

On How to Produce Entangled States Violating Bell's Inequalities in Quantum Theory

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Feynman's path integrals provide a hidden variable description of quantum mechanics (and quantum field theories). The time evolution kernel is unitary in Minkowski time, but generically it becomes real and non-negative in Euclidean time. It follows that the entangled state correlations, that violate Bell's inequalities in Minkowski time, obey the inequalities in Euclidean time. This observation emphasises the link between violation of Bell's inequalities in quantum mechanics and unitarity of the theory. Search for an evolution kernel that cannot be conveniently made non-negative leads to effective interactions that violate time reversal invariance. Interactions giving rise to geometric phases in the effective description of the theory, such as the anomalous Wess-Zumino interactions, have this feature. I infer that they must be present in any set-up that produces entangled states violating Bell's inequalities. Such interactions would be a crucial ingredient in a quantum computer.

I. FEYNMAN'S PATH INTEGRAL AS A HIDDEN VARIABLE THEORY

It is common, and often more convenient, to study quantum mechanics using the Schrödinger/Dirac equations and the Heisenberg operator algebra. Feynman's path integral formulation of quantum mechanics [1], nonetheless, is an alternative approach from which all the known results of quantum mechanics can be derived. In fact, quantum field theories are studied, more often than not, using the path integral formulation. For simplicity, let us consider quantum mechanics in one space dimension. The total amplitude (or wavefunction) for the system to be in the state $\psi(x_f, T)$ at time $t = T$, given an initial state $\psi(x_i, 0)$, is defined in terms of the transition kernel $K(x_f, T; x_i, 0)$:

$$\psi(x_f, T) = \int_{-\infty}^{\infty} K(x_f, T; x_i, 0) \psi(x_i, 0) dx_i \quad , \quad (1.1)$$

$$K(x_f, T; x_i, 0) = \int_0^T [Dx(t)] \exp\left[\frac{i}{\hbar} S(x(t))\right] \quad . \quad (1.2)$$

Here the action for a particular trajectory is the time integral of the corresponding Lagrangian,

$$S(x(t)) = \int_0^T L(x(t)) dt \quad . \quad (1.3)$$

The integration measure $[Dx(t)]$ can be defined precisely by discretising the time interval:

$$[Dx(t)] \propto \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} dx_j \quad , \quad x_j \equiv x(t = jT/N) \quad . \quad (1.4)$$

Transition matrix elements and other expectation values are defined with the straightforward prescription:

$$\begin{aligned} \langle \mathcal{O}(x(t)) \rangle &\equiv \langle \psi(x_f, T) | \mathcal{O}(x, t) | \psi(x_i, 0) \rangle \\ &= \int [Dx(t)] \rho_M(x(t)) \mathcal{O} \quad , \end{aligned} \quad (1.5)$$

$$\rho_M(x(t)) = \frac{\exp\left[\frac{i}{\hbar} S(x(t))\right]}{\int [Dx(t)] \exp\left[\frac{i}{\hbar} S(x(t))\right]} \quad . \quad (1.6)$$

These definitions provide a hidden variable description of quantum mechanics. Indeed, x_j ($j = 1, \dots, N-1$) are the hidden variables which are integrated over [2]. There is no need to worry about ordering of various factors, because there are no non-commuting operators in this Lagrangian description, only complex numbers [3]. The functional integration measure $[Dx(t)]$ represents a sum over all paths connecting the fixed initial and final states; such a sum over paths is inherently a non-local object [4]. $\exp\left[\frac{i}{\hbar} S(x(t))\right]$ is the statistical weight of the path $x(t)$ which depends on the interaction amongst the particles in the system. It is well-known that the typical paths contributing to the functional integral are highly irregular and non-differentiable. The individual paths characterised by $\{x_j\}$ follow local history/dynamics [5], but they do not obey constraints of causality. The virtual intermediate states, for example, can be "off-shell" with no relation between their energies and momenta, and they can propagate at a speed faster than that of light. They can even propagate backwards in time which is interpreted as pair creation and annihilation in relativistic field theories. The total amplitude (i.e. the sum over all paths), however, obeys all the constraints of causality and conservation laws.

Note that $\{x_j\}$ corresponding to one particle are totally independent of those for another. For a system of non-interacting particles, the path integral completely factorises, e.g. $[Dx(t)] = [Dx^{(1)}(t)] [Dx^{(2)}(t)]$ and $S(x(t)) = S(x^{(1)}(t)) + S(x^{(2)}(t))$. In such a case, all the correlations/entanglements amongst the particles are built into the specification of the initial state coordinates, $x_i^{(n)}$. The subsequent evolution of the system and

measurement of single particle properties clearly separate into independent components corresponding to each particle. Such a separation, often dubbed the “Einstein locality” property of the hidden variables, is an important ingredient in the proof of Bell’s inequalities [6]. It ensures that what is measured for one particle is not all influenced by either the property being measured or the choice of the measurement apparatus for the other particle. It has to be stressed that even though the path integrals are non-local while describing single particle evolution, they comply with Einstein locality while describing inter-particle correlations [7].

The measurement process here corresponds to restricting the sum over all possible paths to only those paths which are consistent with the measured observable having a specific value in the final state. This is obviously a contextual process, once all the hidden variables are integrated out; if a second measurement were to be carried out on the system, the sum over all paths would be restricted to paths which are consistent with the results of both the first and the second measurements—the paths which would be consistent with the second measurement but not with the first have been discarded by the act of the first measurement. Thus the measured value of an observable may depend on the results of other measurements carried out prior to it on the same system, when these other measurements correspond to commuting but correlated observables [8]. Note that the order in which the restriction of paths is carried out is immaterial, so there is no ambiguity regarding the final state of the system when measurements are carried out at space-like separations.

All put together, path integrals describe a contextual and non-local hidden variable theory; as a matter of fact, a non-zero value of \hbar and Bell’s theorems ensure that any hidden variable theory describing quantum mechanics has to have such peculiarities [9]. It is similar in many aspects to the de Broglie-Bohm theory [10], but with the clear advantage that it is describable in a simpler language and that it is much more amenable to detailed calculations [11]. The crucial feature in this description is that the integration weight, $\rho_M(x(t))$, is a complex number in general. Therefore, although it is bounded, it cannot be interpreted as a probability density. It is this fact which allows the path integral description to bypass Bell’s inequalities and give a true definition of quantum mechanics [12].

II. WIGNER FUNCTION AND PHASE SPACE FORMULATION

It is often considered desirable (although it is not at all necessary for the proof of Bell’s inequalities) that a hidden variable description of quantum mechanics would provide simultaneous reality to non-commuting physical observables. For example, simultaneous statistical

weight can be given to positions and momenta by formulating the theory in the phase space. Feynman actually showed how one can reconcile the EPR paradigm [13] with quantum mechanics using hidden variables, provided that probabilities are allowed to become negative [14]. He used the well-known Wigner function [15], which is a particular realisation of the density matrix distribution in the phase space:

$$W(x, p) = \int dy \psi^*(x + \frac{1}{2}y) \exp(\frac{i}{\hbar}py) \psi(x - \frac{1}{2}y) . \quad (2.1)$$

Wigner functions are real [16] and obey all the usual manipulations of probability theory, except that they are not always non-negative everywhere in the phase space. Since the expectation value for any physical observable is just

$$\langle \mathcal{O} \rangle = \int dx dp W(x, p) O(x, p) , \quad (2.2)$$

where $O(x, p)$ is the (Hermitian) operator weight corresponding to the observable \mathcal{O} , it is necessary that for situations violating Bell’s inequalities the Wigner function be negative somewhere in the phase space.

Path integrals can also be formulated in the phase space. The transition kernel is expressed as

$$K(x_f, T; x_i, 0) = \int_0^T [Dx][Dp] \exp[\frac{i}{\hbar}S(x, p)] , \quad (2.3)$$

with the action rewritten in terms of the Hamiltonian as

$$S(x, p) = \int_0^T [p \frac{dx}{dt} - H(x, p)] dt . \quad (2.4)$$

Though this is the more general formulation, it has become customary (and convenient) to completely integrate out either the $\{x_j\}$ or the $\{p_j\}$ variables and work with the Lagrangian description. Generic operators $O(x, p)$ need to be converted to either the x - or the p -language, using Fourier transforms wherever necessary. There is no loss of predictive power, but some care is needed in time-ordering products of non-commuting operators. For instance, the commutation relation, $[\hat{x}, \hat{p}] \neq 0$, is realised as [1]:

$$\lim_{N \rightarrow \infty} \langle x_j m \frac{x_j - x_{j-1}}{T/N} - m \frac{x_{j+1} - x_j}{T/N} x_j \rangle \neq 0 . \quad (2.5)$$

Apart from having a fewer number of variables to deal with, a striking advantage of the Lagrangian formulation is that the initial states of variables can be specified freely, subject to only the normalisation constraint. On the other hand, initial state phase space distributions cannot be arbitrary for physical situations; they have to satisfy restrictions following for example from the uncertainty principle, $\langle \Delta x \rangle \langle \Delta p \rangle \geq \frac{\hbar}{2}$.

III. ROLE OF UNITARITY

Now let us apply the familiar trick of rotating to Euclidean (imaginary) time, $\tau = it$. This Wick rotation converts the quantum theory to the language of statistical mechanics. The integration weight,

$$\rho_E(x(\tau)) = \frac{\exp[-\frac{1}{\hbar}S(x(\tau))]}{\int [Dx(\tau)] \exp[-\frac{1}{\hbar}S(x(\tau))]} \quad , \quad (3.1)$$

is now real and non-negative, and can be interpreted as a probability density. This formal property has been exploited before, for vector-like gauge theories at zero chemical potential [17], to derive rigorous inequalities among correlation functions and particle masses [18,19]. The point I want to emphasise in this article is that a non-negative integration weight must obey Bell's inequalities [20].

Suppose that we can unambiguously predict the Minkowski time results from Euclidean time ones. Then we have a hidden variable prescription for describing quantum mechanics. This cannot be correct. We have to identify ways out of this situation, and they will lead us to the origin of the violations of Bell's inequalities. There are two options: (1) Something is lost in the analytic continuation from Minkowski to Euclidean time, which prevents complete reconstruction of Minkowski time results from Euclidean time ones. (2) The integration weight obtained by analytic continuation to Euclidean time is not non-negative, may be even complex. I explore this possibility in section V.

Let us first note that the above mentioned analytic continuation is routinely employed in quantum field theories in dealing with divergent loop integrals and renormalisation. As a matter of fact, there are strong theorems governing such an analytic continuation, e.g. the Wightman axioms [21] and the Osterwalder-Schrader positivity of the transfer matrix [22]. In particular, in the complex energy plane, these theorems rely on a sufficiently fast decrease of the amplitudes at infinity and on there being no singularities in the region covered by the rotation [23]. The integration contours can then be freely rotated without affecting the value of the integrals.

Let us also note that statistical mechanics has many features in common with quantum theory. The density matrix description allows probabilistic interpretation of superposed and mixed states. The presence of the heat-bath [24] gives rise to vacuum fluctuations and unconstrained behaviour of the virtual states. Non-zero commutators (Poisson brackets) leading to the uncertainty principle, e.g. $[\hat{x}, \hat{p}] \neq 0$, exist in the Hamiltonian description of the theory. Non-zero tunnelling amplitudes exist in the Euclidean time theory, and so do “grotesque” states with infinitesimal probabilities [25]. These shared features cannot be responsible for the violation of Bell's inequalities in quantum mechanics.

What the Euclidean time theory lacks is the key concept of unitarity. The Minkowski time transi-

tion kernel, $K_M(x_f, T; x_i, 0)$, corresponds to a unitary transformation—the familiar S -matrix [26]. It is easy to see that a unitary matrix with only real and positive matrix elements has to be the identity matrix (or its row-wise permutation corresponding to a shuffling of the states). The Euclidean time transition kernel is less restricted—it does not preserve the norm, though it maintains orthogonality of the states—and can be represented by a diagonal non-negative definite matrix. The loss of normalisation is not critical, since it can be taken care of following the LSZ prescription [27]. The information about the relative phases of the states, however, is lost. In fact this is the crucial quantum mechanical feature which makes the requirements of unitarity and a real non-negative integration weight mutually incompatible. One can pick situations where the information contained in the relative phases cannot be made arbitrary small, because there are constraints on the complex amplitudes following, for example, from analyticity and dispersion relations [28]. In such cases, a statistical mechanics description cannot provide an arbitrarily close approximation to quantum mechanics. Correlations violating Bell's inequalities can be present in such situations.

Let us look at the problem again from a slightly different angle. The Euclidean time correlation functions defined along the imaginary time axis are real, and a mere analytic continuation of them cannot produce non-trivial complex phase shift factors that are an essential part of Minkowski time scattering amplitudes [29]. *A priori* one does not know whether in analytical continuation of the Euclidean time results the Euclidean time axis should be rotated by $+90^\circ$ or -90° to reach the Minkowski time axis. This ambiguity automatically gets resolved for internal loop variables of Feynman diagrams, just due to the necessity of not crossing any singularities while rotating the integration contours. For the variables corresponding to the external legs (incoming and outgoing states), however, the choice must be enforced as an additional condition; it amounts to the difference between choosing advanced or retarded propagators. Note that given the Euclidean time results, one can construct either the advanced or the retarded (or a linear combination thereof) amplitudes. The criterion of retarded amplitudes only, though physically motivated by principle of causality, is absent in the Euclidean time theory and has to be enforced as an additional input.

More explicitly, the principle of causality is embedded in the use of $i\epsilon$ -prescription for retarded propagators. For example, the naive analytic continuation of the Euclidean time scalar boson propagator, $(k^2 + m^2)^{-1}$, to $(k^2 - m^2)^{-1}$ is incorrect. The correct physical prescription is:

$$\begin{aligned} \frac{1}{k^2 + m^2} &\rightarrow \lim_{\epsilon \rightarrow 0} \frac{1}{k^2 - m^2 + i\epsilon} \\ &= \text{P}\left(\frac{1}{k^2 - m^2}\right) - i\pi\delta(k^2 - m^2) \quad . \end{aligned} \quad (3.2)$$

The non-unitary Euclidean time theory fixes the off-shell

amplitude (the principle value part) completely, but it is necessary to add an on-shell contribution (exemplified by the δ -function at the pole) to comply with unitarity. This subtlety is unimportant in many cases: location of poles and branch cuts, matrix elements with external legs amputated according to the LSZ prescription, scattering amplitudes at threshold which are real etc. can be determined without recourse to the $i\epsilon$ -prescription. Indeed Monte Carlo simulations of quantum field theories on the lattice calculate all such quantities in the Euclidean time framework. On the other hand, the features that can differentiate between advanced and retarded amplitudes (e.g. S -matrix phase shifts, discontinuities across branch cuts etc.) are not directly accessible in the Euclidean time framework.

IV. ANALYSIS OF FAMILIAR EXAMPLES

A. Original EPR correlations

The conventional set up illustrating Bell's inequalities is the case where there is just free propagation of particles after initial creation of the state. Let us first look at the original EPR example [13]—a system of two identical non-relativistic particles freely propagating in one space dimension, where the initial state of the two particles is perfectly correlated in space (hence anti-correlated in momentum) with the constraint

$$\delta(x_i - y_i) = \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}p(x_i - y_i)\right) . \quad (4.1)$$

For a single free particle of mass m , the propagation kernel for Minkowski time T is

$$K_M(x_f, T; x_i, 0) = \sqrt{\frac{m}{2\pi i\hbar T}} \exp\left(i\frac{m(x_f - x_i)^2}{2\hbar T}\right) , \quad (4.2)$$

while for Euclidean time T it is

$$K_E(x_f, T; x_i, 0) = \sqrt{\frac{m}{2\pi\hbar T}} \exp\left(-\frac{m(x_f - x_i)^2}{2\hbar T}\right) . \quad (4.3)$$

Thus the final state wavefunction for the two particle system is

$$\psi_M(x_f, y_f, T) = \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}p(x_f - y_f)\right) \exp\left(\frac{ip^2 T}{\hbar m}\right) \quad (4.4)$$

in Minkowski time, and

$$\psi_E(x_f, y_f, T) = \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}p(x_f - y_f)\right) \exp\left(-\frac{p^2 T}{\hbar m}\right) \quad (4.5)$$

in Euclidean time. In both cases the theory maintains its contextual character; the structure of the kernel ensures that if one particle is detected with momentum p_f , the other is bound to be found with momentum $-p_f$.

The relative probability of observing various values of momentum is different in the two cases, however, which is just due to the difference in normalisation of states. This difference is not small as can be seen by appealing to the uncertainty relation: In order to detect the two particles distinctly, their separation has to be much larger than their de Broglie wavelengths,

$$|x_f - y_f| \sim p_f T / m \gg \hbar / p_f , \quad (4.6)$$

implying that $p^2 T \gg \hbar m$ in the exponent. In this particular example, the exponent can be removed by just following the LSZ prescription, and there is no conflict with any inequality.

The reason behind no conflict with any inequality in the above example is that the δ -function correlation is non-negative. A more general case [31] will have initial state correlations such that the Wigner function becomes negative somewhere in the phase space [32]. The density matrix evolves linearly in time according to:

$$\begin{aligned} \frac{dW_M}{dt} &= -\frac{i}{\hbar}[H, W_M] \implies \\ W_M(t) &= \exp(-\frac{i}{\hbar}Ht)W_M(0)\exp(\frac{i}{\hbar}Ht) , \\ \frac{dW_E}{d\tau} &= -\frac{1}{\hbar}[H, W_E] \implies \\ W_E(\tau) &= \exp(-\frac{1}{\hbar}H\tau)W_E(0)\exp(\frac{1}{\hbar}H\tau) , \end{aligned} \quad (4.7)$$

where H is the Hamiltonian for the system. In case of a non-negative time evolution kernel, interference effects can only annihilate the negative density matrix regions in the phase space; a limit is reached when the Wigner function becomes non-negative everywhere and thereafter no regions of negative Wigner function can be regenerated [33]. Explicitly

$$\begin{aligned} f(\tau) &= \int dx dp |W(x, p; \tau)| / \int dx dp W(x, p; \tau) , \\ f(\tau) &\geq 1 , \quad df(\tau)/d\tau \leq 0 . \end{aligned} \quad (4.8)$$

No such reduction of negative density matrix regions is expected in case of a unitary time evolution; in fact the Minkowski time evolution of the density matrix obeys the continuity equation [15]:

$$\frac{\partial W(x, p)}{\partial t} + \frac{p}{m} \frac{\partial W(x, p)}{\partial x} = 0 . \quad (4.9)$$

Thus, after factoring out the normalisation of states, the residual correlations which typify Bell's inequalities, become substantially different in the Euclidean time case from the corresponding Minkowski time values. It can be argued that a sense of probability description can be retained in Euclidean time, if one replaces $\psi(t) \rightarrow \psi(\tau)$

and $\psi^*(t) \rightarrow \psi^*(-\tau)$ [34]. But the resultant expression is so non-local in Euclidean time that it is not possible to assign physical meaning to it.

To summarise, the Euclidean time evolution cannot dynamically create correlations violating Bell's inequalities if such correlations are absent in the initial state. Moreover, it wipes out such correlations even when they are inserted by hand in the initial state.

B. Bohm-Aharanov correlations

Violation of Bell's inequalities in quantum mechanics is often demonstrated using two non-relativistic spin- $\frac{1}{2}$ particles in a singlet state [35]. It is well-known that the Wigner function for this state has negative elements [14,36]. A path integral representation to $SU(2)$ -spin dynamics can be given in terms of coherent states [37]. These coherent states are represented by unit vectors \vec{n} from the origin to points on a two-dimensional unit sphere S_2 [38]. The conventional spin- $\frac{1}{2}$ eigenstates are represented by the unit vectors pointing to the north and south poles:

$$|\uparrow\rangle = |\vec{n}_0\rangle, \quad |\downarrow\rangle = |-\vec{n}_0\rangle. \quad (4.10)$$

A general coherent state is obtained by rotating the reference vector \vec{n}_0 to \vec{n} ,

$$|\vec{n}\rangle = \exp(\frac{i}{2}\theta\vec{a}\cdot\vec{\sigma})|\uparrow\rangle, \quad (4.11)$$

where \vec{a} is the unit vector in the direction $\vec{n}_0 \times \vec{n}$, and σ_i are the usual Pauli matrices. The set of coherent states is overcomplete, with a uniform and positive integration measure over S_2 . One can revert back to the conventional basis by integrating over the polar angles θ, ϕ parametrising S_2 .

In this coherent state picture, the spin-dynamics is described by the time evolution $|\vec{n}(t)\rangle$. For every spin, the Minkowski time action contains a topological Wess-Zumino term:

$$S_M(\vec{n}) = \frac{1}{2}S_{WZ}(\vec{n}) + \frac{i\delta t}{8} \int_0^T dt (\partial_t \vec{n}(t))^2 + S_{interaction}. \quad (4.12)$$

For a spin in a magnetic field, $S_{interaction} \propto \frac{1}{2} \int_0^T dt \vec{n} \cdot \vec{B}$. The Wess-Zumino term is best expressed by embedding the system into one higher dimension,

$$S_{WZ}(\vec{n}) = \int_0^1 ds \int_0^T dt [\vec{n}(t, s) \cdot \partial_t \vec{n}(t, s) \times \partial_s \vec{n}(t, s)], \quad (4.13)$$

where $\vec{n}(t, 0) \equiv \vec{n}(t)$, $\vec{n}(t, 1) \equiv \vec{n}_0$. The evolution kernel for a single free spin is given by the propagator,

$$K_M(\vec{n}_f, T; \vec{n}_i, 0) = \langle \vec{n}_f | \vec{n}_i \rangle = \exp(\frac{i}{2}\Phi(\vec{n}_i, \vec{n}_f, \vec{n}_0)) \sqrt{\frac{1}{2}(1 + \vec{n}_f \cdot \vec{n}_i)}, \quad (4.14)$$

where $\Phi(\vec{n}_i, \vec{n}_f, \vec{n}_0)$ is the area of the geodesic spherical triangle formed by $\vec{n}_i, \vec{n}_f, \vec{n}_0$. The first factor in the kernel is a unitary phase coming from the Wess-Zumino term, while the second factor is non-negative.

A peculiarity of the Wess-Zumino term is that it contains an odd number of time derivatives (the action in the previous subsection had only even number of time derivatives). As a result, its contribution to the exponent of the path integral weight is imaginary in both Minkowski and Euclidean times. After Wick rotation, the Euclidean time kernel is still a complex number and cannot be interpreted as a probability density. As a matter of fact, for a single free spin, the Minkowski and Euclidean time evolution kernels coincide, $K_M(\vec{n}_f, T; \vec{n}_i, 0) = K_E(\vec{n}_f, T; \vec{n}_i, 0)$.

Going back from the coherent state basis to the conventional one, it is easily seen that a free spin state remains unchanged under time evolution. Thus the entangled singlet state violating Bell's inequalities, $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$, remains the same under both Minkowski and Euclidean time evolutions.

At a deeper level, we can enquire how the unusual Wess-Zumino term came about. Spin is a property of the Poincaré group transformations describing free particles. Expressing the spin operator as the Pauli-Lubanski 4-vector, $ms^\mu = \frac{1}{2}\epsilon^{\mu\nu\sigma\tau}p_\nu J_{\sigma\tau}$, (here $J_{\sigma\tau}$ are the Lorentz generators), we notice that it has to transform as an axial vector under Wick rotation. A complete implementation of Minkowski to Euclidean time rotation described above would transform $\vec{s} \rightarrow -i\vec{s}$ (or $\vec{n} \rightarrow -i\vec{n}$), as appropriate for an angular momentum. Although such a transformation would apparently make the Euclidean time kernel non-negative, it totally destroys the underlying unitary structure of the rotation group $SU(2)$ [39]. (The same transformation is also required to make the interaction term of the spin with a magnetic field real in Euclidean time; under Wick rotation the magnetic field, generated by currents, transforms as $\vec{B} \rightarrow -i\vec{B}$.) It is not at all clear whether such an analytic continuation has any mathematical meaning, or whether such a positive counterpart to $SU(2)$ has any physical interpretation.

Historically, Wess-Zumino type of interactions have been labeled anomalous. This is largely because they violate apparently good symmetries; resolution of the puzzle requires a proper account of the internal properties of the variables. It is worthwhile to recollect that anomalies express global properties of the system and cannot be eliminated by local transformations [40], a property also shared by entangled states. The classic example is that of a neutral pion decaying into two entangled photons in a singlet state. A Wess-Zumino term describes this decay [41]. The properties of this term under Wick rotation are linked with the facts that the pion field is a pseudoscalar and parametrises a unitary group manifold.

V. ORIGIN OF ENTANGLEMENT

We are now ready to characterise what type of interactions would give rise to the entanglements that violate Bell's inequalities. Normally Bell's inequalities are not studied from this view point. They are instead specified by certain correlations in the initial state of a non-interacting system. In the real world, say in any S -matrix set-up where there are no correlations whatsoever in the initial state, we have to analyse how the interactions dynamically generate the correlations appearing in the final state. The question becomes much more important, when one takes the view that correlations are all there is to quantum theory [42].

An initial state without correlations can easily be described by a Wigner function non-negative everywhere in the phase space. In such a case, we have seen above that a non-negative evolution kernel cannot create correlations violating Bell's inequalities. It is easy to see that with a time reversal invariant Lagrangian, the Wick rotation converts a Minkowski time kernel to a non-negative Euclidean time kernel:

$$\begin{aligned} iS_M &= i \int dt L_M(t) , \quad L_M(t) = L_M(-t) \\ \implies -S_E &= - \int d\tau L_E(\tau) , \quad L_E(\tau) = L_E(\tau^*) . \end{aligned} \quad (5.1)$$

Needless to say, complexification of time provides a continuous route to reach a time-reversed configuration; Wick rotation is just proceeding half way along this route.

We note that though violation of time reversal symmetry, \mathcal{T} , is present in the standard model of interactions of fundamental particles [43], it is tiny and inconsequential in problems involving violations of Bell's inequalities. There also exist Boltzmann's famous H -theorem describing monotonic increase of entropy, and dissipative terms that come about from interaction of the system with its environment, but such irreversibility is of no concern for the problem at hand (unitary evolution in quantum theory is reversible). We thus restrict ourselves to situations where the fundamental underlying theory is time reversal invariant. It is important to realise that this assumption does not forbid localised arrangements that do not respect time reversal symmetry. For example, a magnetic field is not time reversal invariant, although the underlying QED is, and we can very well produce magnetic fields in our laboratories.

Several examples of interactions that respect unitarity of the evolution kernel, and yet violate \mathcal{T} , are known. Generically they can be described in terms of geometric phases (also called Berry's adiabatic phases) [44,45]:

$$iS_{Geom}(T) = i \int_0^T dt \vec{A}(\vec{R}) \cdot \frac{d\vec{R}}{dt} = i \int_0^T \vec{A}(\vec{R}) \cdot d\vec{R} . \quad (5.2)$$

Here \vec{R} labels the states in the quantum Hilbert space of the system, and \vec{A} is an effective gauge potential in this space. Different physical situations are described by different group theoretical structure and holonomy of \vec{A} . Examples are [45]: Pancharatnam's and Guoy's phases in optics, correction phase to Born-Oppenheimer approximation in atomic/molecular physics, Aharonov-Bohm phase in electrodynamics, and Wess-Zumino terms describing anomalous interactions. All these phases encode global properties of the system, a feature that is mandatory for describing entangled states.

The situation is best described in the framework of effective theories; all the degrees of freedom of the fundamental theory that are not observed and that are of no direct interest are summed over (say using the path integral formalism). The remaining observed degrees of freedom may be composite in terms of the fundamental ones. The underlying fundamental theory dictates the nature of these effective degrees of freedom and the effective interactions they have amongst themselves. Effective interactions describable by a geometric phase become possible, only when the effective degrees of freedom possess a unitary symmetry as well as nontrivial \mathcal{T} transformation property in the effective theory. Within the domain of the effective theory, the internal properties of the effective degrees of freedom are not available for external manipulations. We can therefore redefine \mathcal{T} as acting only on the effective interactions, while leaving the internal properties of the effective degrees of freedom untouched. The effective interactions can then create correlations violating Bell's inequalities amongst the effective degrees of freedom. Wess-Zumino terms are an important example of such interactions. Given the existence of these anomalous interactions in the examples above, we can even say that \mathcal{T} -invariance of Poincaré group transformations for free particles requires the spin to be an axial vector, and \mathcal{T} -invariance of QCD requires the pion to be a pseudoscalar.

Correlations violating Bell's inequalities can also be in the internal symmetry space of a system, e.g. isospin and colour correlations amongst quarks making up mesons and baryons. In such cases, the charge conjugation symmetry of the appropriate internal symmetry group, \mathcal{C} , plays the same role that the time reversal symmetry, \mathcal{T} , did in the above analysis. (In essence, the effective theory description above combined a unitary symmetry with the internalised time reversal property, in such a way that the combination transforms the same way under \mathcal{T} as it would under \mathcal{C} for the unitary symmetry.) The crucial connection with unitarity is maintained because the symmetry groups in quantum theory are unitary or subgroups of unitary groups [46]. These internal space correlations, unfortunately, are not directly accessible for creation/verification in an experimental set-up.

Finally we observe that the scattering phase shifts discussed in section III change sign under time reversal; a feature of the amplitude that can distinguish between

advanced and retarded behaviour must be odd under \mathcal{T} .

VI. SUMMARY AND OUTLOOK

The purpose of this article is to highlight some mathematical features of quantum mechanics crucial to creation of entangled states that violate Bell's inequalities. The deeper philosophical issues have been kept in the background, and explanation of technicalities has been relegated to footnotes. It is impertinent to ask why nature opted for unitarity or the Minkowski metric. What I have aimed to emphasise is that it is the ubiquitous appearance of “ i ” together with a non-zero value of “ \hbar ”, so characteristic of quantum physics, that sets it apart from classical physics: “ i ” is responsible for correlations violating Bell's inequalities, while “ $\hbar \neq 0$ ” is responsible for non-locality. A lesson to be learned is that care must be exercised in converting results of Euclidean time field theory to physical amplitudes; the restrictions following from unitarity have to be kept in mind.

From the practical viewpoint, effective interactions that do not obey the discrete \mathcal{T} symmetry would be indispensable in a quantum computer. After all, the end result of a quantum computation is nothing but a correlation between the input and the output. In brief, a quantum computer operates on qubits (2-state quantum systems) with unitary operations, and achieves speed-up over its classical counterpart by clever implementation of superposition/phase-rotation and entanglement/interference. While superposition can be easily achieved by preparing the quantum state in one basis and afterwards using/observing it in another, entanglement of initially uncorrelated quantum bits would require the type of “anomalous” interactions discussed above. A typical component of a quantum computer is the 2-qubit controlled-not operation (quantum generalisation of the Boolean exclusive-OR), which can convert superposition into entanglement:

$$(|\uparrow\rangle + |\downarrow\rangle)|\downarrow\rangle \xrightarrow[\text{not}]{\text{controlled}} |\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle. \quad (6.1)$$

It has been implemented using scattering phase shifts [48], and effective spin- $\frac{1}{2}$ state representations [49,50].

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- [1] R. Feynman, Rev. Mod. Phys. **20**, 367 (1948); R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
- [2] Hidden variables are commonly interpreted to be as yet undiscovered internal degrees of freedom. But this need not be so. As far as Bell's theorems are concerned, hidden variables can be “any” set of variables, internal or external, that are summed over and cannot be observed.
- [3] Fermions can be dealt with along similar lines using Grassmann variables, but I do not consider them here.
- [4] The integration measure can be made local by suppressing the quantum fluctuations in the formal limit, $\hbar \rightarrow 0$. When referring to a single object, my use of the adjective “local” or “non-local” means whether at a fixed instant of time it is point-like or spread over a region.
- [5] This is true in quantum mechanics only for non-interacting particles, since long range inter-particle interaction potentials would produce instantaneous action-at-a-distance. The more general framework of quantum field theories can produce instantaneous action-at-a-distance from local Lagrangian densities by exchange of virtual particles.
- [6] J. Bell, Physics (NY) **1**, 195 (1964); J. Bell, *Speakable and Unsayable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).
- [7] The use of the same adjective “local” to describe both single particle evolution and inter-particle correlations is unfortunate, but I have no option short of inventing a new phrase.
- [8] This way of restricting the sum over paths is similar to Bayes' rule for conditional probabilities.
- [9] A recent review on the implications of Bell's theorems can be found in: N. D. Mermin, Rev. Mod. Phys. **65**, 803 (1993). “Physical locality” is obeyed in quantum mechanics for all observable quantities (i.e. expectation values and correlations), and no real signal can propagate faster than the speed of light. “Quantum non-locality” refers to the behaviour of presumed unobservable objects used in the mathematical formulation of quantum mechanics (e.g. states, wavefunctions, path trajectories etc.). Such hidden variables crop up when actual results of experiments are compared with hypothetical results of experiments that could have been performed but were not performed, and Bell's theorems demand that these hidden variables be non-local. See also Ref. [42].
- [10] D. Bohm, Phys. Rev. **85**, 166 (1952); 180 (1952).
- [11] In particular, we know how to generalise from non-relativistic to relativistic case, and from quantum mechanics to quantum field theory.
- [12] For example, there can be cancellation between different paths giving rise to interference effects.
- [13] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935); A. Einstein, in *Albert Einstein, Philosopher Scientist*, edited by P. Schilp, Library of Living Philosophers (Evanston, Illinois, 1949).
- [14] R. Feynman, Int. J. Theor. Phys. **21**, 467 (1982); R. Feynman, “Negative probability” in *Quantum Implications: Essays in honour of David Bohm*, edited by B.

- Hiley and F. David Peat, (Routledge & Kegan Paul, London, 1987), p. 235.
- [15] E. Wigner, Phys. Rev. **40**, 749 (1932).
 - [16] The density matrix is a Hermitian operator, and can always be made real by a suitable choice of basis. Thereafter, orthogonal transformations are sufficient to describe its evolution.
 - [17] QED and QCD belong to this class, in particular the example of para-positronium or neutral pion decaying into two correlated photons.
 - [18] D. Weingarten, Phys. Rev. Lett. **51**, 1830 (1983).
 - [19] C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984); E. Witten, Phys. Rev. Lett. **51**, 2351 (1983).
 - [20] I use the term “Bell’s inequalities” in this article to generically denote the type of inequalities proved by Bell as well as by others.
 - [21] R. Streater and A. Wightman, *PCT, spin and statistics, and all that* (Benjamin/Cummings, New York, 1964).
 - [22] K. Osterwalder and R. Schrader, Commun. Math. Phys. **31**, 83 (1973); Commun. Math. Phys. **42**, 281 (1975); K. Osterwalder and E. Seiler, Ann. Phys. **110**, 440 (1978).
 - [23] The poles corresponding to unstable states or resonances lie on the second Riemann sheet.
 - [24] $\hbar \neq 0$ corresponds to non-zero temperature.
 - [25] For instance, there is a non-zero probability for all the molecules of air in a room to collect themselves in one corner, but it would be silly to wait for such a thing to happen spontaneously.
 - [26] Throughout this article, I stick to discussion of pure quantum states. This is with a view that mixed states can be made pure by enlarging the Hilbert space of the system to include extra degrees of freedom (e.g. those of the environment).
 - [27] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento **1**, 1425 (1955).
 - [28] In fact, imposition of unitarity constraints on Euclidean time results provides a way to quantify/bound the violations of Bell’s inequalities.
 - [29] It is still possible to extract phase shifts from Euclidean time results using indirect methods, e.g. finite volume shifts in energy levels of a multi-particle system [30]. Conversion of cross sections to phase shifts using the optical theorem presupposes unitarity.
 - [30] M. Lüscher, Commun. Math. Phys. **105**, (1986) 153; Nucl. Phys. **B354**, 531 (1991).
 - [31] See for example: J. Bell, in *New Techniques and Ideas in Quantum Measurement Theory* (1986); reproduced as Chapter 21 in *Speakable and Unspeakable in Quantum Mechanics*, Ref. [6].
 - [32] A density matrix with negative entries is perfectly all right for quantum systems, though it would be anathema in statistical physics.
 - [33] Though we are talking about the density matrix here, the situation is quite analogous to the evolution of the wavefunction in Euclidean time: It is common knowledge that the ground state wavefunction in quantum mechanics has no nodes. Euclidean time evolution concentrates the wavefunction towards its ground state component, cutting down the excited state contributions responsible for nodes in the wavefunction.
 - [34] I thank Yigal Shamir for this remark.
 - [35] D. Bohm and Y. Aharonov, Phys. Rev. **108**, 1070 (1957).
 - [36] The Wigner function for a spin- $\frac{1}{2}$ particle (or any other two state system) can be written as a 2×2 matrix. The first index of this matrix can be the eigenstate index (e.g. σ_3), and then the second index would be the index for the operator generating transitions among the eigenstates (e.g. σ_1).
 - [37] See for instance: E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, New York, 1991); A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).
 - [38] $S^2 = SU(2)/U(1)$ forms the phase space for the dynamics of the “classical spin”. The $U(1)$ bundle is responsible for the geometric phase associated with the spherical area enclosed during the spin’s motion.
 - [39] When the generators are multiplied by a factor of $-i$, the transformed set of elements don’t even form a group. They can be embedded in a bigger group such as $SL(2, C)$, however.
 - [40] Extensive discussion of anomalies in field theories can be found in: S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies* (World Scientific, Singapore, 1985).
 - [41] E. Witten, Nucl. Phys. **B223**, 422 (1983).
 - [42] N. D. Mermin, Am. J. Phys. **66**, 753 (1998), [quant-ph/9801057](#).
 - [43] With the standard feature of CPT invariance in relativistic quantum field theories, T -violation and CP -violation are equivalent.
 - [44] M.V. Berry, Proc. R. Soc. London **A392**, 45 (1984).
 - [45] Many instances can be found in: A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1989).
 - [46] According to Wigner’s celebrated result [47], symmetry transformations in quantum theory have to be either unitary or antiunitary. The only known instance of antiunitary transformation is time reversal, properties of which are exploited in this article.
 - [47] E. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra* (Academic Press, New York, 1959).
 - [48] Q.A. Turchette, C.J. Hood, W. Lange, H. Mabuchi and H.J. Kimble, Phys. Rev. Lett. **75**, 4710 (1995).
 - [49] C. Monroe, D.M. Meekhof, B.E. King, W.M. Itano and D.J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).
 - [50] I.L. Chuang, L.M.K. Vandersypen, X. Zhou, D.W. Leung and S.L. Lloyd, Nature **393**, 143 (1998).
 - [51] A. Patel, “Another Look at Bell’s Inequalities and Quantum Mechanics”, Contributed to the 16th International Symposium on Lepton and Photon Interactions, Ithaca, August 1993, [hep-th/9309077](#).