Relativistic Operator Description of Photon Polarization

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We present an operator approach to the description of photon polarization, based on Wigner's concept of elementary relativistic systems. The theory of unitary representations of the Poincarè group, and of parity, are exploited to construct spinlike operators acting on the polarization states of a photon at each fixed energy momentum. The nontrivial topological features of these representations relevant for massless particles, and the departures from the treatment of massive finite spin representations, are highlighted and addressed.

I. INTRODUCTION

In the framework of relativistic quantum mechanics, elementary systems are described by unitary irreducible representations (UIR's) of the proper orthochronous inhomogeneous Lorentz group, or Poincarè group $\mathcal{P}$ [1]. This is the symmetry group of special relativistic space-time. The adjective 'elementary' here implies that for such systems every physical observable can in principle be constructed as a function of the operators implementing the transformations of $\mathcal{P}$. However, as is well known, not all the mathematically constructible UIR's of $\mathcal{P}$ are physically acceptable. Only the finite mass, finite spin UIR's containing positive time-like energy momenta, and the positive lightlike energy momentum finite helicity UIR's, are realised in nature. These two classes of UIR's are closely linked to the rotation subgroup $SO(3)$ of the homogeneous Lorentz group $SO(3,1)$, and to an $E(2)$ subgroup of $SO(3,1)$, respectively. UIR's of $\mathcal{P}$ containing space-like energy momenta, and the light-like continuous spin or infinite helicity UIR's, are unphysical [2].

On account of the special role played by the $SO(3)$ subgroup in the timelike UIR's, there is a clean kinematic separation between orbital and spin angular momenta in these representations. Indeed the concept of spatial position is well-defined [3], and all the generators of $\mathcal{P}$ can be built up as functions of kinematically independent position, momentum and spin operators. Moreover in these UIR's of $\mathcal{P}$, action by the parity operator P can be accommodated without any enlargement of the representation space. Many of these simplifying features can also be linked to the fact that the coset space $SO(3,1)/SO(3)$ (equivalently $SL(2,C)/SU(2)$) is $\mathcal{R}^3$ which is topologically trivial.

In the lightlike finite helicity UIR's of $\mathcal{P}$, such as one uses to describe photons, the situation is markedly different in many respects. Here the role of $SO(3)$ is taken over by the $E(2)$ subgroup mentioned earlier, and this is not directly linked to transformations on space alone. Moreover the coset space $SO(3,1)/E(2)$ is topologically nontrivial. For these reasons, there is no longer a clear cut kinematic separation of anything analogous to spin from variables related to space [4]. Indeed for nonzero helicity even position in space is not a well-defined physical observable; and the parity operator P cannot be defined within such a UIR of $\mathcal{P}$.

On the other hand, beginning with classical optics, the treatment of the polarization states of a plane electromagnetic wave, and its extension to the possible polarization states of a single photon, are both physically very well founded. In polarization optics per se, intensity preserving linear optical systems - rotators, birefringent media, quarter wave plates, half-wave plates,...- are effectively treated as realising various elements of an $SU(2)$ or $U(2)$ group acting on the two-dimensional complex linear space of polarization states [5]. However, from the point of view of the theory of elementary systems, the relevant UIR's of $\mathcal{P}$ lead only to a single polarization state - helicity ± 1 corresponding to right or left circular polarization respectively - and helicity is relativistically invariant. One must bring in the action by parity P in an essential manner to connect the states of opposite helicities and so create a two-dimensional space of polarization states, allow for the definitions of linear polarization, general elliptic polarization etc.

The purpose of this paper is to examine this complex of questions taking the relevant UIR's of $\mathcal{P}$ as a fundamental starting point. We wish to bring out the essential role of parity in this context, something not often emphasized,

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and develop the necessary operator machinery to deal with transformations acting solely on the polarization degree of freedom of a photon. Since one deals here ultimately with an irreducible representation of $\mathcal{P}$ extended by parity, in principle all (physically important) operators can be built up out of the generators of $\mathcal{P}$, finite transformations of $\mathcal{P}$ where necessary, and P. We show that there is an unavoidable momentum dependence in these constructions, including in the building up of generators of $SU(2)$ acting on the polarization states of a photon at fixed energy-momentum.

The contents of this paper are arranged as follows. Section 2 sets up basic notational conventions for dealing with UIR's of $\mathcal{P}$, and includes the statement of the homomorphism from $SL(2, C)$ to the homogeneous Lorentz group. The use of $SL(2, C)$ makes many later calculations much simpler than otherwise. The algebraic relations involving parity, the Casimir invariants for $\mathcal{P}$, and the structure of the positive energy time like UIR's with finite spin are briefly reviewed for the convenience of the reader and for later comparison. In particular the emergence of spin as a separately existing degree of freedom in these UIR's, independent of space variables, is emphasized. In Section 3 we take up the mass zero finite helicity UIR's of $\mathcal{P}$. Particular attention is paid to the nontrivial topological features that emerge here, as compared to the finite mass case. Alternative ways to pass from a standard energy-momentum to a general energy momentum, in a singularity-free manner, and the structure of Hilbert space basis vectors including their transition rules, are developed. Finally, the doubling of the Hilbert space to accommodate parity, a feature absent in the massive case, is described. Section 4 shows how one can construct an $SU(2)$ Lie algebra of operators, in a momentum dependent way, to act on the polarization states of a photon for each fixed energy momentum. Here again the important role of the parity operator P is seen. This construction too has to be done avoiding singularities which would naively occur due to the nontrivial topological features involved. What emerges is that there is no universal or global $SU(2)$ lying behind these momentum dependent constructions, and at the same time there is considerable freedom in the details of the constructions. Section 5 contains some concluding remarks.

II. NOTATIONS AND THE TIMELIKE UIR'S OF $\mathcal{P}$

We begin with some notational preliminaries. The Lorentzian metric will be chosen to be spacelike, with $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, $\mu, \nu = 0, 1, 2, 3$. The two-to-one homomorphism from $SL(2, C)$ to $SO(3, 1)$ is given as follows:

$$
\begin{aligned}
A \epsilon\ SL(2, C): \quad & A\sigma_\mu A^\dagger = \Lambda(A)^\nu{}_\mu\ \sigma_\nu\ , \\
& A\ p \cdot \sigma A^\dagger = (\Lambda(A)p) \cdot \sigma\ , \\
& p \cdot \sigma = p^0 \cdot 1 + \underline{p} \cdot \underline{\sigma}\ , \\
& \Lambda(A)\ \epsilon\ SO(3, 1)\ , \\
& \Lambda(A')\Lambda(A) = \Lambda(A'A) \cdot
\end{aligned}
\tag{2.1}
$$

Here $\sigma_0 = 1$ and $\underline{\sigma}$ are the usual Pauli matrices, and p^μ is any (real) four-vector. In any unitary representation (UR) or UIR of $\mathcal{P}$, we have ten hermitian generators $M_{\mu\nu} = -M_{\nu\mu}, P_\mu$ obeying the standard commutation relations

$$
\begin{aligned}
& [M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu})\ , \\
& [M_{\mu\nu}, P_\rho] = i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu)\ , \\
& [P_\mu, P_\nu] = 0 \cdot
\end{aligned}
\tag{2.2}
$$

The six components of $M_{\mu\nu}$ generate homogeneous Lorentz transformations, and the four P_μ generate space-time translations. In our work we will also have to deal directly with unitary operators $\overline{U}(A)$ representing finite elements of $SL(2, C)$. Using the split notation $M_{jk} = \epsilon_{jk\ell}J_\ell, M_{0j} = K_j, j, k, \ell = 1, 2, 3$, we have for any real three-vectors $\underline{\alpha}, \underline{v}$ the identifications:

$$
\begin{aligned}
& \overline{U}\left(e^{i\underline{\alpha} \cdot \underline{\sigma}/2}\right) = e^{i\underline{\alpha} \cdot \underline{J}}\ , \\
& \overline{U}\left(e^{-\underline{v} \cdot \underline{\sigma}/2}\right) = e^{i\underline{v} \cdot \underline{K}}\ ,
\end{aligned}
\tag{2.3}
$$

and the general transformation law for P^μ:

$$
\overline{U}(A)^{-1}P^\mu\overline{U}(A) = \Lambda(A)^\mu{}_\nu\ P^\nu\ \cdot
\tag{2.4}
$$

Thus if $|p, \dots>$ is an eigenstate of the energy momentum operators P^μ with eigenvalues p^μ, we have the general rule (upto possible phases and normalisation)

$$\overline{U}(A)|p,\ldots> \,=\, |p',\ldots> \ ,$$
$$p' = \Lambda(A)p \cdot \tag{2.5}$$

When the action by parity P is defined, it has the following effects:

$$P\overline{U}(A)P^{-1} = \overline{U}\left(A^{\dagger -1}\right) \ ,$$
$$P\underline{J}P^{-1} = \underline{J} \ ,$$
$$P\underline{K}P^{-1} = -\underline{K} \ ,$$
$$P(P^0,\underline{P})P^{-1} = (P^0, -\underline{P}) \cdot \tag{2.6}$$

For a numerical four-vector $p = (p^0, \underline{p})$, we shall always write $\tilde{p} = (p^0, -\underline{p})$. Then to accompany eqn. (2.5) we have, when P is defined,

$$P|p,\ldots\rangle = |\tilde{p},\ldots\rangle \ . \tag{2.7}$$

Given any UR of $\mathcal{P}$, the Pauli Lubanski pseudo vector W_μ is defined by

$$W_\mu = \frac{1}{2} \in_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \ ,$$
$$\in_{0123} \,=\, +1 \ , \tag{2.8}$$

and is orthogonal to P^μ. Then the two Casimir invariants for $\mathcal{P}$ are

$$\mathcal{C}_1 = -P^\mu P_\mu \ ,$$
$$\mathcal{C}_2 = W^\mu W_\mu \cdot \tag{2.9}$$

In a UIR of $\mathcal{P}$, both reduce to numbers.

The positive energy timelike UIR's may be labelled by the pair $\{m,s\}$, where $m > 0$ is the rest mass and $s = 0, 1/2, 1, \ldots$ the intrinsic spin. The Casimir invariants have the values:

$$\{m,s\} : \mathcal{C}_1 = m^2, \ \mathcal{C}_2 = m^2 s(s+1) \ \cdot \tag{2.10}$$

In this UIR, P^μ and W^μ are respectively positive timelike and spacelike four-vectors with, however, noncommuting components for W^μ (unless $s = 0$, when W_μ vanishes). The rest-frame energy-momentum, from which all others in this UIR can be obtained by suitable Lorentz transformations, will be written as $p^{(0)}$:

$$p^{(0)} = (m, 0, 0, 0) \ \cdot \tag{2.11}$$

The corresponding stability subgroup of $SL(2,C)$ is $SU(2)$:

$$Ap^{(0)} \cdot \sigma A^\dagger = p^{(0)} \cdot \sigma \Leftrightarrow A = a \ \epsilon \ SU(2) \ \cdot \tag{2.12}$$

An (ideal) basis of momentum eigenstates for the Hilbert space $\mathcal{H}(\{m,s\})$ carrying this UIR may be built up as follows, starting with the vectors $|p^{(0)}, s_3\rangle$ describing a spin s particle at rest [6]. These vectors are characterised by their behaviour under rest-frame rotations, ie., elements of $SU(2)$:

$$a \ \epsilon \ SU(2) : \overline{U}(a)\big|p^{(0)}, s_3\rangle = \sum_{s_3'} D^{(s)}_{s_3' s_3}(a)\big|p^{(0)}, s_3'\rangle \ , \tag{2.13}$$

where $D(a)$ is the unitary matrix representing $a \ \epsilon \ SU(2)$ in the $(2s+1)$-dimensional spin $s \quad UIR$ of $SU(2)$. Going on now to general energy-momentum we have:

$$p = (p^0, \underline{p}), p^0 = (m^2 + \underline{p}^2)^{1/2} \ :$$
$$|p, s_3\rangle = \overline{U}(\ell(p))|p^{(0)}, s_3\rangle, s_3 = s, s-1, \ldots, -s \ ,$$
$$\ell(p) = \frac{1}{[2m(m+p^0)]^{1/2}}(m + p \cdot \sigma)$$
$$= \left(p + p^{(0)}\right) \cdot \sigma / [2m(m+p^0)]^{1/2} \ ,$$
$$\ell\left(p^{(0)}\right) = 1 \ ;$$
$$\ell(p)p^{(0)} \cdot \sigma \ell(p)^\dagger = p \cdot \sigma \ ;$$
$$P^\mu|p, s_3\rangle = p^\mu|p, s_3\rangle \ ;$$
$$\langle p', s_3'|p, s_3\rangle = p^0 \cdot \delta_{s_3' s_3} \delta^{(3)}\left(\underline{p}' - \underline{p}\right) \ \cdot \tag{2.14}$$

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A general vector $|\psi\rangle \epsilon \mathcal{H}(\{m,s\})$ has a $(2s+1)$-component momentum space wave function $\psi_{s_3}(\underline{p})$ and squared norm given by

$$\langle p, s_3 | \psi \rangle = \psi_{s_3}(\underline{p}) \ ,$$

$$\langle \psi | \psi \rangle = \int \frac{d^3p}{p^0} \sum_{s_3} |\psi_{s_3}(\underline{p})|^2 \ . \tag{2.15}$$

The action of $\overline{U}(A)$ on $|p, s_3\rangle$ for general $A \epsilon SL(2,C)$ involves the Wigner rotation, an element of $SU(2)$ acting on the spin projection s_3:

$$\overline{U}(A)|p, s_3\rangle = \sum_{s_3'} D^{(s)}_{s_3' s_3}(a(p,A))|p', s_3'\rangle \ ,$$

$$p' = \Lambda(A)p \ ,$$

$$a(p, A) = \ell(p')^{-1} A \ \ell(p) \ \epsilon \ SU(2) \ . \tag{2.16}$$

In case $p = p^{(0)}$ and $A \epsilon SU(2)$, we have the simplifications $p' = p^{(0)}$, $a\left(p^{(0)}, A\right) = A$, so we recover eqn.(2.13).

We notice (as is well known) that the pure Lorentz transformation $\ell(p)$ given in eqn.(2.14) is globally well-defined and singularity free for all $\underline{p} \epsilon \mathcal{R}^3$. Related to this is the fact that in the UIR $\{m,s\}$ of $\mathcal{P}$, one can introduce well-defined hermitian position, momentum and spin operators $\underline{q}, \underline{p}, \underline{S}$ out of which all the generators $M_{\mu\nu}, P_\mu$ of $\mathcal{P}$ can be constructed. The nonvanishing fundamental or primitive commutation relations are:

$$[q_j, p_k] = i\delta_{jk}, \ [S_j, S_k] = i \ \epsilon_{jk\ell}S_\ell \tag{2.17}$$

In the momentum basis subject to the normalisation given in eqn.(2.14) we have

$$\underline{q} = i\left(\frac{\partial}{\partial \underline{p}} - \frac{1}{2}\frac{\underline{p}}{(p^0)^2}\right) \ . \tag{2.18}$$

Starting from the irreducible set $\underline{q}, \underline{p}, \underline{S}$ (where $\underline{S}$ generate the spin s UIR of $SU(2)$) we can reconstruct the generators of $\mathcal{P}$ via

$$P^\mu = \left(\left(m^2 + \underline{p}^2\right)^{1/2}, \ \underline{p}\right) \ ,$$

$$\underline{J} = \underline{q} \ \wedge \ \underline{p} + \underline{S} \ ,$$

$$\underline{K} = \frac{1}{2}\{p^0, \underline{q}\} + \frac{\underline{p} \wedge \underline{S}}{m + p^0} \ . \tag{2.19}$$

Conversely, $\underline{q}, \underline{p}$ and $\underline{S}$ can be recovered from $M_{\mu\nu}, P_\mu$. This is the so-called Shirokov-Foldy form for the generators of $\mathcal{P}$ in the UIR $\{m,s\}$ [7]. (In the massless finite helicity UIR's, however, no such clean separation of primitive dynamical variables and generators is possible). The action of parity, P, can be taken to be

$$P|p, s_3\rangle = \eta|\tilde{p}, s_3\rangle \ , \tag{2.20}$$

where $\eta = \pm 1$ is the intrinsic parity. No enlargement of the representation space $\mathcal{H}(\{m,s\})$ is needed, and we have consistency with eqns.(2.6.7).

III. THE LIGHTLIKE UIR'S OF $\mathcal{P}$ AND THE PARITY DOUBLING

Now we turn to the mass zero finite helicity UIR's of $\mathcal{P}$, to be denoted $\{0, \lambda\}$ with the helicity $\lambda = 0, \pm 1/2, \pm 1, \ldots$. In such a UIR, λ has a fixed value; for definiteness we assume it is nonzero and integral. For photons we need just $\lambda = \pm 1$. In the UIR $\{0, \lambda\}$ both Casimir operators $\mathcal{C}_1$ and $\mathcal{C}_2$ vanish, while the pseudovector W_μ becomes a multiple of P_μ:

$$\{0, \lambda\} \ : \ \ \mathcal{C}_1 = \mathcal{C}_2 = 0 \ ,$$

$$W_\mu = \lambda \ P_\mu \ . \tag{3.1}$$

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This explains why parity P cannot be defined within the space of a single UIR $\{0, \lambda\}$. For the present we will work with a single UIR with fixed λ, and at the end of this Section turn to the question of accommodating parity.

Towards setting up a basis of energy-momentum eigenfunctions for the Hilbert space $\mathcal{H}(\{0, \lambda\})$ carrying the UIR $\{0, \lambda\}$, analogous to eqns.(2.14) for $\mathcal{H}(\{m, s\})$, we begin by noting that the set $\sum$ of all positive lightlike energy-momentum four-vectors,

$$\Sigma = \left\{ p \,\epsilon\, \mathcal{R}^4 | p^\mu p_\mu = 0, \ p^0 > 0 \right\} \tag{3.2}$$

is topologically nontrivial, since it is essentially $\mathcal{R}^3 - \{\underline{0}\}$ [8]. It is therefore convenient to express Σ as the union of two overlapping open subsets Σ_N, Σ_S, each of which is topologically trivial. Using the light cone combinations $p_\pm = p^0 \pm p_3$, we define:

$$\Sigma_N = \{p\epsilon\Sigma | p_+ > 0\} \ ,$$
$$\Sigma_S = \{p\epsilon\Sigma | p_- > 0\} \ ,$$
$$\Sigma = \Sigma_N \cup \Sigma_S \ ;$$
$$\Sigma_N \cap \Sigma_S = \{p\epsilon\Sigma | p_\perp = (p_1, p_2) \neq 0\} \cdot \tag{3.3}$$

The subscripts N, S indicate that the North pole on S^2 is included in Σ_N, the South pole in Σ_S.

Now we need to choose a standard or fiducial energy-momentum four-vector $p^{(0)}$, to replace the choice (2.11) in the time like case. We take $p^{(0)}$ to be

$$p^{(0)} = (1, 0, 0, 1) \cdot \tag{3.4}$$

(No confusion is likely to arise in using the same symbol $p^{(0)}$ as before). Then we have:

$$\tilde{p}^{(0)} \ = \ (1, 0, 0, -1) \ ;$$
$$p^{(0)}\epsilon \ \Sigma_N, \ \not\epsilon \ \Sigma_S; \tilde{p}^{(0)} \ \epsilon \ \Sigma_S, \ \not\epsilon \ \Sigma_N \cdot \tag{3.5}$$

Indeed, the p's omitted from $\Sigma_N(\Sigma_S)$ are all positive multiples of $\tilde{p}^{(0)} \left(p^{(0)}\right)$. The stability subgroup of $p^{(0)}$ is an $E(2)$ subgroup in $SL(2, C)$:

$$A \ p^{(0)} \cdot \sigma A^\dagger = p^{(0)} \cdot \sigma \Leftrightarrow$$
$$A = h(\varphi, \alpha)\epsilon \ E(2) \subset SL(2, C) \ ,$$
$$h(\varphi, \alpha) = \begin{pmatrix} e^{i\varphi/2} & \alpha \\ 0 & e^{-i\varphi/2} \end{pmatrix} \ ,$$
$$0 \leq \varphi \leq 4\pi \ , \quad \alpha \ \epsilon \ \mathcal{C} \cdot \tag{3.6}$$

The topological nontriviality of Σ is the same as that of the coset space $SL(2, C)/E(2)$, since $\Sigma \simeq SL(2, C)/E(2)$.

In the space of the UIR $\{0, \lambda\}$ the fiducial energy momentum eigenvector $|p^{(0)}, \lambda\rangle$ is characterised by the fact that it provides a one-dimensional representation of $E(2)$:

$$\overline{U}(h(\varphi, \alpha))|p^{(0)}, \lambda\rangle = e^{i\lambda\varphi}|p^{(0)}, \lambda\rangle \ \cdot \tag{3.7}$$

In terms of the infinitesimal generators $\underline{J}$, $\underline{K}$ of rotations and pure Lorentz transformations, this means

$$J_3|p^{(0)}, \ \lambda\rangle = \lambda|p^{(0)}, \ \lambda\rangle \ ,$$
$$(J_1 + K_2)|p^{(0)}, \lambda\rangle = (J_2 - K_1)|p^{(0)}, \lambda\rangle = 0 \cdot \tag{3.8}$$

These eqns.(3.7,8) are the replacements for the earlier eqn.(2.13) in the timelike case. Now we need to find, for each $p \ \epsilon \ \Sigma$, an $SL(2, C)$ element whose associated Lorentz transformation will carry $p^{(0)}$ to p: this will enable us to set up other energy-momentum eigenvectors $|p, \lambda\rangle$, and so build up a basis for $\mathcal{H}(\{0, \ \lambda\})$, similar to eqn.(2.14). However in contrast to the timelike case this cannot be done in a globally smooth manner for all $p\epsilon \ \Sigma$ [8]. This again is a consequence of the nontrivial topology of $\Sigma \simeq SL(2, C)/E(2)$. The problem has to be handled separately over each of Σ_N, Σ_S. To prepare for this, we employ the usual spherical polar angles θ, ϕ on S^2 and define the unit vector $\underline{n}(\theta, \phi)$ and an element $a(\theta, \phi)\epsilon \ SU(2)$ as follows:

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$$0 \leq \theta \leq \pi , \quad 0 \leq \phi \leq 2\pi :$$
$$\underline{n}(\theta, \phi) = \underline{n}(2\pi - \theta, \pi + \phi) = -\underline{n}(\pi - \theta, \pi + \phi)$$
$$= (\sin\theta \cos\phi, \ \sin\theta \sin\phi, \ \cos\theta) ;$$
$$a(\theta, \phi) = a(-\theta, \pi + \phi) = \exp\left[\frac{i\theta}{2}(\sigma_1 \sin\phi - \sigma_2 \cos\phi)\right] \ \epsilon \ SU(2) .$$

$$(3.9)$$

We express a general p as $p^0(1, \underline{n}(\theta, \phi))$ and see that Σ_N, Σ_S correspond to $0 \leq \theta < \pi$, $0 < \theta \leq \pi$ respectively. Whereas $\underline{n}(\theta, \phi)$ is well-defined all over S^2, $a(\theta, \phi)$ is undefined at $\theta = \pi$ (south pole). For a general $\underline{n}\epsilon \ S^2$ we have

$$a(\theta, \phi)\underline{n} \cdot \underline{\sigma}a(\theta, \phi)^\dagger = \underline{n}' \cdot \underline{\sigma} ,$$
$$\underline{n}' = (\text{right handed rotation by angle } \theta \text{ about}(-\sin\phi, \cos\phi, 0))\underline{n} ,$$

$$(3.10)$$

so in particular we get the useful relation

$$a(\theta, \phi)\underline{n}(\theta', \phi) \cdot \underline{\sigma}a(\theta, \phi)^\dagger = \underline{n}(\theta' + \theta, \phi) \cdot \underline{\sigma} .$$

$$(3.11)$$

Now a possible solution to the problem of constructing Lorentz transformations connecting $p^{(0)}$ to all $p\epsilon \ \Sigma$ is given by using separate boost and rotation factors in a step-by-step manner:

$$p\epsilon\Sigma_N : \ell(p) = a(\theta, \phi) \exp\left(\frac{1}{2}\ln p^0 \cdot \sigma_3\right) ,$$
$$\ell(p)p^{(0)} \cdot \sigma\ell(p)^\dagger = p \cdot \sigma ; \tag{3.12a}$$
$$p\epsilon\Sigma_S : \ell'(p) = a(\theta - \pi, \phi) \exp\left(-\frac{1}{2}\ln p^0 \cdot \sigma_3\right) i\sigma_2 ,$$
$$\ell'(p)p^{(0)} \cdot \sigma\ell'(p)^\dagger = p \cdot \sigma \tag{3.12b}$$

(Once again, the use of the symbol $\ell(p)$ here should not cause any confusion with its use earlier in Section 2). In the structure of $\ell'(p)$, the purpose of the first factor $i \sigma_2$ is to switch $p^{(0)}$ to $\tilde{p}^{(0)}$, and then the rest follows easily. As is to be expected, in the overlap $\Sigma_N \cap \Sigma_S$, $\ell(p)$ and $\ell'(p)$ differ by an $E(2)$ element on the right:

$$p\epsilon\Sigma_N \cap \Sigma_S : \ \ell'(p) = \ell(p)h(2(\pi - \phi), \ 0) .$$

$$(3.13)$$

We also have the particular values

$$\ell\left(p^{(0)}\right) = 1 ,$$
$$\ell'\left(\tilde{p}^{(0)}\right) = i \ \sigma_2 .$$

$$(3.14)$$

With the aid of these definitions we can set up a basis of energy-momentum eigenvectors for $\mathcal{H}(\{0, \lambda\})$:

$$p\epsilon\Sigma_N : |p, \lambda\rangle = \overline{U}(\ell(p))|p^{(0)}, \lambda\rangle = \overline{U}(a(\theta, \phi))e^{-iK_3 \ln p^0} \cdot |p^{(0)}, \lambda\rangle ;$$
$$p\epsilon\Sigma_S : |p, \lambda\rangle' = \overline{U}(\ell'(p))|p^{(0)}, \lambda\rangle = \overline{U}(a(\theta - \pi, \phi)) \cdot e^{iK_3 \ln p^0} \cdot e^{i\pi J_2}|p^{(0)}, \lambda\rangle ;$$
$$p\epsilon\Sigma_N \cap \Sigma_S : |p, \lambda\rangle' = e^{-2i\lambda\phi}|p, \lambda\rangle ;$$
$$P^\mu(|p, \lambda\rangle \text{ or } |p, \lambda\rangle') = p^\mu(|p, \lambda\rangle \text{ or } |p, \lambda\rangle') .$$

$$(3.15)$$

The overlap or transition rule results from eqns.(3.7,13). These definitions may be supplemented by the inner products

$$\langle p', \lambda|p, \lambda\rangle = p^0\delta^{(3)}\left(\underline{p}' - \underline{p}\right) ,$$
$$'\langle p', \lambda|p, \lambda\rangle' = p^0\delta^{(3)}\left(\underline{p}' - \underline{p}\right) .$$

$$(3.16)$$

It is always implied that in $|p, \lambda\rangle(|p, \lambda\rangle')$ the argument p is restricted to $\Sigma_N(\Sigma_S)$. A general $|\psi\rangle\epsilon\mathcal{H}(\{0, \lambda\})$ has a single component momentum space wave function and squared norm given by

$$\psi(\underline{p}) = \langle p, \lambda | \psi \rangle \ ,$$

$$\langle \psi | \psi \rangle = \int \frac{d^3 p}{p^0} |\psi(\underline{p})|^2 \ . \tag{3.17}$$

Here since the part of Σ omitted in Σ_N is a set of measure zero as far as the integral is concerned, we have used only the basis kets $|p, \lambda\rangle$. Moreover, starting from eqn.(3.7) and following through the definitions (3.15), one checks that

$$J_3 |p^{(0)}, \lambda\rangle = \lambda |p^{(0)}, \lambda\rangle \ ;$$
$$\underline{J} \cdot \underline{p}/p^0 |p, \lambda\rangle = \lambda |p, \lambda\rangle \ ,$$
$$\underline{J} \cdot \underline{p}/p^0 |p, \lambda\rangle' = \lambda |p, \lambda\rangle' \ . \tag{3.18}$$

For a general $A \epsilon \ SL(2, C)$, the action of $\overline{U}(A)$ on $|p, \lambda\rangle$ or $|p, \lambda\rangle'$ can be computed in a manner similar to the steps leading to eqn.(2.16). If $p' = \wedge(A)p$ is in the overlap $\Sigma_N \cap \Sigma_S$, the result can be expressed as a "Wigner phase" times $|p', \lambda\rangle$ or equally well as another "Wigner phase" times $|p', \lambda\rangle'$; on the other hand, if p is a multiple of either $p^{(0)}$ or $\tilde{p}^{(0)}$, the result can be written in only one way. Since we do not need these results explicitly, we omit the details.

The choices for $\ell(p), \ell'(p)$ in eqns.(3.12) were based on simple step-by-step constructions to lead from $p^{(0)}$ to p. Alternative choices, $\tilde{\ell}(p)$ and $\tilde{\ell}'(p)$ say, more in the spirit of the expressions in (2.14) in the timelike case, are also available [9]:

$$p \epsilon \Sigma_N : \tilde{\ell}(p) = \frac{1}{\sqrt{2p_+}}(1 - \sigma_3 + \sigma \cdot p) = \frac{1}{\sqrt{2p_+}} \left(\tilde{p}^{(0)} + p \right) \cdot \sigma \ ;$$

$$p \epsilon \Sigma_S : \tilde{\ell}'(p) = \frac{1}{\sqrt{2p_-}}(1 + \sigma_3 - \sigma \cdot p)\sigma_1 = \frac{1}{\sqrt{2p_-}} \left(p^{(0)} - p \right) \cdot \sigma \sigma_1 \ ;$$

$$p \epsilon \Sigma_N \cap \Sigma_S : \tilde{\ell}'(p) = \tilde{\ell}(p) h(2(\pi - \phi), 2(1 - p_3)/|p_\perp|) \ . \tag{3.19}$$

Again, these choices are $E(2)$-related to the earlier ones:

$$\tilde{\ell}(p) = \ell(p) h \left(0, (1 + 1/p^0) \cdot \tan \theta/2 \cdot e^{-i\phi} \right) \ ,$$
$$\tilde{\ell}'(p) = \ell'(p) h \left(0, (1 - 1/p^0) \cdot \cot \theta/2 \cdot e^{i\phi} \right) \tag{3.20}$$

Since $0 \leq \theta < \pi$ in the first case and $0 < \theta \leq \pi$ in the second, these expressions are well-defined in their respective domains. Here the angle arguments in $h(\ldots, \ldots)$ turn out to vanish, hence we get <u>exactly the same</u> energy-momentum eigenvectors as before:

$$\overline{U}(\tilde{\ell}(p)) |p^{(0)}, \lambda\rangle = \overline{U}(\ell(p)) |p^{(0)}, \lambda\rangle = |p, \lambda\rangle \ ,$$
$$\overline{U}(\tilde{\ell}'(p)) |p^{(0)}, \lambda\rangle = \overline{U}(\ell'(p)) |p^{(0)}, \lambda\rangle = |p, \lambda\rangle' \ . \tag{3.21}$$

Having exhibited these alternative choices, we now revert to the earlier ones.

So far the analysis has been limited to the single UIR $\{0, \lambda\}$ of $\mathcal{P}$ for a fixed value of λ. To accommodate parity P we have to adjoin the inequivalent UIR $\{0, -\lambda\}$ and work in the doubled Hilbert space $\mathcal{H}(\{0, \lambda\}) \oplus \mathcal{H}(\{0, -\lambda\})$. This entails bringing in additional basis vectors $|p^{(0)}, -\lambda\rangle$, $|p, -\lambda\rangle$ and $|p, -\lambda\rangle'$ for $\mathcal{H}(\{0, -\lambda\})$ exactly as in eqn.(3.15). In this extended space we fix the action of P by assuming $P^2 = 1$ and setting:

$$P|p^{(0)}, \pm\lambda\rangle = |\tilde{p}^{(0)}, \mp\lambda\rangle' \ . \tag{3.22}$$

We need not include an intrinsic parity factor η here as was done in eqn.(2.20) since in any case P switches vectors in the two subspaces $\mathcal{H}(\{0, \pm\lambda\})$. One can now follow through the consequences of eqn.(3.22) by exploiting the basic relations (2.6) and the constructions (3.15) in each subspace to obtain:

$$P|p, \pm\lambda\rangle = |\tilde{p}, \mp\lambda\rangle' \ ,$$
$$P|p, \pm\lambda\rangle' = |\tilde{p}, \mp\lambda\rangle \ . \tag{3.23}$$

To these we adjoin the helicity statements in the extended space:

$$\underline{J} \cdot \underline{p}/p^0 (|p, \pm\lambda\rangle \text{ or } |p, \pm\lambda\rangle') = \pm\lambda(|p, \pm\lambda\rangle \text{ or } |p, \pm\lambda\rangle') \ . \tag{3.24}$$

For photons, we take $\lambda = 1$: then $|p, \pm1\rangle$, $|p, \pm1\rangle'$ correspond respectively to right and left circular polarizations.

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IV. $SU(2)$ GENERATORS ON THE POLARIZATION SPACE

For fixed $p\epsilon\ \Sigma$, the two polarization states $|p,\pm\lambda\rangle$ (or $|p,\pm\lambda\rangle'$) form a basis for a two-dimensional polarization space. The existence of a group of $SU(2)$ transformations acting on this space is obvious. Our aim is to see how to construct the generators of this $SU(2)$ out of the generators of $\mathcal{P}$ and parity P. From eqn.(3.24) the helicity operator is already diagonal in this space and acts like the third Pauli matrix σ_3. We need to build up analogues to σ_1 and σ_2.

Now we notice that parity P switches helicity $\pm\lambda$ to $\mp\lambda$, but at the same time changes $p = (p^0, \underline{p})$ to $\tilde{p} = (p^0, -\underline{p})$. We must therefore supplement action by P with a spatial rotation by amount π, about some axis perpendicular to $\underline{p}$, which will bring $-\underline{p}$ back to $\underline{p}$ but leave helicity unaltered. Here we face the same topological problem which has appeared earlier in another guise - it is impossible to find a $\underline{p}$-dependent (unit) vector perpendicular to $\underline{p}$, for all $p\epsilon\ S^2$, in a singularity - free manner. However such nonsingular choices are available on Σ_N, Σ_S separately. We define $\underline{e}(p), \underline{e}'(p)$ as follows:

$$p\epsilon\ \Sigma_N : \underline{e}(p) = \underline{n}\left(\frac{\pi}{2},0\right) - 2\sin\frac{\theta}{2}\cos\phi\ \underline{n}\left(\frac{\theta}{2},\phi\right)\ ,$$

$$|\underline{e}(p)| = 1, \quad \underline{p}\cdot\underline{e}(p) = 0\ ; \tag{4.1a}$$

$$p\epsilon\ \Sigma_S : \underline{e}'(p) = \underline{n}(\pi/2,0) - 2\cos\frac{\theta}{2}\cos\phi\ \underline{n}\left(\frac{1}{2}(\pi+\theta),\phi\right)\ ,$$

$$|\underline{e}'(p)| = 1, \quad \underline{p}\cdot\underline{e}'(p) = 0\ \cdot \tag{4.1b}$$

These are by no means unique but suffice for our purposes; each is also unambiguously defined in the corresponding domain. Starting now with $p\epsilon\ \Sigma_N$ we develop:

$$e^{i\pi\underline{e}(p)\cdot\underline{J}}P|p,\pm\lambda\rangle = e^{i\pi\underline{e}(p)\cdot\underline{J}}|\tilde{p},\mp\lambda\rangle'$$
$$= e^{i\pi\underline{e}(p)\cdot\underline{J}}\overline{U}(a(-\theta,\pi+\phi))\cdot e^{iK_3\ln p^0}\cdot e^{i\pi J_2}|p^{(0)},\mp\lambda\rangle$$
$$= e^{i\pi\underline{e}(p)\cdot\underline{J}}\overline{U}(a(\theta,\phi))e^{i\pi J_2}\cdot e^{-iK_3\ln p^0}\cdot|p^{(0)},\mp\lambda\rangle$$
$$= \overline{U}(a(\theta,\phi))\cdot\overline{U}\left(a(\theta,\phi)^{-1}\cdot e^{\frac{i\pi}{2}\underline{e}(p)\cdot\underline{\sigma}}\cdot a(\theta,\phi)\cdot e^{\frac{i\pi}{2}\sigma_2}\right)\cdot e^{-iK_3\ln p^0}|p^{(0)},\mp\lambda\rangle$$

$$\tag{4.2}$$

The $SU(2)$ element appearing here can be simplified after some algebra and use of eqn.(3.11):

$$a(\theta,\phi)^{-1}\cdot e^{\frac{i\pi}{2}\underline{e}(p)\cdot\underline{\sigma}}\cdot a(\theta,\phi)\cdot e^{\frac{i\pi}{2}\sigma_2}$$
$$= a(\theta,\phi)^{-1}i\underline{e}(p)\cdot\underline{\sigma}a(\theta,\phi)i\sigma_2$$
$$= -i\sigma_3 = e^{-\frac{i\pi}{2}\sigma_3}\ . \tag{4.3}$$

Using this in eqn.(4.2) we get:

$$e^{i\pi\underline{e}(p)\cdot\underline{J}}P|p,\pm\lambda\rangle = \overline{U}(a(\theta,\phi))\cdot e^{-i\pi J_3 - iK_3\ln p^0}\cdot|p^{(0)},\mp\lambda\rangle$$
$$= e^{i\pi\lambda}|p,\mp\lambda\rangle\ \cdot \tag{4.4}$$

By analogous calculations for $p\epsilon\ \Sigma_S$ we again find:

$$e^{i\pi\underline{e}'(p)\cdot\underline{J}}P|p,\pm\lambda\rangle' = e^{i\pi\lambda}|p,\mp\lambda\rangle'\ \cdot \tag{4.5}$$

Thus these operator expressions act essentially like the first Pauli matrix σ_1 in the polarization space.

For photons we set $\lambda = 1$. At each fixed p, we may then make the following identifications:

$$\underline{J}\cdot\underline{p}/p_0\ \longrightarrow\ \sigma_3\ ,$$
$$-e^{i\pi\underline{e}(p)\cdot\underline{J}}P\ \longrightarrow\ \sigma_1\ ,$$
$$i\frac{\underline{J}\cdot\underline{p}}{p^0}\ e^{i\pi\underline{e}(p)\cdot\underline{J}}P\ \longrightarrow\ \sigma_2 \tag{4.6}$$

Here for definiteness we assumed $p\epsilon\ \Sigma_N$. The meaning is that we are working in a basis of circular polarization states, and in that basis the Hilbert space operators standing on the left reduce in their actions to the Pauli matrices on

the right. Apart from factors of $\frac{1}{2}$, these then are the generators of the group of $SU(2)$ transformations familiar in polarization optics.

In the correspondence (4.6) only $\underline{J}$ and P are to be treated as Hilbert space operators while p^μ are c-numbers. One may wonder whether the latter could be replaced by the operators P^μ, and whether one would then somehow obtain an $SU(2)$ algebra not tied down to basis states $|p,\pm\lambda\rangle$ for specific p^μ. This however does not work out due to operator ordering problems. One is obliged to $\underline{first}$ pick some numerical p^μ, and then use the correspondence (4.6) $\underline{only}$ for action on the states $|p,\pm\lambda\rangle$, $|p,\pm\lambda\rangle'$.

V. CONCLUDING REMARKS

In this paper we have focussed on the operator aspects of the description of photon polarization states, taking as primary inputs the concerned UIR's of $\mathcal{P}$, the operators available in such representations, and the parity operation. This is in the spirit of the definition of elementary systems in relativistic quantum mechanics. We have highlighted the many novel features that arise in the treatment of massless particles as compared to massive ones, which can all be traced back to the nontrivial topology of the coset space $SL(2,C)/E(2)$, or of the positive light cone with tip removed. We have emphasized the crucial role played by the parity operation in being able to create a two-dimensional polarization space, and in the construction of operators realising the $SU(2)$ Lie algebra on this space, at each fixed energy-momentum. In a general sense, we can say that while for electrons the operator description of spin precedes the description in terms of spin states, for photons it is usually the description of various polarization states that is physically immediate. We have tried here to supplement this by an operator description in as straightforward a manner as possible.

Our handling of the topological features involved, and avoidance of singularities in expressions, leaves considerable freedom in the choices of coset representatives $\ell(p)$, $\ell'(p)$, fields of vectors $\underline{e}(p)$, $\underline{e}'(p)$ perpendicular to $\underline{p}$, etc. What must be clear is that there is an essential momentum dependence in these constructions, which cannot be eliminated. For each $p\epsilon\ \Sigma$, we do have an $SU(2)$ group acting on the corresponding polarization space; however these various $SU(2)$'s are not representatives of any single natural globally defined $SU(2)$ at all. In particular there is no relation to the geometrical group of rigid rotations in physical space, as there is in the definition of spin for massive particles.

This helps us answer a question which is not as naive as one may at first imagine. Suppose we have two photons with distinct energy momenta p, p' respectively. Can one treat their separate two-dimensional polarization state spaces as though they were like spin half particle states, couple the two photon polarizations to "total spins" 1 or 0, and handle them just as one would handle the spins of two electrons? The answer is that this is not physically well founded, since the $SU(2)$ groups involved are momentum dependent; there is little meaning to the action of "one and the same $SU(2)$ element" on both photon polarizations on account of the conventions and freedoms involved in identifying the $SU(2)$ generators for each p.

[1] E.P.Wigner, Ann. Math. <u>40</u>, 149(1939); see also R.F.Streater and A.S.Wightman, "PCT, Spin and Statistics, and all that", W.A. Benjamin, Inc., New York (1964), Chapter 1.

[2] These have been discussed in, for example, N.Mukunda, Ann. Phys. (N.Y.) <u>61</u>, 329 (1970) and other references cited therein.

[3] T.D. Newton and E.P. Wigner, Rev. Mod. Phys.<u>21</u>, 400 (1949).

[4] See, for instance, ref. (2).

[5] For general introductions to polarization optics one may see R.M.A. Azzam and N.M. Bashara, "Ellipsometry and Polarized light", North-Holland, Amsterdam (1977); W.Swindell, ed., "Polarized Light", Dowden, Hutchinson and Ross, Stroudsburg, PA (1975).

[6] See, for instance, A.S.Wightman in "Relations de dispersion et particules elementaires", Les Houches, (Paris: Hermann), 1960.

[7] Iu. M. Shirokov, Dokl. Adad. Nauk. SSSR <u>94</u>, 857 (1954); <u>97</u>, 737 (1954); L.L. Foldy, Phys. Rev. <u>102</u>, 568 (1956).

[8] Some aspects of this situation have been examined in B.Dutta and N.Mukunda, Pramana. J.Phys. <u>29</u>, 437 (1987).

[9] These have been developed in refs. (2) and (8), though the notations are slightly different.