

Higher Threshold Parameters in $\pi\pi$ Scattering

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Abstract

A family of threshold parameters which probe the stability of chiral predictions is considered. The relevant criteria for the choice of threshold parameters are discussed. Sum rules for these quantities are derived from dispersion relations and evaluated from effective range formulae. Good agreement with two-loop chiral estimates for many of these quantities is found and interesting discrepancies are discussed.

Keywords: Sum rules, $\pi\pi$ scattering, chiral perturbation theory

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1. Dispersion relations for $\pi\pi$ scattering amplitudes with two subtractions have been rigorously established in axiomatic field theory [1]. It is convenient to consider dispersion relations for s -channel amplitudes of definite iso-spin, $T_s^I(s, t, u)$, $I = 0, 1, 2$, where s, t, u are the Mandelstam variables. Furthermore, each of the amplitudes may be written down in terms of a unique function as $T_s^0(s, t, u) = 3A(s, t, u) + A(t, u, s) + A(u, s, t)$, $T_s^1(s, t, u) = A(t, u, s) - A(u, s, t)$, $T_s^2(s, t, u) = A(t, u, s) + A(u, s, t)$. Unitarity, analyticity and crossing symmetry have been used extensively to study this fundamental process of elementary particle physics. Introducing a partial wave expansion for these amplitudes via $T_s^I(s, t, u) = 32\pi\Sigma(2l+1)f_l^I(s)P_l((t-u)/(s-4))$, elastic unitarity implies above threshold and below the four-pion threshold that the partial wave amplitudes may be described in terms of the phase shifts $\delta_l^I(s)$ by $f_l^I(s) = \sqrt{s/(s-4)} \exp(i\delta_l^I(s)) \sin \delta_l^I(s)$, where we have set the pion mass ($m_\pi = 139.6$ MeV) to unity. Note the threshold expansion $\text{Re}f_l^I(\nu) = \nu^l(a_l^I + b_l^I\nu + c_l^I\nu^2 + d_l^I\nu^3 + \dots)$, $\nu = (s-4)/4 > 0$ being the square of the three momentum in the centre of mass frame, which defines the scattering lengths a_l^I , the effective ranges b_l^I and the higher threshold parameters c_l^I and d_l^I , etc. Sum rules have been established for (combinations of) scattering lengths and effective ranges in the past employing analyticity and crossing symmetry constraints. One culmination of the dispersion relation approach to $\pi\pi$ scattering has been the Roy equations [2, 3] which is a system of coupled integral equations for partial wave amplitudes which further trades the two unknown t -dependent functions in fixed- t dispersion relations for the scattering lengths a_0^0 and a_0^2 . Roy equation fits to phase shift data [4, 5] have been extensively studied. Best fits to data give $a_0^0 = 0.26 \pm 0.05$ [6].

2. Chiral perturbation theory [7] is the effective low energy theory of the standard model and describes processes involving pionic degrees of freedom, viewed as the approximate Goldstone bosons of spontaneously broken axial-vector symmetry of massless QCD. At leading order the $\pi\pi$ scattering amplitude is given by the Weinberg result from PCAC: $A(s, t, u) = (s-1)/F_\pi^2$ [8]. (In chiral perturbation theory this amplitude has now been computed to one-loop [7] and even two-loop order [9, 10].) Furthermore this implies that the only non-vanishing threshold parameters are $a_0^0 = 7/(32\pi F_\pi^2)$, $a_0^2 = -1/(16\pi F_\pi^2)$, $a_1^1 = 1/(24\pi F_\pi^2)$, $b_0^0 = 1/(4\pi F_\pi^2)$ and $b_0^2 = -1/(8\pi F_\pi^2)$. It is important to note the well known result that the set of functions: $t^0 = (2\beta(s-4/3) + 5\alpha/3)/(F_\pi^2)$, $t^1 = \beta(t-u)/(F_\pi^2)$, $t^2 =$

$(-\beta(s - 4/3) + 2\alpha/3)/(F_\pi^2)$, where α and β are arbitrary real constants, verifies dispersion relations with two subtractions and vanishing absorptive parts. The Weinberg amplitude is a special case of this general linear amplitude, with $\alpha = \beta = 1$. It may be noted that a generalized version of chiral perturbation theory that is motivated by considerations of a small quark condensate in the QCD vacuum allows α to vary over a range between unity and as much as three, and reorganizes the chiral power counting [11]. In this discussion we confine ourselves to the more predictive standard chiral perturbation theory with $\alpha = 1$ for much of our discussion.

Dispersion relations with two subtractions have been used to write down sum rules for (combinations) of some of these threshold parameters, for example the Wanders sum rules which maybe written down as:

$$\begin{aligned} 18a_1^1 &= 2a_0^0 - 5a_0^2 - \frac{1}{4\pi^2} \int_0^\infty \frac{d\nu}{\sqrt{(\nu(\nu+1))^3}} [2\nu\sigma^0(\nu) - 3(3\nu+2)\sigma^1(\nu) - 5\nu\sigma^2(\nu)], \\ 3b_0^0 &= 2a_0^0 - 5a_0^2 + \frac{1}{4\pi^2} \int_0^\infty \frac{d\nu}{\sqrt{(\nu(\nu+1))^3}} [(4\nu+3)\sigma^0(\nu) - 3(2\nu+1)\sigma^0(0) - 3\nu\sigma^1(\nu) + 5\nu\sigma^2(\nu)], \\ 6b_0^2 &= -2a_0^0 + 5a_0^2 + \frac{1}{4\pi^2} \int_0^\infty \frac{d\nu}{\sqrt{(\nu(\nu+1))^3}} [2\sigma^0(\nu) + 3\nu\sigma^1(\nu) + (7\nu+6)\sigma^2(\nu) - 6(2\nu+1)\sigma^2(0)], \end{aligned}$$

where $\sigma^I(\nu) \equiv (4\pi/\sqrt{\nu(\nu+1)}) \cdot \sum (2l+1) \text{Im} f_l^I(\nu)$ are the cross-sections. The Weinberg predictions for the quantities involved in these sum rules obey these relations identically with all the cross-sections set to zero, since the original amplitude obeys dispersion relations with vanishing absorptive parts. Furthermore, it may be noted that since these are relations at leading order in the chiral expansion, it would be fair to expect that the relevant dispersion integrals make numerically less significant contributions to $18a_1^1$, $3b_0^0$ and $6b_0^2$ compared to $2a_0^0 - 5a_0^2$, even at higher orders in the chiral expansion. The presence of two subtractions in the dispersion relations where a_0^0 and a_0^2 play effectively the role of subtraction constants, render them difficult to pin down on the basis of dispersion relation phenomenology alone: *vital chiral inputs are required to make sharp predictions for these quantities.*

3. At one-loop order, chiral perturbation theory requires the introduction of several coupling constants in $\mathcal{L}^{(4)}$, which account for the non-renormalizable character of the non-linear sigma model Lagrangian, $\mathcal{L}^{(2)}$, which is the basis of the Weinberg result. Four of these constants \bar{l}_i , $i = 1, 2, 3, 4$ enter $\pi\pi$ scattering and in particular \bar{l}_1 and \bar{l}_2 enter predictions for $c_0^0, c_0^2, b_1^1, a_2^0$ and a_2^2 , while all higher threshold

parameters will receive no contributions from the trees generated by $\mathcal{L}^{(4)}$ at this order. In the past experimentally known values for a_2^0 and a_2^2 [6] were used to fix these quantities [7], while no values have been reported in the literature for c_l^I and b_1^1 . For completeness we note the one-loop formulas for c_l^I since they have been reported earlier:

$$\begin{aligned} c_0^0 &= \frac{1}{2304\pi^3 F_\pi^4} (-295 + 88\bar{l}_1 + 112\bar{l}_2) \\ c_0^2 &= \frac{1}{5760\pi^3 F_\pi^4} (-193 + 40\bar{l}_1 + 160\bar{l}_2) \end{aligned} \quad (1)$$

Recently, rapidly converging sum-rules were written down [12] in order to estimate b_1^1 which in principle could be used to fix the values of the two chiral coupling constants of interest, indeed, as one could from the values of c_0^I should they be known. However, systematic ambiguities inherent to one-loop predictions for these quantities have also been discussed recently. \bar{l}_1 and \bar{l}_2 were fixed instead by rewriting the chiral and axiomatic representations of the scattering amplitudes to the appropriate order in the momentum expansion [13]. This method is now being extended to the two-loop case in order to fix the coupling constants that enter at that order in CHPT. Note that the trees generated by $\mathcal{L}^{(6)}$ will now contribute to the threshold parameters d_0^I, c_1^1, b_2^I and a_3^1 . At the one-loop level they are given by the expressions: $d_0^0 = -1643/(40320\pi^3 F_\pi^4)$, $d_0^2 = -893/(40320\pi^3 F_\pi^4)$, $c_1^1 = -23/(13440\pi^3 F_\pi^4)$, $b_2^0 = -481/(201600\pi^3 F_\pi^4)$, $b_2^2 = -277/(201600\pi^3 F_\pi^4)$ and $a_3^1 = 11/(94080\pi^3 F_\pi^4)$.

To summarize, we note that the trees generated by $\mathcal{L}^{(2)}$, contribute to

$$\begin{aligned} &a_0^I \quad b_0^I \\ &a_1^1 \end{aligned} \quad (2)$$

while those generated by $\mathcal{L}^{(4)}$ contribute to

$$\begin{aligned} &a_0^I \quad b_0^I \quad c_0^I \\ &a_1^1 \quad b_1^1 \\ &a_2^I \end{aligned} \quad (3)$$

and those of $\mathcal{L}^{(6)}$ contribute to

$$\begin{aligned}
& a_0^I \quad b_0^I \quad c_0^I \quad d_0^I \\
& a_1^1 \quad b_1^1 \quad c_1^1 \\
& a_2^I \quad b_2^I \\
& a_3^1.
\end{aligned} \tag{4}$$

4. The main purpose of this letter is precisely to provide estimates to those quantities appearing in eq.(3) and eq.(4) for which no information is available in the literature and to compare whenever possible with the predictions of CHPT. Such a consistency check may be viewed as a probe into the range of validity in energy of chiral predictions. Indeed, when Roy equation fits to the planned precision experiments are performed, all the quantities discussed here may be evaluated afresh, which would then amount to a high precision experimental determination of these numbers. Such Roy equation fits may then be employed to evaluate chiral parameters determined from dispersion relation phenomenology, and such a consistency check may then be performed again.

5. We work in the approximation that the absorptive parts are modeled entirely by the S- and P- waves (In the numerical analysis we further assume that the contribution to the dispersion integrals from the S- and P- waves also above the $K\bar{K}$ threshold may be neglected as in Ref. [12].) which has been found to be justified phenomenologically in the past and is supported today by chiral power counting. In this approximation it is particularly convenient to represent the iso-spin amplitudes as [3, 14]:

$$\begin{aligned}
T^I(s, t, u) = & 32\pi \sum_{I'=0}^2 \left(\frac{1}{4} (s\mathbf{I}^{II'} + tC_{st}^{II'} + uC_{su}^{II'}) a_0^{I'} + \right. \\
& \frac{1}{\pi} \int_4^\infty \frac{dx}{x(x-4)} \left\{ \left[\frac{s(s-4)\mathbf{I}^{II'}}{x-s} + \frac{t(t-4)C_{st}^{II'}}{x-t} + \frac{u(u-4)C_{su}^{II'}}{x-u} \right] \text{Im } f_0^{I'}(x) \right. \\
& \left. \left. + 3 \left[\frac{s(t-u)\mathbf{I}^{II'}}{x-s} + \frac{t(s-u)C_{st}^{II'}}{x-t} + \frac{u(t-s)C_{su}^{II'}}{x-u} \right] \text{Im } f_1^{I'}(x) \right\} .
\end{aligned} \tag{5}$$

Projecting these amplitudes onto partial waves will yield the Roy equations for each of the waves in the S- and P- wave approximation. [This approximation is equivalent to setting the so-called *driving terms* of

the Roy equations to zero. However, in the numerical work described below, we evaluate the contribution of the $f_2(1270)$ resonance [15] to the threshold parameters of interest as a measure of the contribution of the driving terms.] The most convenient manner in which the sum rules of interest may be computed is to consider the Roy equations for each of the partial waves f_0^I , f_1^1 , f_2^I and f_3^1 in the neighborhood of the threshold, as power series in ν . Furthermore, in order to isolate the quantities of interest we need to consider the Cauchy Principal Value of the relevant integrals. The Principal Value singularity occurs due to self-coupling of the waves and must be removed during the process of computing the power series of the real parts of the waves. An example is worked out for the $I = 0$ S- wave in Appendix A and the other waves may be treated analogously. The complete set of sum rules for the quantities of interest that have not been published in the past is listed in Appendix B.

6. The spirit of this work will closely follow that of Ref. [14] wherein a modified effective range formula was employed in order to model the absorptive parts of the amplitudes. Note that the dispersion integrals we encounter for (most of) the quantities of interest are very rapidly converging which implies that it is the near threshold region that needs to be modeled accurately, the region where the effective range formula is applicable. The modified effective range formula for the S- and P- wave phase shifts is of the type first proposed by Schenk [16] and is:

$$\begin{aligned}\tan \delta_0^I(\nu) &= \sqrt{\frac{\nu}{\nu+1}} \left\{ a_0^I + [b_0^I - a_0^I/\nu_0^I + (a_0^I)^3]\nu \right\} \frac{\nu_0^I}{\nu_0^I - \nu} \\ \tan \delta_1^1(\nu) &= \sqrt{\frac{\nu^3}{\nu+1}} \left\{ a_1^1 + [b_1^1 - a_1^1/\nu_1^1]\nu \right\} \frac{\nu_1^1}{\nu_1^1 - \nu}\end{aligned}\quad (6)$$

7. Our numerical work requires inputs to the effective range formulae. Much of these inputs are guided by one-loop chiral perturbation theory. We also perform a sample computation with a set of input parameters with a_0^0 corresponding to the best fits to the experimental data of K_{e4} of 0.26 and the rest of the quantities computed for that optimal Roy equation fit. In Table 1 the complete set of inputs is tabulated, besides the conventional choice $\nu_0^0 = 8.5$, $\nu_1^1 = 6.6$, $\nu_0^2 = -5.0$.

In Table 2 we present the computed values of the threshold parameters of interest for the inputs of Table 1. The contribution of the f_2 resonance [15] in the narrow width approximation to the quantities

of interest, calculated by plugging in the appropriate absorptive part into the relevant Roy equation and computed in the appropriate limit, yields the results tabulated along f_2 in Table 2. This should be considered as setting the scale of the corrections arising from all the higher waves and the higher energy tail that would be described in terms of Regge parameterization of the absorptive parts and possible Pomeron contributions. It may be judged from this that the bulk of the contribution is received from the S- and P-wave low energy absorptive parts.

In Table 3 we tabulate the one-loop and two-loop predictions for these quantities whenever available [9, 17]. Let us first consider the one loop predictions for c_0^I which receive contributions from \bar{l}_1 and \bar{l}_2 . The latter computed from optimal Roy equation fits of Ref. [13] have been inserted into eq.(1) to produce the entries in the first two columns of Table 3. The rest of the quantities are of the pure one-loop variety and their values numerically tabulated. Alternatively, one may take the values of the c_0^I and solve for $\bar{l}_{1,2}$ via. eq.(1). Taking the extreme values [including the contribution of the f_2] for $-0.0049 \leq c_0^0 \leq 0.010$ and $0.012 \leq c_0^2 \leq 0.015$ we obtain $-4.7 \leq \bar{l}_1 \leq 0.08$ and $3.8 \leq \bar{l}_2 \leq 5.7$. While these numbers are not to be taken literally since the contributions of the high energy tail are completely neglected and the effective range formula is not a real substitute for a Roy equation solution of the lowest partial waves, they continue to provide an important consistency check on the values of these coupling constants. These are consistent with several prior determinations for these quantities.

The comparison of the values of c_0^I with the two-loop predictions continue to be encouraging as we observe from Tables 2 and 3. Some attention may be paid to the quantity c_1^1 : at one-loop order it is negative where as the two-loop prediction is positive and larger in magnitude. This value creeps up towards the value provided by the sum rules. It could be that the presence of the ρ as a non-perturbative feature of hadron dynamics is responsible for the mismatch between even the two-loop prediction and the sum rule result for c_1^1 .

Furthermore, the Taylor series of $\text{Re}f_l^I(\nu) = \sqrt{\frac{\nu+1}{\nu}} \sin(2\delta_l^I(\nu))/2$ computed with the effective range formulae for the phase shifts, itself yields coefficients c_l^I and d_0^I which are comparable with the quantities computed from the sum-rules. This proves to be a check on the effective range parameterization itself. For instance, for the choice I in Table 1, we find $c_0^0 = 0.0092$, $c_0^2 = 0.011$, $c_1^1 =$

0.00092, $d_0^0 = -0.013$, $d_0^2 = -0.0012$, whereas for that of choice IV, we have $c_0^0 = -0.0025$, $c_0^2 = 0.011$, $c_1^1 = 0.00084$, $d_0^0 = -0.015$, $d_0^2 = -0.0013$.

Thus we see that detailed considerations of unitarity, analyticity and crossing in the near threshold region is in excellent agreement with chiral predictions — these, however, do not suffice to discriminate between the standard and generalized scenarios of chiral perturbation theory.

8. In order to come to grips with these numbers, we have also chosen to compare the predictions for the P- wave threshold parameters with numbers arising from resonance saturation with the ρ . In particular, the narrow width formula for the absorptive part generated by the ρ :

$$\text{Im}f_1^1(x) = \pi\Gamma_\rho m_\rho \sqrt{\frac{x}{x-4}} \delta(x - m_\rho^2) \quad (7)$$

gives rise to an amplitude $A(s, t, u)$ by inverting eq.(5) via $A(s, t, u) = (T^0(s, t, u) - T^2(s, t, u))/3$,

$$A^\rho(s, t, u) = \frac{(48\pi)\Gamma_\rho}{m_\rho^2(m_\rho^2 - 4)^{3/2}} \left(\frac{t(s-u)}{m_\rho^2 - t} + \frac{u(s-t)}{m_\rho^2 - u} \right). \quad (8)$$

This is identical to the result in eq. (C.9) of Ref. [7] obtained from effective Lagrangian techniques with the appropriate identification of the relevant coupling constant when we take into account the identities, $1/(m_\rho^2 - t) = 1/m_\rho^2 \cdot (1 + t/(m_\rho^2 - t))$ and $[t(s-u) + u(s-t)] = [-2(s-2)^2 + 1/2(s^2 + (t-u)^2)]$. This is a consequence of two subtractions in the dispersion relations used here. The formula eq. (8) when inserted into the $I = 1$ amplitude and projected onto the P- wave yields for $a_1^1 - (2a_0^0 - 5a_0^2)/18 = 0.0073$, $b_1^1 = 0.0077$, $c_1^1 = 0.0006$, when parameters of the resonance [15] are inserted into the formulas for these quantities. Note the ρ contribution to \bar{l}_1 and \bar{l}_2 are $-\pi m_\rho/(2\Gamma_\rho)$ and $\pi m_\rho/(4\Gamma_\rho)$ respectively [see, e.g., eq. (C.10) in [7]].

Another treatment of resonance saturation [18] with the ρ is based on writing an unsubtracted dispersion relation for the $I = 1$ t-channel amplitude divided by $(s-u)$. This is expected to converge under the assumption of the validity of the Pomeranchuk theorem, supported by the behavior of the relevant Regge trajectory. The ρ contribution to \bar{l}_1 and \bar{l}_2 are, however, $-\pi m_\rho/(4\Gamma_\rho)$ and $\pi m_\rho/(4\Gamma_\rho)$ respectively [as read off from eq. (28) and eq. (32) of [18]]. This is in variance with eq. (C.10) of [7]. This must result from considering dispersion relations without a sufficient number of subtractions that guarantee the absence of crossing constraints on the absorptive parts due to the P- wave. [An analogous situation arises

with the resonance saturation with states of $l \geq 2$ of fixed-t dispersion relations with two subtractions; this does not yield a crossing symmetric amplitude and must be treated with care.]

9. In the preceding sections we have established a program of comparison between families of threshold parameters for which sum rules have been derived and the quantities numerically estimated, and then compared with one- and two-loop predictions for these quantities. The role of the number of subtractions in dispersion relations is crucial and has been emphasized in a variety of ways. Other recent efforts in this confrontation and comparison between data, dispersion relations and chiral perturbation theory use extrapolation to subthreshold regions reaching analogous conclusions regarding the interplay of analyticity and unitarity [19]. Our work is also in keeping with the expectation expressed in Ref. [20] that sensitive tests of QCD in low energy $\pi\pi$ scattering should use all theoretical constraints and pertinent low energy observables. It should, however, be emphasized that we have considered only quantities that do not require vital chiral inputs such as a_0^I : work is in progress to extract sharp predictions for these quantities.

After this work was completed, we have received a preprint [22] where several $O(p^6)$ coupling constants have been evaluated. It would be important to study the implications of this evaluation to the program discussed here and *vice versa*.

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A Derivation of the Sum rules

The three lowest partial waves can be written as:

$$\begin{aligned}
 \text{Re } f_0^0(\nu) &= a_0^0 + (2a_0^0 - 5a_0^2) \frac{\nu}{3} \\
 &\quad + 4 \sum_{I'=0}^2 \sum_{I''=0}^{\infty} P \int_0^{\infty} d\nu' K_{00}^{II'}(\nu, \nu') \text{Im } f_{I'}^{I''}(\nu') \\
 \text{Re } f_1^1(\nu) &= (2a_0^0 - 5a_0^2) \frac{\nu}{18}
 \end{aligned} \tag{9}$$

$$+4 \sum_{I'=0}^2 \sum_{l'=0}^{\infty} P \int_0^{\infty} d\nu' K_{11}^{l'I'}(\nu, \nu') \text{Im } f_{\nu'}^{l'}(\nu') \quad (10)$$

$$\begin{aligned} \text{Re } f_0^2(\nu) &= a_0^2 - (2a_0^0 - 5a_0^2) \frac{\nu}{6} \\ &+ 4 \sum_{I'=0}^2 \sum_{l'=0}^{\infty} P \int_0^{\infty} d\nu' K_{02}^{l'I'}(\nu, \nu') \text{Im } f_{\nu'}^{l'}(\nu') \end{aligned} \quad (11)$$

As an example we give the derivation of the higher threshold parameters for $I = 0, l = 0$.

Working in the S - and P -wave approximation, the summation over the angular momentum l runs from zero to one. Besides, this is equivalent to projecting eq.(5) on the relevant partial wave. The contributions of the three lowest partial waves to $f_0^0(\nu)$ are explicitly written down as:

$$\begin{aligned} f^0(\nu) &\equiv \frac{1}{\pi} P \int_0^{\infty} d\nu' \left\{ \frac{1}{\nu' - \nu} - \frac{2\nu}{3\nu'(1 + \nu')} - \frac{3 + 5\nu'}{3\nu'(1 + \nu')} + \frac{2}{3\nu} \ln \frac{\nu + \nu' + 1}{\nu' + 1} \right\} \text{Im } f_0^0(\nu'), \\ f^1(\nu) &\equiv \frac{1}{\pi} P \int_0^{\infty} d\nu' \left\{ -\frac{3(3\nu + 2\nu' + 4)}{\nu'(1 + \nu')} + \frac{6(2\nu + \nu' + 2)}{\nu\nu'} \ln \frac{\nu + \nu' + 1}{\nu' + 1} \right\} \text{Im } f_1^1(\nu'), \\ f^2(\nu) &\equiv \frac{1}{\pi} P \int_0^{\infty} d\nu' \left\{ \frac{5(\nu - 2\nu')}{3\nu'(1 + \nu')} + \frac{10}{3\nu} \ln \frac{\nu + \nu' + 1}{\nu' + 1} \right\} \text{Im } f_0^2(\nu'), \end{aligned} \quad (12)$$

such that

$$\text{Re } f_0^0(\nu) = a_0^0 + (2a_0^0 - 5a_0^2) \frac{\nu}{3} + f^0(\nu) + f^1(\nu) + f^2(\nu). \quad (13)$$

The singularity of the integrand resides in the first term of $f^0(\nu)$. Adding and subtracting $\text{Im } f_0^0(\nu')/(\pi(\nu + \nu' + 1))$ does not change the integral and we may write

$$\begin{aligned} f^0(\nu) &= G(\nu) + \\ &\frac{1}{\pi} \int_0^{\infty} d\nu' \left\{ \frac{2}{3\nu} \ln \frac{\nu + \nu' + 1}{\nu' + 1} - \frac{2}{3\nu'(1 + \nu')} - \frac{3 + 5\nu'}{3\nu'(1 + \nu')} - \frac{1}{\nu + \nu' + 1} \right\} \text{Im } f_0^0(\nu'), \end{aligned} \quad (14)$$

where

$$G(\nu) = \frac{1}{\pi} P \int_0^{\infty} d\nu' \frac{2\nu' + 1}{(\nu' - \nu)(\nu' + \nu + 1)} \text{Im } f_0^0(\nu'). \quad (15)$$

Consider the difference

$$\begin{aligned} f_0^0(\nu) - f_0^0(0) &= (2a_0^0 - 5a_0^2) \frac{\nu}{3} + G(\nu) - G(0) \\ &+ \bar{f}^0(\nu) - \bar{f}^0(0) + f^1(\nu) - f^1(0) + f^2(\nu) - f^2(0) \\ \bar{f}^0(\nu) &\equiv f_0^0(\nu) - G(\nu) \end{aligned} \quad (16)$$

We note that the integrals not involving G in eq. (16) are free of singularities. Furthermore $G(\nu) - G(0)$ is also free of singularities as we show below. Consider

$$G(\nu) - G(0) = \frac{1}{\pi} P \int_0^\infty \frac{(2\nu' + 1)(\nu + \nu'^2)}{(\nu' - \nu)(\nu + \nu' + 1)(\nu' + 1)\nu'} \text{Im } f_0^0(\nu') \quad (17)$$

which still contains the Principal Value singularity. With

$$P \int_0^\infty \frac{d\tau'}{\tau'^2 - \tau^2} = 0 \quad (18)$$

and $\tau' = \sqrt{\nu'(\nu' + 1)}$ and $\tau = \sqrt{\nu(\nu + 1)}$ we can write

$$P \int_0^\infty d\nu' \frac{2\nu' + 1}{\sqrt{\nu'(\nu' + 1)}(\nu' - \nu)(\nu' + \nu + 1)} = 0. \quad (19)$$

Furthermore, in the S- (and P-) wave approximation, assuming normal threshold behavior, we have:

$$\text{Im } f_0^0(\nu) = \frac{1}{4\pi} \sqrt{\nu(\nu + 1)} \sigma^0(\nu) \quad (20)$$

and therefore we can write

$$G(\nu) - G(0) = \frac{1}{4\pi^2} \int_0^\infty d\nu' \frac{(2\nu' + 1)(\nu + \nu'^2)}{\sqrt{\nu'(\nu' + 1)}(\nu' - \nu)(\nu + \nu' + 1)} \left\{ \sigma^0(\nu') - \sigma^0(\nu) \right\} \quad (21)$$

which is seen to be free from the Principal Value singularity.

The higher threshold parameters are now given by the Taylor coefficients of the difference $f_0^0(\nu) - f_0^0(0)$.

B Sum rules for the higher threshold parameters

For the higher threshold parameters considered in the text and which have not been considered in the literature [21], we find

$$\begin{aligned} c_0^0 &= \frac{64}{\pi} \int_0^\infty d\nu \left[\frac{5}{288(1 + \nu)^3} \text{Im } f_0^2(\nu) - \frac{1 + 2\nu}{32\nu(1 + \nu)^3} \text{Im } f_1^1(\nu) \right. \\ &\quad \left. + \left(\frac{1}{64\nu^3} + \frac{1}{288(1 + \nu)^3} \right) \text{Im } f_0^0(\nu) - \frac{\sqrt{\nu(\nu + 1)}}{256\pi\nu^3(1 + \nu)^3} (1 + 2\nu) \times \right. \\ &\quad \left. \left\{ \sigma^0(0)(1 + \nu + \nu^2) + \frac{d}{d\nu} \sigma^0(\nu) \Big|_{\nu=0} \right\} \right] \quad (22) \end{aligned}$$

$$\begin{aligned}
d_0^0 &= \frac{256}{\pi} \int_0^\infty d\nu \left[-\frac{5}{1536(1+\nu)^4} \text{Im} f_0^2(\nu) + \frac{2+5\nu}{512\nu(1+\nu)^4} \text{Im} f_1^1(\nu) \right. \\
&\quad + \left(\frac{1}{256\nu^4} - \frac{1}{1536(1+\nu)^4} \right) \text{Im} f_0^0(\nu) - \frac{\sqrt{\nu(1+\nu)}}{1024\pi\nu(\nu+1)} (1+2\nu) \\
&\quad \left. \left\{ \frac{1+\nu+\nu^2}{\nu^2(1+\nu)^2} \frac{d}{d\nu} \sigma^0(\nu)|_{\nu=0} + \frac{1+2\nu+2\nu^2}{\nu^3(1+\nu)^3} \sigma^0(0) + \frac{1}{2(\nu+\nu^2)} \frac{d^2}{d\nu^2} \sigma^0(\nu)|_{\nu=0} \right\} \right] \quad (23)
\end{aligned}$$

Similar one gets the threshold parameters for other waves:

$$\begin{aligned}
c_1^1 &= \frac{256}{\pi} \int_0^\infty d\nu \left[\frac{1}{2560(1+\nu)^4} \text{Im} f_0^0(\nu) - \frac{1}{1024(1+\nu)^4} \text{Im} f_0^2(\nu) \right. \\
&\quad \left. + \left(\frac{1}{256\pi} - \frac{2+11\nu}{5120(1+\nu)^4} \right) \text{Im} f_1^1(\nu) - \frac{\sqrt{\nu(1+\nu)}}{3072\pi\nu(1+\nu)} \frac{1+2\nu}{\nu+\nu^2} \frac{d^2}{d\nu^2} \sigma^1(\nu)|_{\nu=0} \right] \quad (24)
\end{aligned}$$

$$\begin{aligned}
c_0^2 &= \frac{64}{\pi} \int_0^\infty d\nu \left[\frac{1}{288(1+\nu)^3} \text{Im} f_0^0(\nu) + \frac{1+2\nu}{64\nu(1+\nu)^3} \text{Im} f_1^1(\nu) \right. \\
&\quad + \left(\frac{1}{64\nu^3} + \frac{1}{576(1+\nu)^3} \right) \text{Im} f_0^2(\nu) - \frac{\sqrt{\nu(\nu+1)}}{256\pi\nu^3(1+\nu)^3} (1+2\nu) \times \\
&\quad \left. \left\{ (1+\nu+\nu^2) \sigma^2(\nu) + (\nu+\nu^2) \frac{d}{d\nu} \sigma^2(\nu)|_{\nu=0} \right\} \right] \quad (25)
\end{aligned}$$

$$\begin{aligned}
d_0^2 &= \frac{256}{\pi} \int_0^\infty d\nu \left[-\frac{1}{1536(1+\nu)^4} \text{Im} f_0^0(\nu) - \frac{2+5\nu}{1024\nu(1+\nu)^4} \text{Im} f_1^1(\nu) \right. \\
&\quad + \left(\frac{1}{256\nu^4} - \frac{1}{3072(1+\nu)^4} \right) \text{Im} f_0^2(\nu) - \frac{\sqrt{\nu(1+\nu)}}{1024\pi\nu(1+\nu)} (1+2\nu) \times \\
&\quad \left. \left\{ \frac{1+\nu+\nu^2}{\nu^2(1+\nu)^2} \frac{d}{d\nu} \sigma^2(\nu)|_{\nu=0} + \frac{1+2\nu+2\nu^2}{\nu^2(1+\nu)^3} \sigma^2(0) + \frac{1}{2(\nu+\nu^2)} \frac{d^2}{d\nu^2} \sigma^2(\nu)|_{\nu=0} \right\} \right] \quad (26)
\end{aligned}$$

In the same way one gets the threshold parameters for $I = 0, l = 2, I = 2, l = 2$:

$$b_2^0 = \frac{1}{30\pi} \int_0^\infty d\nu \frac{1}{\nu(1+\nu)^4} \left[-\nu \text{Im} f_0^0(\nu) + (3\nu-6) \text{Im} f_1^1(\nu) - 5\nu \text{Im} f_0^2(\nu) \right] \quad (27)$$

$$b_2^2 = \frac{1}{60\pi} \int_0^\infty d\nu \frac{1}{\nu(1+\nu)^4} \left[-2\nu \text{Im} f_0^0(\nu) - (3\nu-6) \text{Im} f_1^1(\nu) - \nu \text{Im} f_0^2(\nu) \right] \quad (28)$$

$$(29)$$

A check on the sum rules for d_0^l, c_1^l and b_2^l is to saturate the right hand sides with the lowest order chiral phase shifts which should then yield the pure one-loop formulae for these quantities due to the perturbative unitarity of the chiral expansion. The sum rules for c_0^l do not converge fast enough; one may consider appropriate linear combinations with say the D- wave scattering lengths a_2^l in which the

$\bar{l}_{1,2}$ dependence drop out and to saturate the corresponding sum rules with the lowest order chiral phase shifts and reproduce the one-loop formulae for these combinations.

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#	a_0^0	b_0^0	a_0^2	b_0^2	a_1^1	b_1^1
I	0.19	0.237	-0.04	-0.074	0.035	0.006
II	0.20	0.236	-0.036	-0.074	0.035	0.006
III	0.21	0.236	-0.035	-0.075	0.035	0.006
IV	0.26	0.231	-0.021	-0.076	0.036	0.006

Table 1. List of inputs for the effective range formula

#	c_0^0	c_0^2	d_0^0	d_0^2	c_1^1	b_2^0	b_2^2	a_3^1
I	0.0084	0.012	-0.016	-0.0045	0.00080	-0.00031	-0.00028	0.000049
II	0.0061	0.013	-0.016	-0.0047	0.00084	-0.00031	-0.00029	0.000051
III	0.0041	0.013	-0.016	-0.0050	0.00087	-0.00032	-0.00030	0.000053
IV	-0.0049	0.015	-0.017	-0.0061	0.00096	-0.00036	-0.00035	0.000060
f_2	0.0016	0.00024	-0.000030	-0.0000304	$5.7 \cdot 10^{-6}$	0.000016	$2.2 \cdot 10^{-6}$	$3.2 \cdot 10^{-7}$

Table 2. Higher threshold parameters computed from sum rules for the inputs of Table 1. The contribution of f_2 is also listed.

#	c_0^0	c_0^2	d_0^0	d_0^2	c_1^1	b_2^0	b_2^2	a_3^1
O-L	0.0075	0.015	-0.0066	-0.0036	-0.00028	-0.00039	-0.00022	0.000020
	0.0081	0.015						
	0.0087	0.015						
T-L(S)	0.0068	0.013	-0.017	-0.0049	0.00032	-0.00031	-0.00031	0.000050
T-L(G)					0.00046	-0.00034	-0.00035	0.000058

Table 3. Table of values of quantities computed at one- and two-loop (standard and generalized) order.

For the C_l^I the quantities are computed from \bar{l}_1 and \bar{l}_2 listed in Table 1 of Ref. [13]. The two-loop quantities are from [9] and [17].