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Universal Correlations in Random Matrices: Quantum Chaos, the $1/r^2$ Integrable Model, and Quantum Gravity

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Abstract

Random matrix theory (RMT) provides a common mathematical formulation of distinct physical questions in three different areas: quantum chaos, the 1-d integrable model with the $1/r^2$ interaction (the Calogero-Sutherland-Moser system), and 2-d quantum gravity. We review the connection of RMT with these areas. We also discuss the method of loop equations for determining correlation functions in RMT, and smoothed global eigenvalue correlators in the 2-matrix model for gaussian orthogonal, unitary and symplectic ensembles.

1. Introduction

In the last few years it has been realized that matrix models form a bridge between two apparently distinct subjects: the Calogero-Sutherland-Moser integrable model of particles in 1-d interacting via the $1/r^2$ potential (CSM model), and quantum chaos. Correlations of the eigenvalue density in quantum chaotic systems and spacetime dependent particle density correlators in the CSM model both map onto the same eigenvalue correlator in a random matrix model. Furthermore very closely related correlation functions in the matrix model appear in the context of QCD, string theory and 2-d quantum gravity - the correlation functions of 'loop operators', which have a geometric interpretation in terms of surfaces. Thus different physical questions in these three vastly different areas reduce to the same mathematical question formulated in terms of matrix models. In this review we discuss these questions and their common mathematical formulation. We also review the method of loop equations, a technique for calculating these correlators that originated in the QCD/string theory context, and its application to obtain correlation functions in the 2-matrix model that are relevant for quantum chaos and the CSM model.

In section 2 an introduction to RMT is given. We collect some recent results for correlation functions in 1- and 2-matrix models that are relevant to the sequel. Sections 3 and 4 describe the application of RMT in the field of quantum chaos and for obtaining the static correlators of the CSM model, respectively. Section 5 discusses a deeper relationship between quantum chaos and the CSM model, that is, the equivalence of parametric correlations in quantum chaos and time dependent correlations in the CSM model via a mapping onto a 2-matrix model. Two-dimensional quantum gravity and its relationship with matrix models is described in section 6. Section 7 discusses loop equations for determining correlation functions in matrix models and derives some of the results for eigenvalue correlation functions mentioned in section 2. Section 8 contains a brief summary.

2. Random Matrix Models

Consider the partition function

$$Z = \int dA \;\; e^{-S(A)}, \tag{1}$$

where A is an $N \times N$ hermitian matrix, dA is the standard $U(N)$ invariant measure for hermitian matrices, and $S(A)$ is a $U(N)$ invariant action of the form

$$S(A) = N \mathrm{Tr} V(A), \qquad V(A) = \sum_{n=1}^{\infty} \frac{1}{n} g_n A^n. \tag{2}$$

This partition function defines the normalization of a probability distribution for the matrix A, namely $P(A) = (1/Z)e^{-S(A)}$, in which the expectation value of any function $f(A)$ is given by $\langle f \rangle = \int dA\, P(A)\, f(A)$. Typical examples of functions whose expectation values and correlation functions will be of interest are the Green's function or 'loop operator' $\hat{W}(z) \equiv (1/N)\mathrm{Tr}(z-A)^{-1}$, and the eigenvalue density operator $\hat{\rho}(x) \equiv (1/N)\mathrm{Tr}\, \delta(x-A)$. One is mostly interested in this matrix model in the large N limit.

The $U(N)$ symmetry of the above probability distribution (namely symmetry under the transformation $A \to UAU^\dagger$, with U any $N \times N$ unitary matrix) can be used to project it down to a joint probability distribution of the eigenvalues $\lambda_1, ..., \lambda_N$ of A. This probability distribution is given by

$$P(\lambda_1, ..., \lambda_N) = N_0 \prod_{i<j} |\lambda_i - \lambda_j|^\beta \, e^{-N \sum_{i=1}^N V(\lambda_i)}, \tag{3}$$

where $\beta = 2$. The projection is achieved by using the symmetry to diagonalize A : $A = UDU^\dagger$, $D = \mathrm{diag}(\lambda_1, ..., \lambda_N)$. The 'angular variables' U decouple from the eigenvalue integration because of the $U(N)$ symmetry and their integration provides merely an overall factor absorbed in the normalization constant N_0. The Van der Monde determinant factor $\prod_{i<j} |\lambda_i - \lambda_j|^\beta$ is simply the jacobian of the change of variables from A to D and U.

As a consequence of this projection the expectation value of any 'observable' (any $U(N)$ invariant function $f(A)$; in particular $\hat{W}(z)$ and $\hat{\rho}(x)$ above) is given by

$$\langle f \rangle = N_0 \int d\lambda_1 ... d\lambda_N \, e^{-E(\lambda_1, ..., \lambda_N)} \, f(D), \tag{4}$$

with

$$E(\lambda_1, ..., \lambda_N) = N \sum_{i=1}^N V(\lambda_i) - \beta \sum_{i<j} \log |\lambda_i - \lambda_j|. \tag{5}$$

This provides a very useful physical picture of the matrix model: the eigenvalue λ_i can be thought of as the position coordinate of the ith particle in a 1-dimensional gas of N particles (the Dyson gas [1]) in which all particles experience the same external potential $NV(x)$, and repel each other logarithmically, giving rise to a (static) energy function E. $\langle f \rangle$ is merely the Boltzmann weighted average of the function $f(D)$ of the position variables.

If A belonged to the ensemble of $N \times N$ real symmetric matrices rather than hermitian matrices (that is to say, for real symmetric A, $P(A)$ is the same as that given above, but zero otherwise), equations (3-5) remain unchanged, except that now $\beta = 1$. Similarly, if A belonged to the ensemble of $N \times N$ real self-dual quaternions, then $\beta = 4$. The three ensembles corresponding to $\beta = 1, 2, 4$ are referred to as the orthogonal, unitary and symplectic ensembles respectively, corresponding to the symmetry group in each case For details, see, e.g., [3]. β, which determines the strength of the eigenvalue repulsion, simply measures the unlikelyhood of finding a matrix in the ensemble with a pair of equal eigenvalues.

The probability distribution (3) depends upon the parameters g_n of the potential V. The simplest cases are the gaussian ensembles (where $g_2 \equiv \mu > 0, \quad g_1 = g_n = 0$ for $n \geq 3$) for which the one-point function (or eigenvalue density) $\rho(x) \equiv \langle \hat{\rho}(x) \rangle$ obeys the semicircle law [4]

$$\rho(x) = \frac{2}{\pi a^2}\sqrt{a^2 - x^2}, \quad |x| \leq a, \qquad \text{and} \qquad \rho(x) = 0, \quad |x| > a, \tag{6}$$

with $a^2 = 2\beta/\mu$. In general, when $V(x)$ has a single minimum, $\rho(x)$ is nonvanishing on a single segment $[a, b]$ of the real line and has the form $\rho(x) = P(x)\sqrt{(x-a)(b-x)}$, where $P(x)$ is a polynomial in x whose degree depends upon the degree of V. Both a and b as well as the coefficients of $P(x)$ depend upon the g_n, thereby making $\rho(x)$ nonuniversal.

However, it turns out that higher correlation functions of the eigenvalue density and the loop operator exhibit a remarkable universality (independence of the choice of g_n) in certain domains. For example, the exact two point density-density correlator $\langle \hat{\rho}(x)\hat{\rho}(y) \rangle$ for an arbitrary even polynomial potential V for the unitary ensemble ($\beta = 2$), computed in [5], though not universal in general, is universal in two physically interesting domains. The first domain is the 'local' region where the interval $[x, y]$ contains a finite number ($O(N^0)$) of eigenvalues and is located a finite distance from the endpoints. Then, the result, for arbitrary V, is

$$\frac{\langle \hat{\rho}(x)\hat{\rho}(y) \rangle}{\rho(x)\rho(y)} = 1 - \frac{\sin^2 \delta}{\delta^2}, \qquad \delta \equiv \pi N(x-y)\rho(\frac{x+y}{2}), \tag{7}$$

which is universal when expressed in terms of variables that are rescaled by the factor $\rho(\frac{x+y}{2})$ that determines the local mean level spacing. This is a 'fine grained' correlator that oscillates over distances of the order of mean eigenvalue spacing, and has a nonsingular limit as $x - y \to 0$. (7) was derived originally for the circular unitary ensemble (CUE: ensemble of unitary matrices, whose eigenvalues lie on a circle) in [6] and for the gaussian unitary ensemble (GUE: ensemble of hermitian matrices with a quadratic V) in [7]. Its universality for the unitary ensemble ($\beta = 2$) was extended to all V's that correspond to the classical orthogonal polynomials in [8] and to all even polynomial V's in [5]. Expressions analogous to (7) exist for the other two ensembles ($\beta = 1, 4$) [9,3]. The universality of all these local fine grained correlators with respect to arbitrary potentials has been shown in [10]. For other extensions of the universality of (7) and the higher point local correlators see [11].

While the universality of local correlators has long been expected on empirical grounds (see next section), it is somewhat more of a surprise that even certain 'global' correlators are universal. This is the domain where $[x, y]$ contains a large number of eigenvalues including upto $O(N)$, but the correlator is 'smoothed' by independently averaging over x and y over intervals much greater than the mean eigenvalue spacing. In this domain the result for the connected two point function in the large N limit is

$$\langle \hat{\rho}(x) \hat{\rho}(y) \rangle_c = -\frac{1}{\pi^2 N^2 \beta} \frac{1}{(x-y)^2} \frac{a^2 - xy}{[(a^2 - x^2)(a^2 - y^2)]^{1/2}}. \tag{8}$$

This is universal in that it depends on the g_n only through the endpoint of the spectrum a. (For even potentials the endpoints are $\pm a$. For non-even potentials, the result is still (8), but now a represents half the width of the support, and x, y on the r.h.s. of (8) are shifted by the midpoint of the support.) This is a 'global' correlator in the sense that x and y could be anywhere in the interval $[-a, a]$, close to the endpoints or far away (provided only that $x - y$ contains a large enough number of eigenvalues for 'smoothing'). (8) was derived originally for the gaussian ensembles in ref. [12]. Its universality with respect to arbitrary potentials for $\beta = 2$ follows from the results of ref. [13] obtained for the two-point function of loop operators. In [5] (8) was shown to be the smoothed version of the exact two-point density-density correlator for arbitrary even polynomial potentials in the unitary ensemble. (8) has been proven to be universal for arbitrary potentials V which give rise to an eigenvalue density with support in a single segment for all three ensembles [14]. For these potentials the three and higher point smoothed correlators vanish to leading order in $1/N$ [5,14,15] and are in general nonuniversal in higher orders [16].

There is yet another type of universality of smoothed correlators at multicritical points [17] that arises in the double scaling limit [18] studied in the context of 2-d gravity and noncritical string theory [19]. Here only the region close to the endpoint of the support of $\rho(x)$ matters, and correlations depend on the g_n only through the number of zeroes of the polynomial $P(x)$ that coincide with the endpoint a (the level of multicriticality) [20]. Further, there is evidence of a universality of finegrained correlators at the spectrum edge even in the absence of a double scaling limit [21].

The 2-matrix model is defined by the partition function

$$Z = \int dA \ \ dB \ \ e^{-S} \tag{9}$$

where A and B are $N \times N$ matrices drawn from any of the above mentioned ensembles, and we will restrict the discussion to S of the type $S = N\mathrm{Tr}\ (V_1(A) + V_2(B) - cAB)$. When A and B are drawn from different ensembles, e.g., A is drawn from the gaussian orthogonal or symplectic ensemble ($\beta = 1$ or 4) and B from the gaussian unitary ensemble ($\beta = 2$), the eigenvalue correlations (relevant for parametrizing time reversal symmetry breaking in chaotic systems) have been computed in [22]. Here we will be primarily interested in the case where both A and B are drawn from the same ensemble.

The smoothed, global expression for the 2-point function $\rho_{AB}(x,y) \equiv \langle \hat{\rho}_A(x)\hat{\rho}_B(y)\rangle_c \equiv \langle \hat{\rho}_A(x)\hat{\rho}_B(y)\rangle - \langle \hat{\rho}_A(x)\rangle\langle \hat{\rho}_B(y)\rangle$, (where $\hat{\rho}_A(x) \equiv \frac{1}{N}\mathrm{Tr}\ \delta(x-A)$ and similarly $\hat{\rho}_B(x)$) is, in the large N limit,

$$\begin{aligned}\rho_{AB}(x,y) = &-\frac{1}{2\pi^2 N^2}\frac{1}{\beta a^2}\frac{1}{\cos\theta\cos\alpha} \\ &[\frac{1+\mathrm{ch}u\cos(\theta+\alpha)}{(\mathrm{ch}u+\cos(\theta+\alpha))^2} + \frac{1-\mathrm{ch}u\cos(\theta-\alpha)}{(\mathrm{ch}u-\cos(\theta-\alpha))^2}].\end{aligned} \tag{10}$$

Here $V_1(x) = V_2(x) = (1/2)\mu x^2$, $u \equiv \ln(\frac{\mu}{c})$, $x \equiv a \sin\theta$, $y \equiv a \sin\alpha$. The one point function $\rho(x) \equiv \langle\hat{\rho}_A(x)\rangle = \langle\hat{\rho}_B(x)\rangle$ is still given by the semicircle law except that the endpoint of its support is now $a = (\frac{2\beta\mu}{\mu^2 - c^2})^{1/2}$, and (10) is valid for $|x|, |y| < a$. (10) was computed in [23] for $\beta = 2$ using a diagrammatic technique and in [24] for $\beta = 1, 2, 4$ using the method of loop equations [25].

The local fine-grained correlators in the 2-matrix model for the gaussian ensembles were computed in [26] ($\beta = 1, 2$) and [27] ($\beta = 4$) using the the supersymmetry technique [28]. For $\beta = 2$, global fine-grained correlations are known [29] (see also [30]) using the method of orthogonal polynomials [31]. The local correlators near the centre of the eigenvalue distribution have been shown to be universal (independent of the form of the potential) in [10] (see [32] for numerical evidence and [33] for a physical argument). The smoothed global correlator (10) although derived for a quadratic potential reduces, in two different limits, to expressions that are universal. The first is when x, y are restricted to the region near the centre of the cut; then the expression (see section 5) is the local smoothed correlator [34] [35] which is universal. The second is when μ and c are tuned such that the probability distribution e^{-S} has vanishing weight over configurations in which $A \neq B$, in which case (10) reduces to the universal global 1-matrix result (8). However, the universality of (10) in general remains unproven.

Correlators in the double scaling limit of the $\beta = 2$ 2-matrix model with non-quadratic potentials have been studied in [36,37]

3. Quantum Chaos

The subject of quantum chaos is at present defined as the study of the quantum properties of a system that is classically chaotic. Consider a classically chaotic system described by some hamiltonian $H = H(p_1, q_1, p_2, q_2, ...)$. A signature of classical chaos is that trajectories are extremely sensitive to initial conditions. The question is, what can one say about the quantum system? Here we restrict ourselves to questions regarding the spectrum of the quantum hamiltonian. In general, the analytic calculation of the energy eigenvalues is a hopeless task. However, there is a substantial amount of data on the discrete eigenvalues E_i obtained numerically for model systems and experimentally for real systems. From this data one obtains various statistical properties of the spectrum. For example, one can in principle determine the correlation functions of the density of eigenvalues $\langle \rho(E_1)\rho(E_2)...\rho(E_n)\rangle$, where $\rho(E) \equiv \sum_i \delta(E-E_i)$. ($\langle\ \rangle$ is an average over many energy levels of the hamiltonian in question keeping the differences $E_i - E_j$ fixed.) In practice one computes various statistical measures from the data that are related to these correlation functions (for reviews see [38,39,3,40]), for example, the distribution of level spacings, (i.e., the probability for the spacing between consecutive energy levels to lie in a specified range). The result of these empirical analyses is that while the one-point function $\langle \rho(E)\rangle$ varies from system to system, the local two and higher point correlation functions as well as the level spacing distribution, for sufficiently high energy levels not related by any symmetry, exhibit a universal (i.e., system independent) behaviour for a wide variety of systems.

This observed universality of local eigenvalue statistics suggests that there might be a probability distribution of energy eigenvalues with universal features characterizing generic chaotic systems. It was Wigner's insight [4], in the context of nuclear physics, that the statistical spectra of complex nuclei should be obtainable by taking the extreme view that one knows essentially nothing about the hamiltonian, that is, the hamiltonian is drawn completely at random from the space of all possible hamiltonians. This led him to propose gaussian random matrix ensembles for theoretically calculating the eigenvalue statistics, wherein the random matrix A of the 1-matrix model in the previous section corresponds to a hamiltonian, and correlation functions of the 'observable' $\hat{\rho}(x)$ in the matrix ensembles are to be compared with empirically obtained correlators of the eigenvalue density $\rho(E)$ (x is identified with E upto a rescaling to be discussed later). Dyson [1] showed that the three different ensembles in random matrix theory should correspond to hamiltonians in three different universality classes identified by symmetry: $\beta = 2$ corresponds to hamiltonians with no time reversal invariance; if H has time reversal invariance, then $\beta = 1$, except if the system has no rotational invariance and has odd spin, in which case $\beta = 4$. It was proposed that the same statistical hypothesis should apply to complex atoms [41] and to electrons in disordered cavities (mesoscopic systems) [42]. Later, it was suggested [43] and numerically observed [44] that spectra of classically chaotic systems obey level repulsion as in RMT, and a proposal was made in [45] that that local correlations in RMT should provide the eigenvalue statistics for any hamiltonian system in the strongly chaotic regime.

These proposals have been tested and verified in an amazing diversity of systems. The list includes the empirical spectra of complex nuclei [46], atoms [41,47], molecules [48], and microwave cavities [49], the numerically obtained spectra of chaotic billiards [45] and other chaotic systems [50], and analytic calculations from microscopic models of electrons in weakly disordered mesoscopic systems [28]. The empirically observed universality mimics the universality mentioned for RMT in section 2, in that the eigenvalue density is nonuniversal but higher point local correlators and the spacing distribution is universal. For systems in which one can tune parameters in the hamiltonian to go from the classically integrable to the chaotic regime, agreement with RMT is seen in the strongly chaotic regime, and the statistics departs from RMT in the integrable or near integrable domain.

Recently, the smoothed global correlators have also found applications. An application of the universality of (8) is an argument [14] for the universality of conductance fluctuations of weakly disordered mesoscopic conductors using the transfer matrix formalism [51]. Other examples include possible application to the spectrum of the Dirac operator in strongly coupled QCD [52] and to the quantum Hall effect.

It seems that while being classically chaotic is usually a good indication of a system being 'generic' enough for Wigner's statistical hypothesis to apply, there are some exceptions to this rule. One class of exceptions are conductors with strong disorder. Classically, these systems represent chaotic billiards since an electron can bounce repeatedly from randomly placed impurities. In practice their spectrum disagrees with random matrix theory. In this case the departure can be explained in terms of a new quantum phenomenon, localization, that arises due to strong disorder, which disallows quantum states that are analogues of the classical ballistic chaotic trajectories [53]. Other exceptions are discussed in [54]. Therefore, until a better understanding of the precise domain of applicability of RMT is reached (see [55] for efforts using semiclassical methods [56]), the agreement of energy level statistics of classically chaotic systems with random matrix theory remains a fairly ubiquitous, but essentially empirical, fact.

4. The $1/r^2$ Integrable Model: Static Correlators

Consider the system of N fermions in one dimension with the hamiltonian (the Calogero-Sutherland-Moser (CSM) system [57–59])

$$H = \sum_i^N (p_i^2 + \omega^2 x_i^2) + \frac{1}{2}\beta(\beta - 2) \sum_{i<j} \frac{1}{(x_i - x_j)^2}. \tag{11}$$

The particles are in a harmonic potential and interact with each other via the $1/r^2$ potential with a strength determined by β ($\beta = 2$ corresponds to free fermions). This is an integrable model with N conserved quantitities. Its exact ground state wave function is given [58] by

$$\psi(x_i, ..., x_N) = C e^{-\omega \sum_i x_i^2/2} \prod_{i<j} |x_i - x_j|^{\frac{\beta}{2}}. \tag{12}$$

Therefore the expectation value of any position dependent operator $F(x_1, ..., x_N)$ in the ground state is given by

$$\langle F \rangle = \int dx_1 ... dx_N \psi^* F \psi = C^2 \int dx_1 ... dx_N \prod_{i<j} |x_i - x_j|^\beta e^{-\omega \sum_i x_i^2} F(x_1, ..., x_N). \tag{13}$$

This has the same form as the expectation value of an observable in the Wigner-Dyson distribution (4) with $V(x) = \frac{1}{N}\omega x^2$, $C^2 = N_0$, at $\beta = 1$, 2, and 4, with the position coordinates of the particles corresponding to the eigenvalues of the random matrix. Thus the matrix model (1) describes free fermions for $\beta = 2$ and interacting fermions for $\beta = 1, 4$. In particular, for F we may choose the n-point function of the density operator in the CSM model, $\hat{\rho}(x) \equiv \sum_i \delta(x - x_i)$. Then, for example the connected *spatial* equal-time density correlator in the $1/r^2$ model, exactly matches the matrix model and hence the quantum chaos *eigenvalue* density correlator: $\langle \hat{\rho}(x)\hat{\rho}(y) \rangle|_{\text{CSM model}} = N \langle \hat{\rho}(x/\sqrt{N})\hat{\rho}(y/\sqrt{N}) \rangle|_{\text{matrix model}}$, with the r.h.s. given by (7) or (8).

5. Dynamical Correlators, Perturbed Chaotic Hamiltonians, and the 2-Matrix Model

We now discuss a deeper connection between quantum chaos and the CSM model, established through the intermediary of a 2-matrix model. Consider a chaotic hamiltonian H consisting of an unperturbed part proportional to H_0 and a perturbation proportional to H_1:

$$H = H_0 \cos \Omega\phi + H_1 \frac{\sin \Omega\phi}{\Omega}. \tag{14}$$

ϕ is a measure of the strength of the perturbation (Ω is a constant, introduced for convenience, to be specified shortly). Given H_0 and H_1, the eigenvalues of H are a function of ϕ. One can think of a 1-d gas of particles whose positions are the eigenvalues and ϕ as the time on which they depend. An interesting fact is that the time evolution of this gas (ϕ dependence of the positions of the eigenvalues) is given by a classical hamiltonian which is closely related to the CSM hamiltonian with some additional degrees of freedom. This is known as the Pechukas gas [60]. We do not discuss this further; instead we focus on the 'quenched' averages that arise when the spectral statistics of H is discussed using matrix model ensembles. In particular we discuss the ϕ dependent correlator $\langle \rho_{H_0}(E_1)\rho_H(E_2)\rangle_c$ which is of interest in quantum chaos, since it measures the correlations between the eigenvalues of H_0 and H as a function of the strength of the perturbation ($\rho_H(E) \equiv \sum \delta(E - E_i)$ where E_i are the eigenvalues of H).

To see how this correlator is connected to the 2-matrix model, note that the probability $P(H, \phi)dH$ that the full hamiltonian lies in the volume element dH located at H (in the space of $N \times N$ matrices of the appropriate class given by β) is given by $P(H, \phi)dH = \int dH_0 P_0(H_0)P(H, \phi|H_0)dH$, where $P_0(H_0)dH_0$ is the probability that the unperturbed hamiltonian lies in the volume element dH_0 located at H_0, and $P(H, \phi|H_0)dH$ is the conditional probability that full hamiltonian lies in the volume element dH located at H given a fixed unperturbed part H_0. But now $P(H, \phi|H_0)dH = P_1(H_1)dH_1 = const.\ P_1(H_1)dH$, where $P_1(H_1)dH_1$ is the probability of the perturbation being in a volume element dH_1 located at H_1, and we have assumed that H_1 and H come from the same class of ensemble (same β), whereby, for fixed H_0, $dH_1 = const.\ dH$. Then $P(H,\phi)dH = const. \int dH_0 P_0(H_0) P_1(H_1) dH$, where H_1 is understood to be given in terms of H_0 and H by (14). Thus the average of any H dependent quantity will be given by a 2-matrix integral (the two matrices to be integrated over being H_0 and H), in which the factor $P_1(H_1)$, when expressed in terms of H_0 and H, will involve (ϕ dependent) terms that couple the two matrices together. In particular if we take both probability distributions P_0 and P_1 to be gaussian: $P_0(H_0) \sim \exp(-\frac{\Omega^2}{2}\mathrm{Tr} H_0^2)$, $P_1(H_1) \sim \exp(-\frac{1}{2}\mathrm{Tr} H_1^2)$, it follows that

$$P(H,\phi)dH = const. \int dH_0 dH \, \exp(-\mathrm{Tr}[V(H_0) + V(H) - cH_0 H]), \tag{15}$$

$$V(x) = \frac{1}{2}\mu x^2, \qquad \mu = \frac{\Omega^2}{\sin^2 \Omega\phi}, \qquad \text{and} \qquad c = \frac{\Omega^2 \cos \Omega\phi}{\sin^2 \Omega\phi}. \tag{16}$$

It is convenient to rescale $H_0 = \sqrt{N} A$, $H = \sqrt{N} B$, and define the 2-matrix model partition function (9) with $S = N\mathrm{Tr}\ (V(A) + V(B) - cAB)$. From the symmetry between H_0 and H in (15) it follows that the two eigenvalue densities are equal, $\langle \hat{\rho}_A(x) \rangle = \langle \hat{\rho}_B(x) \rangle \equiv \rho(x)$, and are given by the semicircle law for gaussian P_0 and P_1. The constant Ω is chosen to be $\Omega = \sqrt{\pi^2 \beta / 2N}$, which means that μ and c are $O(1)$ when $\phi \sim O(1)$. It is evident from the above that under the random matrix hypothesis the desired 2-point correlator in quantum chaos is given by [22,39,61]

$$\langle \rho_{H_0}(E_1) \rho_H(E_2) \rangle_c |_{\text{quantum chaos}} = N \rho_{AB}(E_1', E_2'), \tag{17}$$

with $E_i' = N^{-\frac{1}{2}} E_i$.

Altschuler, Simons, Szafer, and Lee provided evidence based on numerical results for chaotic billiards and the Anderson model for the universality of the l.h.s. of (17) [32,26], computed the matrix model local correlator for the r.h.s. of (17) and exhibited agreement with numerical results [26], and conjectured that the r.h.s. of (17) furthermore equals the *dynamical* density-density correlation function in the CSM model [34], with the perturbation strength ϕ in the quantum chaos problem playing the role of time in the CSM model (and E_i playing the role of the positions x_i of the CSM particles).

We now outline the proof of the latter conjecture [61] refering the reader to [33] for a detailed review. (See also [39] for related observations and for a proof that holds for smoothed correlators only, see [35].) It is not difficult to see that the probability distribution $P(H,\phi|H_0) = const.\ exp(-\mathrm{Tr}[V(H_0)+V(H)-cH_0H])$ for the matrix elements of H which appears in (15) is a solution of a Fokker-Planck equation describing the independent Brownian motion of the matrix elements of H, where the time t is given by

$$t = -\frac{\log(\cos(\Omega\phi))}{\Omega^2}, \tag{18}$$

and the initial condition is that at $t = \phi = 0$, the distribution is localized on H_0, $P(H,0|H_0) = \delta(H-H_0)$. Dyson [62] had shown that when the elements of a matrix execute independent Brownian motion, its eigenvalues execute a correlated Brownian motion in which they repel each other by a logarithmic interaction, and derived the corresponding Fokker-Planck equation for the time dependent probability distribution of the eigenvalues. Sutherland [63] showed that the eigenfunctions of this latter Fokker-Planck hamiltonian were in one-to-one correspondence with the eigenfunctions of the CSM hamiltonian. It is this correspondence, which extends the relationship between the matrix model and the CSM model at the *ground state* level (responsible for static correlators as discussed in the previous section) to all the *excited states*, that is at the heart of the mapping between the dynamical correlators of the CSM model and matrix models. One can also map the CSM model to a continuous matrix model in which the matrix A is a function of a continuous variable t [64]. For the two-point function the precise mapping is [61]

$$\langle\rho(x,0)\rho(y,t)\rangle_c|_{\mathrm{CSM\,model}} = N\rho_{AB}(x',y'), \tag{19}$$

where $\hat{\rho}(x,t) = \sum_i \delta(x-\hat{x}_i(t))$, and $\hat{x}_i(t)$ is the time dependent position operator of the CSM model in the Heisenberg picture. The l.h.s. of (19) as a function of x, y, t, and parameters ω, β, N is thus expressed as a correlator of the 2-matrix model (9) with couplings μ, c expressed in terms of t via eqs. (16,18), $\omega = \Omega^2/2$, and the mapping works for the values $\beta = 1, 2, 4$ for which the r.h.s. can be defined in terms of the appropriate matrix ensemble.

Eq. (10), substituted in (19), therefore gives the smoothed global spacetime dependent density-density correlator of the CSM model. If one specializes (10) to the region near the centre of the semicircle ($x, y \ll a$), and for $t \approx \phi^2 \ll O(N)$ one recovers the previously known 'translationally invariant' result [34] [35]

$$\langle \hat{\rho}(x,0)\hat{\rho}(y,t)\rangle_c = -\frac{2}{\pi^2\beta^2}\frac{[(x-y)^2 - \pi^2\beta^2 t^2]}{[(x-y)^2 + \pi^2\beta^2 t^2]^2}. \tag{20}$$

This identifies the sound velocity as $v_s = \pi\beta$. We mention that for the CSM model at other rational values of β not covered by known matrix ensembles, local fine-grained dynamical correlators have been obtained by the method of Jack Polynomials [65]. See [66] for a proof of (8) for all even β.

Together with (17) the equation (19) establishes a connection between perturbation strength dependent correlators in quantum chaos and dynamical correlators in the CSM model via the 2-matrix model. The r.h.s. of (20) therefore also equals the parametric eigenvalue correlator $\langle \rho_{H_0}(x)\rho_H(y)\rangle_c$ in the quantum chaos problem $H = H_0 + \phi H_1$, when H_0 and H_1 belong to the same symmetry class. We emphasize that the mapping from the quantum chaos problem to the matrix model is an ansatz that has essentially empirical support. On the other hand the mapping between the matrix model and the CSM model discussed above is an exact mathematical mapping.

6. Two Dimensional Quantum Gravity

Matrix models have long been studied in quantum chromodynamics, where the basic degree of freedom is a matrix valued field. The observation [67] that the $1/N$ expansion of SU(N) QCD groups the Feynman diagrams of the theory according to the topological classes of 2-dimensional surfaces led to a vigorous study of the random matrix models in the large-N limit [19]. This has also led to matrix models being proposed as models of 2-dimensional quantum gravity and string theory [68], where one needs to sum over all 2-dimensional surfaces keeping track of their topology.

A 2-dimensional surface can be latticized by triangulation, which is a representation of the surface in terms of small, flat, equilateral triangles (all of the same size) joined along their edges. In this discrete representation, all information about the local geometry of the surface is contained in the local 'coordination number', the number of triangles that meet at a vertex. The area of the surface is equal to the total number of triangles (in units of the area of each elementary triangle), the length of any boundary is equal to the number of edges in that boundary, and the topology of the surface, (given by its Euler characteristic $\chi \equiv 2 - 2h - b$ where h is the number of handles and b the number of boundaries) is $\chi = N_0 - N_1 + N_2$, where N_0 = number of vertices, N_1 = number of edges, and N_2 = number of triangles in the triangulation.

In a path integral formulation of 2-dimensional quantum gravity one needs to sum over all surfaces, assigning a specific weight to each surface. This sum is defined to be the sum over all distinct triangulations in the lattice version of the theory [68]. For example the partition function for pure gravity (without matter fields) is defined to be

$$Z(\kappa, \lambda) = \sum_{h=0}^{\infty} e^{\kappa(2-2h)} \sum_{A=4}^{\infty} e^{-\lambda A} \tilde{Z}(h, A), \tag{21}$$

where $\tilde{Z}(h, A)$ is the number of distinct triangulations of connected oriented surfaces with area A, number of handles h, and no boundary. κ is referred to as the gravitational constant since it multiplies the Einstein action (which is proportional to $2 - 2h$ for closed surfaces in two dimensions), and λ the cosmological constant.

The above partition function can be obtained from the partition function of a hermitian 1-matrix model: $Z(\kappa, \lambda) = \log Z$ where the Z on the r.h.s. is given by (1), with $V(A) = (1/2)A^2 - gA^3$, $N = e^{\kappa}$ and $g = e^{-\lambda}$. This is apparent upon expanding (1) as a power series in g, and collecting all the Feynman diagrams which appear with the same power of $1/N$. Since every Feynman diagram is a vacuum bubble made of propagators and cubic vertices only, its dual diagram (obtained by joining the centres of adjacent loops) is a triangulated surface. Since the perturbative evaluation of the partition function involves summing over all Feynman diagrams, one automatically gets a sum over triangulations. The weight factor appearing with every Feynman diagram containing V vertices, P propagators and L loops is $N^{V-P+L}g^V$, which becomes $e^{\kappa(N_0-N_1+N_2)-\lambda N_2}$ as needed, since every dual diagram has $N_0 = L$ vertices, $N_1 = P$ edges, and $N_2 = V$ triangles. The logarithm of Z appears because in (21) we need only connected surfaces. The leading large N contribution to $\log Z$ comes from closed connected surfaces with the spherical topology ($h = 0$, $\chi = 2$). The orthogonal and symplectic ensembles lead to sums over non-orientable surfaces (see, e.g., [69]).

Consider now the insertion of $\mathrm{Tr}\, A^l$ in the matrix model path integral, i.e., the quantity $Z\langle \mathrm{Tr}\, A^l\rangle \equiv \int dA\, e^{-S}\mathrm{Tr}A^l$, with l a positive integer. Expanding the r.h.s. as a perturbation expansion in g, one notes that as before all vertices of the Feynman diagrams are cubic, except one (corresponding to the external insertion of $\mathrm{Tr}A^l$) where l propagators meet. Thus all the corresponding dual diagrams are triangulated surfaces as before, but with one elementary plaquette which has l sides. This distinguished plaquette can be considered to be absent, i.e., the triangulated surface can be thought of as having one boundary of length l. Thus the insertion of $\mathrm{Tr}A^l$ in the path integral has the interpretation of inserting a 'loop' of length l on the surface [70]. Similarly the insertion of $\mathrm{Tr}A^l\mathrm{Tr}A^m$ can be seen to produce two loops of size l and m on the surface. $\hat{W}(z) = (1/N)\sum_{n=0}^{\infty}(1/z^{n+1})\mathrm{Tr}A^n$ is called the loop operator because it is the generating function of loops of all lengths; z is the fugacity for the length of the loop. Thus correlation functions of the type

$$\langle \hat{W}(z_1)...\hat{W}(z_n)\rangle = \frac{1}{Z}\int dA\; e^{-S}\hat{W}(z_1)...\hat{W}(z_n) \tag{22}$$

are of direct physical significance in 2-d gravity and string theory since they determine sums over surfaces with n boundaries of arbitrary length.

Now these correlation functions are closely related to the eigenvalue correlations in the matrix model because of the simple identity $\hat{\rho}(x) = (i/2\pi)\lim_{\epsilon\to 0}[\hat{W}(x+i\epsilon) - \hat{W}(x-i\epsilon)]$. Thus for example $\rho(x)$ is given by the discontinuity across the cut in $W(z)$ and the two point function by

$$\begin{aligned}\langle\hat{\rho}(x)\hat{\rho}(y)\rangle = -\tfrac{1}{4\pi^2}\lim\nolimits_{\epsilon\to 0}[&\langle\hat{W}(x+i\epsilon)\hat{W}(y+i\epsilon)\rangle + \langle\hat{W}(x-i\epsilon)\hat{W}(y-i\epsilon)\rangle\\ &-\langle\hat{W}(x+i\epsilon)\hat{W}(y-i\epsilon)\rangle - \langle\hat{W}(x-i\epsilon)\hat{W}(y+i\epsilon)\rangle].\end{aligned} \tag{23}$$

Therefore, in view of the correspondence discussed in the previous sections, a single mathematical quantity in the 1-matrix model (the r.h.s. of (22)) simultaneously gives three different things: correlation functions of loop operators in 2-d gravity, eigenvalue correlators in quantum chaos, and static spatial correlators in the CSM model.

The 2-matrix model defined by (9) with a cubic potential $V(A) = \frac{1}{2}A^2 - gA^3$ maps onto 2-dimensional gravity coupled to the Ising model [71]. The two matrices A and B correspond to the two spin configurations of the Ising model and the coupling constant c in the action $S(A, B)$ provides the extra Boltzmann factor when adjacent spins are unequal.

7. Loop Equations and Eigenvalue Correlators

Loop equations [72,19] are identities relating different correlation functions of the loop operators. There is a hierarchy of identities through which successively higher point correlation functions get coupled to the lower (e.g., one or two point) correlation functions. In the large N limit, the connected higher point correlation functions are associated with higher powers of $1/N$ as compared to the disconnected parts; this causes the infinite hierarchy of loop equations to break up into finite subsets of algebraic equations, from which all correlation functions can be solved for by an inductive procedure.

As a simple example of this procedure in the 1-matrix model consider the identity

$$0 = \int dA \ \ \frac{\partial}{\partial A_{ij}}(e^{-S}(\frac{1}{z-A})_{ij}), \tag{24}$$

where A is a real symmetric matrix, $S = N\mathrm{Tr}V(A)$, $V(A) = (1/2)\mu A^2$ (gaussian orthogonal ensemble, $\beta = 1$), and i and j are not summed over. Expanding the action of the derivative on its argument and summing over i and j one gets the identity (for details see [24])

$$0 = -\mu(zW(z) - 1) + \frac{1}{2}\langle \hat{W}(z)\hat{W}(z)\rangle - \frac{1}{2N}\frac{\partial}{\partial z}W(z). \tag{25}$$

This is an exact loop equation where the one and two point correlators are both present. Now decompose $\langle \hat{W}(z)\hat{W}(z)\rangle$ into its disconnected and connected parts: $\langle \hat{W}(z)^2\rangle = \langle \hat{W}(z)\rangle^2 + \langle \hat{W}(z)\hat{W}(z)\rangle_c$. In the large N limit $\langle \hat{W}(z)\rangle = W(z)$ is $O(1)$ and $\langle \hat{W}(z)\hat{W}(z)\rangle_c$ is $O(\frac{1}{N^2})$ (see below). Then to leading order in $1/N$ we can suppress the latter as well as the last term in

(25), obtaining the closed large N loop equation $0 = -\mu(zW(z) - 1) + (1/2)W(z)^2$ with the solution $W(z) = (2/a^2)[z - \sqrt{z^2 - a^2}]$ which has a branch cut between $z = \pm a$, $a^2 = 2\beta/\mu$. Using the identity $\rho(x) = (i/2\pi)\lim_{\epsilon\to 0}[W(x + i\epsilon) - W(x - i\epsilon)]$ in the above solution for $W(z)$, one immediately reproduces the Wigner semicircle law (6).

To prove 'large N factorization' one observes that in (5) when g_n are $O(1)$, then generically, $E \sim O(N^2)$ (e.g., for the gaussian case when the endpoint $a = \sqrt{2\beta/\mu}$ is $O(1)$, all the eigenvalues λ_i are in an $O(1)$ region around the origin, and then both the terms in (5) are $O(N^2)$). Thus to leading order $F \equiv \ln Z$ is $O(N^2)$, say $F \approx N^2\Gamma$, where Γ is a function of g_n only, independent of N. From this it follows that $\langle \mathrm{Tr}A^n \mathrm{Tr}A^m \rangle_c = \partial_{g_n}\partial_{g_m}\Gamma$ is $O(1)$, while the disconnected part $\langle \mathrm{Tr}A^n \rangle \langle \mathrm{Tr}A^m \rangle$ is $O(N^2)$ since $\langle \mathrm{Tr}A^n \rangle = -N\partial_{g_n}\Gamma$ is $O(N)$. Alternatively one can understand factorization geometrically. As indicated earlier, the N dependence coming from various types of surfaces is proportional to N^χ, where χ is the Euler characteristic of the surface. Let us restrict our consideration to surfaces with $h = 0$ (no handles) which will always give the leading large N contributions. $\langle \hat{W}(z)\hat{W}(z)\rangle_c$, since it contains two insertions of the loop operator, corresponds to connected surfaces with two boundaries whose Euler characteristic is therefore $\chi = 2 - 2h - b = 0$, while the disconnected part of $\langle \hat{W}(z)^2 \rangle$, namely $\langle \hat{W}(z)\rangle^2$, corresponds to a pair of surfaces each with one boundary (and hence each with $\chi = 1$), thereby representing a total $\chi = 2$. Thus the disconnected part is a factor N^2 larger than the connected part.

To obtain the 2-point function of the loop operators, start from the identity

$$0 = \int dA \ \ \frac{\partial}{\partial A_{ij}}(e^{-S}(\frac{1}{z - A})_{ij}\hat{W}(w)). \tag{26}$$

Expanding the action of the derivative on its argument yields an identity connecting 2 and 3-point functions of the loop operator. Expanding that identity into connected and disconnected parts as before, one finds that terms proportional to (25) exactly cancel, while the connected 3-point correlator piece is suppressed by factors of $1/N$ or $1/N^2$ compared to other terms. In the large N limit one gets the closed loop equation

$$0 = (W(z) - \mu z)\langle \hat{W}(z)\hat{W}(w)\rangle_c + \frac{1}{N^2}\frac{\partial}{\partial w}(\frac{W(w) - W(z)}{w - z}), \tag{27}$$

which yields upon simplification

$$\langle \hat{W}(z)\hat{W}(w)\rangle_c = \frac{1}{N^2}\frac{a^2}{2\beta}\frac{1}{(z-w)^2}\frac{(W(z)-W(w))^2}{(1-\frac{a^2}{4}W^2(z))(1-\frac{a^2}{4}W^2(w))}. \tag{28}$$

We have outlined the proof of this equation for the gaussian orthogonal ensemble, $\beta = 1$. The same steps repeated for the other two ensembles yields the β dependence displayed above. The difference enters at the point where action of the derivative is expanded. Concretely, for $\beta = 1$, $\frac{\partial A_{kl}}{\partial A_{ij}} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ whereas for $\beta = 2$, $\frac{\partial A_{kl}}{\partial A_{ij}} = \delta_{ik}\delta_{jl}$, and a slightly more complicated expression exists for $\beta = 4$. The difference in these expressions simply reflects the different number of independent degrees of freedom in the three ensembles. The extension of the method of loop equations to the $\beta = 1, 4$ ensembles was discussed in [24]. From (28) and (23) it immediately follows that the 2-point connected correlator of the eigenvalue density operator is given by (8). In ref. [13] (28) is derived for an arbitrary potential from the method of loop equations for $\beta = 2$. That also establishes the universality of (8) for the unitary ensemble. We expect that this proof of universality of (28) would also extend to the other ensembles.

For the 2-matrix model, since the two matrices are coupled, a larger number of identities are needed to get closed equations. A procedure similar to the one outlined above yields for the three gaussian ensembles the global correlator [24]

$$\begin{aligned}\langle \hat{W}_A(z)\hat{W}_B(w)\rangle_c = &\frac{1}{N^2}\frac{ca^2}{2\mu\beta}\frac{1}{(1-\frac{ca^2}{4\mu}W(z)W(w))^2}\\ &[\frac{W(z)^2}{(1-\frac{a^2}{4}W^2(z))}][\frac{W(w)^2}{(1-\frac{a^2}{4}W^2(w))}],\end{aligned} \tag{29}$$

with $a = (\frac{2\beta\mu}{\mu^2-c^2})^{1/2}$. (10) follows from (29) upon using (23). This method bypasses the problem of angular integrations. For other applications of loop equations in hermitian, non-gaussian 2-matrix models, see [37].

Finally we remark that (28), (8), (29), and (10) provide 'smoothed' correlation functions because we first compute loop operator correlation functions for z and w far from the cut which automatically averages over the oscillations on the scale of the eigenvalue spacing.

8. Summary

We have described some elementary mathematical properties of random matrix models and their connection with three distinct areas: two-dimensional quantum gravity, the $1/r^2$ integrable model, and quantum chaos (the last connection includes applications to nuclear physics and mesoscopic systems). The loop operators are a link between these areas since they provide eigenvalue correlations in quantum chaos and the CSM model on the one hand, and also have a direct geometric significance in quantum gravity and string theory. The method of loop equations is a powerful method for calculating smoothed global correlators. This method treats all ensembles on the same footing because the procedure for deriving the loop equations remains the same. The only distinction between the ensembles enters at the level of counting the degrees of freedom, that is, in the expression for $\frac{\partial A_{kl}}{\partial A_{ij}}$. The extension of loop equations to the orthogonal and symplectic ensembles should also find application to sums over unoriented random surfaces.

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