DIAGONALIZABLE GENERALIZED ABEL INTEGRAL OPERATORS*

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Abstract. We find a large class of integral operators which allow a systematic and explicit diagonalization. These operators generalize Abel's integral operator, for which such a diagonalization has recently been found.

Key words. Abel integral operator, inverse problems, diagonalization

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1. Introduction and definitions. In connection with some problems of stellar winds in astronomy, Knill and others (see [2] and [3]) have recently solved an Abel integral equation by the method of diagonalization of Abel's operator of a special type. The beauty and advantage of the diagonalization method have been explained in these papers. They have also pointed out the surprising fact that such a diagonalization result appears only more than 160 years after Abel's original work. The Abel integral equation arises in many areas of mathematical sciences, and it has varieties of applications (see [1], [4], and [5]). In the present paper, we have made a systematic study of the art of diagonalization of a general integral operator, of which Abel's operator is a special case, and utilized it to the corresponding inversion problems. For the presentation of our results, we require the following definitions.

DEFINITION 1. Let S_r be the space of all real analytic functions $f(x)$, defined on the interval $[0, r)$, spanned by the polynomials $\{x^n\}_{n \in \mathbb{N}}$, where r is a positive real number. Every function in S_r has the Taylor expansion

$$(1.1) \quad f(x) = \sum_{n=0}^{\infty} f_n x^n,$$

which converges for all $x \in [0, r)$.

DEFINITION 2. Let S_H be the space of all real functions $f(x)$, defined on the interval $[0, r)$, spanned by the sequence of functions $\{h^n(x)\}_{n \in \mathbb{N}}$, where $h^n(x) = [h(x)]^n$, with $h(x)$ representing a strictly monotonically increasing differentiable function having the property that $h(0) = 0$ and r is a positive real number. Every function in S_H has an expansion

$$(1.2) \quad f(x) = \sum_{n=0}^{\infty} f_n h^n(x),$$

which converges for all $x \in [0, r)$.

DEFINITION 3. Let $q(x) = \sum_{n=1}^{\infty} q_n x^{-n}$ be a function defined on the interval $(0, \infty]$, which is analytic at infinity. We assume that for all $x > 0$ the Taylor series

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converges to a finite value $q(x)$, and $q(\infty) = 0$. Let R_0 denote the class of all such functions.

DEFINITION 4. An Abel-type integral operator G is defined by the relation

$$(1.3) \quad G(g)(x) = \int_0^x \frac{k(x, t)}{(x^\beta - t^\beta)^\alpha} g(t) dt, \quad 0 < \alpha < 1, \beta > 0,$$

where

$$(1.4) \quad k(x, t) = \sum_{j=1}^m a_j x^{\alpha\beta-j} t^{j-1},$$

with constants a_j ($j = 1, 2, \dots, m$).

In the case when $\beta = \frac{1}{\alpha}$, we denote the operator G by $\hat{G}$.

DEFINITION 5. An operator F_α is defined by the relation

$$(1.5) \quad F_\alpha(q)(x) = \int_x^\infty \frac{q(t)}{(t^{\frac{1}{\alpha}} - x^{\frac{1}{\alpha}})^\alpha} dt, \quad x > 0, \quad 0 < \alpha < 1.$$

DEFINITION 6. An operator H is defined by the relation

$$(1.6) \quad H(g)(x) = \int_0^x \frac{k(x, t)g(t)}{(h^\beta(x) - h^\beta(t))^\alpha} dt, \quad 0 < \alpha < 1, \beta > 0,$$

where $h(x)$ is a strict monotonically increasing differentiable function, with $h(0) = 0$, and

$$(1.7) \quad k(x, t) = [h(x)]^{\alpha\beta-1} h'(t),$$

the prime sign denoting differentiation with respect to the argument.

We have presented the diagonalization method applicable to the operators G , F_α , and H , generalizing the class of kernels and operators involved in the study of Abel-type integral equations. Specific examples have been taken up to support the general method developed.

2. The diagonalization method. In this section, we have developed the diagonalization method for the general Abel-type integral equation

$$(2.1) \quad G(g)(x) = f(x).$$

We define

$$(2.2) \quad \hat{k}(s) = \sum_{j=1}^m a_j s^{j-1}$$

so that

$$(2.3) \quad k(x, xs) = x^{\alpha\beta-1} \hat{k}(s).$$

We also assume that the known function $f(x)$ belongs to the class S_r .

To diagonalize the operator G we have to find suitable eigenfunctions $\varphi_p(x)$ and eigenvalues μ_p such that

$$(2.4) \quad \int_0^x \frac{\varphi_p(t)k(x, t)dt}{\varphi_p(x)(x^\beta - t^\beta)^\alpha} := \mu_p$$

holds.

Suppose that

$$(2.5) \quad \varphi_p(x) = x^p \text{ for any } p \geq 0.$$

Using the relations (2.2) and (2.3) we obtain that

$$(2.6) \quad \begin{aligned} \mu_p &= \int_0^x \frac{t^p k(x, t)}{x^p (x^\beta - t^\beta)^\alpha} dt \\ &= \int_0^1 \frac{s^p \hat{k}(s)}{(1 - s^\beta)^\alpha} ds \\ (2.7) \quad &= \frac{\Gamma(1 - \alpha)}{\beta} \sum_{j=1}^m a_j \frac{\Gamma(\frac{p+j}{\beta})}{\Gamma(\frac{p+j}{\beta} + 1 - \alpha)}. \end{aligned}$$

To evaluate the above integral, we have made use of the well-known result

$$(2.8) \quad \int_0^1 y^{\frac{p+j}{\beta}-1} (1-y)^{-\alpha} dy = \frac{\Gamma(1-\alpha) \Gamma(\frac{p+j}{\beta})}{\Gamma(\frac{p+j}{\beta} + 1 - \alpha)}.$$

This shows that the function $\varphi_p(x) = x^p (p \geq 0)$ is an eigenfunction of the operator G with the eigenvalue μ_p .

In particular, the function $\varphi_n(x) = x^n \in S_r$ will satisfy the equation

$$(2.9) \quad G^{\pm 1}(\varphi_n)(x) = \mu_n^{\pm 1} \varphi_n(x),$$

where

$$(2.10) \quad \mu_n = \frac{\Gamma(1-\alpha)}{\beta} \sum_{j=1}^m a_j \frac{\Gamma(\frac{n+j}{\beta})}{\Gamma(\frac{n+j}{\beta} + 1 - \alpha)}.$$

Using Stirling's formula, we get the asymptotic relation

$$\mu_p = \frac{\Gamma(1-\alpha)}{\beta(\frac{p}{\beta})^{1-\alpha}} \sum_{j=1}^m a_j \left\{ 1 + O\left(\frac{1}{p}\right) \right\},$$

giving

$$\mu_p \sim \beta^{-\alpha} \Gamma(1-\alpha) p^{\alpha-1} \sum_{j=1}^m a_j \quad \text{for } p \rightarrow \infty,$$

from which the relations

$$(2.11) \quad \lim_{p \rightarrow \infty} \mu_p = 0$$

and

$$(2.12) \quad \lim_{p \rightarrow \infty} \frac{\mu_{p+1}}{\mu_p} = 1$$

follow.

Now we consider any function $f(x) \in S_r$. The Taylor series expansion along with result (2.9) leads to the diagonalization of the operator G . We obtain that

$$(2.13) \quad G^{\pm 1}(f)(x) = \sum_{n=0}^{\infty} f_n \mu_n^{\pm 1} x^n.$$

Also, results (2.11) and (2.12) show that the radius of convergence of the Taylor series does not change while applying G or G^{-1} . The above observations can be put in the form of a theorem as given below.

THEOREM 1. *For any integer $n(\geq 0)$ the function $\varphi_n(x) = x^n \in S_r$ is an eigenfunction of the operator G with eigenvalue μ_n . The linear operators G and G^{-1} defined on S_r and the Taylor expansion lead to the diagonalization*

$$(2.14) \quad G^{\pm 1} : \sum_{n=0}^{\infty} f_n x^n \mapsto \sum_{n=0}^{\infty} f_n \mu_n^{\pm 1} x^n.$$

The operator G as well as its inverse G^{-1} map S_r onto S_r .

Remark 1. Theorem 1 can be applied to the functions in the space S_r . Let us consider an integral equation

$$(2.15) \quad G(g)(x) = f(x),$$

where $f(x) \in S_r$. Clearly, the solution function $g(x)$ also belongs to the class S_r . The solution of equation (2.15) can be expressed as

$$(2.16) \quad g(x) = \sum_{n=0}^{\infty} \frac{f_n}{\mu_n} x^n, \quad 0 \leq x < r.$$

A perturbation δf in the function $f(x)$ makes the error in the function $g(x)$ as given by

$$(2.17) \quad \delta g(x) = \sum_{n=0}^{\infty} \frac{\delta f_n}{\mu_n} x^n.$$

Since $\mu_n \rightarrow 0$ as $n \rightarrow \infty$, δg can be arbitrarily large even for an arbitrarily small perturbation δf , leading to instability in the solution. An easy way to make solution (2.16) stable is to truncate the series before the eigenvalues become very small. The rapid convergence of the scaled eigenfunctions $r^{-n}\varphi_n(x)$ to a very small value can be attained, at any point $x = x_0 (0 \leq x_0 < r)$, in accordance with our desirability.

For numerical purposes, the solution of the inversion problem (2.15) can be written as

$$(2.18) \quad g^{(N)}(x) = \sum_{n=0}^N \frac{f_n}{\mu_n} x^n,$$

which is an approximation to the unknown function $g(x)$, if N is not very large.

Remark 2. Let the function $k(x, t)$ involved in the kernel of the operator G be a more general function, which satisfies the following properties:

(i)

$$(2.19) \quad k(x, xs) = x^{\alpha\beta-1} \hat{k}(s) \quad \text{for } t = xs, \quad 0 < \alpha < 1, \quad \beta > 0$$

and

(ii)

$$(2.20) \quad \int_0^1 \frac{s^p \hat{k}(s)}{(1-s^\beta)^\alpha} ds \quad \text{exists for } p \geq 0, \quad 0 < \alpha < 1, \beta > 0.$$

Then it gives rise to an operator, which can be diagonalized as explained above.

Remark 3. If we take $\beta = \frac{1}{\alpha}$ and $k(x, t) \equiv 1$ in (1.3), we obtain the Abel operator

$$(2.21) \quad \hat{G}(g)(x) = \int_0^x \frac{g(t) dt}{(x^{\frac{1}{\alpha}} - t^{\frac{1}{\alpha}})^\alpha}.$$

We can easily apply Theorem 1 with the eigenvalue

$$\hat{\mu}_p = \alpha \Gamma(1 - \alpha) \frac{\Gamma(\alpha p + \alpha)}{\Gamma(\alpha p + 1)}.$$

Remark 4. In the case of a new operator F_α of Definition 5, we obtain that

$$(2.22) \quad \lambda_p := \int_x^\infty \frac{t^{-p}}{x^{-p}} \frac{dt}{(t^{\frac{1}{\alpha}} - x^{\frac{1}{\alpha}})^\alpha} = \frac{\alpha \Gamma(1 - \alpha) \Gamma(\alpha p)}{\Gamma(\alpha p + 1 - \alpha)} \quad \text{for } p > 0,$$

and this leads to the result

$$(2.23) \quad F_\alpha^{\pm 1}(\chi_p)(x) = \lambda_p^{\pm 1} \chi_p(x),$$

where

$$(2.24) \quad \chi_p(x) = x^{-p}.$$

We also have

$$(2.25) \quad \lambda_p \sim \alpha^\alpha \Gamma(1 - \alpha) p^{\alpha-1} \quad \text{for } p \rightarrow \infty,$$

giving

$$(2.26) \quad \lim_{p \rightarrow \infty} \frac{\lambda_{p+1}}{\lambda_p} = 1.$$

The above discussion leads to the following theorem.

THEOREM 2. *For any $n \geq 1$, the function $\chi_n(x) = x^{-n} \in R_0$ is an eigenfunction of the operator F_α , with the eigenvalue $\lambda_n = \frac{\alpha \Gamma(1-\alpha) \Gamma(\alpha n)}{\Gamma(\alpha n + 1 - \alpha)}$. The linear operators F_α and F_α^{-1} are defined on R_0 , and the Taylor expansion of $q(x) \in R_0$ leads to the diagonalization*

$$(2.27) \quad F_\alpha^{\pm 1} : \sum_{n=1}^{\infty} q_n x^{-n} \mapsto \sum_{n=1}^{\infty} \lambda_n^{\pm 1} q_n x^{-n}.$$

The operator F_α as well as its inverse F_α^{-1} map R_0 onto R_0 .

Remark 5. The operator A , defined in the paper of Knill (see [2]), is a special case ($\alpha = \frac{1}{2}$) of the operator F_α ; i.e.,

$$(2.28) \quad F_{\frac{1}{2}} \equiv A.$$

Remark 6. It is rather straightforward to derive the diagonalization of the operator H of Definition 6.

In order to come out with an eigenpair $(\varphi_p(x), \tilde{\mu}_p)$ for the operator H , we select

$$(2.29) \quad \varphi_p(x) = [h(x)]^p = h^p(x) (p \geq 0)$$

and then evaluate the integral

$$(2.30) \quad \int_0^x \frac{\varphi_p(t)k(x,t)dt}{(h^\beta(x) - h^\beta(t))^\alpha \varphi_p(x)} := \tilde{\mu}_p$$

and show that the value of the integral is a constant for fixed p .

Using conditions (1.7) and (2.29) in relation (2.30), we obtain that

$$(2.31) \quad \tilde{\mu}_p = \int_0^x \frac{[h(t)]^p}{[h(x)]^p} \frac{[h(x)]^{\alpha\beta-1} h'(t)}{(h^\beta(x) - h^\beta(t))^\alpha} dt.$$

Putting $h(x) = \xi$, $h(t) = \eta$ and using the condition (1.6) in equation (2.31), we obtain that

$$(2.32) \quad \begin{aligned} \tilde{\mu}_p &= \int_0^\xi \frac{\eta^p \xi^{\alpha\beta-1}}{\xi^p (\xi^\beta - \eta^\beta)^\alpha} d\eta \\ &= \int_0^1 \frac{s^p ds}{(1 - s^\beta)^\alpha} \\ &= \frac{\Gamma(1-\alpha)}{\beta} \frac{\Gamma(\frac{p+1}{\beta})}{\Gamma(\frac{p+1}{\beta} + 1 - \alpha)}, \end{aligned}$$

derived by using the result

$$(2.33) \quad \int_0^1 y^{\frac{p+1}{\beta}-1} (1-y)^{-\alpha} dy = \frac{\Gamma(1-\alpha)\Gamma(\frac{p+1}{\beta})}{\Gamma(\frac{p+1}{\beta} + 1 - \alpha)}.$$

This leads to the relation

$$(2.34) \quad H^{\pm 1}(\varphi_p)(x) = \tilde{\mu}_p^{\pm 1} \varphi_p(x),$$

where $\varphi_p(x)$ is given by relation (2.29) and $\tilde{\mu}_p$ is given by relation (2.32). Any $f(x) \in S_H$ has Taylor expansion as given by the relation (1.2). The Taylor expansion along with the result given by relation (2.34) leads to the diagonalization of the operator H . It is also very clear that the radius of convergence of the Taylor series does not change while applying H or H^{-1} . These observations can be put in the form of the following theorem.

THEOREM 3. *For any integer $n(\geq 0)$, the function $\varphi_n(x) = h^n(x) \in S_H$ is an eigenfunction of the operator H with the eigenvalue $\tilde{\mu}_n$. The linear operators H and H^{-1} defined on S_H and the Taylor expansion lead to the diagonalization*

$$(2.35) \quad H^{\pm 1} : \sum_{n=0}^{\infty} f_n h^n(x) \longmapsto \sum_{n=0}^{\infty} f_n \tilde{\mu}_n^{\pm 1} h^n(x).$$

The operator H as well as its inverse H^{-1} map S_H onto S_H .

Example 1. Let

$$(2.36) \quad h(x) = x, \quad \beta = \frac{1}{\alpha}, \quad \text{and } 0 < \alpha < 1.$$

Then

$$k(x, t) = [h(x)]^{1-\alpha} h'(t) \equiv 1,$$

and the operator $H \equiv \hat{G}$, as defined by relation (2.21), with its eigenvalues

$$(2.37) \quad \tilde{\mu}_p = \alpha \Gamma(1 - \alpha) \frac{\Gamma(\alpha p + \alpha)}{\Gamma(\alpha p + 1)} = \hat{\mu}_p$$

and

$$(2.38) \quad \tilde{\mu}_p \sim \alpha^\alpha \Gamma(1 - \alpha) p^{\alpha-1} \quad \text{for } p \rightarrow \infty.$$

Example 2. In the situation when

$$(2.39) \quad h(x) = \sin x, \quad \beta = \frac{1}{\alpha}, \quad 0 \leq x \leq \frac{\pi}{2}, \quad \text{and } 0 < \alpha < 1,$$

we obtain that

$$(2.40) \quad k(x, t) = \cos t.$$

The operator H is then given by the relation

$$(2.41) \quad H(g)(x) = \int_0^x \frac{g(t) \cos t}{(\sin^{\frac{1}{\alpha}} x - \sin^{\frac{1}{\alpha}} t)^\alpha} dt.$$

In this case

$$(2.42) \quad \tilde{\mu}_n = \frac{\alpha \Gamma(1 - \alpha) \Gamma(\alpha n + \alpha)}{\Gamma(\alpha n + 1)} \quad (n \geq 0),$$

and

$$\left(\sin^n x, \frac{\alpha \Gamma(1 - \alpha) \Gamma(\alpha n + \alpha)}{\Gamma(\alpha n + 1)} \right)$$

is the eigenpair.

The solution of the equation

$$(2.43) \quad \int_0^x \frac{g(t) \cos t}{(\sin^{\frac{1}{\alpha}} x - \sin^{\frac{1}{\alpha}} t)^\alpha} dt = f(x)$$

is then given by the formula

$$(2.44) \quad g(x) = \frac{1}{\alpha \Gamma(1 - \alpha)} \sum_{n=0}^{\infty} \frac{f_n \Gamma(\alpha n + 1)}{\Gamma(\alpha n + \alpha)} \sin^n x,$$

provided

$$(2.45) \quad f(x) = \sum_{n=0}^{\infty} f_n \sin^n x \in S_H.$$

Conclusion. Abel integral equations with a rather general class of kernels have been shown in this paper to be solvable by the method of diagonalization which is an idea of recent origin and concern. Specific examples are taken up to support the general method.

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