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Baby Universes in 2d Quantum Gravity

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Abstract

We investigate the fractal structure of $2d$ quantum gravity, both for pure gravity and for gravity coupled to multiple gaussian fields and for gravity coupled to Ising spins. The roughness of the surfaces is described in terms of baby universes and using numerical simulations we measure their distribution which is related to the string susceptibility exponent γ_{string}.

1 Introduction

The fractal and selfsimilar structure of $2d$ quantum gravity is related to the entropy- (or string susceptibility-) exponent γ. This has been discussed in a recent paper [1] where the structure of so-called baby universes was analyzed. It is convenient in the following discussion to consider $2d$ quantum gravity with an ultraviolet cut-off and we will consider the surfaces entering in the path integral as triangulated surfaces built out of equilateral triangles [3, 4, 2]. In the case of surfaces of spherical topology a closed, non-intersecting loop along the links will separate the surface in two parts. The smallest such loop will be of length 3. It will split the surface in two parts. If the smallest part is different from a single triangle we call it (following the notation of [1]) a "minimum neck baby universe", abbreviated "mimbu". The smallest possible area of a mimbu is 3 and the largest possible area will be $N_T/2$, where N_T is the number of triangles constituting the surface.

In the case of pure 2d quantum gravity it is known that the number of distinct surfaces of genus zero made out of N_T triangles has the following asymptotic form

$$Z(N_T) \sim e^{\mu_c N_T} N_T^{\gamma-3} \tag{1.1}$$

where $\gamma = -1/2$. For the models which can be solved explicitly and where $c < 1$ we have the following partition function:

$$Z(\mu) = \sum_{N_T} Z(N_T) e^{-\mu N_T} \tag{1.2}$$

where $Z(N_T)$ for large N_T is of the form (1.1), just with a different $\gamma = \gamma(c)$. For $c = 1$ it is known that there are logarithmical corrections to (1.1), while the asymptotic form of $Z(N_T)$ is unknown for $c > 1$, although it can be proven that it is exponentially bounded ([2]). If we *assume* (1.1) one can prove that the average number of mimbu's of area B on a closed surface of spherical topology and with area N_T (we use the notation *area* $\equiv$ *#triangles*) is given by

$$\bar{n}_{N_T}(B) \sim (N_T - B)^{\gamma-2} B^{\gamma-2} \tag{1.3}$$

provided N_T and B are large enough.

The above formalism is well suited for numerical simulations. The measurement of the exponent γ has always been somewhat difficult. The first attempts used a grand canonical updating ([6, 7, 8]), which generated directly the distribution (1.2). The disadvantage is that one has to fine-tune the value of μ to μ_c. Later improved versions allowed one to avoid this [9], but one still had to perform independent Monte Carlo simulations for a whole range of N_T and γ still appeared as a subleading correction to the determination of the critical point μ_c. These disadvantages disappear when we use (1.3). γ does not appear as a subleading correction to μ_c and one can use canonical Monte Carlo simulations (the so-called link-flip algorithm [4]) which keeps N_T fixed, and still in a single Monte Carlo simulation get a measurement of the distribution of mimbu's all the way up to $N_T/2$. One can therefore take a large N_T and make one very long run. This allows us to avoid the problems with long thermalization time. In addition the actual measurement of the mimbu distribution is easy. For a given thermalized configuration one has to identify all possible mimbu's associated with the configuration. This is done by picking up one link, l_0, and checking whether any links which have a vertex in common with l_0 have a vertex in common which do not belong to l_0. This being the case we will have a minimal neck of length 3. For a given l_0 there will always be two such, corresponding to the two triangles sharing l_0. But there might be additional ones and they will divide the surface into a mimbu and its "mother". By scanning over l_0's, avoiding double counting and repeating the process for independent configurations we can construct the distribution of mimbu's.

In the rest of this paper we report on the results of such numerical simulations.

2 Numerical simulations

2.1 Pure gravity

The simulations were done on lattices of size ranging from 1000 to 4000 triangles N_T and we used the standard "link flip" algorithm [4] to update the geometry. We used of the order of 10^7 sweeps, where each sweep consists of N_T link flips. After thermalization we measured for each 10^{th} sweep the distribution of mimbu's, that is we counted all areas $B > 1$ enclosed by boundaries of length 3. The reason for performing the measurements so often is simply that they are not time consuming (the time it takes to make one measurement is comparable to the time it takes to perform one sweep). The distributions are shown in fig. 1. In order to extract γ the distributions are fitted to equation (1.3). But as eq. (1.3) is only asymptotically correct deviations can be expected for small B. Thus a lower cut-off B_0 has to be introduced in the data to avoid the effects of this deviations. Moreover we have added the simplest type of correction term which arises from the replacement

$$B^{\gamma-2} \to B^{\gamma-2}\left(1+\frac{C}{B}+\mathcal{O}(1/B^2)\right) \tag{2.1}$$

in (1.3) and fitted to the form

$$\ln(\bar{n}_{N_T}) = A + (\gamma - 2)\ln(B(1 - \frac{B}{N_T})) + \frac{C}{B} \tag{2.2}$$

for $B \geq B_0$. A and C are some fit parameters. Comparison of the results with and without this correction term can be seen in fig. 2 where we plot the value of γ extracted with different cut-off's B_0. We see that including the correction improves the results considerable.

Let us assume that the values γ_{B_0} extracted from (2.2) appproach exponentially a limiting value for large B_0:

$$\gamma_{B_0} = \gamma - c_1 e^{-c_2 B_0}. \tag{2.3}$$

The result of such a fit is shown in fig. 2. It is clear from fig. 2 that the assumption of an exponential approach of γ_{B_0} to γ is not essential for the extraction of γ. We have introduced it at this point in order to treat all measurements consistently. For the matter fields coupled to gravity the finite size effects will be larger and extrapolation to large B_0 more important.

The γ extracted in this way for different lattice sizes is:

N_T	γ
1000	-0.496 ± 0.005
2000	-0.501 ± 0.004
3000	-0.504 ± 0.004

which is in good agreement with the expected value of $\gamma = -0.5$. It shows that this kind of simulations are indeed well suited to measure γ and it is thus natural to try apply them to the case of matter couple to $2d$ gravity.

2.2 The Ising model

The next non-trivial test of the method is to study the Ising model coupled to $2d$ gravity. It has been solved analytically [5] and was found to have a $3rd$-order phase transition. The coupling to gravity is in a sense weak as it only changes the string susceptibility *at* the critical point (from $\gamma = -1/2$ to $\gamma = -1/3$). For this reason it has until now been considered very difficult to measure γ directly, since it required a fine-tuning of both the bare cosmological constant μ and the spin coupling constant β. On the other hand it has been verified that it is indeed possible to extract the other known critical exponents [10] since for these exponents it is possible to use the canonical ensemble in the simulations.

The Ising spins are placed in the center of the triangles and they interact with the spins on neighbouring triangles. This corresponds to placing them on vertices in the dual graph. In that case the critical point has been found explicitly and is $\beta_c = 0.7733...$ [11]. The (canonical) partition function of the model is

$$Z_{N_T}(\beta) = \sum_{T\sim N_T} \sum_{\{\sigma_i\}} e^{\beta \sum_{<i,j>} \sigma_i \sigma_j} \tag{2.4}$$

where the summation is over all triangulations with N_T triangles.

In the simulations we used a Swendsen-Wang cluster algorithm [12] to update the Ising spins and lattices sizes $N_T = 1000$ and 2000. We made runs for several values of the coupling in the interval $0.6 \leq \beta \leq 0.95$ and then fitted the distributions to eq. (2.2). In this way we could extract values $\gamma_{B_0}(\beta)$ and by assuming a relation like (2.3):

$$\gamma_{B_0}(\beta) = \gamma(\beta) - c_1(\beta) e^{-c_2(\beta) B_0} \tag{2.5}$$

we have extracted the values for $\gamma(\beta)$ shown in fig. 3. Examples of $\gamma_{B_0}(\beta)$ and the exponential fit (2.5) for different values of β are shown in fig. 4. We observe a marked increase in the dependence on B_0 when β approaches β_c.

We get, as expected, the pure gravity value of -0.5 for couplings far below and above β_c. In the vicinity of the phase transition we see on the other hand a clear peak and the peak values agree well with the exact value $\gamma = -1/3$. We conclude that the method for extracting γ works well in this case too, although it should be clear that the amount of numerical work needed is much larger in this case than in the case of pure gravity.

2.3 Gaussian fields

The gaussian fields x^μ, $\mu = 1, \ldots, D$ are placed on the sites i of the triangulation T. They can be viewed as representing an immersion $i \to x_i^\mu$ of our abstract triangulation T into R^D, i.e. a model for non-critical strings and they also represent a coupling of matter with central charge $c = D$ to gravity. The multiple gaussian models do not interact directly with each other but only through their mutual interaction with the geometry. The (canonical) partition function is given by

$$Z_{N_T} = \sum_{T\sim N_T} \int \prod_{i \in T \backslash \{i_0\}} d^D x_i \, e^{-\sum_{<i,j>} (x_i^\mu - x_j^\mu)^2} \tag{2.6}$$

where the summation is over all triangulations T with N_T triangles. One site is kept fixed in order to eliminate the translation mode. No coupling constant appears in the action as it can be absorbed in a redefinition of the gaussian fields.

Again we have performed simulations with up to 10^7 sweeps for lattice sizes ranging from 1000 to 4000 triangles. We have used from one to five gaussian fields and a standard Metropolis algorithm to update them. In fig. 5 we show how the distributions of baby universes change with increased c (normalized with the distribution for pure gravity). Fitting these distributions to the functional form (2.2) and extracting γ as above yields the results shown in fig. 6. The results are compatible with earlier estimates [9].

It is seen that γ is too small for $c = 1$ where it is known that $\gamma = 0$. But in the case $c = 1$ we know that the asymptotic form (1.1) is not correct. It should be multiplied with logarithmic corrections. If we include these we get for $c = 1$ that (1.3) is replaced by [1]

$$\bar{n}_{N_T}(B) \sim [(N_T - B)B]^{\gamma-2} \left[\ln(N_T - B) \ln B\right]^{\alpha}. \tag{2.7}$$

In this formula we have left γ and α as variables. Model calculations give $\alpha = -2$, but it is not known whether this power is universal and the model has not been solved analytically in the case of one gaussian field.

If we fit to (2.7) in the way described above (including also the $1/B$ correction) we extract for $c = 1$ the following values of γ and α for different lattice sizes:

N_T	γ	α
1000	-0.22 ± 0.05	-0.5 ± 0.4
2000	-0.14 ± 0.07	-1.0 ± 0.4
4000	-0.09 ± 0.08	-1.2 ± 0.4

Both γ and α moves towards the expected values 0 and -2 as a function of N_T, but the finite size effects are clearly larger here than for pure gravity.

In fig. 6 the results of a fit to (2.7) for $c > 1$ is included. It is seen that γ extracted in this way exceeds the theoretical upper bound $\gamma = 1/2$ ([6]). In addition the power α decreases from -1.2 for $c = 1$ to -5 for $c = 5$. We conclude that either logaritmic corrections are not the right ones to include for $c > 1$ or finite size effects are so large that they make the fits unreliable.

What *is* clear from the analysis is that γ increases with c for c in the range $0 - 5$. According to (1.3) this means that the number of baby universes of a given size will increase, i.e. the fractal structure will be more pronounced with increasing c. We have illustrated this in fig. 7, which shows two "typical" surfaces corresponding to $c = 0$ and $c = 5$. It should be emphasized that the pictures are only intended to visualize the *internal* structure, i.e. the connectivity of the surfaces[1]

[1]The surfaces are constructed in the following way: Given the connectivity matrix of the trian-

3 Discussion

We have verified that the technique of extracting the entropy exponent γ directly from the distribution of baby universes is superior to the methods used until now from a practical point of view. A single (although long) Monte Carlo run for a fixed value of N_T is sufficient for extracting γ and we get the correct results for $c < 1$. On the other hand the situation in the case $c > 1$ has not really improved much compared to the earlier measurements [9]. We get in fact similar results, and this shows that the method also works in the case $c > 1$ and the ambiguity in extracting γ for $c > 1$ is that we do not know the correct functional form to be used in the fits. It is clear that it would be most interesting if we could reverse the procedure and use the data to obtain knowledge about the corrections to (1.1) for $c > 1$. Our data are not yet good enough to do this in a convincing way, but the problem is clearly not due to the baby universe technique introduced in this paper, but due to the inefficiency of the flip algorithm used to update the triangulations.

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gulation we choose arbitrary coordinates for the vertices in R^3. Next we introduce an attraction between neighbouring vertices in order to keep the surface together and extrinsic curvature to smooth out the surface during a Monte Carlo simulation. When the surface reach a configuration without self-intersection we put a pressure in the interior and a Coulomb repulsion between distant vertices and in this way we blow up the surface as a balloon.

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Figure captions

Fig.1 The distribution of baby universes in the case of pure gravity.

Fig.2 Fitted values of γ_{B_0} for different cutoff's B_0 for pure gravity. Values are shown for fits with and without the correction term included. The curve shows a fit using (2.3) resulting in $\gamma = -0.496 \pm 0.005$ (errors are 95% confidence limits for a χ^2-test).

Fig.3 Fitted values of γ vs the coupling β in the case of one Ising model coupled to gravity. Results are shown for two lattice sizes, $N_T = 1000$ and 2000.

Fig.4 Fitted values of $\gamma_{B_0}(\beta)$ for various β as a function of the cut-off B_0. The curves represent fits to (2.5).

Fig.5 The distributions of baby universes for up to five Gaussian fields coupled to $2d$ gravity. The values are normalized with the distribution for pure gravity.

Fig.6 Fitted values of γ vs central charge in the case of multble Gaussian fields. Results are shown for fits without (circles) and with (squares) a logarithmic correction term included.

Fig.7 3d illustration of the fractal structure of the surfaces for $c = 0$ (fig. 7a) and $c = 5$ (fig. 7b). $N_T = 200$ is used.