

PERTURBATIVE CORRECTIONS FOR STAGGERED FOUR-FERMION OPERATORS

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Abstract

We present results for one-loop matching coefficients between continuum four-fermion operators, defined in the Naive Dimensional Regularization scheme, and staggered fermion operators of various types. We calculate diagrams involving gluon exchange between quark lines, and “penguin” diagrams containing quark loops. For the former we use Landau gauge operators, with and without $O(a)$ improvement, and including the tadpole improvement suggested by Lepage and Mackenzie. For the latter we use gauge-invariant operators. Combined with existing results for two-loop anomalous dimension matrices and one-loop matching coefficients, our results allow a lattice calculation of the amplitudes for $K\bar{K}$ mixing and $K \rightarrow \pi\pi$ decays with all corrections of $O(g^2)$ included. We also discuss the mixing of $\Delta S = 1$ operators with lower dimension operators, and show that, with staggered fermions, only a single lower dimension operator need be removed by non-perturbative subtraction.

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1 INTRODUCTION

Many of the low-energy effects of the electroweak interactions can be expressed as matrix elements of four-fermion operators between hadronic states. For example, CP-violation in $K - \bar{K}$ mixing is proportional to such a matrix element (parameterized by B_K). Knowledge of these matrix elements is required to extract the parameters of the electroweak theory. Since the matrix elements involve non-perturbative physics, lattice QCD is well suited for their calculation.³ A necessary ingredient, however, is the relationship between continuum and lattice four-fermion operators. This “matching” can be done using perturbation theory, since the operators differ only for momenta close to the lattice cut-off.

The general form of the relationship between lattice and continuum four-fermion operators is

$$\mathcal{O}_i^{cont} = \mathcal{O}_i^{lat} + \frac{g^2}{16\pi^2} \sum_j \left(\gamma_{ij}^{(0)} \ln(\pi/\mu a) + c_{ij} \right) \mathcal{O}_j^{lat} + O(g^4) + O(a), \quad (1.1)$$

where μ is the renormalization scale used to define the continuum operators, and a is the lattice spacing. In other words, to calculate the matrix elements of a continuum operator \mathcal{O}_i^{cont} , we must use the linear combination of lattice operators given by the right-hand-side of this equation. $\gamma_{ij}^{(0)}$ is the one-loop anomalous dimension matrix (using the conventions of Ref. [2]), and c_{ij} is the finite part of the one-loop matching coefficient. The finite part depends on the continuum regularization scheme, on the type of lattice fermions used, and on the precise definition of the lattice operators.

In this paper we calculate c_{ij} for two classes of lattice four-fermion operators composed of staggered fermions. In general, such operators involve quark and antiquark fields at different positions, and must be made gauge-invariant. Our first calculation concerns operators which are defined by fixing to Landau gauge and dropping all gauge link factors. We compute the one-loop matching coefficients for such “Landau-gauge” operators coming from diagrams in which a gluon is exchanged between quark lines (see Fig. 1), which we refer to as X diagrams. These are the only contributions for operators transforming as $\mathbf{27}$'s under flavor SU(3), e.g. the $\Delta S = 2$ operator leading to $K - \bar{K}$ mixing, and the $\Delta I = 3/2$ part of the $K \rightarrow \pi\pi$ amplitude. We consider both the simplest Landau-gauge operators and “smeared” operators having no $O(a)$ corrections at tree-level. These operators have been used extensively in numerical simulations [3, 4, 5, 6], and the results have small enough statistical errors that it is important to include the perturbative corrections.

Our second calculation is of the contribution of “penguin” diagrams to the matching coefficients (see Fig. 3). At one-loop, the contributions of X diagrams and penguin diagrams add, $c_{ij} = c_{ij}^X + c_{ij}^{pen}$, and can be treated separately. Penguin diagrams are present for operators transforming as octets under flavor SU(3). Such operators contribute to the $\Delta I = 1/2$ part of the $K \rightarrow \pi\pi$ amplitude, and in particular to CP-violation in these decays. Numerical calculations of their matrix elements are possible [7], although less advanced than those for $\mathbf{27}$'s. A technical problem with the calculations is that there is mixing not only with four-fermion operators but also with bilinears of lower dimension. These must be subtracted non-perturbatively, e.g. using the method of Ref. [8]. If one uses operators made gauge-invariant by the inclusion of gauge links (“gauge-invariant” operators), then the mixing is

³For a recent review using lattice matrix elements to constrain standard model parameters see Ref. [1].

constrained by gauge invariance. In contrast, Landau-gauge four-fermion operators can mix with gauge non-invariant bilinears, and require many more subtractions. For this reason, we present results for penguin diagrams only for gauge-invariant operators.

In the continuum, the mixing of octet operators with bilinears of lower dimension is restricted by chiral symmetry and CP. It turns out that only a single such bilinear can appear, with just the overall factor undetermined. This result is crucial for the subtraction method of Ref. [8] to work. Complications arise with lattice operators, however, because of the fermion doubling problem. The issue is well understood for Wilson fermions, where the breaking of chiral symmetry allows mixing with a number of lower dimension bilinears [9, 10, 11], necessitating the use of more elaborate subtraction schemes. With staggered fermions, on the other hand, mixing is restricted by the axial $U(1)$ symmetry. While a large number of lower dimension bilinears can appear, only one of these contributes to physical matrix elements. This allows one to use the subtraction method of Ref. [8]. The fact that only one operator need be subtracted has been either implicitly or explicitly assumed in previous discussions [12, 13, 14], but is demonstrated here.

The mixing coefficients c_{ij} depend on the choice of scheme used to define the continuum operators. In physical matrix elements this dependence cancels with that of the two-loop anomalous dimension matrix. Thus for our results to be useful they must be given for a continuum scheme in which the two-loop anomalous dimension has been calculated. Results for both **27** and **8** operators are available in both Naive Dimensional Regularization (NDR) and 't Hooft-Veltman (HV—Ref. [15]) schemes. We choose to give results for the NDR scheme.

This work extends previous studies using staggered fermions. Gauge-invariant four-fermion operators were discussed by Daniel and Sheard [16], and the complete results for some of these operators were given by Sheard [17]. The calculation for Landau-gauge four-fermion operators turns out to be simpler than that for gauge-invariant operators, requiring only the matching coefficients for bilinears. These have been given in our previous work [18], henceforth referred to as PS. Ishizuka and Shizawa have independently calculated the corrections from X diagrams for unsmearred Landau-gauge four-fermion operators [6, 19], and our results are in complete agreement with theirs. This is a non-trivial check since the calculations are done using somewhat different methods. They have also calculated the corrections from X diagrams for gauge-invariant four-fermion operators. No previous calculations have studied penguin diagrams. The complete one-loop results for gauge-invariant $SU(3)$ octets can thus be obtained by combining c_{ij}^X from Ref. [19] with those for c_{ij}^{pen} given here.

A single staggered fermion corresponds to $N_f = 4$ Dirac fermions in the continuum limit. We follow Ref. [12] and introduce a staggered fermion “species” for each continuum flavor, i.e. χ_u, χ_d, χ_s etc. Thus the continuum theory to which we match has N_f degenerate copies of each quark in QCD. It is straightforward to relate the matrix elements in this continuum theory to those in continuum QCD [12, 13, 14].

The outline of this paper is as follows. In sect. 2 we explain our notation for four-fermion operators, and discuss their symmetries. Section 3 gives the details of the computations of the X diagrams for Landau-gauge operators, and sect. 4 the corresponding results for operators of phenomenological interest. In sect. 5 we present the calculation of the penguin diagrams, and give results for the matching of the continuum operators which contribute to kaon decays. We close in sect. 6 with a discussion of the mixing with lower dimension operators.

Two appendices collect results on Fierz transformations and the matching coefficients for bilinears.

Preliminary accounts of this work have been given in Refs. [14] and [20]. Some of the numerical results in these two works are wrong, and are corrected here.

2 NOTATION AND DEFINITIONS

In this section we set up the notation required to specify the four-fermion operators, and discuss their symmetries. While this section is self-contained, it draws heavily on the presentation given in PS, which should be consulted for further details.

2.1 Bilinears

We use a gamma matrix basis for both spin and flavor matrices. To enumerate this basis we use “hypercube vectors”, four-vectors whose components are 0 or 1, which are combined using modulo-2 operations. A general gamma matrix is labeled by a hypercube vector

$$\gamma_S = \gamma_1^{S_1} \gamma_2^{S_2} \gamma_3^{S_3} \gamma_4^{S_4} . \quad (2.1)$$

We use Euclidean space gamma matrices, which are hermitian, and satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. An alternative basis is built out of the complex conjugate matrices $\xi_\mu = \gamma_\mu^*$,

$$\xi_F = \xi_1^{F_1} \xi_2^{F_2} \xi_3^{F_3} \xi_4^{F_4} , \quad (2.2)$$

where F is another hypercube vector. To keep the notation clear, we always use ξ matrices for flavor, and γ matrices for spin. Useful shorthand notations are

$$S = \hat{\mu} \Rightarrow \gamma_S = \gamma_\mu , \quad S = (1, 1, 1, 1) \Rightarrow \gamma_S = \gamma_5 , \quad S = (0, 0, 0, 0) \Rightarrow \gamma_S = I , \quad (2.3)$$

$$\gamma_S \gamma_{S'} = \gamma_{SS'} , \quad \xi_S \xi_{S'} = \xi_{SS'} . \quad (2.4)$$

We label quark fields in a continuum theory with four degenerate fermions using upper case letters, e.g. $Q_{\alpha,a}$. Here α is the spinor index, and a the flavor index, both running from 1 to 4. For the moment we suppress color indices. A general bilinear is specified by a spin and a flavor matrix

$$\mathcal{O}_{SF}^{cont} = \overline{Q}_{\alpha,a} \gamma_S^{\alpha\beta} \xi_F^{ab} Q_{\beta,b} = \overline{Q}_{\alpha,a} (\gamma_S \otimes \xi_F)^{\alpha a, \beta b} Q_{\beta,b} \equiv \overline{Q} (\gamma_S \otimes \xi_F) Q . \quad (2.5)$$

In the second step we combine spin and flavor matrices into a single 16×16 matrix $(\gamma_S \otimes \xi_F)$.

We use a variety of lattice transcriptions of these continuum operators. All make use of a set of matrices which are related to $(\gamma_S \otimes \xi_F)$ by a unitary transformation [16, 21] :

$$\overline{(\gamma_S \otimes \xi_F)}_{AB} \equiv \frac{1}{4} \text{Tr}[\gamma_A^\dagger \gamma_S \gamma_B \gamma_F^\dagger] \quad (2.6)$$

$$= \sum_{\alpha,a,\beta,b} \left(\frac{1}{2}\xi_A\right)^{\alpha a} (\gamma_S \otimes \xi_F)^{\alpha a, \beta b} \left(\frac{1}{2}\gamma_B\right)^{\beta b} . \quad (2.7)$$

In the new basis we have traded the indices $\{\alpha, a\}$ for a hypercube vector A . Perturbative calculations in momentum space require another unitarily equivalent set of matrices

$$\overline{(\overline{\gamma_S \otimes \xi_F})}_{AB} \equiv \sum_{CD} \frac{1}{4} (-)^{A.C} \overline{(\gamma_S \otimes \xi_F)}_{CD} \frac{1}{4} (-)^{D.B} . \quad (2.8)$$

All three sets of matrices satisfy the multiplication rule exemplified by

$$\sum_B \overline{(\gamma_S \otimes \xi_F)}_{AB} \overline{(\gamma_{S'} \otimes \xi_{F'})}_{BC} = \overline{(\gamma_{SS'} \otimes \xi_{FF'})}_{AC} . \quad (2.9)$$

To define lattice bilinears we divide the lattice into 2^4 hypercubes in one of the 16 possible ways. Points on the original lattice are then specified by a vector y labeling the hypercubes (with all components even), and a hypercube vector C determining the position within the hypercube. The 16 components of the staggered fermion field χ for a given y are collected in to a single hypercube field

$$\chi(y)_C = \frac{1}{4} \chi(y + C) . \quad (2.10)$$

In the continuum limit this becomes equal to Q , when expressed in the appropriate basis [22]

$$\chi(y)_C \longrightarrow Q(y)_C = (\frac{1}{2} \xi_C)^{\beta b} Q_{\beta, b} . \quad (2.11)$$

Thus the lattice operator

$$\mathcal{O}_{SF}(y) = \sum_{C,D} \overline{\chi}(y)_C \overline{(\gamma_S \otimes \xi_F)}_{CD} \chi(y)_D \equiv \overline{\chi}(\overline{\gamma_S \otimes \xi_F}) \chi . \quad (2.12)$$

has the same flavor, spin and normalization as $\mathcal{O}_{SF}^{cont} = \overline{Q}(\gamma_S \otimes \xi_F)Q$ in the continuum limit.

An important property of a bilinear is its “distance”,

$$\Delta = \sum_{\mu=1}^4 |S_\mu - F_\mu|^2 . \quad (2.13)$$

For the lattice bilinear of Eq. 2.12 this is the number of links between the quark and antiquark field. From this we construct the “distance-parity” of the bilinear, $(-1)^\Delta$, which is +1 if the distance is even, -1 if it is odd. This parity is useful because of the $U(1)_A$ symmetry, valid when $ma = 0$,

$$\chi \rightarrow \exp \left[i\alpha \overline{(\gamma_5 \otimes \xi_5)} \right] \chi , \quad \overline{\chi} \rightarrow \overline{\chi} \exp \left[i\alpha \overline{(\gamma_5 \otimes \xi_5)} \right] , \quad (2.14)$$

where α is the rotation angle. Under this transformation, even distance bilinears rotate

$$(-)^\Delta = 1 : \quad \overline{\chi}(\overline{\gamma_S \otimes \xi_F}) \chi \rightarrow \cos \alpha \overline{\chi}(\overline{\gamma_S \otimes \xi_F}) \chi + i \sin \alpha \overline{\chi}(\overline{\gamma_{S5} \otimes \xi_{F5}}) \chi , \quad (2.15)$$

while odd distance bilinears are unchanged. It follows that even and odd distance bilinears cannot mix, and that the renormalization of even distance bilinears related by multiplication by $\overline{(\gamma_5 \otimes \xi_5)}$ is identical [23]. It has been noticed in Ref. [16] and in PS that, at one-loop, the latter result holds also for odd distance bilinears, although this does not follow from $U(1)_A$ symmetry. We now understand this result. As long as we consider diagrams in which

the $\bar{\chi}$ and χ fields are not contracted, then we can do separate $U(1)_A$ rotations on the two fields. In particular, if the rotation angles α for χ and $\bar{\chi}$ are opposite, then odd distance bilinears rotate as in Eq. 2.15, while even distance bilinears are unchanged. It follows that odd distance bilinears related by multiplication by $(\gamma_5 \otimes \xi_5)$ will renormalize identically.

As mentioned in the introduction, we consider both gauge-invariant and Landau-gauge operators. In the former, we insert between $\bar{\chi}$ and χ a matrix given by the average of the products of gauge links along the shortest paths connecting the quark and antiquark. In the latter, we fix to Landau gauge, and then use Eq. 2.12 without any gauge links. Landau-gauge operators have the advantages of being simple to implement numerically, and of having Feynman rules with no coupling to gluons. Their disadvantage is that, nonperturbatively, there may be Gribov copies, in which case the definition of the operators is ambiguous. In perturbation theory there is, however, no ambiguity, and we do not consider this problem further here.

As shown in PS, both gauge-invariant and Landau-gauge bilinears differ from the corresponding continuum bilinears at $O(a)$. For Landau-gauge bilinears, however, it is straightforward to “improve” the operators so that they have only $O(a^2)$ corrections at tree level [14]. One simply replaces the quark field in Eq. 2.12 by

$$\chi(y)_A \rightarrow \frac{1}{4} \sum_{\mu} \chi(y + 2\hat{\mu}[1 - 2A_{\mu}])_A = \frac{1}{16} \sum_{\mu} \chi(y + A + 2\hat{\mu}[1 - 2A_{\mu}]) , \quad (2.16)$$

and performs a similar replacement for the antiquark field. The resulting “smeared” operators are spread over a 4^4 hypercube. Since the fields are shifted by an even number of lattice spacings, smearing does not affect the spin-flavor content or the distance-parity of the operator.

We also consider a further improvement of the various operators. Lepage and Mackenzie have shown that a large part of the difference between lattice and continuum operators is due to fluctuations of the gauge links, and in particular due to tadpole diagrams [24]. These corrections can be partially summed using a rescaled gauge link $U'_{\mu} = U_{\mu}/u_0$ in both the action and the operators. As shown in PS, the prescription for staggered fermion bilinears is to multiply them by $u_0^{1-n_U}$ in order to match more closely with the continuum operators. Here n_U is the number of gauge links in the operator ($n_U = 0$ for Landau-gauge operators). We call operators including this factor “tadpole-improved”. There are various choices for u_0 ; for reasons given in PS we choose $u_0 = \langle \frac{1}{3} \text{Tr}(U)_L \rangle$, the normalized trace of the link in Landau gauge.

2.2 Four-fermion operators

The general four-fermion operator is specified by two spins and two flavors

$$\overline{\chi(\gamma_{S'} \otimes \xi_{F'})} \chi \overline{\chi(\gamma_S \otimes \xi_F)} \chi \equiv \overline{(\gamma_{S'} \otimes \xi_{F'})} \overline{(\gamma_S \otimes \xi_F)} . \quad (2.17)$$

The second form is a useful abbreviation for the operator which we only use when there is no danger of confusion with the product of two single-barred matrices such as that in Eq. 2.9. The two bilinears in Eq. 2.17 can be of any type (gauge-invariant, smeared Landau-gauge, etc.), although we only use operators in which both are of the same type. For a given type

of bilinear, there are 256^2 four-fermion operators, so that the matching matrices are huge: $65536 \times 65536!$ Fortunately we need only calculate small blocks of this matrix. In particular, as explained below, matrix elements of phenomenological interest can be obtained using operators for which $S = S'$ and $F = F'$, which we refer to as “diagonal” operators, and the associated operators $\overline{(\gamma_S \otimes \xi_F)} \overline{(\gamma_{S5} \otimes \xi_{F5})}$.

For certain calculations we can restrict our attention further, and consider only the 35 linear combinations of the diagonal operators which transform as the identity representation of the Euclidean lattice rotation group.⁴ Such “diagonal scalar” operators suffice, for example, in a calculation of B_K [3, 14]. There are few enough such operators that we can display some explicit results, rather than simply giving numerical answers.

To enumerate the 35 diagonal scalar operators we use a compact notation. Since the spin and flavor of both bilinears are the same, we need only specify the spin and flavor of one bilinear. The first 25 operators are those in which the spin and flavor independently form scalars. For the spin we use the basis

$$S = I \cdot I, \quad V = \sum_{\mu} \gamma_{\mu} \cdot \gamma_{\mu}, \quad T = \sum_{\mu < \nu} \gamma_{\mu} \gamma_{\nu} \cdot \gamma_{\nu} \gamma_{\mu},$$

$$A = \sum_{\mu} \gamma_{\mu} \gamma_5 \cdot \gamma_5 \gamma_{\mu}, \quad P = \gamma_5 \cdot \gamma_5,$$

where the dot separates the matrices in the two bilinears.⁵ Note that the spin matrix in the second bilinear is the hermitian conjugate of that in the first. We use exactly the same basis for flavor except that $\gamma \rightarrow \xi$. The first 25 scalar operators are obtained by combining the five possible spins with the five flavors. Our notation is exemplified by

$$[A \times P] = \sum_{\mu} \overline{(\gamma_{\mu 5} \otimes \xi_5)} \overline{(\gamma_{5\mu} \otimes \xi_5)}, \quad (2.18)$$

$$[V \times T] = \sum_{\mu} \sum_{\nu < \rho} \overline{(\gamma_{\mu} \otimes \xi_{\nu\rho})} \overline{(\gamma_{\mu} \otimes \xi_{\rho\nu})}. \quad (2.19)$$

In the continuum limit, these operators become Lorentz scalars.

In the remaining 10 operators the spin and flavor are not independent. We denote these operators as

$$[V_{\mu} \times V_{\mu}] = \sum_{\mu} \overline{(\gamma_{\mu} \otimes \xi_{\mu})} \overline{(\gamma_{\mu} \otimes \xi_{\mu})}, \quad (2.20)$$

with $[V_{\mu} \times A_{\mu}]$, $[A_{\mu} \times V_{\mu}]$ and $[A_{\mu} \times A_{\mu}]$ defined similarly;

$$[V_{\mu} \times T_{\mu}] = \sum_{\substack{\mu, \nu \\ \mu \neq \nu}} \overline{(\gamma_{\mu} \otimes \xi_{\mu\nu})} \overline{(\gamma_{\mu} \otimes \xi_{\nu\mu})} - \overline{(\gamma_{\mu} \otimes \xi_{\mu\nu 5})} \overline{(\gamma_{\mu} \otimes \xi_{5\nu\mu})}, \quad (2.21)$$

and the analogous $[T_{\mu} \times V_{\mu}]$;

$$[A_{\mu} \times T_{\mu}] = \sum_{\substack{\mu, \nu \\ \mu \neq \nu}} \overline{(\gamma_{\mu 5} \otimes \xi_{\mu\nu 5})} \overline{(\gamma_{5\mu} \otimes \xi_{5\nu\mu})} - \overline{(\gamma_{\mu 5} \otimes \xi_{\mu\nu})} \overline{(\gamma_{5\mu} \otimes \xi_{\nu\mu})}, \quad (2.22)$$

⁴Among the 65536 operators of Eq. 2.17 there are many other operators transforming as the identity representation, but they are off-diagonal, i.e. $S' \neq S$ and/or $F' \neq F$ [17].

⁵We are using S to denote both a hypercube vector representing all possible spins, and a scalar operator. Where confusion might arise, we denote the scalar operator by I .

and the analogous $[T_\mu \times A_\mu]$; and finally

$$[T_- \times T_-] = \sum_{\mu < \nu} \overline{(\gamma_{\mu\nu} \otimes \xi_{\mu\nu})} \overline{(\gamma_{\nu\mu} \otimes \xi_{\nu\mu})} - \overline{(\gamma_{\mu\nu} \otimes \xi_{\mu\nu 5})} \overline{(\gamma_{\nu\mu} \otimes \xi_{5\nu\mu})} \quad (2.23)$$

with $[T_+ \times T_+]$ defined similarly.⁶ Linear combinations of these operators are written

$$[(V + A)_\mu \times T_\mu] = [V_\mu \times T_\mu] + [A_\mu \times T_\mu] . \quad (2.24)$$

These 10 operators are lattice scalars, but do not become Lorentz scalars in the continuum limit.

Each four-fermion operator comes in two ‘‘color-types’’. The color indices can be contracted either between bilinears or within each bilinear

$$\mathcal{O}_I = \overline{(\gamma_S \otimes \xi_F)_{ab}} \overline{(\gamma_S \otimes \xi_F)_{ba}} ; \quad \mathcal{O}_{II} = \overline{(\gamma_S \otimes \xi_F)_{aa}} \overline{(\gamma_S \otimes \xi_F)_{bb}} . \quad (2.25)$$

This notation indicates the number of color loops when each of the bilinears is contracted with an external color singlet meson: operators of type (I) give rise to a single color trace, while those of type (II) produce two traces.

Lastly, we consider the implications of the $U(1)_A$ symmetry for the mixing of four-fermion operators. For diagrams in which the fields in the operator are not contracted with each other (i.e. X diagrams), we can do independent $U(1)_A$ rotations on each of the four fields. Several results follow from this.

- The distance-parity of each bilinear cannot be changed by mixing. Diagonal operators can be classified as even or odd distance because both bilinears have the same distance-parity. Thus it follows that even and odd distance diagonal four-fermion operators cannot mix.
- The mixing coefficients for the operators $\overline{(\gamma_S \otimes \xi_F)} \overline{(\gamma_{S5} \otimes \xi_{F5})}$ can be obtained from those for $\overline{(\gamma_S \otimes \xi_F)} \overline{(\gamma_S \otimes \xi_F)}$ by applying an appropriate axial rotation to the second bilinear. This is equivalent to multiplying the second bilinear by $\overline{(\gamma_5 \otimes \xi_5)}$ from either the right or left.
- The corrections are also unchanged if we multiply both bilinears by $\overline{(\gamma_5 \otimes \xi_5)}$ from, say, the left. We call this operation \mathcal{P} . This invariance is derived by applying an appropriate axial rotation to both bilinears.

These results are valid to all orders in perturbation theory, up to corrections of $O(ma)$. We have checked them explicitly in our one-loop calculations. They are not valid for penguin diagrams, as we discuss in more detail in sects. 5 and 6.

3 CALCULATING X DIAGRAMS

⁶The definitions of $[T_\pm \times T_\pm]$ differ from those used in Ref. [14] by a factor of two.

3.1 Overview of method

In this section we describe the calculation of the contribution to one-loop matching coefficients from X diagrams. We do this only for Landau-gauge operators. As mentioned in the introduction, we first match operators in QCD onto those in a continuum theory with N_f degenerate versions of each quark. We consider the following operators in the latter theory

$$\mathcal{O}_i^{cont} = \overline{S}(\gamma_S \otimes \xi_F) D \overline{S}(\gamma_S \otimes \xi_F) D . \quad (3.1)$$

This choice of flavors puts the operator into a **27** of flavor $SU(3)$, and ensures that only X diagrams contribute to the matching calculation. This operator will match onto lattice operators having the same flavor

$$\mathcal{O}_j = \overline{\chi}_s(\gamma_{S'} \otimes \xi_{F'}) \chi_d \overline{\chi}_s(\gamma_{S''} \otimes \xi_{F''}) \chi_d . \quad (3.2)$$

Throughout this section, and the following, we assume that such a choice of flavors has been made, but do not show the flavor indices explicitly.

We consider only diagonal operators in Eq. 3.1, because the continuum operators that we are interested in are diagonal in spin, and because actual numerical calculations take matrix elements between states having the same flavor F . There is no technical difficulty in extending the calculations to off-diagonal operators. The diagonal continuum operators do, however, match onto lattice operators which are both diagonal and off-diagonal, as indicated in Eq. 3.2. In particular, at one-loop, operators appear having $F' \neq F''$, although $S' = S''$. The matrix elements of such operators vanish, however, between lattice states of the same flavor, up to corrections suppressed either by powers of a , or by additional factors of g^2 . Thus we need only consider matching with diagonal lattice operators.

We are also interested in the matching matrices for odd parity operators of the form

$$\overline{S}(\gamma_S \otimes \xi_F) D \overline{S}(\gamma_{S5} \otimes \xi_{F5}) D . \quad (3.3)$$

As explained in the previous section, however, we can obtain the matching coefficients for these with no extra work.

The relation between continuum and lattice operators, Eq. 1.1, is derived by comparing the matrix elements of lattice and continuum operators between external states consisting of two quarks and two antiquarks. The coefficients $\gamma_{ij}^{(0)}$ and c_{ij} do not depend on the external momenta or quark masses except through terms suppressed by powers of the lattice spacing. Thus we set the external momenta and the quark masses to zero. Infrared divergences are regulated by adding a gluon mass, which can be set to zero in the final result. Ultraviolet divergences are removed in the continuum using a variety of renormalization schemes.

The diagrams which contribute at one-loop are those of Fig. 1, with the gluon propagator being in Landau gauge. Following Refs. [16] and PS, we call the first three Xa, Xb and Xc diagrams, and the remaining two Z and ZT diagrams. For tadpole-improved operators the contribution of ZT diagram cancels with the $O(g^2)$ term in the expansion of u_0^{-1} , so one can exclude these diagrams from the start. This is an example of how the tadpole-improved operators are similar to continuum operators, for the latter are corrected by all except the ZT diagrams.

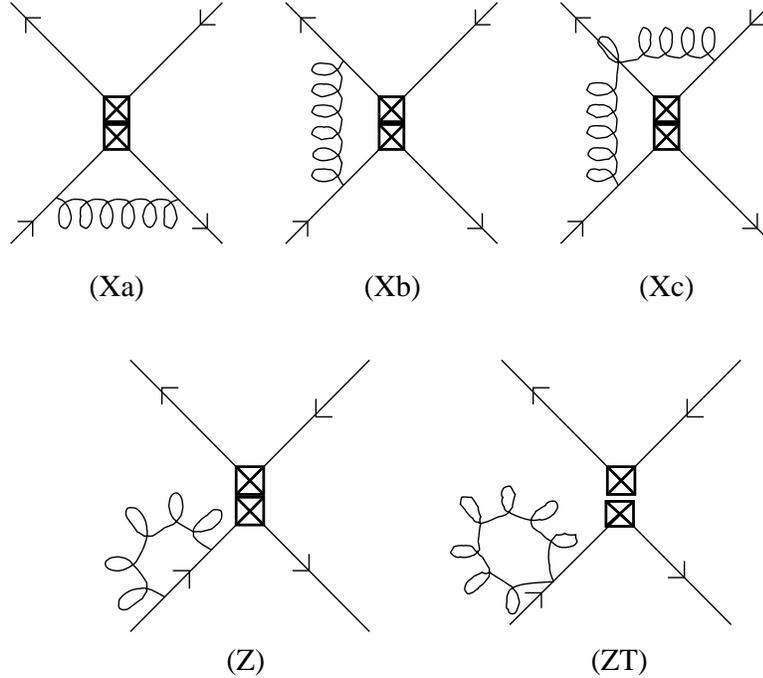


Figure 1: Diagrams contributing to renormalization of four-fermion operators for Landau-gauge operators. The two boxes represent the two bilinears which make up the operators. Only half of the X diagrams are shown; in the others the gluon exchanges are between the remaining two fermion lines. Similarly there are additional Z and ZT diagrams.

We first consider the one-loop lattice matrix element. We take the operator to be of color-type (II), and explain at the end how the results are modified for color-type (I) operators.

The Xa diagrams combine with the self-energy corrections Z and ZT just as in the calculation for bilinears. The corrections can thus be obtained from the results for Landau-gauge bilinears. One must simply add the results for each of the two bilinears contained in the four-fermion operator. We discuss this in more detail in Appendix B.

It turns out that one can also obtain the results for the Xb and Xc diagrams using only those for bilinears. This is done in a number of steps. First, we note that to bring the Xc diagrams into canonical form one of the quark propagators has to be conjugated, since the gluon connects two quarks (or antiquarks). The fermion-gluon vertex remains unaffected, but conjugation reverses the momentum flow along the fermion propagator and thus changes its sign when $m = 0$. Z and ZT diagrams, which are unaffected by conjugation, can now be added to the Xb diagrams, and subtracted from the Xc diagrams, such that each calculation is like that for a bilinear. Schematically

$$Xb + Xc = Xb - Xc^\dagger = (Xb + Z + ZT) - (Xc + Z + ZT)^\dagger. \quad (3.4)$$

Note that these manipulations are possible because the color factors are the same for the Xb and Xc diagrams.

For the Xb diagrams, the next step is to do a simultaneous Fierz transformation on the spin and flavor indices. This is necessary because the gluon connects a quark and antiquark which are part of different bilinears in the original form of the operator. The Fierz

transformations for spin (or flavor) alone are standard. We use the notation

$$\Gamma_{\alpha\beta}\Gamma_{\gamma\delta}^\dagger = \sum_{\Gamma'} f(\Gamma, \Gamma') \Gamma'_{\alpha\delta} \Gamma_{\gamma\beta}^\dagger, \quad (3.5)$$

where Γ is a general Dirac matrix. Combining two such transformations, we find

$$\overline{(\gamma_S \otimes \xi_F)_{AB}} \overline{(\gamma_S^\dagger \otimes \xi_F^\dagger)_{CD}} = \sum_{S', F'} f(S, S') f(F, F') \overline{(\gamma_{S'} \otimes \xi_{F'})_{AD}} \overline{(\gamma_{S'}^\dagger \otimes \xi_{F'}^\dagger)_{BC}}. \quad (3.6)$$

For the diagonal scalars, which Fierz transform into one another, we give the Fierz table in Appendix A. In general we use a computer program to calculate the Fierz table. Having Fierz transformed, we can use the results for the bilinear corrections, and then Fierz transform back to the original basis.

For the Xc diagrams, an additional step is required. First, one of the bilinears is conjugated using

$$\overline{(\gamma_S \otimes \xi_F)_{AB}} = \overline{(\gamma_S^\dagger \otimes \xi_F^\dagger)_{BA}}. \quad (3.7)$$

One then does a Fierz transformation, calculates the correction, Fierz transforms back, and finally conjugates again. All these steps are straightforward, but, given the number of operators involved, they rapidly lead to an explosion of terms. In practice we have written a computer program to do the various steps, and have checked it by working out some examples by hand.

The same method works for the four-fermion operators of color type (I). The only difference is that the Z and ZT diagrams combine with the Xb diagrams to form a bilinear correction, while one must add and subtract Z and ZT diagrams to the Xa and Xc diagrams respectively.

For continuum operators the steps in the calculation are the same, except for one subtlety which we return to shortly. The end result is that lattice and continuum corrections combine just as in the calculation for bilinears. Thus what we require from our previous work is the bilinear matching coefficients defined by

$$\mathcal{B}_i^{cont} = \mathcal{B}_i^{lat} \left(1 + C_F \frac{g^2}{16\pi^2} [2(1 - \sigma_S) \ln(\pi/\mu a) + c_i] \right) + \text{off-diagonal}. \quad (3.8)$$

Here \mathcal{B} denotes a bilinear, $C_F = 4/3$ is the color factor, and σ_S , which depends only on the spin of the bilinear, is $(4, 1, 0, 1, 4)$ for spin tensors (S, V, T, A, P) . The coefficients c_i depend upon the spin and flavor matrices in the bilinear, upon its type (e.g. smeared *vs.* unsmeared), and upon the continuum renormalization scheme. Numerical values are given in PS.

We do not need to keep the off-diagonal components of the one-loop mixing matrix in Eq. 3.8. These cause mixing between operators having the same spin but different flavor. For example, the continuum vector current $\overline{Q}(\gamma_\mu \otimes \xi_\nu)Q$ ($\mu \neq \nu$) matches onto a linear combination of the lattice operators $\overline{\chi}(\gamma_\mu \otimes \xi_\nu)\chi$ and $\overline{\chi}(\gamma_\mu \otimes \xi_\mu)\chi$ (no sum on μ). Such matching coefficients imply that continuum four-fermion operators constructed of two bilinears with the same flavor and spin will match in part with lattice operators in which the spins of the two bilinears are the same but the flavors differ. As discussed above, the

contribution of such operators can be dropped in the matrix elements between states having the same flavor. We note, however, that in a two-loop perturbative calculation one must retain such off-diagonal mixing at intermediate stages.

We now return to the subtlety concerning continuum regularization schemes. Fierz transformation and charge conjugation apply only if the gamma matrices are in 4-dimensions, as in Dimensional Reduction (DRED) and Pauli-Villars (PV) schemes. For schemes involving n -dimensional gamma matrices, such as NDR, the method described above will not work. In such schemes, one has to enlarge the basis of operators to include “evanescent” operators which vanish when restricted to 4-dimensions. There is considerable ambiguity in how one defines evanescent operators, and different choices lead to different results for the finite parts of diagrams, and thus for the coefficients c_i . The method described above amounts to a particular, and unconventional, choice of evanescent operators. We denote the result with this choice as being in the NDR' scheme. To relate the results to the conventional choice of evanescent operators of Refs. [2, 25, 26], we must make a finite correction. We explain this further in subsect. 4.5.

In the “correction” step of the above calculations we need to convert the corrections for bilinears into those for four-fermion operators, keeping only the diagonal part. For a general diagonal four-fermion operator of the form of Eq. 2.17, we simply sum the corrections to each of the constituent bilinears. In practice we implement this numerically. For the 35 diagonal scalars, which are linear combinations of operators receiving different corrections, the results are worked out in Appendix B.

3.2 Color factors

Each gluon exchange gives rise to a color factor

$$\sum_{\alpha=1}^8 T_{ab}^{\alpha} T_{cd}^{\alpha} = \frac{1}{2} \delta_{ad} \delta_{cb} - \frac{1}{6} \delta_{ab} \delta_{cd} . \quad (3.9)$$

In general this mixes operators of the two color types. Since the color mixing does not depend on the spin or flavor of the operator, it can be factored out. To display the result we use a vector notation

$$\vec{\mathcal{O}}_i = \begin{pmatrix} \mathcal{O}_{i(I)} \\ \mathcal{O}_{i(II)} \end{pmatrix} , \quad (3.10)$$

where i only labels the spin and flavor of the operator. The matching equation can then be written

$$\vec{\mathcal{O}}_i^{cont} = \vec{\mathcal{O}}_i^{lat} + \frac{g^2}{16\pi^2} \sum_j \left(\mathcal{M}_{ij}^a \vec{C}_a + \mathcal{M}_{ij}^b \vec{C}_b + \mathcal{M}_{ij}^c \vec{C}_c \right) \vec{\mathcal{O}}_j^{lat} , \quad (3.11)$$

where $\mathcal{M}_{ij}^{a,b,c}$ are the contributions from the Xa,b,c diagrams, respectively, together with associated self-energy diagrams, with color factors (and $g^2/16\pi^2$) removed. $C_{a,b,c}$ are the corresponding color matrices

$$\vec{C}_a = \frac{1}{6} \begin{pmatrix} -1 & 3 \\ 0 & 8 \end{pmatrix} , \quad \vec{C}_b = \frac{1}{6} \begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix} , \quad \vec{C}_c = \frac{1}{6} \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} . \quad (3.12)$$

3.3 An example: $[V \times S]$

To explain the method in more detail we calculate the lattice operator which matches onto the continuum operator $[V \times S]$. This is part of the matching calculation needed to extract B_K . For simplicity of presentation, we adopt the following notation for rows of the matching matrices

$$\mathcal{M}^a([V \times S]) \equiv \left(\sum_j \mathcal{M}_{ij}^a \mathcal{O}_j \right)_{i=[V \times S]} . \quad (3.13)$$

3.3.1 Xa diagrams

For the Xa diagrams, we can take the result directly from appendix B

$$\mathcal{M}^a([V \times S]) = 2c_{VS} [V \times S] . \quad (3.14)$$

3.3.2 Xb diagrams

We must perform the operations Fierz-correct-Fierz. Using the appendices we find

$$\begin{aligned} 16\mathcal{M}^b([V \times S]) = & (c_{SS} - c_{SP} + 3c_{VV2} + c_{VV0} - 3c_{VA2} - c_{VA4})([S \times V] + [P \times A]) \\ & + (-2c_{SV} + 2c_{SA} - 2c_{VS} + 2c_{VP})([S \times V] - [P \times A]) \\ & + (c_{SS} - c_{SP} - 3c_{VV2} - c_{VV0} + 3c_{VA2} + c_{VA4})([S \times A] + [P \times V]) \\ & + (2c_{SV} - 2c_{SA} - 2c_{VS} + 2c_{VP})([S \times A] - [P \times V]) \\ & + (c_{SS} + 6c_{ST} + c_{SP} + 3c_{VV2} + c_{VV0} + 3c_{VA2} + c_{VA4})([V \times S] + [A \times P]) \\ & + (4c_{SV} + 4c_{SA} + c_{VS} + 3c_{VT1} + 3c_{VT3} + c_{VP})([V \times S] - [A \times P]) \\ & + (c_{SS} - 2c_{ST} + c_{SP})[(V + A) \times T] \\ & + (c_{VS} - c_{VT1} - c_{VT3} + c_{VP})[(V - A) \times T] \\ & + (c_{SS} + 6c_{ST} + c_{SP} - 3c_{VV2} - c_{VV0} - 3c_{VA2} - c_{VA4})([V \times P] + [A \times S]) \\ & + (-4c_{SV} - 4c_{SA} + c_{VS} + 3c_{VT1} + 3c_{VT3} + c_{VP})([V \times P] - [A \times S]) \\ & + (c_{SS} - c_{SP})[T \times (V + A)] \\ & + (-2c_{SV} + 2c_{SA})[T \times (V - A)] \\ & + (c_{VV0} - c_{VV2} + c_{VA4} - c_{VA2})[(V + A)_\mu \times T_\mu] \\ & + (c_{VV0} - c_{VV2} - c_{VA4} + c_{VA2})[T_\mu \times (V + A)_\mu] \\ & + (2c_{VT3} - 2c_{VT1})[T_\mu \times (V - A)_\mu] \\ & + \text{off-diagonal} \end{aligned}$$

We have simplified the answer by setting $\mu = \pi/a$, so that the anomalous dimension factor is absent. It is simple to reinstate this factor, however, as it depends only on the spin of the bilinear being corrected (Eq. 3.8), and thus can be determined by replacing c_{SF} with $2(1 - \sigma_S) \ln(\pi/\mu a)$ in the above equation. For the present example, this gives

$$\mathcal{M}^b([V \times S]; \text{anom. dim.}) = -6 \ln(\pi/\mu a) ([V \times S] + [A \times S]) . \quad (3.15)$$

Note that no even-distance operators appear in \mathcal{M}_b , as required by the axial symmetries.

3.3.3 Xc diagrams

The first step is to conjugate, which for $[V \times S]$ has no effect. Thus the ensuing Fierz-correct-Fierz steps give the same result as for the Xb diagrams. Finally, we must conjugate again, and multiply the total result by -1 . This gives

$$\begin{aligned}
-16\mathcal{M}^c([V \times S]) = & (c_{SS} - c_{SP} + 3c_{VV2} + c_{VV0} - 3c_{VA2} - c_{VA4})([S \times V] - [P \times A]) \\
& + (-2c_{SV} + 2c_{SA} - 2c_{VS} + 2c_{VP})([S \times V] + [P \times A]) \\
& - (c_{SS} - c_{SP} - 3c_{VV2} - c_{VV0} + 3c_{VA2} + c_{VA4})([S \times A] - [P \times V]) \\
& - (2c_{SV} - 2c_{SA} - 2c_{VS} + 2c_{VP})([S \times A] + [P \times V]) \\
& + (c_{SS} + 6c_{ST} + c_{SP} + 3c_{VV2} + c_{VV0} + 3c_{VA2} + c_{VA4})([V \times S] - [A \times P]) \\
& + (4c_{SV} + 4c_{SA} + c_{VS} + 3c_{VT1} + 3c_{VT3} + c_{VP})([V \times S] + [A \times P]) \\
& - (c_{SS} - 2c_{ST} + c_{SP})[(V - A) \times T] \\
& - (c_{VS} - c_{VT1} - c_{VT3} + c_{VP})[(V + A) \times T] \\
& + (c_{SS} + 6c_{ST} + c_{SP} - 3c_{VV2} - c_{VV0} - 3c_{VA2} - c_{VA4})([V \times P] - [A \times S]) \\
& + (-4c_{SV} - 4c_{SA} + c_{VS} + 3c_{VT1} + 3c_{VT3} + c_{VP})([V \times P] + [A \times S]) \\
& - (c_{SS} - c_{SP})[T \times (V - A)] \\
& - (-2c_{SV} + 2c_{SA})[T \times (V + A)] \\
& - (c_{VV0} - c_{VV2} + c_{VA4} - c_{VA2})[(V - A)_\mu \times T_\mu] \\
& - (c_{VV0} - c_{VV2} - c_{VA4} + c_{VA2})[T_\mu \times (V - A)_\mu] \\
& - (2c_{VT3} - 2c_{VT1})[T_\mu \times (V + A)_\mu] \\
& + \text{off-diagonal}
\end{aligned}$$

The anomalous dimension term can be reconstructed from this result in the same way as for \mathcal{M}_b

$$\mathcal{M}^c([V \times S]; \text{anom. dim.}) = 6 \ln(\pi/\mu a)([V \times S] - [A \times S]) . \quad (3.16)$$

Once again, only odd-parity operators appear.

4 RESULTS FOR X DIAGRAMS

4.1 General features

Using the method explained in the previous section it is straightforward to obtain the 256×256 block of the matching matrices $\mathcal{M}^{a,b,c}$ connecting diagonal four-fermion operators to each other. Both the Fierz transformations and the (diagonal part of) the bilinear corrections can be represented as 256×256 matrices. To obtain numerical results, we have written a computer program to multiply these matrices appropriately and put in the color factors. We have done this both for the full matrix and for the 35×35 block connecting diagonal scalar operators to each other. We have checked these programs by comparing them to each other, and to various analytical results such as those of the previous section.

We do not need the full matching matrix if we take matrix elements between states of definite external flavor. For example, in the mixing of $[V \times S]$, we are only interested in

the matching coefficients for the operators with flavor S . Operators with other flavors will contribute only at $O(g^2a)$ or $O(g^4)$. Because odd-distance operators mix only with each other, it suffices to consider the mixing of $[V \times S]$ with $[A \times S]$. Thus we want only a 2×2 block of the matrices $\mathcal{M}^{a,b,c}$. Including color types, we then have a 4×4 matching matrix. We give results only for such sub-matrices.

We quote results for two types of operator: unsmeared and smeared, both tadpole improved. The results for operators without tadpole improvement are the same except that diagonal matching matrix elements are shifted: $c_{ii} \rightarrow c_{ii} - 24.4661$. This simple change follows from the fact that tadpoles appear only on external lines.

To present numerical results we must choose a continuum regularization scheme. We want to use NDR, but as explained above, the method of the previous section leads to results in a slightly different scheme, NDR' . Since we have to add a finite correction to convert from NDR' to NDR, we might as well use an intermediate scheme for which the methods of the previous section apply, i.e. a scheme involving 4-dimensional gamma matrices. We choose Dimensional Reduction [27] using the ‘‘easy subtraction’’ scheme of Ref. [28] ($\text{DRE}\overline{\text{Z}}'$). The prime indicates that, for four-fermion operators, our method differs slightly from that of Ref. [28]. Another virtue of using $\text{DRE}\overline{\text{Z}}'$ is that the numerical results for bilinears are given in PS using this scheme. This allows the diligent reader to check the results given here more easily.

It is straightforward to convert from $\text{DRE}\overline{\text{Z}}'$ to other schemes. Changing schemes introduces a spin dependent, but flavor independent shift in the bilinear matching coefficients

$$c_i(\text{scheme 2}) = c_i(\text{scheme 1}) + t_S(\text{scheme 2}) - t_S(\text{scheme 1}) . \quad (4.1)$$

For the schemes we consider $t_I = t_P$ and $t_V = t_A$. The values of t_S , for $\text{DRE}\overline{\text{Z}}'$, dimensional reduction using the subtraction scheme of Ref. [29] (DRED), naive dimensional regularization with an anticommuting γ_5 (NDR'), and Pauli-Villars using the subtraction scheme of Ref. [16] (PV)⁷, are

$$t_S = \begin{cases} (& 0.5, & 0, & 0.5) & (\text{DRE}\overline{\text{Z}}') \\ (& 0.5, & 0.5, & 0.5) & (\text{DRED}) \\ (& -0.5, & 0, & 1.5) & (\text{NDR}') \\ (& 0, & 0, & 0) & (\text{PV}) \end{cases} \quad (4.2)$$

for spin tensors (I, V, T), respectively. These results are taken from PS, except that we have added the result for the PV scheme. The shift in c_i introduces a change in $\mathcal{M}^{a,b,c}$ which depends on the spins of the four-fermion operators but not on their flavor. Including color factors one finds the change in the matching matrices to be

$$c_{ij}(\text{scheme 2}) = c_{ij}(\text{scheme 1}) + \sum_{S=I,V,T} d_{ij}^S [t_S(\text{scheme 2}) - t_S(\text{scheme 1})] , \quad (4.3)$$

where the three matrices $d^{I,V,T}$ are independent of flavor. We quote results for these below, in addition to those for c_{ij} .

⁷For the PV scheme we use $M_g = \mu\sqrt{e}$ for the gluon regulator mass.

A change in renormalization scale μ is a special case of changing schemes. Using Eq. 3.8 it is simple to show that

$$\gamma_{ij}^{(0)} = -6d_{ij}^I + 2d_{ij}^T. \quad (4.4)$$

We now present results for various operators that are used in present calculations. Following PS, to give some idea of the size of the corrections we set $\mu = \pi/a$ and use a boosted coupling $g^2 = 1.8$. With these choices, the anomalous dimension contribution to matching vanishes, and the matching matrix is to be multiplied by $g^2/16\pi^2 \approx 1/90$.

4.2 $[V \times S]$ and $[A \times S]$

This is the part of the matching matrix needed for a calculation of B_K using external pseudo-Goldstone pions. We are interested in the submatrix of c_{ij} with indices

$$i, j = ([V \times S]_I, [V \times S]_{II}, [A \times S]_I, [A \times S]_{II}) . \quad (4.5)$$

Using symmetry operation \mathcal{P} (defined in the previous section), this matrix is the same as that with indices

$$i, j = -([A \times P]_I, [A \times P]_{II}, [V \times P]_I, [V \times P]_{II}) . \quad (4.6)$$

The minus sign arises because, under \mathcal{P} , $V \leftrightarrow -A$. (The other transformations are $S \leftrightarrow P$ and $T \leftrightarrow T$.) The sign is, however, irrelevant here, since it is an overall sign for the operators and does not affect c_{ij} .

The matrices needed for changing schemes are

$$d^I = \frac{1}{6} \begin{pmatrix} 9 & -3 & 7 & 3 \\ 0 & 0 & 6 & -2 \\ 7 & 3 & 9 & -3 \\ 6 & -2 & 0 & 0 \end{pmatrix}, \quad d^V = \frac{1}{6} \begin{pmatrix} 7 & 3 & -7 & -3 \\ 0 & 16 & -6 & 2 \\ -7 & -3 & 7 & 3 \\ -6 & 2 & 0 & 16 \end{pmatrix}, \quad d^T = 0, \quad (4.7)$$

so that $\gamma^{(0)} = -6d^I$. For the finite part of the coefficients, the results for the (tadpole-improved) Landau-gauge operators are⁸

$$c(\text{unsmeared}) = \begin{pmatrix} 2.0450 & -0.6817 & 6.8467 & 2.9343 \\ 0 & 0 & 5.8686 & -1.9562 \\ 6.8467 & 2.9343 & 2.5726 & -2.2643 \\ 5.8686 & -1.9562 & 0 & -4.2203 \end{pmatrix}, \quad (4.8)$$

$$c(\text{smeared}) = \begin{pmatrix} 10.0757 & -4.6508 & 12.5283 & 5.3693 \\ 0 & -3.8768 & 10.7385 & -3.5795 \\ 12.5283 & 5.3693 & 10.2457 & -5.1609 \\ 10.7385 & -3.5795 & 0 & -5.2371 \end{pmatrix}. \quad (4.9)$$

These corrections are quite small—multiplying by 1/90 one sees that they are 10% or less. Without tadpole improvement the corrections would be somewhat larger.

⁸The accuracy of these and following results is $\approx \pm 0.0001$.

Ishizuka and Shizawa have reported results for a variety of unsmeared operators, both Landau-gauge and gauge-invariant [19]. They quote results for a rectangular block of the mixing matrix for Landau-gauge operators: c_{ik} , with i given by Eq. 4.5, and k running over all operators. This contains the square matrix Eq. 4.8. Using the conversion formula

$$-c_{ij}^{lat}(\text{Ref. [19]}) = c_{ij} + (\ln \pi + \frac{7}{12})\gamma_{ij}^{(0)}, \quad (4.10)$$

and accounting for our opposite definition of A and T , we find that the results agree within the quoted errors.

4.3 $[S \times S]$, $[T \times S]$ and $[P \times S]$

For calculations of ϵ' , matrix elements of the operators $[S \times S]$, $[P \times S]$, $[P \times P]$ and $[S \times P]$ are required. The matching coefficients for these operators also involve $[T \times S]$ and $[T \times P]$. Thus we present results for the submatrices having indices

$$i, j = ([S \times S]_I, [S \times S]_{II}, [T \times S]_I, [T \times S]_{II}, [P \times S]_I, [P \times S]_{II}), \quad (4.11)$$

which are equal to the submatrices having indices

$$i, j = ([P \times P]_I, [P \times P]_{II}, [T \times P]_I, [T \times P]_{II}, [S \times P]_I, [S \times P]_{II}). \quad (4.12)$$

For the anomalous dimension parts we find

$$d^I = \frac{1}{24} \begin{pmatrix} 1 & 21 & 7 & 3 & 9 & -3 \\ 0 & 64 & 6 & -2 & 0 & 0 \\ 42 & 18 & 54 & -18 & 42 & 18 \\ 36 & -12 & 0 & 0 & 36 & -12 \\ 9 & -3 & 7 & 3 & 1 & 21 \\ 0 & 0 & 6 & -2 & 0 & 64 \end{pmatrix}, \quad (4.13)$$

$$d^V = \frac{1}{2} \begin{pmatrix} 3 & -1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.14)$$

$$d^T = \frac{1}{24} \begin{pmatrix} 27 & -9 & -7 & -3 & 27 & -9 \\ 0 & 0 & -6 & 2 & 0 & 0 \\ -42 & -18 & 10 & 18 & -42 & -18 \\ -36 & 12 & 0 & 64 & -36 & 12 \\ 27 & -9 & -7 & -3 & 27 & -9 \\ 0 & 0 & -6 & 2 & 0 & 0 \end{pmatrix}, \quad (4.15)$$

while the finite parts are

$$\begin{pmatrix} 1.3864 & -20.1799 & 2.0878 & 0.8948 & 0.75 & -0.25 \\ 0 & -59.1533 & 1.7895 & -0.5965 & 0 & 0 \\ 12.5267 & 5.3686 & 6.7342 & -5.5910 & 12.5267 & 5.3686 \\ 10.7371 & -3.5791 & 0 & -10.7371 & -10.7371 & -3.5791 \\ 0.75 & -0.25 & 2.0878 & 0.8948 & -9.3253 & 11.9549 \\ 0 & 0 & 1.7895 & -0.5965 & 0 & 26.5395 \end{pmatrix}, \quad (4.16)$$

$$\begin{pmatrix} -5.9359 & 2.4296 & 3.9817 & 1.7064 & 0.75 & -0.25 \\ 0 & 1.3531 & 3.4128 & -1.1376 & 0 & 0 \\ 23.8899 & 10.2385 & 17.1791 & -10.3096 & 23.8899 & 10.2385 \\ 20.4771 & -6.8257 & 0 & -13.7499 & 20.4771 & -6.8257 \\ 0.75 & -0.25 & 3.9817 & 1.7064 & -9.2877 & 12.4850 \\ 0 & 0 & 3.4128 & -1.1376 & 0 & 28.1675 \end{pmatrix} \quad (4.17)$$

respectively for unsmearred and smeared operators.

These corrections are larger than those for the vector and axial operators. For the unsmearred operators, the range is roughly -0.65 to 0.3 , which is sufficiently large that it may not be possible to do reliable computations. Calculations with smeared operators should be more reliable, since the corrections range only from -0.15 to 0.3 .

4.4 Flavor ξ_{35} and ξ_{345} operators

Numerical calculations have also been performed with lattice pions having flavors other than ξ_5 . For example, Ref. [30] uses flavors ξ_{35} , ξ_{45} and ξ_{345} . The continuum operators that are required, e.g.

$$\sum_{\mu} \overline{(\gamma_{\mu 5} \otimes \xi_{35})} \overline{(\gamma_{5\mu} \otimes \xi_{53})}, \quad (4.18)$$

are diagonal but not scalar. Thus we must use the full basis of the 256 diagonal operators. We present results only for flavors ξ_{35} and ξ_{345} , as one can extract the results for all other flavors using the \mathcal{P} operation and the lattice rotation symmetry,

In this subsection we give results only for the c_{ij} . The d^S matrices, which are independent of flavor, have already been given above. The only subtle point is that one must reexpress them in the different bases that we use here.

We begin with the results for the vector and axial operators needed for B_K . For external flavor ξ_{35} states, the continuum operators we need are

$$i = \left[\sum_{\mu} \overline{(\gamma_{\mu 5} \otimes \xi_{35})} \overline{(\gamma_{5\mu} \otimes \xi_{53})}_{(I,II)}, \sum_{\mu} \overline{(\gamma_{\mu} \otimes \xi_{35})} \overline{(\gamma_{\mu} \otimes \xi_{53})}_{(I,II)} \right]. \quad (4.19)$$

to match onto which we need the lattice operators

$$j = \left[\sum_{\mu \neq 3} \overline{(\gamma_{\mu 5} \otimes \xi_{35})} \overline{(\gamma_{5\mu} \otimes \xi_{53})}_{(I,II)}, \sum_{\mu \neq 3} \overline{(\gamma_{\mu} \otimes \xi_{35})} \overline{(\gamma_{\mu} \otimes \xi_{53})}_{(I,II)}, \right. \\ \left. \overline{(\gamma_{35} \otimes \xi_{35})} \overline{(\gamma_{53} \otimes \xi_{53})}_{(I,II)}, \overline{(\gamma_3 \otimes \xi_{35})} \overline{(\gamma_3 \otimes \xi_{53})}_{(I,II)} \right].$$

The matching matrix for the unsmeared operators is

$$\begin{pmatrix} 2.6784 & -2.5817 & 6.8467 & 2.9343 & 3.9134 & -6.2867 & 6.8467 & 2.9343 \\ 0.0000 & -5.0667 & 5.8686 & -1.9562 & 0.0000 & -14.9467 & 5.8686 & -1.9562 \\ 6.8467 & 2.9343 & 2.6723 & -2.5636 & 6.8467 & 2.9343 & 2.6358 & -2.4541 \\ 5.8686 & -1.9562 & 0.0000 & -5.0184 & 5.8686 & -1.9562 & 0.0000 & -4.7264 \end{pmatrix}, \quad (4.20)$$

while that for the smeared operators is

$$\begin{pmatrix} 10.2345 & -5.1274 & 12.5283 & 5.3693 & 10.7478 & -6.6672 & 12.5283 & 5.3693 \\ 0.0000 & -5.1477 & 10.7385 & -3.5795 & 0.0000 & -9.2539 & 10.7385 & -3.5795 \\ 12.5283 & 5.3693 & 10.3081 & -5.3481 & 12.5283 & 5.3693 & 10.2013 & -5.0277 \\ 10.7385 & -3.5795 & 0.0000 & -5.7363 & 10.7385 & -3.5795 & 0.0000 & -4.8819 \end{pmatrix}. \quad (4.21)$$

For flavor ξ_{345} pions, we need continuum operators

$$i = \left[\sum_{\mu} \overline{(\gamma_{\mu 5} \otimes \xi_{345})} \overline{(\gamma_{5\mu} \otimes \xi_{543})}_{(I,II)}, \sum_{\mu} \overline{(\gamma_{\mu} \otimes \xi_{345})} \overline{(\gamma_{\mu} \otimes \xi_{543})}_{(I,II)} \right]. \quad (4.22)$$

and the lattice operators

$$j = \left[\sum_{\mu=3,4} \overline{(\gamma_{\mu 5} \otimes \xi_{345})} \overline{(\gamma_{5\mu} \otimes \xi_{543})}_{(I,II)}, \sum_{\mu=3,4} \overline{(\gamma_{\mu} \otimes \xi_{345})} \overline{(\gamma_{\mu} \otimes \xi_{543})}_{(I,II)}, \right. \\ \left. \sum_{\mu=1,2} \overline{(\gamma_{\mu 5} \otimes \xi_{345})} \overline{(\gamma_{5\mu} \otimes \xi_{543})}_{(I,II)}, \sum_{\mu=1,2} \overline{(\gamma_{\mu} \otimes \xi_{345})} \overline{(\gamma_{\mu} \otimes \xi_{543})}_{(I,II)} \right].$$

The matching matrices are

$$\begin{pmatrix} 3.1770 & -4.0775 & 6.8467 & 2.9343 & 2.6615 & -2.5312 & 6.8467 & 2.9343 \\ 0.0000 & -9.0555 & 5.8686 & -1.9562 & 0.0000 & -4.9320 & 5.8686 & -1.9562 \\ 6.8467 & 2.9343 & 2.6615 & -2.5312 & 6.8467 & 2.9343 & 3.1770 & -4.0775 \\ 5.8686 & -1.9562 & 0.0000 & -4.9320 & 5.8686 & -1.9562 & 0.0000 & -9.0555 \end{pmatrix}, \quad (4.23)$$

$$\begin{pmatrix} 10.4955 & -5.9104 & 12.5283 & 5.3693 & 10.2191 & -5.0812 & 12.5283 & 5.3693 \\ 0.0000 & -7.2357 & 10.7385 & -3.5795 & 0.0000 & -5.0245 & 10.7385 & -3.5795 \\ 12.5283 & 5.3693 & 10.2191 & -5.0812 & 12.5283 & 5.3693 & 10.4955 & -5.9104 \\ 10.7385 & -3.5795 & 0.0000 & -5.0245 & 10.7385 & -3.5795 & 0.0000 & -7.2357 \end{pmatrix}. \quad (4.24)$$

for unsmeared and smeared operators respectively.

For scalar, pseudoscalar and tensor operators of flavor ξ_{35}

$$i = \left[\overline{(I \otimes \xi_{35})} \overline{(I \otimes \xi_{53})}_{(I,II)}, \overline{(\gamma_5 \otimes \xi_{35})} \overline{(\gamma_5 \otimes \xi_{53})}_{(I,II)}, \sum_{\mu < \nu} \overline{(\gamma_{\mu\nu} \otimes \xi_{35})} \overline{(\gamma_{\nu\mu} \otimes \xi_{53})}_{(I,II)} \right], \quad (4.25)$$

matching requires the following operators

$$j = \left[\overline{(I \otimes \xi_{35})} \overline{(I \otimes \xi_{53})}_{(I,II)}, \overline{(\gamma_5 \otimes \xi_{35})} \overline{(\gamma_5 \otimes \xi_{53})}_{(I,II)}, \right. \\ \left. \sum_{\mu \neq 3} \overline{(\gamma_{3\mu} \otimes \xi_{35})} \overline{(\gamma_{\mu 3} \otimes \xi_{53})}_{(I,II)}, \sum_{\mu \neq 3} \overline{(\gamma_{3\mu 5} \otimes \xi_{35})} \overline{(\gamma_{5\mu 3} \otimes \xi_{53})}_{(I,II)} \right].$$

The results for unsmeared and smeared operators, respectively, are

$$\begin{pmatrix} -8.9254 & 10.7553 & 0.7500 & -0.2500 & 2.0878 & 0.8948 & 2.0878 & 0.8948 \\ 0.0000 & 23.3405 & 0.0000 & 0.0000 & 1.7895 & -0.5965 & 1.7895 & -0.5965 \\ 0.7500 & -0.2500 & -5.7497 & 1.2283 & 2.0878 & 0.8948 & 2.0878 & 0.8948 \\ 0.0000 & 0.0000 & 0.0000 & -2.0648 & 1.7895 & -0.5965 & 1.7895 & -0.5965 \\ 12.5267 & 5.3686 & 12.5267 & 5.3686 & 6.8395 & -5.9070 & 5.2835 & -1.2389 \\ 10.7371 & -3.5790 & 10.7371 & -3.5790 & 0.0000 & -10.8816 & 0.0000 & 1.5667 \end{pmatrix}, \quad (4.26)$$

$$\begin{pmatrix} -9.1536 & 12.0827 & 0.7500 & -0.2500 & 3.9817 & 1.7064 & 3.9817 & 1.7064 \\ 0.0000 & 27.0947 & 0.0000 & 0.0000 & 3.4128 & -1.1376 & 3.4128 & -1.1376 \\ 0.7500 & -0.2500 & -8.1815 & 9.1665 & 3.9817 & 1.7064 & 3.9817 & 1.7064 \\ 0.0000 & 0.0000 & 0.0000 & 19.3181 & 3.4128 & -1.1376 & 3.4128 & -1.1376 \\ 23.8899 & 10.2385 & 23.8899 & 10.2385 & 17.2133 & -10.4124 & 17.1090 & -10.0995 \\ 20.4771 & -6.8257 & 20.4771 & -6.8257 & 0.0000 & -14.0240 & 0.0000 & -13.1896 \end{pmatrix}. \quad (4.27)$$

Finally, for operators with flavor ξ_{345} , it is convenient to first present results for scalars and pseudoscalars, i.e.

$$i = [\overline{(I \otimes \xi_{345})} \overline{(I \otimes \xi_{543})}_{(I,II)}, \overline{(\gamma_5 \otimes \xi_{345})} \overline{(\gamma_5 \otimes \xi_{543})}_{(I,II)}], \quad (4.28)$$

which requires the lattice operators

$$j = [\overline{(I \otimes \xi_{345})} \overline{(I \otimes \xi_{543})}_{(I,II)}, \overline{(\gamma_5 \otimes \xi_{345})} \overline{(\gamma_5 \otimes \xi_{543})}_{(I,II)}], \sum_{\mu < \nu} \overline{(\gamma_{\mu\nu} \otimes \xi_{345})} \overline{(\gamma_{\nu\mu} \otimes \xi_{543})}_{(I,II)}. \quad (4.29)$$

The results for unsmeared and smeared operators, respectively are

$$\begin{pmatrix} -8.1285 & 8.3646 & 0.7500 & -0.2500 & 2.0878 & 0.8948 \\ 0.0000 & 16.9653 & 0.0000 & 0.0000 & 1.7895 & -0.5965 \\ 0.7500 & -0.2500 & -8.1285 & 8.3646 & 2.0878 & 0.8948 \\ 0.0000 & 0.0000 & 0.0000 & 16.9653 & 1.7895 & -0.5965 \end{pmatrix}, \quad (4.30)$$

$$\begin{pmatrix} -8.8985 & 11.3174 & 0.7500 & -0.2500 & 3.9817 & 1.7064 \\ 0.0000 & 25.0539 & 0.0000 & 0.0000 & 3.4128 & -1.1376 \\ 0.7500 & -0.2500 & -8.8985 & 11.3174 & 3.9817 & 1.7064 \\ 0.0000 & 0.0000 & 0.0000 & 25.0539 & 3.4128 & -1.1376 \end{pmatrix}. \quad (4.31)$$

For the continuum tensor operator

$$i = [\sum_{\mu < \nu} \overline{(\gamma_{\mu\nu} \otimes \xi_{345})} \overline{(\gamma_{\nu\mu} \otimes \xi_{543})}_{(I,II)}], \quad (4.32)$$

the following lattice operators are required

$$j = [\overline{(I \otimes \xi_{345})} \overline{(I \otimes \xi_{543})} + \overline{(\gamma_5 \otimes \xi_{345})} \overline{(\gamma_5 \otimes \xi_{543})}_{(I,II)}, \sum_{\mu < \nu} \overline{(\gamma_{\mu\nu} \otimes \xi_{345})} \overline{(\gamma_{\nu\mu} \otimes \xi_{543})}_{(I,II)}, \overline{(\gamma_{34} \otimes \xi_{345})} \overline{(\gamma_{43} \otimes \xi_{543})}_{(I,II)}, \overline{(\gamma_{12} \otimes \xi_{345})} \overline{(\gamma_{21} \otimes \xi_{543})}_{(I,II)}].$$

Here $\sum'_{\mu<\nu}$ means the sum excluding the terms with $\mu = 1, \nu = 2$ and $\mu = 3, \nu = 4$. The results are

$$\begin{pmatrix} 12.5267 & 5.3686 & 6.8395 & -5.9070 & 7.1507 & -6.8404 & 5.2835 & -1.2389 \\ 10.7371 & -3.5790 & 0.0000 & -10.8816 & 0.0000 & -13.3707 & 0.0000 & 1.5667 \end{pmatrix}, \quad (4.33)$$

$$\begin{pmatrix} 23.8899 & 10.2385 & 17.1940 & -10.3544 & 17.2254 & -10.4486 & 16.8368 & -9.2828 \\ 20.4771 & -6.8257 & 0.0000 & -13.8693 & 0.0000 & -14.1205 & 0.0000 & -11.0117 \end{pmatrix}, \quad (4.34)$$

for smeared and unsmeared operators respectively.

In summary, we see that the corrections for operators with flavors other than ξ_5 are of moderate size for all spins, not exceeding 20% for unsmeared operators, and 30% for smeared operators.

4.5 Converting results to NDR

To convert our results for c_{ij} from $\overline{\text{DREZ}}$ to NDR, we proceed in two steps. First, we convert to NDR' , using Eq. 4.3

$$c_{ij}(\text{NDR}') = c_{ij}(\overline{\text{DREZ}}) + d_{ij}^T - d_{ij}^I. \quad (4.35)$$

Note that this conversion depends only on the spin, and not on the flavor, of the operators. In the second step we convert from NDR' to NDR. We do this in the standard way by comparing the one-loop matrix elements in the two schemes. Since this is entirely a continuum calculation, it also does not depend on the flavor of the operators. We use the notation of Refs. [25, 26]

$$c_{ij}(\text{NDR}) = c_{ij}(\text{NDR}') + \Delta r_{ij}. \quad (4.36)$$

For the bilinears, $i, j = [(V - A)_I, (V - A)_{II}]$, $i, j = [(V + A)_I, (V + A)_{II}]$, and $i, j = [(S - P)_I, (S - P)_{II}]$, only the Xc diagrams contribute, and the result is the same for all three cases

$$\Delta r_{ij} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}. \quad (4.37)$$

This notation means that the result applies for any flavor, e.g. $i, j = ([V \times S]_I, [V \times S]_{II})$. For $i, j = [(S + P)_I, (S + P)_{II}, T_I, T_{II}]$, both Xb and Xc diagrams contribute and we find

$$\Delta r_{ij} = \frac{1}{6} \begin{pmatrix} 9 & -3 & 14 & 6 \\ 0 & 0 & 12 & -4 \\ -28 & -12 & -5 & -9 \\ -24 & 8 & -12 & 4 \end{pmatrix}. \quad (4.38)$$

This last result deserves some explanation. Previous calculations using NDR have considered only four-fermion operators with spins $V \pm A$ and $S - P$. To extend these calculations to $S + P$ and T we must decide how to project against evanescent operators. In the notation of Ref. [26], we use the projectors

$$P_1 = \frac{1}{32}(1 + \gamma_5) \otimes (1 + \gamma_5), \quad P_2 = \frac{1}{128} \sum_{\mu\nu} \gamma_\mu \gamma_\nu (1 + \gamma_5) \otimes \gamma_\nu \gamma_\mu (1 + \gamma_5), \quad (4.39)$$

where the sums run over $n = 4 - 2\epsilon$ dimensions. Given a positive parity operator \mathcal{O} , whose composition in terms of $S + P$ and T is unknown,

$$\mathcal{O} = a(S + P) + b(T) + \text{evanescent operators} , \quad (4.40)$$

but whose projections are known, we use

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 0 & 18 + 18\epsilon \\ 24 + 28\epsilon & -6 - 13\epsilon \end{pmatrix} \begin{pmatrix} P_1(\mathcal{O}) \\ P_2(\mathcal{O}) \end{pmatrix} . \quad (4.41)$$

We have dropped terms of $O(\epsilon^2)$.

We close this section with the example of B_K . When calculated on the lattice using the pseudo-Goldstone kaons as external states, this is the ratio of the matrix element of

$$\mathcal{O}_B = [(V - A) \times P]_I + [(V - A) \times P]_{II} , \quad (4.42)$$

between a K and a \bar{K} , to the square of the matrix element of the bilinear $A_4 = \overline{(\gamma_{45} \otimes \xi_5)}$ between a K and the vacuum [3, 14].

Using the above, we find

$$\mathcal{O}_B^{cont}(\text{NDR}) = \left[1 + \frac{g^2}{16\pi^2} (4 \ln(\pi/a\mu) - \frac{4}{3}) \right] \mathcal{O}_B^{lat} + \frac{g^2}{16\pi^2} \delta\mathcal{O} , \quad (4.43)$$

where

$$\delta\mathcal{O} = 10.6702 [A \times P]_I + 1.6598 [A \times P]_{II} - 10.1426 [V \times P]_I - 7.4626 [V \times P]_{II} \quad (4.44)$$

for unsmearred operators, and

$$\delta\mathcal{O} = 13.1912 [A \times P]_I + 10.3174 [A \times P]_{II} - 13.0211 [V \times P]_I - 12.1878 [V \times P]_{II} \quad (4.45)$$

for smeared operators. In these results, the effect of changing from $\text{DR}\overline{\text{E}}\overline{\text{Z}}'$ to NDR is the coefficient $-4/3$ multiplying \mathcal{O}_B^{lat} . This conversion factor agrees with that of Ref. [19].

For the denominator in B_K , the matching relation is the same in NDR and $\text{DR}\overline{\text{E}}\overline{\text{Z}}'$ schemes

$$(A_4^{cont})^2 = (A_4^{lat})^2 \left(1 + \frac{g^2}{16\pi^2} 2C_{FCVS} \right) , \quad (4.46)$$

with $2C_{FCVS} = 0, -3.8769$ for unsmearred and smeared operators respectively. Without tadpole improvement, each of the numerical coefficients in both numerator and denominator is augmented by -24.4661 .

We can combine the corrections to the numerator and denominator of B_K , since the latter are an overall factor. This amounts to subtracting the coefficient of g^2 in the denominator from each of the numbers in the results for the numerator. Doing this, we see that the results with and without tadpole-improvement are the same. This is because, for Landau-gauge operators, the tadpole diagrams are only on the external legs, of which there are the same number for both numerator and denominator.

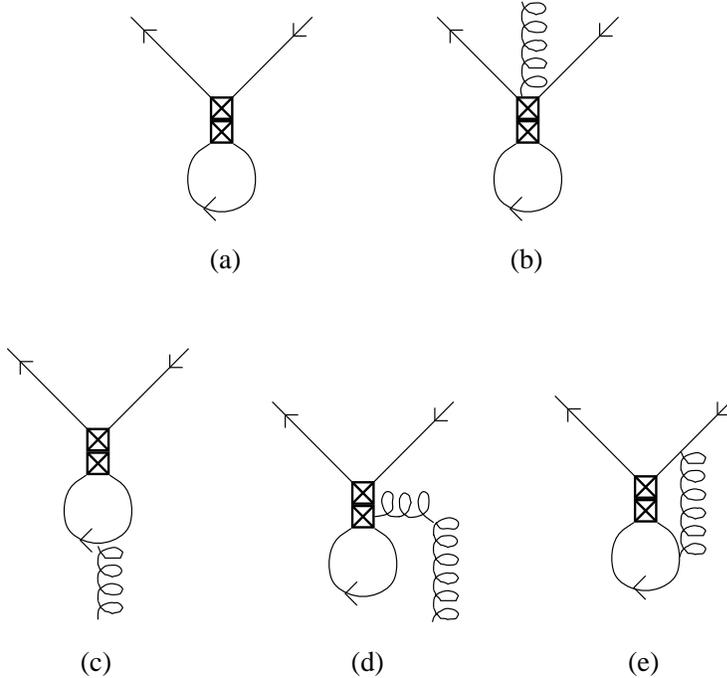


Figure 2: Examples of diagrams leading to mixing of four-fermion operators with bilinears. Each box represents a quark bilinear.

5 PENGUIN DIAGRAMS

5.1 Overview

We now turn to diagrams in which one of the quarks in the four-fermion operator is contracted with one of the antiquarks to form a closed loop. Examples are shown in Fig. 2. They give rise to mixing of four-fermion operators with quark bilinears (possibly containing gluon fields and derivatives). The mixing coefficients are proportional to $a^{d'-d}$, where $d = 6$ and $d' \geq 3$ are the dimensions of the four-fermion operators and bilinears, respectively. The appropriate treatment depends on d' :

- For $d' < 6$, i.e. mixing with lower dimension operators, the coefficients are proportional to inverse powers of the lattice spacing. Such ultraviolet divergent coefficients cannot be calculated to the required accuracy using perturbation theory [9]. One must, instead, use a non-perturbative method to subtract any lower dimension operators that are produced. In the following section we provide a catalogue of such operators, and show that only one of them contributes to the matrix elements that are used in actual numerical calculations.
- For $d' = 6$, a reliable perturbative calculation is possible. We perform a one loop computation in this section.
- For $d' > 6$, the calculation is sensitive to infrared physics and thus unreliable. The mixing vanishes in the continuum limit, however, and can be ignored.

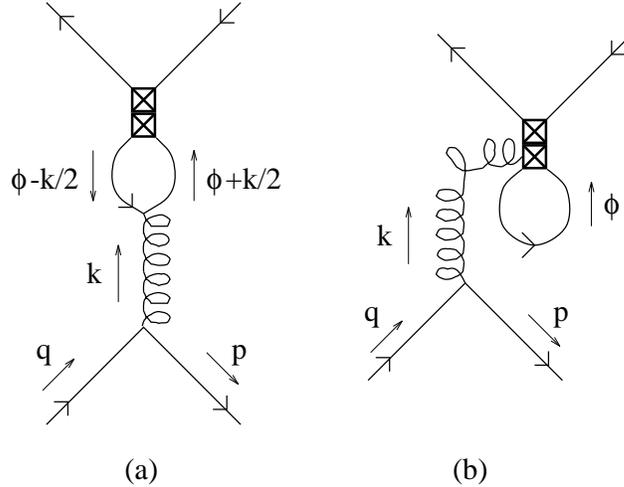


Figure 3: “Penguin” contributions to the renormalization of four-fermion operators. Diagram (b) is absent for Landau gauge operators. Each box represents a quark bilinear.

The one-loop diagrams are those shown in Figs. 2(a-d), together with diagrams of the same form having additional gluons attached to the bilinears. These latter diagrams add no new features to the following analysis, and we will not discuss them further. Note that we have chosen to attach the quark loop to a single bilinear. All continuum operators can be transcribed onto the lattice in this way, and we use this method both because it is simpler, and because it has been used in all numerical simulations to date. Given this choice, it turns out that the diagrams of Figs. 2(a) and (b) lead only to mixing with bilinears having $d' < 6$. The only one-loop diagrams giving $d' = 6$ operators are those of Figs. 2(c) and (d), and these we calculate in this section. They also give rise to mixing with a $d' = 5$ operator, as discussed below.

In the continuum a considerable simplification occurs upon using the equations of motion. In the chiral limit, the $d' = 6$ bilinears either vanish, or can be converted into four-fermion operators. This use of the equations of motion is equivalent to attaching a quark line to the gluon, converting Figs. 2(c) and (d) into the “penguin” diagrams shown in Fig. 3. On the lattice, not all of the $d' = 6$ bilinears that occur at one-loop can be rewritten as four-fermion operators using the equations of motion. Nevertheless, it turns out that the operators of interest can be so rewritten, so we couch our discussion in terms of a calculation of lattice penguin diagrams. We first do the general calculation, and then describe some applications in sect. 5.5.

We concentrate on gauge-invariant operators in this and the following section. Such operators receive corrections from both Fig. 3(a), in which a gluon connects the closed quark loop with a spectator quark line, and from Fig. 3(b), in which the gluon connects directly to the bilinear. The latter diagram is absent in the continuum; on the lattice it serves to cancel mixing with lower dimension gauge non-invariant operators induced by Fig. 3(a). This cancellation does not occur for Landau-gauge operators, for which Fig. 3(b) is absent. This illustrates the problem with using Landau-gauge operators for diagrams involving closed quark loops: they mix with gauge non-invariant lower dimension bilinears. This requires additional subtractions, and makes it difficult to extract phenomenologically

interesting matrix elements. We explain this further in sect. 5.7.

At higher orders, diagrams of the type shown in Fig. 2(e) lead to mixing with bilinears both of lower dimension and with a large number of additional operators having $d' = 6$. Although we have not calculated the mixing coefficients, we discuss in sect. 5.6 the most interesting of the operators that arise. Of course, there is also mixing with $d' = 6$ operators due to X diagrams, which we have considered above.

Finally, we note that the results of this section are unaffected by tadpole improvement, because one-loop tadpole diagrams are already accounted for when calculating X-diagrams.

5.2 Notation

Penguin diagrams occur for operators which transform as octets under the continuum flavor SU(3) symmetry (or which contain a part which so transforms). Aside from overall factors, which we discuss in sect. 5.5, the calculation is the same for all octet operators, so we consider in detail only the operator

$$\mathcal{O}_{S'F',SF} = \overline{\chi}_s(\gamma_{S'} \otimes \xi_{F'})\chi_d \overline{\chi}_u(\gamma_S \otimes \xi_F)\chi_u , \quad (5.1)$$

where color indices are suppressed. The continuum flavors have been chosen so that there is only one possible contraction leading to a penguin diagram, that having a u quark in the loop. Note that we consider the most general form of the four-fermion operator, in which the spins and flavors of the two bilinears are different. This requires no extra work for penguin diagrams because the first bilinear is a spectator in the calculation.

The continuum operator corresponding to $\mathcal{O}_{S'F',SF}$ is

$$\mathcal{O}_{S'F',SF}^{cont} = \overline{S}(\gamma_{S'} \otimes \xi_{F'})D \overline{U}(\gamma_S \otimes \xi_F)U . \quad (5.2)$$

By construction, the tree level matrix elements of these two operators, calculated respectively on the lattice and in the continuum, are the same for physical momenta⁹

$$\langle \mathcal{O}_{S'F',SF}^{cont} \rangle^{(0)} = \langle \mathcal{O}_{S'F',SF} \rangle^{(0)}(1 + O(p_{\text{phys}}a)) . \quad (5.3)$$

The superscript refers to the number of loops. Both lattice and continuum theories differ from QCD by having an additional factor of $N_f = 4$ flavors. In particular, there are N_f quarks running around the loop, rather than a single quark in QCD. To match onto QCD, one must divide the diagram by N_f . To make it possible to see the effect of this correction, we keep the factor of N_f explicit below.

One-loop diagrams lead to mixing with a single class of SU(3) octet operators, which we call ‘‘penguin operators’’. The continuum and lattice forms are, respectively,

$$\begin{aligned} \mathcal{P}_{S'F',S''F''} &= \overline{\chi}_s(\gamma_{S'} \otimes \xi_{F'})\chi_d \sum_q \overline{\chi}_q(\gamma_{S''} \otimes \xi_{F''})\chi_q , \\ \mathcal{P}_{S'F',S''F''}^{cont} &= \overline{S}(\gamma_{S'} \otimes \xi_{F'})D \sum_Q \overline{Q}(\gamma_{S''} \otimes \xi_{F''})Q . \end{aligned} \quad (5.4)$$

⁹As discussed in PS, this is true when the continuum matrix element is expressed in the appropriate spin-flavor basis.

The sums run over the number of active light flavors, typically u, d, s and c in present lattice simulations.

The color factors are simple to calculate. All the operators come in two color types, and the matching matrix is a 2×2 matrix in color space. Using the notation of sect. 3.2, we find

$$\overrightarrow{\mathcal{O}}_i^{cont} = \overrightarrow{\mathcal{O}}_i^{lat} + \frac{g^2}{16\pi^2} \sum_j \rho_{ij} \overleftarrow{C}_{pen} \overrightarrow{\mathcal{P}}_j + \text{X diagram contributions} , \quad (5.5)$$

where i and j represent both the spin and flavor indices of the operators, and

$$\overleftarrow{C}_{pen} = \frac{1}{6} \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.6)$$

is the color matrix. The first row follows from Eq. 3.9, while the second row vanishes because the gauge generators are traceless.

To calculate ρ we need the lattice and continuum one-loop matrix elements. These have the general form (in a notation based on that of Ref. [28])

$$\langle \mathcal{O}_{S'F',SF} \rangle_{pen}^{(1)} = \frac{g^2}{16\pi^2} \sum_{S''F''} \rho_{SF,S''F''}^{lat} \langle \mathcal{P}_{S'F',S''F''} \rangle^{(0)} \quad (5.7)$$

(the subscript pen indicating that only the penguin part of the matrix element is included), and

$$\langle \mathcal{O}_{S'F',SF}^{cont} \rangle_{pen}^{(1)} = \frac{g^2}{16\pi^2} \sum_{S''F''} \rho_{SF,S''F''}^{cont} \langle \mathcal{P}_{S'F',S''F''}^{cont} \rangle^{(0)} , \quad (5.8)$$

respectively. This is a rather elaborate notation for the continuum result, because, when $ma = 0$, almost all of the one-loop coefficients vanish

$$\rho_{SF,S''F''}^{cont} = \sum_{\mu} \delta_{S\hat{\mu}} \delta_{FI} \delta_{S'\hat{\mu}} \delta_{F'I} \rho_{\hat{\mu}I,\hat{\mu}I}^{cont} . \quad (5.9)$$

Such a notation is required, however, for the lattice matrix elements. Combining Eqs. 5.7 and 5.8, and using Eq. 5.3, we find

$$\rho_{ij} = \rho_{ij}^{cont} - \rho_{ij}^{lat} . \quad (5.10)$$

This notation is slightly redundant in that ρ_{ij} depend only on S, F, S'' and F'' , but not on S' and F' .

5.3 Calculation

Since the first bilinear in Fig. 3 is simply a spectator, we are, in effect, calculating the mixing of the second bilinear, $(\gamma_S \otimes \xi_F)$, into other bilinears. We first consider this mixing on the lattice. Using the Feynman rules given in PS, and the momentum definitions from Fig. 3, we find¹⁰

$$\langle \overline{(\gamma_S \otimes \xi_F)} \rangle_{pen}^{(1)} = \frac{ig}{16\pi^2} \sum_{AB} \frac{1}{16} \overline{(\gamma_S \otimes \xi_F)}_{AB} e^{-ik \cdot (A+B)/2} \sum_{\mu\nu} P_{\mu}(k; B, A) D_{\mu\nu}(k) V_{\nu}(k) . \quad (5.11)$$

¹⁰In this subsection the sum over quark flavors q (see Eq. 5.4) for the lower quark line in Fig. 3 is implicit.

Here and in the following we have suppressed all color matrices and indices, since the color factors have already been calculated. The factors P , D and V represent respectively the penguin loop, the gluon propagator and the lower quark-antiquark-gluon vertex. Expressions for D and V are given below. We first focus on the penguin loop, for which we find

$$P_\mu(k; B, A) = \frac{N_f}{4} \int_\phi (I_1 - I_2) e^{i\phi \cdot (A-B)} , \quad (5.12)$$

$$I_1 = \sum_{CD} S(\phi + \frac{1}{2}k)_{BC} c_\mu \overline{(\gamma_\mu \otimes I)}_{CD} S(\phi - \frac{1}{2}k)_{DA} \quad (5.13)$$

$$I_2 = \sum_{CD} S(\phi)_{BC} c_\mu \overline{(\gamma_\mu \otimes I)}_{CD} S(\phi)_{DA} , \quad (5.14)$$

where S is the quark propagator

$$S(\phi)_{AB} = F(\phi) \left[m \overline{(I \otimes I)} + i \sum_\mu s_\mu \overline{(\gamma_\mu \otimes I)} \right]_{AB} , \quad F(\phi) = \left[m^2 + \sum_\mu s_\mu^2 \right]^{-1} , \quad (5.15)$$

and we are using the notations

$$\int_\phi \equiv 16\pi^2 \int_{-\pi}^\pi \frac{d^4\phi}{(2\pi)^4} , \quad s_\mu \equiv \sin(\phi_\mu) , \quad c_\mu \equiv \cos(\phi_\mu) . \quad (5.16)$$

Various features of these results require explanation.

- The factor of $1/16$ multiplying $\overline{(\gamma_S \otimes \xi_F)}$ in Eq. 5.11 arises from the fact that we integrate the momenta over the entire Brillouin zone *and* trace over the spin-flavor indices. To avoid “double” counting we have to divide the result by 16.
- Infrared regularization is provided by the quark mass, which we can only set to zero after the lattice and continuum results are combined.
- We have transformed the matrices to the single-bar representation, which corresponds to physical position on a hypercube. This is the most convenient prescription since it allows one to easily determine the constraints on A and B . We have used these constraints to simplify Eq. 5.12.
- The I_2 term in Eq. 5.12 is the contribution of Fig. 3(b). We have written it so as to show explicitly that P_μ vanishes when $k = 0$.
- This vanishing is one of the consequences of the Ward identity,

$$\sum_\mu \sin(\frac{1}{2}k_\mu) P_\mu(k; B, A) = 0 , \quad (5.17)$$

which is satisfied by our expression for all values of k . This identity follows from gauge invariance, and is necessary for the gluon to remain massless.

It turns out that contributions to operator mixing come not only from $k \sim 0$, as in the continuum, but also from $k \sim C\pi$, where C is a non-zero hypercube vector. This is because a momentum close to π is physical for staggered fermions. We first consider $k \sim 0$. The integrand I_1 has to be expanded out to $O(k^2)$. Tedious algebra leads to the result

$$P_\mu(k; B, A) = i\epsilon_{\mu\rho\sigma\nu}k_\rho \overline{(\gamma_{\sigma\nu} \otimes \xi_{\sigma 5})}_{BA} I_a - i\frac{1}{2}mk_\rho \left[\overline{(\gamma_{\mu\rho} \otimes I)}_{BA} - \overline{(\gamma_{\rho\mu} \otimes I)}_{BA} \right] I_b + (k_\mu k_\nu - \delta_{\mu\nu}k^2) \overline{(\gamma_\nu \otimes I)}_{BA} I_c + O(k^3), \quad (5.18)$$

where the indices ρ , σ and ν are summed. The integrals are

$$I_a = \frac{N_f}{4} \int F^2 c_1^2 c_2^2 s_3^2 = \frac{N_f}{4} \times 11.2291, \quad I_b = \frac{N_f}{4} \int F^2 c_1^2 c_2^2 = \frac{N_f}{4} \left(\int F^2 - 40.7767 \right), \\ I_c = \frac{N_f}{4} \int \frac{1}{6} F^2 [2 - 2s_1^2 - s_1^2 s_2^2] = \frac{N_f}{4} \left(\frac{1}{3} \int F^2 - 9.5146 \right). \quad (5.19)$$

Note that I_a is finite, while I_b and I_c diverge logarithmically as $m \rightarrow 0$.

The first two terms in Eq. 5.18 correspond to mixing of the bilinear with gluonic operators which cannot be rewritten using the equations of motion. In other words, for these terms we are calculating Figs. 2(c) and (d) rather than penguin diagrams. For these terms we should drop the factor of $D \times V$ in Eq. 5.11, and multiply by the gluon polarization tensor ϵ_μ .

To analyze the I_a term we must expand the factor $e^{-ik \cdot (A+B)/2}$ in Eq. 5.11 up to linear order in k around the center of the hypercube. The leading term is unity, and gives rise to mixing between the bilinear $\overline{\chi}(\gamma_{\sigma\nu} \otimes \xi_{\sigma 5})\chi$ and the lower dimension gluonic operator $\tilde{F}_{\sigma\nu}$ (the dual of the gluon field strength). Reinstating the spectator bilinear, this is an example of mixing with a $d' = 5$ operator. It is consistent with the lattice symmetries, but, as explained in sect. 6, it does not contribute to the matrix elements of practical interest. For the $O(k)$ term in the expansion, one finds after some algebra, that it leads to mixing between $\overline{\chi}(\gamma_{\sigma 5} \otimes \xi_{\sigma\nu})\chi$ and $D_\nu \tilde{F}_{\sigma\nu}$. Including the spectator bilinear, this corresponds to mixing with a $d' = 6$ operator. At one-loop, this mixing does not affect the operators needed to study the $\Delta I = 1/2$ rule (which are listed below in sect. 5.5), since none contain bilinears having flavor $\xi_F = \xi_{\sigma\nu}$. It will affect these operators at two-loop order. However, as explained in sect. 6, its contribution in matrix elements will be suppressed by powers of a due to its flavor.

The I_b term corresponds to mixing of the operator $\overline{\chi}(\gamma_{\mu\rho} \otimes I)\chi$ ($\mu \neq \rho$) with $mF_{\mu\rho}$, both of which have dimension 3. Such mixing also occurs in the continuum, and when one combines the lattice and continuum corrections the infrared divergence in I_b cancels with that in the corresponding continuum integral. Since the operator vanishes in the chiral limit, it is dropped in renormalization group analyses [2], and we follow this practice here. In fact, for the operators needed to study the $\Delta I = 1/2$ rule, the initial operator contains no terms with tensor bilinears, so mixing with $mF_{\mu\rho}$ begins only at two loop order. Thus its contribution is suppressed both by the quark mass and by an additional power of $g^2/16\pi^2$. The factor of m appears because of the chiral symmetry of staggered fermions, and is not present for Wilson fermions [11].

We are thus interested only in the term proportional to I_c , which corresponds to the mixing of $\overline{(\gamma_\nu \otimes I)}$ with $D_\mu F_{\mu\nu}$, and which can be rewritten using the equations of motion.

To extract the infrared divergence we use

$$\int F^2 = 16(-\ln(4m^2a^2) - \gamma_E + F_{0000}) + O((ma)^2) . \quad (5.20)$$

Numerical values of the constants are given, for example, in PS. We calculate the remaining finite piece of I_c numerically, yielding, in total

$$I_c = -\frac{8N_f}{3} \ln(ma/\pi) - 8.8945 + O((ma)^2) . \quad (5.21)$$

The error on the numerical constant is roughly 0.0001.

To complete the amplitude, the gluon must be joined onto the external fermion line. The result in Feynman and Landau gauges is the same because of the Ward identity Eq. 5.17, so we use the Feynman gauge propagator

$$D_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{\sum_\rho 4 \sin^2(k_\rho/2)} . \quad (5.22)$$

For $k \sim 0$ the vertex is (see PS)

$$V_\mu(p) = -ig\bar{\delta}(p' + k - q') \overline{(\gamma_\mu \otimes I)}_{A'B'} \cos(q'_\mu - \frac{1}{2}k_\mu) . \quad (5.23)$$

The external momenta have been decomposed into physical and flavor-related parts: $p = p' + A'\pi$ and $q = q' + B'\pi$, where $p', q' \in [-\pi/2, \pi/2]$. We assume that p' and $q' = k + p'$ are both close to zero, so that the cosine can be set equal to unity. Using the equations of motion on the vertex (i.e. $\sum_\mu \sin(k_\mu/2)V_\mu = 0$ for massless fermions), we see that only the k^2 term in the coefficient of I_c in Eq. 5.18 survives. This cancels the denominator of the gluon propagator, so that there is no longer any dependence on k , and we have arrived at a local operator. We can now do the integral over k to remove the delta function in Eq. 5.23, and sum over A and B in Eq. 5.11. We can set the factor $\exp(-ik \cdot (A + B)/2)$ to unity, since the corrections lead to operators with $d' > 6$. Thus the only non-zero matrix element from gluon momenta $k \approx 0$ is

$$\langle \overline{(\gamma_\mu \otimes I)} \rangle_{pen}^{(1)} = -\frac{g^2}{16\pi^2} I_c \overline{(\gamma_\mu \otimes I)}_{A'B'} . \quad (5.24)$$

Since $\overline{(\gamma_\mu \otimes I)}_{A'B'}$ is the tree level matrix element of the bilinear $\overline{(\gamma_\mu \otimes I)}$, we can read off one contribution to ρ^{lat}

$$\rho_{\hat{\mu}I, \hat{\mu}I}^{lat} = -I_c . \quad (5.25)$$

Note that this contribution is diagonal in spin and flavor.

The corresponding continuum calculation is straightforward (see Ref. [28] for more details), with the result

$$\langle \mathcal{O}_{SF}^{cont} \rangle_{pen}^{(1)} = \frac{ig}{16\pi^2} \sum_{\mu\nu\rho} (k_\mu k_\rho - \delta_{\mu\rho} k^2) D_{\mu\nu}(k) V_\nu(k) \frac{\text{Tr}(\gamma_S \gamma_\rho)}{4} \frac{\text{Tr}(\xi_F)}{N_f} I_c^{cont} \quad (5.26)$$

$$= -\frac{g^2}{16\pi^2} I_c^{cont} \sum_\mu \delta_{S, \hat{\mu}} \delta_{F, I} (\gamma_\mu \otimes I) . \quad (5.27)$$

The continuum integral is

$$I_c^{cont} = -\frac{4N_f}{3} \ln\left(\frac{m^2}{\mu^2}\right), \quad (5.28)$$

in $\overline{\text{DR}}\overline{\text{Z}}$, DRED, NDR, HV and PV schemes¹¹. Thus the only non-zero elements of the continuum mixing matrix are

$$\rho_{\widehat{\mu I}, \widehat{\mu I}}^{cont} = -I_c^{cont}. \quad (5.29)$$

The infrared logarithms cancel when we combine lattice and continuum results

$$\rho_{\widehat{\mu I}, \widehat{\mu I}} = I_c - I_c^{cont} = -\frac{8N_f}{3} \ln(\mu a/\pi) - 8.8945. \quad (5.30)$$

The additional contributions discussed below do not change this mixing coefficient.

5.4 Contributions from high momentum gluons

As mentioned earlier, this is not the whole story. Unlike in the continuum, gluons with momenta $k \approx C\pi$ ($C \neq 0$) also contribute on the lattice. This is because, while p' and q' must be close to zero, $k = q' - p' + \pi(B' - A')$ need not be. These contributions result in a local operator because both the gluon propagator and the fermion loop are both short ranged.

It is instructive to begin by considering the vertex connecting the gluon to the external line

$$V_\nu(C\pi) = -ig\bar{\delta}(p' - q') \frac{(1 + (-1)^{C_\nu})}{2} \overline{(\gamma_\nu \otimes I)_{A', B'+C}}. \quad (5.31)$$

Clearly we must choose $C_\nu = 0$ for the vertex to be non-vanishing. Notice the appearance of the subscript $B' + C$ in place of B' . It is simple to show that

$$V_\nu(C\pi) = -ig\bar{\delta}(p' - q') \frac{(1 + (-1)^{C_\nu})}{2} \overline{(\gamma_\nu \tilde{C} \otimes \xi_{\tilde{C}})_{A'B'}} , \quad (5.32)$$

where $\tilde{C}_\nu =_2 \sum_{\rho \neq \nu} C_\rho$. This shows that the external quarks “see” this vertex as having a different spin and flavor.

The expressions 5.31 and 5.32 differ from the simplified Feynman rules listed in PS. The rules given there are appropriate for propagators and vertices involved in loop integrations, but are inapplicable if more than one of the legs of the vertex is an external line. This is because, to derive the rules, one has to use the freedom to redefine loop variables. For tree level couplings no rearrangement is allowed and the correct vertex is as given above. Note that similar changes occur for matrix elements of operators when they are inserted at non-zero momenta.

The penguin loop is straightforward to calculate using the fact that the integration variable can be shifted freely. Furthermore, since the Ward Identity Eq. 5.17 holds for all k , the longitudinal part of the gluon propagator does not contribute, so that $D_{\mu\nu} \propto \delta_{\mu\nu}$. Thus in

¹¹In the PV scheme μ is the fermion regulator mass.

evaluating P_μ we can assume that $C_\mu = 0$. Including the momentum factor from the bilinear vertex, the loop becomes

$$\begin{aligned}
e^{-ik \cdot (A+B)/2} P_\mu(C\pi; B, A) &= -mI_d \sum_{\rho \neq \mu} (1 - (-1)^{C_\rho}) \overline{(\gamma_{\mu 5\tilde{C}} \otimes \xi_{\rho 5\tilde{C}})}_{BA} \\
&\quad + \frac{1}{2} I_e \sum_{\rho \neq \mu, \sigma \neq \mu} ((-1)^{C_\rho} - (-1)^{C_\sigma}) \overline{(\gamma_{\mu\tilde{C}} \otimes \xi_{\sigma\rho\tilde{C}})}_{BA} \\
&\quad + I_d \overline{(\gamma_{\mu\tilde{C}} \otimes \xi_{\tilde{C}})}_{BA} \sum_{\rho \neq \mu} ((-1)^{C_\rho} - 1) . \tag{5.33}
\end{aligned}$$

The integrals needed are

$$I_d = \frac{N_f}{4} \int F^2 c_1^2 s_2^2 = \frac{N_f}{4} \times 16.3107 , \quad I_e = \frac{N_f}{4} \int F^2 c_1^2 s_2^2 s_3^2 = \frac{N_f}{4} \times 5.0816 , \tag{5.34}$$

both of which are infrared finite. The first term in Eq. 5.33 corresponds to mixing with a $d' = 7$ operator, because of the factor of m , and can be dropped. The remaining two terms correspond to mixing with $d' = 6$ operators.

To complete the calculation we need the gluon propagator

$$B^{-1}(C\pi) = \sum_{\rho} 2(1 - (-1)^{C_\rho}) . \tag{5.35}$$

Altogether, after integrating over gluon momenta near $k = C\pi$ to remove the delta-function in Eq. 5.32, we find the following non-zero matrix elements

$$\langle \overline{(\gamma_{\mu\tilde{C}} \otimes \xi_{\tilde{C}})} \rangle_{pen}^{(1)} = -\frac{g^2}{16\pi^2} \frac{1}{2} I_d \overline{(\gamma_{\mu\tilde{C}} \otimes \xi_{\tilde{C}})}_{A'B'} , \tag{5.36}$$

$$\langle \overline{(\gamma_{\mu\tilde{C}} \otimes \xi_{\sigma\rho\tilde{C}})} \rangle_{pen}^{(1)} = \frac{g^2}{16\pi^2} \frac{1}{2} I_e B(C\pi) [(-1)^{C_\rho} - (-1)^{C_\sigma}] \overline{(\gamma_{\mu\tilde{C}} \otimes \xi_{\tilde{C}})}_{A'B'} , \tag{5.37}$$

where μ, ρ and σ can take any values, but are all different, and C is any non-zero hypercube vector satisfying $C_\mu = 0$. The remaining one-loop contributions to ρ_{ij}^{lat} can be read off from this equation. There are no continuum contributions to these coefficients, so $\rho_{ij} = -\rho_{ij}^{lat}$.

These results show the complications due to the use of staggered fermions. The mixing is between bilinears all of which have $F \neq I$; the flavor is transmitted by the highly off-shell gluon. Furthermore, the mixing of Eq. 5.37 is flavor off-diagonal, but is nevertheless consistent with the hypercube symmetry group [31], as we have checked explicitly.

5.5 Applications

The continuum octet operators which are needed to calculate kaon decay amplitudes are

$$\begin{aligned}
Q_1 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) u_b \bar{u}_b \gamma_\mu (1 + \gamma_5) d_a , \\
Q_2 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) u_a \bar{u}_b \gamma_\mu (1 + \gamma_5) d_b , \\
Q_3 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) d_a \sum_q \bar{q}_b \gamma_\mu (1 + \gamma_5) q_b , \\
Q_4 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) d_b \sum_q \bar{q}_b \gamma_\mu (1 + \gamma_5) q_a , \tag{5.38}
\end{aligned}$$

$$\begin{aligned}
Q_5 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) d_a \sum_q \bar{q}_b \gamma_\mu (1 - \gamma_5) q_b , \\
Q_6 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) d_b \sum_q \bar{q}_b \gamma_\mu (1 - \gamma_5) q_a .
\end{aligned}$$

where we follow the notation of Refs. [25, 26], except that we use $(1 + \gamma_5)$ for the left-handed projector. The sum over q runs over the f active flavors: u, d, s and possibly c . For brevity we have left out the electromagnetic penguin operators, Q_{7-10} . It is straightforward to extend the discussion to them.

References [25, 26] have calculated the two-loop anomalous dimension matrices for these operators in the NDR scheme. (They also use the HV scheme which we do not consider here.) We wish to find the lattice operators which match onto Q_{1-6} to one-loop accuracy. We consider here only the contributions to the matching from penguin diagrams; those from X diagrams can be obtained from sect. 4 or, in the case of gauge-invariant operators, from Ref. [19].

The first step is entirely in the continuum. It turns out to be more convenient to change from the basis of operators given above to that in which Q_1, Q_2, Q_3 and Q_5 are replaced by their Fierz transforms

$$\begin{aligned}
\tilde{Q}_1 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) d_a \bar{u}_b \gamma_\mu (1 + \gamma_5) u_b , \\
\tilde{Q}_2 &= \bar{s}_a \gamma_\mu (1 + \gamma_5) d_b \bar{u}_b \gamma_\mu (1 + \gamma_5) u_a , \\
\tilde{Q}_3 &= \sum_q \bar{s}_a \gamma_\mu (1 + \gamma_5) q_b \bar{q}_b \gamma_\mu (1 + \gamma_5) d_a , \\
\tilde{Q}_5 &= -2 \sum_q \bar{s}_a (1 - \gamma_5) q_b \bar{q}_b (1 + \gamma_5) d_a ,
\end{aligned} \tag{5.39}$$

respectively. The two bases are identical in four dimensions, but differ in n -dimensions. In particular, the finite parts of their penguin matrix elements differ. The matching coefficients between these two bases are simple to obtain by generalizing the work of Ref. [25], and we give them below.

When we transcribe these operators onto the lattice, we do so at the level of contractions [13, 14]. For example, the penguin contraction of \tilde{Q}_2 in a $K \rightarrow \pi\pi$ matrix element is obtained using the penguin contraction of the lattice operator

$$\tilde{Q}_2^{lat} = \frac{1}{N_f} \left[(\overline{\chi_s (\gamma_{\mu 5} \otimes \xi_5)}) \chi_d \overline{\chi_u (\gamma_\mu \otimes I)} \chi_u \right]_I + (\overline{\chi_s (\gamma_\mu \otimes \xi_5)}) \chi_d \overline{\chi_u (\gamma_{\mu 5} \otimes I)} \chi_u \right] . \tag{5.40}$$

The $1/N_f$ cancels the extra lattice flavors in the loop. In all cases of practical interest the flavors of the bilinears are either I or ξ_5 [7]. This means that the mixings of Eqs. 5.36 and 5.37 are not relevant, because the flavor of the bilinears in these equations is never I or ξ_5 . The only relevant mixing is that of Eq. 5.24, which affects just the bilinears $\overline{(\gamma_\mu \otimes I)}$. Thus only the first term in \tilde{Q}_2^{lat} receives corrections.

We can now explain why we use \tilde{Q}_2 rather than Q_2 . When we transcribe the former onto the lattice, as in Eq. 5.40, the penguin loop attaches to a single bilinear, and the spin-flavor indices are contracted in two loops. On the other hand, had we transcribed Q_2 onto the lattice, the spin-flavor indices would have been contracted into a single loop. While there is no fundamental reason to prefer one choice over the other, numerical simulations have so far

used the two spin-flavor loop form [7]. Thus we have calculated the corrections in sect. 5 for this form. Given this, we must match onto \tilde{Q}_2 rather than Q_2 . A similar argument applies for Q_1 , except that the penguin contraction vanishes at one-loop because of the color factor, so it turns out to make no difference whether we choose Q_1 or \tilde{Q}_1 .

For the operators Q_{3-6} , the situation is more complicated. Each operator has two types of penguin contraction: either \bar{q} is contracted with q , or the penguin loop is formed by contracting q with \bar{s} or \bar{q} with d . For each operator, one type of contraction has a vanishing color factor.¹² To put the remaining contraction in two spin-flavor loop form, we must match onto \tilde{Q}_3 , Q_4 , \tilde{Q}_5 and Q_6 . The lattice transcriptions of the penguin contractions of these operators are analogous to Eq. 5.40, and we do not give explicit forms.

We now collect the penguin contributions to the matching coefficients. For the negative parity parts of the operators, the general form is

$$Q_i(\text{NDR}) = Q_i^{\text{lat}} + P[Q_i] \frac{g^2}{16\pi^2} Q_P^{\text{lat}}, \quad (5.41)$$

$$Q_P^{\text{lat}} = \frac{2}{N_f} \sum_q \left[(\overline{\chi_s}(\gamma_{\mu 5} \otimes \xi_5) \chi_d \overline{\chi_q}(\gamma_\mu \otimes I) \chi_q)_I - \frac{1}{3} (\overline{\chi_s}(\gamma_{\mu 5} \otimes \xi_5) \chi_d \overline{\chi_q}(\gamma_\mu \otimes I) \chi_q)_{II} \right].$$

For the positive parity parts of the operators, the same result applies with $\overline{(\gamma_{\mu 5} \otimes \xi_5)}$ replaced with $\overline{(\gamma_\mu \otimes I)}$ in Q_P^{lat} . The calculation done in the previous two subsections applies directly to the operator \tilde{Q}_2 . For the other operators we need only count the number of continuum flavors that can run round the penguin loop. Including the matching coefficient between the continuum bases, we find

$$P[Q_1] = 0, \quad P[Q_2] = \frac{1}{3} + \frac{1}{4}\rho, \quad P[Q_3] = 2P[Q_2], \quad P[Q_4] = P[Q_6] = \frac{1}{4}f\rho, \quad P[Q_5] = 0, \quad (5.42)$$

where $\rho \equiv \rho_{\widehat{\mu I}, \widehat{\mu I}}$. One check on Eqs. 5.30 and 5.41 is that we can extract the contribution of penguin diagrams to the one-loop anomalous dimension matrix from the coefficient of $\ln(\mu)$. Our results agree with those of, for example, Ref. [25].

How large are the matching corrections? First we note that $Q_P = -Q_3/3 + Q_4 - Q_5/3 + Q_6$. Thus the operator produced by mixing is of the same “size” as the original operators. To estimate the size of the coefficients, we proceed as above and set $\mu = \pi/a$ and $g^2/16\pi^2 \approx 1/90$. The largest coefficient of Q_P is $(g^2/16\pi^2)f\rho/(2N_f) \approx -1/20$. Thus the penguin contributions to matching are small.

There is a subtlety in the application of Eq. 5.41 to the quenched approximation. Recall that the initial mixing is with a bilinear containing gluon fields, which is then rewritten as a four-fermion operator using the equations of motion. The second step is altered in the quenched approximation. It is simple to show that, when taking matrix elements of the operator produced by mixing one should exclude contractions between the $\overline{\chi}_q$ and χ_q fields.

5.6 Beyond one loop

At higher orders in perturbation theory the continuum operators Q_{1-6} will match onto a linear combination of a much larger number of $d' = 6$ lattice operators. We wish to point

¹²We stress that in an actual numerical simulation one must include both types of contractions—only at one-loop order does one of them vanish.

out some general features of this mixing. Some of the results stated here rely on the analysis of sect. 6.

Mixing produces both four-fermion and bilinear $d' = 6$ operators. We are only interested in those operators whose flavors match those of the external states, i.e. those having flavor $\xi_F = 1$ or γ_5 . One-loop mixing produces essentially all the four-fermion operators satisfying this criterion: higher loops hold no surprises. This is not true, however, for the bilinears. In particular, the lattice transcriptions of the operators

$$\sum_{\mu} \bar{s} \gamma_{\mu} (1 + \gamma_5) \overrightarrow{D}_{\mu} \overrightarrow{D}_{\mu} \overrightarrow{D}_{\mu} d \quad \text{and} \quad \sum_{\mu} \bar{s} \overleftarrow{D}_{\mu} \overleftarrow{D}_{\mu} \overleftarrow{D}_{\mu} \gamma_{\mu} (1 + \gamma_5) d , \quad (5.43)$$

can appear. These are allowed by the lattice symmetries, though not by the continuum symmetries, and cannot be simplified using the equations of motion. Normally such operators are suppressed by powers of a , but this is not true here because they are corrections to $d' = 4$ operators which themselves are multiplied by a^{-2} . The operators of Eq. 5.43 are not produced by one loop graphs of Figs. 2 and 3, because these do not depend on the momenta of the s and d quarks. Presumably they appear in two loop graphs such as that shown in Fig. 2(e).

Because they appear only in two-loop graphs, these operators are likely to be unimportant numerically. We mention them mainly as a curiosity. We note, however, that they also occur with Wilson fermions, yet have so far been overlooked, while other operators appearing at two-loops have been considered [11]. The difference of the two operators in Eq. 5.43 is even under CPS symmetry (see Ref. [8] and the sect. 6), and so does contribute to both even and odd parity matrix elements. Thus the tricks suggested in Ref. [10] do not remove this operator.

We mention also two other $d' = 6$ operators which appear at two (or higher) loops

$$\sum_{\mu, \nu} \partial_{\mu} [\bar{s} \gamma_{\nu} F_{\mu\nu} (1 + \gamma_5) d] \quad \text{and} \quad \sum_{\mu, \nu} \partial_{\mu} [\bar{s} \gamma_{\nu} \gamma_5 \tilde{F}_{\mu\nu} (1 + \gamma_5) d] . \quad (5.44)$$

These are allowed by the continuum symmetries, and by CPS, but are usually dropped because, being total derivatives, they do not contribute to physical matrix elements. On the lattice, however, the standard approaches to calculating $K \rightarrow \pi\pi$ amplitudes use indirect methods based on amplitudes in which the weak Hamiltonian inserts momenta. For example, one method requires the $K \rightarrow \pi\pi$ amplitude at threshold for equal mass quarks, so that energy is inserted by the operator [10]. Thus the operators of Eq. 5.44 can contribute.

5.7 Landau gauge operators

We close this section with a brief discussion of the differences in the results for Landau-gauge and gauge-invariant operators. The only difference in the one-loop calculation is that Fig. 3(b) is absent for Landau gauge operators, leading to a violation of the Ward identity, Eq. 5.17. The contribution of this diagram to $P_{\mu}(k; B, A)$ is (for $ma = 0$)

$$\int_{\phi} S(\phi)_{BA} (A - B)_{\mu} e^{i\phi \cdot (A - B)} = -4\pi^2 \overline{(\gamma_{\mu} \otimes I)}_{BA} , \quad (5.45)$$

which is independent of k . By enforcing gauge invariance the Ward identity restricts the mixing of the bilinears. Eq. 5.45 shows that, without this restriction, there is unwanted mixing of the Landau gauge bilinear with the lower dimensional operator A_μ . Because of this, a continuum operator such as \tilde{Q}_2 will mix with the lattice transcription of $\bar{s}\gamma_\mu(1+\gamma_5)A_\mu d$. As mentioned at the beginning of this section, such mixing cannot be corrected using perturbation theory; it must be eliminated by a non-perturbative constraint. One must also worry about mixing with operators containing two or three factors of A_μ . Thus, Landau-gauge operators require several non-perturbative subtractions, which makes them unattractive for use in numerical calculations of penguin diagrams.

Barring this caveat, the mixing of unsmeared Landau gauge operators with the penguin operator is identical to that for gauge-invariant operators. This follows since the equations of motion for the external fermion line annihilate any contribution from the longitudinal part of the gluon propagator.

For smeared Landau gauge operators additional work is required, but, given the problems with gauge noninvariant operators, we have not done the calculation.

6 LOWER DIMENSION OPERATORS

We close the paper by considering the mixing of $SU(3)$ octet operators with lower dimension operators. We begin by summarizing the results of Ref. [8] for continuum operators. The diagrams of Figs. 2 and 3 lead to mixing of the $d = 6$ operators listed in Eq. 5.38 with quark bilinears of lower dimension, $d' < 6$. The original operators transform in the $(8, 1)$ representation of $SU(3)_L \times SU(3)_R$, are Lorentz (pseudo)scalars, and have positive parity under CPS (CP followed by $s \leftrightarrow d$ interchange). The only lower dimension bilinear with these properties is

$$\mathcal{O}_{sub} = \bar{s}\gamma_\mu(1+\gamma_5) \overleftrightarrow{D}_\mu d, \quad \overleftrightarrow{D}_\mu \equiv (\overrightarrow{D}_\mu - \overleftarrow{D}_\mu). \quad (6.1)$$

In the continuum, one does not need to worry about this operator because, after using the equations of motion, it can be absorbed into the mass matrix by a chiral rotation of the fields. On the lattice, however, one must subtract \mathcal{O}_{sub} by hand [32]. This cannot be done using perturbation theory because the coefficient is proportional to $1/a^2$ [9]. One must use a non-perturbative method such as that suggested in Ref. [8]. This uses chiral perturbation theory, and relies on the fact that the positive and negative parity parts of \mathcal{O}_{sub} are related. The remaining issue is whether lattice artifacts invalidate the subtraction procedure, for example by causing mixing with additional lower dimension operators.

For Wilson fermions the subtraction procedure is, indeed, invalidated by the explicit breaking of chiral symmetry in the action. For example, there is mixing with the $d' = 3$ bilinears $\bar{s}d$ and $\bar{s}\gamma_5 d$, with no relationship between the two coefficients [9]. Staggered fermions, on the other hand, do have a remnant chiral symmetry on the lattice, and it turns out that this is sufficient to allow the method of Ref. [8] to be employed. Part of the argument has been given in Refs. [12, 13, 7, 14]; the remainder is given here. The argument only applies, however, if the original four-fermion operators are gauge-invariant, and we assume this henceforth.

One point has been glossed over in this discussion. We have assumed that the operators produced by mixing have definite transformation properties under chiral $SU(3)$, as is true

in perturbation theory. We must include non-perturbative effects, however, since the mixing is with operators of lower dimension. In particular, chiral symmetry itself is broken by the vacuum. Nevertheless, the assumption remains valid because the chiral transformation properties of the operators imply that their matrix elements satisfy Ward Identities, and these are not affected by the breaking of chiral symmetry.

There are two questions that must be addressed with staggered fermions. First, if we transcribe the (8, 1) operators onto the lattice, using the rules explained in Refs. [12, 7, 14], with which lower dimension operators can they mix? Second, are the positive and negative parity parts of the operators that one obtains related in such a way that one may use the subtraction method of Ref. [8]? The answer to the first question, as we show below, is that the only relevant mixing is with the lattice transcription of \mathcal{O}_{sub} . Given this, Refs. [12, 14] have answered the second question affirmatively. Thus in the following we discuss the first question alone. We consider only the positive parity parts of operators, i.e. those which contribute to the $K \rightarrow \pi$ matrix elements. The extension to the negative parity parts is obtained by multiplying the operators listed below by $(\overline{\gamma_5 \otimes \xi_5})$.

Mixing is restricted by the properties of the operators needed to transcribe the penguin contractions of Q_{1-6} onto the lattice. To illustrate these properties we show two examples. The lattice transcription for the positive parity part of \tilde{Q}_1 is

$$N_f \tilde{Q}_1^{lat} = \left(\overline{\chi_s (\gamma_\mu \otimes I)} \chi_d \overline{\chi_u (\gamma_\mu \otimes I)} \chi_u \right)_{II} + \left(\overline{\chi_s (\gamma_{\mu 5} \otimes I)} \chi_d \overline{\chi_u (\gamma_{\mu 5} \otimes I)} \chi_u \right)_{II} \quad (6.2)$$

Only the penguin contraction of this operator is to be considered, i.e. that in which $\overline{\chi_u}$ and χ_u are contracted. For the operator \tilde{Q}_5 the lattice transcription involves several operators. That giving the largest contribution is

$$\begin{aligned} N_f \tilde{Q}_5^{lat} = & -2 \left(\overline{\chi_s (I \otimes I)} \chi_d \left[\overline{\chi_s (I \otimes I)} \chi_s + \overline{\chi_d (I \otimes I)} \chi_d \right] \right)_I \\ & + 2 \left(\overline{\chi_s (\gamma_5 \otimes \xi_5)} \chi_d \left[\overline{\chi_s (\gamma_5 \otimes \xi_5)} \chi_s + \overline{\chi_d (\gamma_5 \otimes \xi_5)} \chi_d \right] \right)_I . \end{aligned} \quad (6.3)$$

Different contractions of the two parts of the operator are to be kept: for the first line, only those in which the quark fields of the second bilinear are contracted with each other; for the second line, only those where each bilinear is contracted with one of the external mesons. The second contraction is not a penguin diagram, but it is crucial that it be included in order to satisfy the Ward Identity described below. The need for this extra term is related to the fact that all the bilinears in Eq. 6.3 have even distance, in contrast to the odd distance bilinears in Eq. 6.2.

These operators, and all others which arise, have the following properties:

1. They are singlets under the group of reflections and rotations which map the hypercube into itself, the hypercubic group [33]. Operators produced by mixing must also be singlets.
2. They are flavor octets. Thus the bilinears produced by mixing must have the flavor form $\overline{\chi_s (\gamma_S \otimes \xi_F)} \chi_d$.
3. They have positive parity under $C_0 S$, where C_0 is the lattice charge conjugation symmetry (see, e.g. Ref. [12]), and S is the $s \leftrightarrow d$ interchange symmetry. The transformation

rules for C_0S are

$$\begin{aligned} [U_\mu]_{ab} &\longrightarrow [U_\mu^\dagger]_{ba} , & [F_{\mu\nu}]_{ab} &\longrightarrow -[F_{\mu\nu}]_{ba} , & m_s &\leftrightarrow m_d, \\ \overline{[\chi_s]_a (\gamma_S \otimes \xi_F) [\chi_d]_b} &\longrightarrow (-)^\Delta \overline{[\chi_s]_b (\gamma_S^\dagger \otimes \xi_F^\dagger) [\chi_d]_a} , \end{aligned} \quad (6.4)$$

where a, b are color indices, and $F_{\mu\nu}$ is a lattice transcription of the gluon field strength. The lower dimension operators must therefore have positive C_0S parity. This is possible either if the bilinear itself has positive C_0S parity, or if it has negative C_0S parity and is multiplied by $m_s - m_d$.

4. They satisfy a Ward Identity relating their $K^0\pi^0$ correlator ($C_{K\pi}$) to that between the strange scalar $\kappa = \bar{s}d$ and the vacuum (C_κ):

$$\sqrt{2}(m_s + m_d) \sum_{t_K} C_{K\pi}(t_K, t_\pi) = -C_\kappa(t_\pi) . \quad (6.5)$$

(This is equivalent to Eq. 8.10 of Ref. [12].) This relation holds provided that one uses the operators $\overline{(\gamma_5 \otimes \xi_5)}$ and $\overline{(I \otimes I)}$ to create the pseudoscalars and scalar, respectively, and provided that $m_u = m_d$.¹³ It is the lattice analog of the continuum Ward Identity which follows from current algebra. It is straightforward to show that Eq. 6.5 can be satisfied by lower dimension operators in two ways: (a) if the bilinear itself has positive C_0S parity, then it must have odd distance; (b) if the bilinear itself has negative C_0S parity, then it must have even distance. These selection rules play the role of parity in the continuum CPS symmetry.

Verstegen has classified staggered fermion bilinears under the hypercubic group [31].¹⁴ Using his results we find that the following lower dimension bilinears have all the symmetry properties listed above and thus can be produced by mixing

$$\begin{aligned} \text{dim3 : } \quad \mathcal{B}_3 &= \sum_\mu \overline{\chi_s (\gamma_5 \otimes \xi_{\mu 5})} \chi_d \\ \text{dim4 : } \quad \mathcal{B}_{4a} &= \sum_\mu \overline{\chi_s (\gamma_\mu \otimes I)} \overleftrightarrow{D}_\mu \chi_d \\ \mathcal{B}_{4b} &= \sum_\mu \sum_{\nu \neq \mu} \partial_\mu \overline{[\chi_s (\gamma_\mu \otimes \xi_{\mu\nu})} \chi_d] \\ \text{dim5 : } \quad \mathcal{B}_{5a} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \overline{\chi_s (\gamma_{\rho\sigma} \otimes \xi_{\rho 5})} F_{\mu\nu} \chi_d \\ \mathcal{B}_{5b} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \overline{\chi_s (\gamma_{\rho\sigma} \otimes \xi_{\mu 5})} F_{\mu\nu} \chi_d \\ \mathcal{B}_{5c} &= \sum_{\rho, \mu} \overline{\chi_s} \overleftrightarrow{D}_\rho (\overline{\gamma_5 \otimes \xi_{\mu 5}}) \overleftrightarrow{D}_\rho \chi_d \\ \mathcal{B}_{5d} &= \sum_\mu \overline{\chi_s} \overleftrightarrow{D}_\mu (\overline{\gamma_5 \otimes \xi_{\mu 5}}) \overleftrightarrow{D}_\mu \chi_d \end{aligned}$$

¹³For the even distance operators such as that in Eq. 6.3, there is a subtlety concerning the allowed contractions of the operator in C_κ which is discussed in Ref. [14]. Here we simply assume that Eq. 6.5 applies.

¹⁴In applying his results one must note that his flavor matrix T_μ corresponds to our $\xi_{\mu 5}$.

$$\begin{aligned}
\mathcal{B}_{5e} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \overline{\chi}_s \overleftarrow{D}_\mu \left[\overline{(\gamma_{\rho\sigma} \otimes \xi_\mu)} - \overline{(\gamma_{\rho\sigma} \otimes \xi_\nu)} \right] \overrightarrow{D}_\nu \chi_d \\
\mathcal{B}_{5f} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \overline{\chi}_s \left(\overleftarrow{D}_\mu \overleftarrow{D}_\nu - \overrightarrow{D}_\nu \overrightarrow{D}_\mu \right) \left[\overline{(\gamma_{\rho\sigma} \otimes \xi_{\mu 5})} - \overline{(\gamma_{\rho\sigma} \otimes \xi_{\nu 5})} \right] \chi_d \\
\mathcal{B}_{5g} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \partial_\mu [\overline{\chi}_s \overline{(\gamma_{\rho\sigma} \otimes \xi_{\mu 5})}] \overleftarrow{D}_\nu \chi_d \\
\mathcal{B}_{5h} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \partial_\mu [\overline{\chi}_s \overline{(\gamma_{\rho\sigma} \otimes \xi_{\rho 5})}] \overleftarrow{D}_\nu \chi_d \\
\mathcal{B}_{5i} &= \partial^2 \sum_\mu [\overline{\chi}_s \overline{(\gamma_5 \otimes \xi_{\mu 5})}] \chi_d \\
\mathcal{B}_{5j} &= \sum_\mu \partial_\mu^2 [\overline{\chi}_s \sum_\mu \overline{(\gamma_5 \otimes \xi_{\mu 5})}] \chi_d \\
\mathcal{B}_{5k} &= \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \partial_\mu \partial_\nu [\overline{\chi}_s \overline{(\gamma_{\rho\sigma} \otimes \xi_\mu)}] \chi_d
\end{aligned}$$

$F_{\mu\nu}$ is any lattice version of the field strength with appropriate transformation properties under the hypercubic group. The covariant derivatives D_μ will be defined precisely below. We have included total derivatives in the list because they can contribute if one calculates $K \rightarrow \pi$ matrix elements with non-degenerate quarks. All the operators given above can be multiplied by functions of m_s and m_d that have positive C_0S parity. It turns out that no lower dimension operators with explicit factors of $m_s - m_d$ (or any function of m_s and m_d having negative C_0S parity) satisfy all the properties listed above. This is why all the operators in the list have odd distance.

The original four-fermion operators (e.g. Eqs. 6.2 and 6.3) are staggered flavor singlets.¹⁵ Thus, in the continuum limit, only bilinears with $F = I$ can be produced by mixing. In fact, of the fourteen lower dimension operators, only \mathcal{B}_{4a} has $F = I$. It is the lattice transcription of \mathcal{O}_{sub} , and is the operator we expect to be produced. All the other operators are staggered flavor non-singlets, and are allowed because elements of the hypercubic group contain flavor transformations. What we must determine is the conditions for which these operators, which are artifacts of the lattice regularization, do not contribute to matrix elements.

The first condition is that we take matrix elements between states having the same staggered flavor, e.g. between two pseudo-Goldstone pions each having flavor ξ_5 . At first sight this solves the problem because the matrix elements of staggered flavor non-singlet operators should vanish, leaving only \mathcal{B}_{4a} . This is only true, however, up to corrections of $O(p_{ext}a)$, where p_{ext} is an external physical momentum. In effect, the matrix element picks out higher dimension parts of the original operator, and these parts can have different staggered flavors. What one must do is take out sufficient factors of momenta until one obtains a flavor singlet. The flavor singlet components of all the non-singlet operators listed above turn out to be of dimension 6 or higher.

To explain how this works we need some results concerning the full staggered fermion symmetry group including translations [34]. In particular, we need the representations of the zero four-momentum subgroup, \mathcal{G}_0 , which are summarized in Ref. [12]. To discuss these representations, it is convenient to use a different definition of bilinears. Our choice up to

¹⁵We use the expression ‘‘staggered flavor’’ to distinguish from the continuum flavor SU(3) symmetry.

now, $\mathcal{O}_{SF}(y)$, (defined in Eqs. 2.7 and 2.12) has all fields lying in a single hypercube, labeled by y . This definition has the disadvantage that $\sum_y \mathcal{O}_{SF}(y)$ is not a representation of \mathcal{G}_0 , except for operators with $\Delta = 0$.

Instead, following Ref. [35], we proceed as follows. Each term in $\mathcal{O}_{SF}(y)$ is a product $\bar{\chi}(y+A)\chi(y+B)$ multiplied by a phase, where $B =_2 A + S - F$, the subscript indicating mod-2 arithmetic. We first think of this term as if both $\bar{\chi}$ and χ started at site $y+A$, with χ then shifted to site $y+B$. If $(S-F)_\mu \neq 0$, then in the new operators we perform this shift symmetrically

$$S_\mu \chi(y+A) = \frac{1}{2} [\chi(y+A+\mu) + \chi(y+A-\mu)] , \quad (6.6)$$

rather than picking just one of these two terms. We do this symmetric shift in all directions for which $S-F$ is non-vanishing, resulting, in general, in an operator contained in a 4^4 hypercube. To complete the definition we take the average of this enlarged operator with that obtained by shifting $\bar{\chi}$ instead of χ . This last step gives an operator which transforms into itself under C_0 , with the sign given by Eq. 6.4. The resulting operator we call $\mathcal{O}_{SF}^{true}(y)$, because $\sum_y \mathcal{O}_{SF}^{true}(y)$ is part of a true representation of \mathcal{G}_0 . In particular, it transforms into itself under translation in the μ 'th direction, with phase $(-)^{\tilde{F}_\mu}$, where $\tilde{F}_\mu =_2 \sum_{\nu \neq \mu} F_\nu$.

We also need additional ‘‘derivative’’ operators defined as for \mathcal{O}_{SF}^{true} except that one or more of the symmetric shifts is replaced by the antisymmetric shift

$$D_\mu \chi(y+A) = \frac{1}{2} [\chi(y+A+\mu) - \chi(y+A-\mu)] . \quad (6.7)$$

This corresponds to taking a derivative of the quark field, and we write the resulting bilinear as $\overline{\chi(\gamma_S \otimes \xi_F)} D_\mu \chi$. Clearly $(S-F)_\mu$ must be non-zero for there to be the possibility of an insertion of D_μ . Unlike for \mathcal{O}_{SF}^{true} , we distinguish explicitly between the derivatives acting on the quark and antiquark fields. These must be combined to obtain eigenstates of $C_0 S$, as in \mathcal{B}_{4a} . Two properties of derivative operators are important. First, when summed over all y , they give true representations of \mathcal{G}_0 , with the translation phase being $(-)^{\tilde{F}_\mu}$. Second, they transform under the hypercubic subgroup in the same way as $\overline{\chi(\gamma_{\mu 5} \otimes \xi_{\mu 5})} \chi$. This follows because a hypercube operator \mathcal{O}_{SF} can be written as a sum of \mathcal{O}_{SF}^{true} , which transforms exactly as \mathcal{O}_{SF} under the hypercubic group, and a number of derivative operators. For example,

$$\overline{\chi(\gamma_5 \otimes \xi_{\mu 5})} \chi = [\overline{\chi(\gamma_5 \otimes \xi_{\mu 5})} \chi]^{true} - \frac{1}{2} \overline{\chi(\gamma_\mu \otimes I)} \overleftrightarrow{D}_\mu \chi . \quad (6.8)$$

This shows the decomposition of the original hypercube operators into true representations of \mathcal{G}_0 .

We now return to the list of lower dimension operators \mathcal{B}_i . First we complete their definition. Some of them contain covariant derivatives in directions for which $S-F=0$, e.g. \mathcal{B}_{5c} . In these cases the covariant derivative is defined by

$$D_\mu \chi(y+A) = \frac{1}{4} [\chi(j+A+2\hat{\mu}) - \chi(j+A-2\hat{\mu})] .$$

Similarly the total derivatives are defined by shifting two lattice units, which avoids mixing with staggered flavor symmetries. Next we note that, using results such as Eq. 6.8, we can rewrite all the operators so that they become true representations of \mathcal{G}_0 when summed over

y . In fact, the list already includes all the derivative operators of dimension less than 6 that can appear. For example, turning \mathcal{B}_3 into a true representation using Eq. 6.8, we find the derivative operator \mathcal{B}_{4a} . Thus, with no loss of generality, we can take all the bilinears in the list to already be true representations.

We can now make more precise the statement that the matrix elements of the flavor non-singlet operators vanish up to momentum corrections. Since $\tilde{F} = 0$ iff $F = 0$, all such operators have a non-trivial translation phase in at least one direction. Thus their matrix elements vanish at $p = 0$, where p is the total momentum inserted by the operator. In order to obtain, say, a $K \rightarrow \pi$ matrix element at $p = 0$, the K and π must both have the same staggered flavor (since otherwise $p \approx C\pi$ for some non-zero hypercube vector C), and the K and π must be degenerate (for one can then create the states at infinite positive and negative time, respectively, so that the operator can be averaged over all positions). For such a matrix element only \mathcal{B}_{4a} contributes and the argument is complete. In a general lattice matrix element, however, there will be a small physical momentum inserted by the operator, e.g. if the K and π are not degenerate. In this case the flavor non-singlet operators can have non-vanishing matrix elements.

To evaluate these contributions we use two identities concerning matrix elements when $(-)^{\tilde{F}\mu} = -1$. The first is valid when $(S - F)_\mu \neq 0$,

$$\langle (\overline{\chi(\gamma_S \otimes \xi_F)\chi})^{true}(p) \rangle = \tan(p_\mu/2)^2 \langle \frac{1}{2} \overline{\chi(\gamma_{\mu 5 S} \otimes \xi_{\mu 5 F})} \overleftrightarrow{D}_\mu \chi \rangle, \quad (6.9)$$

the second for $(S - F)_\mu = 0$

$$\langle (\overline{\chi(\gamma_S \otimes \xi_F)\chi})^{true}(p) \rangle = i \tan(p_\mu/2) \langle \overline{\chi(\gamma_{\mu 5 S} \otimes \xi_{\mu 5 F})\chi} \rangle. \quad (6.10)$$

There is no implied sum over μ in either equation. The operators are defined on a single hypercube, and the matrix element is between external states which insert momentum p . For small p , the factors of $i \tan(p_\mu/2)$ can be replaced by $-\partial_\mu/2$ acting on the operators. These equations remain valid even if there are covariant derivatives or field strengths in the operators. Since $(-)^{\mu 5 \tilde{F}\mu} = +1$, the operators on the right-hand-sides have trivial translation phases in the μ 'th direction. By using the equations sequentially in all directions in which the translation phase is non-trivial, one can find the flavor singlet component in a given bilinear.

We now apply these equations to the bilinears \mathcal{B}_i . Each application raises the dimension of the operator by either 1 or 3, depending on whether $(S - F)_\mu = 0$ or 1. Recall that all the bilinears \mathcal{B}_i are now taken to be true representations of \mathcal{G}_0 . We work our way up from the $d' = 3$ operators.

$d'=3$ There is only one such operator: \mathcal{B}_3 . It can be replaced by $\frac{1}{4} \partial^2 \overline{\chi_s(\gamma_\mu \otimes I)} \overleftrightarrow{D}_\mu \chi_d$, using Eq. 6.9. This is a dimension 6 operator which, furthermore, vanishes by the equations of motion if the quark masses are set to zero.

$d'=4$ The only flavor non-singlet dimension 4 operator can be written

$$\mathcal{B}_{4b} = \sum_\mu \partial_\mu \mathcal{B}_\mu, \quad \mathcal{B}_\mu = \sum_{\nu \neq \mu} \overline{\chi_s(\gamma_\mu \otimes \xi_{\mu\nu})} \chi_d. \quad (6.11)$$

One application of each of Eqs. 6.9 and 6.10 shows that \mathcal{B}_μ acts as a dimension 7 operator in matrix elements, so that \mathcal{B}_{4b} acts as a dimension 8 operator and can be ignored.

$d'=5$ Dimension 5 operators are converted into flavor-singlets of dimension 6 or higher using Eqs. 6.9 and 6.10. All dimension 6 operators that occur will appear in the mixing calculations at some order. In sect. 5 we calculated this mixing to one-loop, and found that the only staggered flavor singlet operators that appear can be rewritten as four-fermion operators.

We conclude that all the lower dimension operators except \mathcal{B}_{4a} have vanishing matrix elements even if non-zero physical momentum is inserted.

We close by noting that the term proportional to I_a in Eq. 5.18 is an explicit example of mixing with lower dimension operators. At one loop this does not affect the lattice transcriptions of Q_{1-6} , but when inserted in a two loop diagram, it gives rise to mixing with the operator \mathcal{B}_{5a} . This operator has $(-)^{\tilde{F}_\rho} = -1$, and $(S - F)_\rho \neq 0$. Using Eq. 6.9, one finds that it acts in matrix elements as though it has dimension 8, and thus can be ignored.

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A FIERZ TRANSFORMATIONS

In this appendix we give the Fierz transformation rules for the 35 diagonal scalar operators. We do *not* include the sign due to Fermi statistics. Since we always do two Fierz transformations, any such sign would cancel.

The first 25 operators are of the form exemplified by $[V \times T]$, in which the spin and flavor matrices are independent. These form a closed set under Fierz transformation, with the spin and flavor part of the operators being transformed independently. The rules can thus be obtained from those for the spin alone:

$$\begin{aligned} V - A &\leftrightarrow A - V \\ V + A &\leftrightarrow 2(S - P) \\ S + P &\leftrightarrow \frac{1}{2}(S + P + T) \\ T &\leftrightarrow \frac{1}{2}(3S + 3P - T) . \end{aligned}$$

Thus, for example, the transformation needed in the example of Section 3.3 is

$$[V \times S] \rightarrow \frac{1}{8} [(2S - V + A - 2P) \times (S + V + T + A + P)] .$$

The Fierz transformation rules for the remaining 10 operators are

$$\begin{aligned}
[(V + A)_\mu \times (V + A)_\mu] &\leftrightarrow [(S - P) \times (S - P)] + [T_- \times T_-] \\
2[(V + A)_\mu \times (V - A)_\mu] &\leftrightarrow -[(S - P) \times (V - A)] - [T_\mu \times (V + A)_\mu] \\
[(V - A)_\mu \times (V - A)_\mu] &\leftrightarrow [(V - A)_\mu \times (V - A)_\mu] \\
[(V - A)_\mu \times T_\mu] &\leftrightarrow [T_\mu \times (V - A)_\mu] \\
2[(V + A)_\mu \times T_\mu] &\leftrightarrow [(V - A) \times (V + A)] - 4[(V - A)_\mu \times (V + A)_\mu] \\
4[T_- \times T_-] &\leftrightarrow -[(V + A) \times (V + A)] + 4[(V + A)_\mu \times (V + A)_\mu] \\
4[T_+ \times T_+] &\leftrightarrow 4[(S + P) \times (S + P)] - [(S + P + T) \times (S + P + T)] \\
&\quad + 4[T_+ \times T_+] .
\end{aligned}$$

B Corrections from Xa diagrams

In this appendix we use the one-loop matching coefficients for bilinears to calculate those for the diagonal scalar four-fermion operators. We consider only the contributions from Xa diagrams, since, as discussed in the text, we can bring all other one-loop calculations into the form of an Xa diagram. We leave out color factors and an overall $g^2/16\pi^2$, and set $\mu = \pi/a$, so that the anomalous dimension term is absent. Thus, in the notation of Eq. 3.11, we are calculating the matrix \mathcal{M}^a .

The matching coefficients for bilinears (defined in Eq. 3.8) are given in PS. They depend on both the spin and flavor of the bilinear, and this is codified in the labels given to the c_i . For example, the correction for the bilinear $(\overline{\gamma_{\nu 5}} \otimes I)$ is labeled c_{AS} . Since c_{AS} does not depend on the index ν , the correction for the operator $[A \times S]$ is diagonal. In the notation of Eq. 3.13

$$\mathcal{M}^a([A \times S]) = 2c_{AS}[A \times S] . \quad (\text{B.1})$$

The factor of two comes from the fact that both bilinears are corrected.

An important feature of the one-loop corrections for bilinears is that they commute with multiplication by $(\overline{\gamma_5} \otimes \xi_5)$. Thus if one simultaneously changes the spin and flavor of the operators according to $S \leftrightarrow P, V \leftrightarrow A, T \leftrightarrow T$, the correction coefficients remain unchanged. This means, for example, that $c_{AS} = c_{VP}$. As in sect. 2, we define \mathcal{P} to be the operation of multiplying both bilinears in the four-fermion operator by $(\overline{\gamma_5} \otimes \xi_5)$ from the left. Due to the axial symmetry, this operation commutes with multiplication by the correction matrix \mathcal{M}^a . For example, applying \mathcal{P} to Eq. B.1 we find

$$\mathcal{M}^a([V \times P]) = 2c_{AS}[V \times P] = 2c_{VP}[V \times P] . \quad (\text{B.2})$$

The simple diagonal form of the correction in Eqs. B.1 and B.2 applies for those operators whose flavor or spin is either S or P . In this case, the corrections do not break the flavor or spin symmetries. For other operators, e.g. $[V \times V]$, these symmetries are broken. This is because the bilinears in these operators consist of parts of different distances. The bilinears in $[V \times V]$ can have $\Delta = 2$ (e.g. $(\overline{\gamma_\mu} \otimes \xi_\nu)$ with $\mu \neq \nu$) or $\Delta = 0$ (e.g. $(\overline{\gamma_\mu} \otimes \xi_\mu)$), and these are corrected with different coefficients (c_{V2} and c_{V0} respectively). The net result is that,

in terms of the operator basis that we use, the corrections are

$$\mathcal{M}^a([V \times V]) = 2c_{VV2}[V \times V] + 2(c_{VV0} - c_{VV2})[V_\mu \times V_\mu] \quad (\text{B.3})$$

$$\mathcal{M}^a([V_\mu \times V_\mu]) = 2c_{VV0}[V_\mu \times V_\mu]. \quad (\text{B.4})$$

Similarly, we find

$$\mathcal{M}^a([V \times A]) = 2c_{VA2}[V \times A] + 2(c_{VA4} - c_{VA2})[V_\mu \times A_\mu] \quad (\text{B.5})$$

$$\mathcal{M}^a([V_\mu \times A_\mu]) = 2c_{VA4}[V_\mu \times A_\mu]. \quad (\text{B.6})$$

The remaining results for operators containing vector bilinears are

$$\mathcal{M}^a([V \times T]) = (c_{VT1} + c_{VT3})[V \times T] + (c_{VT1} - c_{VT3})[V_\mu \times T_\mu] \quad (\text{B.7})$$

$$\mathcal{M}^a([V_\mu \times T_\mu]) = (c_{VT1} - c_{VT3})[V \times T] + (c_{VT1} + c_{VT3})[V_\mu \times T_\mu]. \quad (\text{B.8})$$

The results for operators containing axial bilinears can be obtained by applying \mathcal{P} to the above results.

Finally, we consider tensor operators. The results for $[T \times V]$ and $[T_\mu \times V_\mu]$ are as in Eqs. B.7 and B.8, with V and T interchanged. The relevant coefficients are thus c_{TV1} and c_{TV3} . Results for $[T \times A]$ can be obtained using \mathcal{P} . The remaining results are

$$\begin{aligned} \mathcal{M}^a([T \times T]) &= (c_{TT0} - c_{TT4})[T_- \times T_-] + (c_{TT0} + c_{TT4} - 2c_{TT2})[T_+ \times T_+] \\ &\quad + 2c_{TT2}[T \times T] \end{aligned} \quad (\text{B.9})$$

$$\mathcal{M}^a([T_- \times T_-]) = (c_{TT0} + c_{TT4})[T_- \times T_-] + (c_{TT0} - c_{TT4})[T_+ \times T_+] \quad (\text{B.10})$$

$$\mathcal{M}^a([T_+ \times T_+]) = (c_{TT0} - c_{TT4})[T_- \times T_-] + (c_{TT0} + c_{TT4})[T_+ \times T_+]. \quad (\text{B.11})$$

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