplex, just with opposite orientation. The smallest of the 4-spheres will be called the minbu, the largest the "mother", in agreement with the 2d notation.

It is almost clear that the above description can be turned into an efficient numerical algorithm (after, admittedly, some pain with double counting etc.) provided one is given the coincidence matrix of the triangulated 4-manifold.

Let us now argue that the counting arguments in [1] extend to the 4d triangulated manifolds as well. Let us by $Z'(\kappa_2, N_4)$ denote the canonical partition function for 4-manifolds consisting of N_4 4-simplices where one 4-simplex is marked. Generically we will get a different manifold for each mark so for large N_4, where accidental symmetries are expected to play no role, we get $Z'(\kappa_2, N_4) \approx N_4 Z(\kappa_2, N_4)$. Such a marked manifold can also be viewed as a manifold with $(N_4 - 1)$ 4-simplexes and a minimal boundary of the kind associated with minbu's (by removing the interior of the marked 4-simplex). A moments reflection will convince the reader that the average number of minbu's of volume V_4 on 4-manifolds of total volume N_4 will be given by

$$\mathcal{N}_{N_4}(\kappa_2, V_4) \approx \frac{60}{Z(\kappa_2, N_4)} Z'(\kappa_2, V_4) Z'(\kappa_2, N_4 - V_4) \tag{9}$$

where 60 is the number of ways one can glue the two boundaries of the minbu and the mother together with opposite orientation of the boundary.

If we *assume* the canonical partition function is given by (5), we get

$$\mathcal{N}_{N_4}(\kappa_2, V_4) \sim \ C(\kappa_2)\ N_4\ V_4^{\gamma(\kappa_2)-2} \tag{10}$$

The scene is now set for a numerical determination of the number of minbu's since it is already by now [7, 8, 10, 11, 12, 13, 14] standard how to generate by Monte Carlo simulations the class of triangulated 4-manifolds corresponding to the partition function (3). It is also well known [8] how one can effectively stay in the neighbourhood of a given volume, N_4, even if one is using the grand canonical partition function

(3), as one is forced to by ergodicity requirements in 4d, contrary to what is the case in 2d.

3 Numerical results

By Monte Carlo simulations we have generated a number of 4d-manifolds according to the distribution dictated by the partition function (3). For details about this procedure we refer e.g. to [8, 12]. We have considered $\kappa_2 = 0.0$, 0.8, 1.0, 1.1 and 1.4 and $N_4 = 4000$, 9000, 16000[2]. For each value of κ_2 and for each value of N_4 we have generated approximately 1000 configurations.

Let us at this point review some of the results of [8]: For $\kappa_2 \approx 1.0 - 1.1$ we observed a transition in geometry from a highly connected phase with a seemingly large Hausdorff dimension to a phase with an elongated, almost one-dimensional geometry. This is why our main efforts have been concerned with this region of coupling constant space.

The thermalization time is short in the highly connected phase, but quite long for $\kappa_2 > 1.0$. In the case of $N_4 = 16000$ we have for these values of the coupling constant used 10000 sweeps[3] for thermalization and performed measurements after each successive fifty sweeps (after each successive hundred sweeps for $\kappa_2 = 1.4$). The result of the measurements of minbu's is shown in fig. 1 for $N_4 = 16000$. We have plotted the number of minbu's on a log-log scale since the formula (10) suggests the dependence:

$$\log \mathcal{N}_{N_4}(\kappa_2, V_4) = \mathrm{Const}(\kappa_2) + (\gamma(\kappa_2) - 2)\log V_4 \tag{11}$$

The straight lines are the result of a least χ^2 fit. In fig. 2 we have shown the results for $k_2 = 1.1$ and different values of N_4 and finally in fig. 3 we have shown $\gamma(\kappa_2)$ as extracted from fig. 1.

A few comments should be made. We have only taken into account minbu's with a volume $V_4 \geq 9$. For small values of V_4 the numbers $\mathcal{N}_{N_4}(\kappa_2, V_4)$ fall into two classes according to whether $V_4 = 4n - 1$ or $V_4 = 4n + 1$. This is a clear finite size effect. We have only included in our analysis the last class of numbers as they seem to fit (11) all the way down to $V_4 = 9$. For large volumes V_4 of the minbu's the results are average values for volumes in the neighbourhood of the plotted V_4 since the statistics for a specific large volume is not good. The errorbars resulting from the binning procedure are smaller than the symbols used in the figures.

[2]We have also made a few runs for $N_4 = 32000$ and $k_2 = 1.0$.

[3]By a sweep we mean N_4 *accepted* updatings.

In general we see a clear deviation from (11) when $\mathcal{N}(V_4) < 0.1$ and when $\kappa_2 < 1.1$. In this case the number of minbu's begins to drop faster than given by (11). We have not shown these data, since they have bad statistics, and we do not know whether the above mentioned behaviour reflects that (11) is not really valid for large V_4 or rather that the statistics is just not good enough (there is typically less than one such large minbu per universe) for the volumes of manifolds we consider. The message we get by comparing different N_4's does not allow us at the moment to answer this question. For κ_2 larger than 1.1 the situation is different: The number of baby universes is much larger and even if the number of very large baby universes is small and errorbars large their numbers seem not to drop below numbers predicted by (11).

It is seen that the curve for $\kappa_2 = 1.4$ bends at $V_4 \approx 30 - 40$ to a different slope. We have used the part with the larger values of V_4 to extract $\gamma(\kappa_2)$. Such a bending is not observed for $\kappa_2 \leq 1.0$, i.e. in the phase with a highly connected geometry. A close look at the data for $\kappa_2 = 1.1$ reveals a slight tendency to such a bending, and since $\kappa_2 = 1.1$ is just at the borderline of the transition to the elongated geometry the bending is probably related to the fact that the very fractal structure observed in this phase is only unfolded for large volumes V_4.

4 Discussion

With the above reservation concerning the interpretation of the numerical data, we still consider the results as somewhat remarkable. It seems possible by numerical simulations to study in detail the fractal structure of quantum gravity. Indeed, our data contains a lot of information about the branching of the geometry which will be published in a more extensive report elsewhere [19]. Let us just here mention that the two phases of 4d simplicial quantum gravity observed in former works appear quite clearly in the minbu measurements. In the phase with highly connected geometry there is one "big mother" which contains the major fraction of the volume (from 95% to 80% depending on κ_2). In the other phase of very elongated geometry this is not so and there seems genuine democracy in "mother size". *Nevertheless the $\gamma(\kappa_2)$ extracted seems perfectly smooth when we pass the transition in geometry*, as is apparent from fig. 3. It is interesting to note that a similar smoothness has been observed in the numerical simulations of 2d quantum gravity [9, 20] when we pass the $c = 1$ barrier. It points to the fact that subleading corrections to (1) might play a very important geometrical role, and it might be possible to understand analytically the interplay between the subleading terms in (1) and the fractal geometry of quantum gravity. The seed to such an analysis is already to be found in [1].

We would like to stress that the results here, especially for $\kappa_2 \leq 1.1$, are obtained for quite small baby universes and we see some deviations for larger volumes, as mentioned above. It would clearly be desirable to let the babies grow, but as is well know from biology this requires time, in our case computer time. Work in this direction is in progress. It is nevertheless somewhat intriguing that the transition between the two types of geometry takes place at a value of $\gamma \approx -0.5 - 0.0$, i.e. the same values of γ which are of main interest in 2d gravity. Note further that the results for γ seem internally consistent in the sense the value in the phase with almost linear geometry (Hausdorff dimension not larger that two) seems close to the value $\gamma = 1/2$ for the so-called branched polymers. In addition, assuming that the extraction of γ for $\kappa_2 < 1.1$ will survive the test of large babies, one could wonder if this reflects that the situation is like in 2d gravity where we have a $\gamma(c)$ depending on the central charge. In our case κ_2 would then play a role somewhat similar to the central charge. The range of γ's is in agreement with such a picture. An interesting continuum approach somewhat related to such an interpretation can be found in a recent paper by Antoniadis, Mazur and Mottola [21], where the authors develop a quantum theory for the conformal mode of gravity.

References

[1] S. Jain and S.D. Mathur, *World-Sheet Geometry and Baby Universes in 2-D Quantum Gravity*, HUTP-92/A002.

[2] F. David, Nucl. Phys. B 257 (1985) 45.

[3] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B 257 (1985) 433; Nucl.Phys. B270 (1986) 457; Nucl. Phys. B275 (1986) 161.

[4] F. David, Nucl. Phys. B257 (1985) 543.

[5] V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. 157B (1985) 295.

[6] D. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Nucl. Phys. B275 (1986) 641.

[7] M.E. Agishtein and A.A. Migdal, Mod. Phys. Lett. A7 (1992) 1039.

[8] J. Ambjørn and J. Jurkiewicz, Phys.Lett B278 (1992) 42.

[9] J. Ambjørn, S. Jain and G. Thorleifsson, *Baby Universes in 2d Quantum Gravity*, NBI-HE-93-4.

[10] S. Varsted,UCSD-PhTh-92-03.

[11] M.E. Agishtein and A.A. Migdal, Nucl. Phys. B385 (1992) 395.

[12] J. Ambjørn, J. Jurkiewicz and C.F. Kristjansen, NBI-HE-92-53 (to appear in Nucl. Phys. B).

[13] B. Brügmann, *Non-uniform Measure in Four-Dimensional simplicial Quamtum Gravity*, SU-GP-92/9-1.

[14] B. Brügmann and E. Marinari, *4d Simplicial Quantum Gravity with a Non-Trivial Measure*, SU-GP-92/9-2.

[15] B. Boulatov and A. Krzywicki, Mod.Phyd.Lett A6 (1991) 3005.

[16] J. Ambjørn, D. Boulatov, A. Krzywicki and S. Varsted, Phys.Lett B276 (1992) 432.

[17] J. Ambjørn and S. Varsted, Phys. Lett (B226), (1991) 285.

[18] J. Ambjørn and S. Varsted, Nucl.Phys. B373 (1992) 557.

[19] J. Ambjørn, J. Jurkiewicz and C.F. Kristjansen, To appear.

[20] J. Ambjørn, D. Boulatov and V.A. Kazakov, Mod. Phys. Lett A5 (1990) 771.

[21] I. Antoniadis, P.O. Mazur and E. Mottola, *Scaling behavior of quantum four-geometries*, Preprint CPTH-A214.1292

Figure Captions

Fig. 1 The number of minbu's for $N_4 = 16000$ and $k_2 = 0.0(\Box)$, $0.8(\bigcirc)$, $1.0(\triangle)$, $1.1(+)$ and $1.4(\times)$. The straight lines represent the best fits to (10).

Fig. 2 The number of mimbues for $k_2 = 1.0$ and $N_4 = 4000(\Box)$, $9000(\bigcirc)$, $16000(\triangle)$ and $32000(+)$, normalized such that $\mathcal{N}_{N_4}(V)$ is multiplied by $16000/N_4$. (log-log scale)

Fig. 3 $\gamma(\kappa_2)$ as extracted from fig.1.