

Yang-Mills, Gravity, and String Symmetries

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(To Appear in Physics Letters B)

In this work we use constructs from the dual space of the semi-direct product of the Virasoro algebra and the affine Lie algebra of a circle to write a theory of gravitation which is a natural analogue of Yang-Mills theory. The theory provides a relation between quadratic differentials in 1+1 dimensions and rank two symmetric tensors in higher dimensions as well as a covariant local Lagrangian for two dimensional gravity. The isotropy equations of coadjoint orbits are interpreted as Gauss law constraints for a field theory in two dimensions, which enables us to extend to higher dimensions. The theory has a Newtonian limit in any space-time dimension. Our approach introduces a novel relationship between string theories and 2D field theories that might be useful in defining dual theories. We briefly discuss how this gravitational field couples to fermions.

I. INTRODUCTION

Throughout the literature, the focus on coadjoint orbits [1–3] for two dimensional theories has been in the construction of geometric actions that enjoy the symmetries of an underlying Lie algebra. In fact, the celebrated WZW [4] model and Polyakov [5] gravity have been shown to be precisely the geometric actions associated with the affine Lie algebra of special unitary groups [6,7] and circle diffeomorphisms [8] respectively [9–11]. Recently we have turned our attention away from theories that live on the orbits and toward theories that are transverse to these orbits [12,13]. This paper reports the latest progress in this direction. We promote the fixed coadjoint vectors to dynamical fields and reinterpret the generator of isotropy on the orbits as Gauss law constraints that arise from a 2D field theory. We should emphasize that this is not the method of coadjoint orbits that is used to construct geometric actions. In particular, the 2D gravitational action that we derive is not the Polyakov action [5] since the Polyakov action is the anomalous contribution to the gravitational action. The analogy between our work and that in Ref. [5] is the same as the relationship between the Yang-Mills action and WZW action. In our action, there is no a priori dimensional restriction. The coadjoint vectors describe a vector potential associated with a gauge theory and another field (the gravitational tensor potential) which we interpret as a possible hitherto missing contribution to gravitation. The new carrier of gravitation is a rank two symmetric tensor $T_{\mu\nu}$, that has units of mass squared. After gauge fixing in two dimensions, one finds residual coordinate transformations that explain the appearance of an inhomogeneous contribution. We are also able to extract the symplectic part of the gravitational Lagrangian. The propagator then has both a quadratic and quartic contributions instead of just quartic as believed earlier [12]. The final action can

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live in *any dimension*, yet is consistent with the constraints that arise in the 2 dimensional theory. Since constraints in the 2 dimensional theory are directly related to symmetries of a string theory, namely the Virasoro algebra, our approach introduces a novel relationship between string theories and field theories that may be generalized for other algebras. Furthermore this approach may provide insight into the dual relationship between string theories and two dimensional field theories. This theory of gravitation might augment General Relativity with the inclusion of a gravitational tensor potential that can be used to study fluctuations in the gravitational field about a fixed metric while maintaining general covariance. However the details of this are unknown at present. Recovery of this putative new contribution which reconciles the different characteristics of two-dimensional and higher-dimensional gravitation is the major thrust of this letter.

To begin [12,13], one treats the adjoint representation of the Virasoro algebra and the affine Lie algebra with adjoint vector $\mathcal{F} = (\xi(\theta), \Lambda(\theta), a)$ as residual coordinate and gauge transformations respectively on a gauge potential A and a gravitational potential T which are components of a centrally extended coadjoint vector, $B = (T(\theta), A(\theta), \mu)$. The adjoint vector \mathcal{F} then transforms B through [14],

$$B_{\mathcal{F}} = (\xi(\theta), \Lambda(\theta), a) * (T(\theta), A(\theta), \mu) = (T(\theta)_{\text{new}}, A(\theta)_{\text{new}}, 0), \quad (1)$$

where the transformed fields are defined by,

$$T(\theta)_{\text{new}} = \underbrace{2\xi' T + T' \xi + \frac{c\mu}{24\pi} \xi'''}_{\text{coordinate trans}} - \underbrace{\text{Tr}(A\Lambda')}_{\text{gauge trans}} \quad (2)$$

and

$$A(\theta)_{\text{new}} = \underbrace{A' \xi + \xi' A}_{\text{coord trans}} - \underbrace{[\Lambda A - A \Lambda] + k \mu \Lambda'}_{\text{gauge trans}}. \quad (3)$$

These transformations are to be thought of as gauge and coordinate transformations on a time-like slice of a 1 + 1 dimensional field theory. They represent the residual transformations that are available after temporal gauge fixing. One recovers the *isotropy equations* of the coadjoint orbits defined by the quadratic differential T , the one dimensional gauge field A , and the central extension μ by setting the quantities in (2) and (3) to zero. These equations will be used to extract the Gauss laws identifying the residual symmetries after gauge fixing in 2D. The isotropy algebra will arise from field equations that have collapsed to Gauss law constraints when the action is varied in two dimensions. With this simple assertion we shall be able to write a theory for any dimension, and possibly augment Einstein's theory of General Relativity.

II. CONSTRUCTION OF THE ACTION

It is our intention to educe a theory, analogous to Yang-Mills, that corresponds to a gravitational action, and which has a natural reduction to the quadratic differential T when gauge fixed in 2D. Yang-Mills theory will serve as an archetype in the construction of the gravitational contributions. To recapitulate our earlier assertion, we claim that the isotropy equations of the coadjoint orbits are just the residual gauge and coordinate transformations after fixing a temporal gauge in 2D. These residual transformation are encoded in a Gauss law that arises from the field equations of one of the components of the gravitational tensor potential in 2D.

To begin the construction of our action, we need to write a Gauss generator that yields the time independent transformation laws that come from the isotropy equations of the orbits. Recall equations (1), (2) and (3).

$$\delta_{\mathcal{F}} \tilde{B} = \{\delta T, \delta A, \delta \mu\} = \{2\xi' T + T' \xi + \frac{c\mu}{24\pi} \xi'' - \text{Tr}(A\Lambda'), A' \xi + \xi' A + [\Lambda A - A \Lambda] + k \mu \Lambda', 0\}. \quad (4)$$

Throughout we shall assume that the constant $k\mu = -1$. From here we can extract the transformation laws and Gauss law constraints in 2D that lead directly to higher dimensional theories which include interactions.

Since we are interested in a picture of pure gravity we will consider just the coordinate transformations on T and set the gauge field to zero. We claim that T is the one remaining component of a rank two symmetric tensor after

gauge fixing in two dimensions. This would be consistent with the fact that in D dimensions there are D , ξ^α fields that serve as “gauge parameters” for coordinate transformations. Therefore we can remove D degrees of freedom.

This would be tantamount to setting $T_{\mu 0} = 0$, leaving only the purely spatial components T_{ij} as dynamical variables. Furthermore, for the two dimensional case, gauge fixing in the temporal gauge leaves an anomalous inhomogeneous transformation on T_{11} or, using world-sheet notation, $T_{\theta\theta}$ that must be exhibited in a Gauss law constraint.

Since the 2D theory will be our bridge to higher dimensions, let us focus on its structure. From (4) we have

$$\delta T = 2\xi' T + T' \xi + q \xi''',$$

where $q = c\mu/24\pi$. In this work we stress that $T_{\mu\nu}$ is a rank two tensor and not a pseudo-tensor as we believed earlier [12]. The inhomogeneous transformation of $T_{\mu\nu}$ is a consequence of gauge fixing in two dimensions. In 2D the field equations of the T_{01} component become constraints; as opposed to dynamical field equations.

Our claim is that the above equation carries these residual gauge transformations after gauge fixing a 2D field theory. We need a Gauss law that will deliver the time independent coordinate transformations for the T_{11} component. Let X^{ij} be the conjugate variable to T_{ij} . This is a symmetric object whose transformation law will be determined by the Gauss law. Let

$$(G_{\text{diff}})_a = X^{lm} \partial_a T_{lm} - \partial_l (X^{lm} T_{am}) - \partial_m (T_{la} X^{lm}) - q \partial_a \partial_l \partial_m X^{lm}. \quad (5)$$

This is the generator of time independent coordinate transformations in 1 + 1 dimensions with an additional inhomogeneous term. We have purposely left in the tensor structure for future use in higher dimensions. From the Poisson brackets, one can recover the transformations laws of X^{ab} and T_{ab} . We have

$$Q_{\text{diff}} = \int d^3x G_a \xi^a, \quad (6)$$

where the ξ^a are the time independent spatial translations. As a result,

$$\begin{aligned} \{Q_{\text{diff}}, T_{lm}\} &= -\xi^a \partial_a T_{lm} - T_{am} \partial_l \xi^a - T_{la} \partial_m \xi^a - q \partial_a \partial_l \partial_m \xi^a = -\delta_\xi T_{lm} \\ \{Q_{\text{diff}}, X^{lm}\} &= \xi^a \partial_a X^{lm} - (\partial_a \xi^l) X^{am} - (\partial_a \xi^m) X^{la} + (\partial_a \xi^a) X^{lm} = \delta_\xi X^{lm}. \end{aligned} \quad (7)$$

The gauge fixing conditions, $T_{\mu 0} = 0$, are preserved under the time independent coordinate transformations.

The transformation law for X^{lm} defines it as a rank two tensor density of weight one. Thus X^{lm} can carry a factor of \sqrt{g} in its definition. In two dimensions, the transformation law for the “space-space” component of X^{lm} reduces to the transformation law of elements in the adjoint representation of the Virasoro algebra, viz. $X' \xi - \xi' X$. The fact that X^{ab} is a tensor density follows since Q must be a scalar. In what is to follow, we shall remove the \sqrt{g} factor from the definition of X and simply multiply our Lagrangian density by the \sqrt{g} factor.

We can now replace the fields ξ^a with T^{a0} and write down the first part of the 1 + 1 Lagrangian as:

$$L_0 = X^{lm} (T^{a0} \partial_a T_{lm} + T_{am} \partial_l T^{a0} + T_{la} \partial_m T^{a0}). \quad (8)$$

Also, one can easily show that L_0 is invariant (up to total derivatives) under time independent spatial transformations in 1 + 1 dimensions.

As can be seen from the above gauge fixed expression, the conjugate momentum X^{ab} comes from a tensor density of the type $X^{\mu\nu}{}_\rho$, where ρ has been evaluated in the time direction, i.e. $X^{ab} = X^{ab}{}_0$ in analogy with E_i and F_{i0} . This tensor is symmetric in its $\mu\nu$ indices. Furthermore, the appearance of spatial derivatives on T^{a0} in L_0 suggests that $X^{\mu\nu}{}_\rho$ comes from a covariant tensor with the structure

$$X_{\mu\nu\rho} = \partial_\rho T_{\mu\nu}.$$

On an arbitrary manifold with fixed metric we may write the fully covariant field $X_{\alpha\beta\gamma}$ as

$$X_{\alpha\beta\gamma} = \nabla_\gamma T_{\alpha\beta}. \quad (9)$$

Also we introduce the tensor

$$K_{\gamma\alpha\beta} = \frac{1}{2} (\nabla_\gamma T_{\alpha\beta} - \nabla_\alpha T_{\gamma\beta}) \quad (10)$$

which we will use later to couple $T_{\alpha\beta}$ to fermions. The tensor $X_{\alpha\beta\gamma}$ is symmetric in its first two indices, while $K_{\alpha\beta\gamma}$ is also acyclic in all of its indices but anti-symmetric in its first two indices.

For the moment let us set $q = 0$, eliminating the 2D inhomogeneous contribution to the constraint equation. Then in any dimension the following covariant version of Eq.(8) can be used to recover the homogeneous part of the constraint equation,

$$\mathcal{L}_0 = X^{\lambda\mu\rho} Y_{\lambda\mu\rho}, \quad (11)$$

where

$$Y_{\lambda\mu\rho} = (T^\alpha{}_\rho \partial_\alpha T_{\lambda\mu} + T_{\lambda\alpha} \partial_\mu T^\alpha{}_\rho + T_{\alpha\mu} \partial_\lambda T^\alpha{}_\rho - \partial_\rho (T^\alpha{}_\lambda T_{\alpha\mu})). \quad (12)$$

Notice that the tensor $Y_{\lambda\mu\rho}$ is covariant and independent of the derivative operator or the metric used to raise and lower the indices of $T_{\alpha\beta}$. This tensor represents a covariant extension of the Lie derivative used in Eq.(7) for the derivation of $T_{\alpha\beta}$ on itself when $q = 0$. Notice that if $T_{\alpha\beta}$ were a metric tensor, $Y_{\lambda\mu\rho}$ would vanish identically. Therefore $Y_{\lambda\mu\rho}$ is unrelated to a connection and cannot be accessed by curvature. In a similar manner the inhomogeneous contributions ($q \neq 0$) to the constraints in Eq.(5) are accessed through the Lagrangian

$$\mathcal{L}_1 = -\frac{q}{4} \nabla_\beta T^\beta{}_\alpha \nabla_\lambda \nabla_\mu X^{\lambda\mu\alpha}. \quad (13)$$

By varying the sum of the above Lagrangians with respect to the $T_{\nu 0}$ components and fixing the temporal gauge on a flat manifold we find

$$X^{lm0} \partial_\nu T^{lm} - \partial_m (X^{ml0} T_{\nu}) - \partial_l (X^{ml0} T_{m\nu}) - q \partial_\nu \partial_l \partial_m X^{lm0} = 0. \quad (14)$$

In the above the Latin indices refer to space components and the Greek space-time components. We see that we have recovered the generator of time independent coordinate transformations, Eq.(5). In 1 + 1 dimensions this can be identified with the isotropy equation, $\xi T' + 2\xi' T + q \xi''' = 0$ where ξ takes the role of $X^{1,1}$ as an adjoint element (recall that it transforms as a tensor density).

However Eqs.(11,13) cannot build the full action as they do not determine the symplectic structure. Another contribution to the Lagrangian should exist that confirms the relation between time derivatives of the dynamical variables and their conjugate momenta. Consider the action,

$$S_I = \int \sqrt{g} \mu^{6-n} \left[(-\nabla_\alpha T_{\beta\gamma}) X^{\beta\gamma\alpha} + \frac{1}{2} X^{\beta\gamma\alpha} X_{\beta\gamma\alpha} \right] d^n x. \quad (15)$$

Here μ is a gravitational coupling constant with inverse mass dimensions. S_I is known only up to a scale factor, relative to S_0 , since the field equations for $T_{\theta 0}$ have no contribution from S_I in the temporal gauge in two dimensions. It is obvious that S_I contains that part of the Lagrangian which determines the symplectic structure of the action. Since a variation of S_I with respect to $\partial_0 T_{\alpha\beta}$ yields $X^{\alpha\beta 0}$ as the conjugate momentum. After gauge fixing only the T_{ij} components have non-trivial conjugate momentum. Furthermore, after gauge fixing, Eqs.(11,13) will not contribute to the conjugate momentum.

Putting this all together and writing partial derivatives on $T_{\alpha\beta}$ in terms of X , Eq.(9), we may write the fully coordinate invariant action in n dimensions as

$$\begin{aligned} S_{\text{AT}} = & \int d^n x \sqrt{g} \mu^{8-n} (X^{\lambda\mu\rho} T^\alpha{}_\rho X_{\mu\lambda\alpha} + 2X^{\lambda\mu\rho} T_{\lambda\alpha} X^\alpha{}_{\rho\mu} - X^{\lambda\mu\rho} X^\alpha{}_{\lambda\rho} T_{\alpha\mu} - X^{\lambda\mu\rho} X_{\alpha\mu\rho} T^\alpha{}_\lambda) \\ & - \frac{q}{4} \int d^n x \sqrt{g} \mu^{8-n} (X^{\alpha\beta}{}_\gamma \nabla_\lambda \nabla_\mu X^{\lambda\mu}{}_\alpha) \\ & - \frac{1}{2} \int d^n x \sqrt{g} \mu^{6-n} (X^{\beta\gamma\alpha} X_{\beta\gamma\alpha}). \end{aligned} \quad (16)$$

This action describes a self-interacting rank two field, $T_{\alpha\beta}$ that in the very least defines classical gravity in two dimensions.

Since the metric $g_{\alpha\beta}$ is omnipresent, we interpret the field $T_{\alpha\beta}$ as further gravitational contributions about a background metric. It may provide a natural bifurcation between local gravitational effects and cosmological effects of gravity due to the background metrics. It may be that without introducing any arbitrary splittings or asymptotics of the metric tensor, one can study fluctuations in gravity (due to diffeomorphisms) by studying S_{AT} . The origins of the

field $T_{\alpha\beta}$ are from the diffeomorphism algebra and necessarily must be related to gravitation. In two dimensions this theory along with the extended Polyakov action, Eq.(17) (which is already gauge fixed), completely defines classical two dimensional gravity and its anomalous contributions coming from the effective action. This is in direct analogy with the way that Yang-Mills and the WZW models complete 2D gauge theories.

The procedure that we have developed is quite general and might be useful in defining dual theories for string theories in two dimensions. For any given algebra of the circle or string, one can construct the coadjoint orbits for the algebra and extract the anomalous contributions. The coadjoint vectors serve as dynamical fields while the adjoint vectors serve as the conjugate momenta. The isotropy equation for the orbits is then reinterpreted as a constraint equation for conjugate variables providing a field theory that will interpolate between the different coadjoint orbits.

III. MATTER COUPLINGS

It is natural to ask how $T_{\alpha\beta}$ might couple to matter. Coupling to point particles is straightforward as one can write $(P^\alpha P^\beta)T_{\alpha\beta}$ and one can see that a Newtonian limit will exist in any space-time dimension. In order to couple this to fermions let us recall the 2D geometric action found in [9,13,14]. There the 1 + 1 dimensional theory admits the action

$$S = \underbrace{\int d^2x T(\theta) \frac{\partial_\tau s}{\partial_\theta s}}_{\text{fermion-gravity}} + \int d^3x \left(\partial_\tau T \frac{\partial_\lambda s}{\partial_\theta s} - \partial_\lambda T \frac{\partial_\tau s}{\partial_\theta s} \right) + \frac{c\mu}{48\pi} \int \left[\frac{\partial_\theta^2 s}{(\partial_\theta s)^2} \partial_\tau \partial_\theta s - \frac{(\partial_\theta^2 s)^2 (\partial_\tau s)}{(\partial_\theta s)^3} \right] d\theta d\tau. \quad (17)$$

The bosonized fermions form the dimensionless rank two field, $\partial_\alpha s / \partial_\beta s$ and the fermion-gravity interaction dictates an interaction term,

$$S_{\Psi T} = \int \sqrt{g} \bar{\Psi} \gamma^\mu (\partial_\mu + \omega_\mu + \mu^2 K^{\alpha\beta}{}_\mu [\gamma_\alpha, \gamma_\beta]) \Psi d^4x, \quad (18)$$

where $K^{\alpha\beta}{}_\mu$ is defined in (10) and ω_μ is the spin connection. Thus $K^{\alpha\beta}{}_\mu$ acts as a contribution to the spin connection. The gauge fixed action of Eq.(16) along with its constraints and Eq.(17) define the theory of chiral fermions interaction with gravitation in two dimensions or if one prefers an effective action of gravity in two dimensions with its anomalous contributions.

IV. ACKNOWLEDGMENTS

This work was partially supported by NSF grant PHY-94-11002, the Carver Foundation and the Obermann Center for Advanced Studies. T.B. and V.G.J.R. are indebted to the Obermann Center for Advanced Studies for support and hospitality. V.G.J.R. would like to thank the Institute for Fundamental Theory and the High Energy Experimental Group at the University of Florida and the Aspen Center for Physics where some of this work was done. This work is dedicated to the memory of Emma Meurice.

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