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Pion-Pion Scattering in Chiral Perturbation and Dispersion Relation Theories ¹

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Abstract

Chiral perturbation theory, the low energy effective theory of the strong interactions for the light pseudoscalar degrees of freedom, is based on effective Lagrangian techniques and is an expansion in the powers of the external momenta and the powers of the quark masses, which correct the soft-pion theorems. Our primary emphasis will be on the problem of $\pi\pi$ scattering. After briefly reviewing these features and some results, we review some features of $\pi - N$ scattering.

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1 Introduction

Pion-pion scattering is a problem that engaged the attention of a generation of elementary particle physicists. Today, many important inputs towards a possible comprehensive understanding of the problem requires inputs from effective Lagrangian or chiral perturbation theory techniques, in addition to the well-known dispersion relation techniques, suitably modified. The purpose of this talk is to present in a concise manner some of the results and techniques of effective field theories that arises in the low energy sector of the strong interactions. The topics are the techniques and results of modern chiral perturbation theory [1, 2, 3, 4, 5, 6]. Much of what will be said can also be found in standard textbooks [7, 8]. The pion was posited to explain the forces between two nucleons. π^\pm and π^0 are the lightest hadrons and are assumed to be degenerate in mass lying in an iso-spin triplet. Their lightness may be understood by regarding them as the Goldstone bosons of spontaneously broken chiral symmetry of massless QCD; the presence of non-vanishing quark masses shifts the pion pole to $M_\pi^2 = 2\hat{m}B$, where \hat{m} is the average mass of the u- and d- quarks, B is a measure of the vacuum expectation value $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = -F_\pi^2 B$ and F_π is the pion decay constant [3]. QCD is the theory of quark and gluon degrees of freedom and exhibits the property of asymptotic freedom in that at large momenta the coupling constant becomes smaller and one may study the theory in a perturbation series in the strong coupling constant $\alpha_S = g^2/(4\pi)$. The Lagrangian is:

$$\mathcal{L}_{QCD} = -\frac{1}{4g^2}G_{\mu\nu}^a G^{a\mu\nu} + \bar{q}i\gamma^\mu D_\mu q - \bar{q}\mathcal{M}q. \quad (1)$$

If the quark mass matrix $\mathcal{M} = 0$ then the left- and right- chiral projections may be rotated independently. Thus associated with say N light quark flavors, we have the chiral symmetry $SU(N)_L \times SU(N)_R$ (the $U(1)_V$ and the anomalous $U(1)_A$ are not considered here). This chiral symmetry is broken spontaneously to $SU(N)_V$, where $V = (L + R)$, and corresponding to the $SU(N)_A$ ($A = R - L$) broken symmetry we have $N^2 - 1$ (pseudoscalar) Goldstone bosons. Although we do not yet know how to obtain the hadronic spectrum from the QCD Lagrangian, nor the mechanism by which chiral symmetry is broken spontaneously, the QCD Lagrangian provides the justification for the successful results obtained from PCAC and current algebra.

The effective low energy theory of the strong interactions at next to leading order requires the knowledge of the underlying theory and the analysis rests on writing down the generating functional for the currents of the theory which is the vacuum to vacuum transition amplitude in the presence of external sources. The low energy expansion is one that involves derivatives of the external sources a well known example being the Euler-Heisenberg method of analyzing QED. Chiral perturbation theory then is the low-energy effective theory of the strong interactions and involves a simultaneous expansion in the mass of the quarks and the external momenta about the chirally symmetric $SU(2) \times SU(2)$ limit of the massless QCD (here we work in a world with two quark flavors) with the the spontaneous breakdown of this symmetry by the ground state to $SU(2)_V$, the pions corresponding to the Goldstone bosons of the broken $SU(2)_A$ generators. The Goldstone theorem yields

$$\langle 0|A_\mu|\pi \rangle = F_\pi p_\mu, \quad (2)$$

and $F_\pi \approx 93$ MeV. To leading order, $O(p^2)$, the effective Lagrangian is that of the non-linear sigma model. The effective action is

$$Z_1 = F^2 \int dx \frac{1}{2} \nabla_\mu U^T \nabla^\mu U, \quad (3)$$

where U is a four component real $O(4)$ (note that $O(4) \equiv SU(2) \times SU(2)$) unit vector. This model is not renormalizable and the loops of the model lead to divergences which cannot be absorbed into the parameters of the model. In order to absorb the divergences, one is led to introduce higher derivative interactions, which then allow one to extend the predictions at leading order in the momentum or derivative to the next order. The price is the proliferation of low energy constants (LEC) that must be extracted from experiment or alternatively from theoretical considerations such as the behavior of the low energy constants in the chiral limit as well as non-perturbative approaches such as large N_c calculations. The effective Lagrangian at $O(p^4)$ [3] and at $O(p^6)$ [9] have been worked out. (When one considers the interactions of pions with nucleons, the chiral power counting is different since the nucleon mass does not go to zero in the chiral limit as the pion mass does.) For completeness we write down the effective Lagrangian at $O(p^4)$ [3]:

$$\begin{aligned} \mathcal{L}_4 = & l_1(\nabla^\mu U^T \nabla_\mu U)^2 + l_2(\nabla^\mu U^T \nabla^\nu U)(\nabla_\mu U^T \nabla_\nu U) + l_3(\chi^T U)^2 + \\ & l_4(\nabla^\mu \chi^T \nabla_\mu U) + l_5(U^T F^{\mu\nu} F_{\mu\nu} U) + l_6(\nabla^\mu U^T F_{\mu\nu} \nabla^\nu U) + l_7(\tilde{\chi}^T U)^2 + \\ & h_1 \chi^T \chi + h_2 \text{tr} F_{\mu\nu} F^{\mu\nu} + h_3 \tilde{\chi}^T \tilde{\chi}, \end{aligned} \quad (4)$$

where $F_{\mu\nu}$ are covariant tensors involving the external fields and their derivatives and the vectors χ and $\tilde{\chi}$ are proportional to the external scalar and pseudoscalar fields. With this effective Lagrangian and with the loops generated by the non-linear sigma model and appropriate renormalization, one may obtain the Green's functions of QCD in the momentum expansion. At this order, 10 additional low energy constants enter the effective Lagrangian.

Although the number of coupling constants at $O(p^6)$ is very large (> 100), those entering the pion-pion scattering amplitude are still limited in number. Of course, once the coupling constants are fixed from a certain class of experiments, at that order, the theory would have predictions for all other processes at the appropriate level of accuracy. Furthermore, the external field technique permits an off-shell analysis of the Green's functions of the theory and allows one to study the quark mass dependence of the Green's functions.

The important processes of $\pi\pi$ and πN scattering have been analyzed in great detail and methods have been described in standard books [10, 11, 12, 13, 14, 15], and will be of interest to us in this discussion. Note that a good deal of the experimental information on the processes of interest to us has been obtained via dispersion relation analysis of phase shift information. In fact, there is a rich interplay between the effective Lagrangian methods of chiral perturbation theory and dispersion relation theory which we will describe below.

In the following sections, we briefly review the status of $\pi\pi$ scattering and $\pi - N$ scattering. A few remarks are listed on other subjects of interest with some references to the literature.

2 $\pi\pi$ Scattering

We describe the $\pi\pi$ process [11, 16] in some detail, although these results are now nearly 40 years old³. In axiomatic field theory the validity of dispersion relations have been proved some time ago. In the case of $\pi\pi$ scattering dispersion relations are particularly simple. Phase shift information has been analyzed, well before chiral perturbation theory or QCD were established: an analysis that employs chiral results *ab initio* is now required to confront experimental data.

³Note that we assume here iso-spin to be a conserved quantity in strong interactions.

Pion-pion scattering is described in terms of a single function $A(s, t, u)$ of the Mandelstam variables s, t, u [17]. The process is schematically represented by

$$\pi^a(p_1) + \pi^b(p_2) \rightarrow \pi^c(p_3) + \pi^d(p_4) \quad (5)$$

and since iso-spin is conserved by the strong interactions, the transition matrix is given by:

$$A(a, b \rightarrow c, d) = A(s, t, u)\delta^{ab}\delta^{cd} + A(t, u, s)\delta^{ac}\delta^{bd} + A(u, s, t)\delta^{ad}\delta^{bc}, \quad (6)$$

where the function $A(s, t, u) = A(s, u, t) \equiv A_s$ due to generalized Bose statistics and $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$ and $u = (p_1 + p_4)^2$, all momenta taken to be incoming. \sqrt{s} represents the centre of mass energy and t and u are related to the cosine of the centre of mass scattering angle via $\cos\theta = (t - u)/(s - 4)$, $s + t + u = 4$, when setting the pion mass to unity. Since the pions lie in an iso-spin triplet, the s-channel amplitudes for definite iso-spin can be written down:

$$\begin{aligned} T_s^0(s, t, u) &= 3A_s + A_t + A_u \\ T_s^1(s, t, u) &= A_t - A_u \\ T_s^2(s, t, u) &= A_t + A_u \end{aligned} \quad (7)$$

which follows from iso-spin coupling (see e.g. Ref. [18]). One convenient representation for dispersion relations for the amplitudes of definite iso-spin in the t-channel with two subtractions is:

$$\begin{aligned} T_t^I(s, t, u) &= \mu_I(t) + \nu_I(t)(s - u) \\ &+ \frac{1}{\pi} \int_4^\infty \frac{ds'}{s'^2} \left(\frac{s^2}{s' - s} + (-1)^I \frac{u^2}{s' - u} \right) \sum_{I'} C_{st}^{II'} A_s^{I'}(s', t), \end{aligned} \quad (8)$$

where $\mu_I(t)$, $\nu_I(t)$ are unknown t-dependent subtraction constants ($\mu_1 = \nu_0 = \nu_2 = 0$) (the number of subtractions is dictated by the Froissart bound), and $A_s^I(s', t) \equiv \text{Im}T_s^I(s', t)$ is the absorptive part of the s-channel amplitude. The matrix C_{st} is a so-called crossing matrix, (embodying the fundamental property of crossing in axiomatic field theory) the entries of which may be written down from the general formula resulting from iso-spin coupling in terms of the Wigner 6-j symbol [19] as:

$$C_{st}(c, d) = (-1)^{(c+d)}(2c + 1) \begin{Bmatrix} 1 & 1 & d \\ 1 & 1 & c \end{Bmatrix}. \quad (9)$$

A very convenient form of dispersion relations has been found which eliminates the unknown functions μ_I, ν_I in favour of the S- wave scattering lengths a_0^0 and a_0^2 [20]. The scattering lengths a_l^I arise in the threshold expansion for the partial wave amplitudes $\text{Re}f_l^I(\nu) = \nu^l(a_l^I + b_l^I\nu + \dots)$, where the latter are defined by:

$$T_s^I(s, t, u) = 32\pi \sum (2l + 1) f_l^I(s) P_l((t - u)/(s - 4)),$$

where $\nu = (s - 4)/4$. This form is:

$$\begin{aligned} T_s^I(s, t) &= \sum_{I'} \frac{1}{4} (s \mathbf{1}^{II'} + t C_{st}^{II'} + u C_{su}^{II'}) T_s^{I'}(4, 0) \\ &+ \int_4^\infty ds' g_2^{II'}(s, t, s') A_s^{I'}(s', 0) + \int_4^\infty ds' g_3^{II'}(s, t, s') A_s^{I'}(s', t), \end{aligned} \quad (10)$$

where $g_2(s, t, s')$ and $g_3(s, t, s')$ are matrices with the crossing matrices C_{st}, C_{su}, C_{tu} as their building blocks [20]. Furthermore, $T_s(4, 0) = 32\pi(a_0^0, 0, a_0^2)$. Approximating the s-channel amplitudes by the S- and P-waves and inverting the iso-spin amplitudes, eq. (8), one ends up, after some algebra, with the representation for $A(s, t, u)$ [21]:

$$\begin{aligned} A(s, t, u) &= \frac{32\pi}{3} (\gamma_0^0 - \gamma_0^2) s^2 + 16\pi\gamma_0^2 (t^2 + u^2) + 4\pi\alpha_0^2 (u + t) \\ &+ \frac{8\pi}{3} (\alpha_0^0 - \alpha_0^2) s + 16\pi\beta_1^1 t (s - u) + 16\pi\beta_1^1 u (s - t) \\ &+ 32\pi \left(\frac{1}{3} \frac{s^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-s)} (\text{Im}f_0^0(x) - \text{Im}f_0^2(x)) \right. \\ &+ \frac{3}{2} (s-u) \frac{t^2}{\pi} \int_4^\infty \frac{dx}{x^2(x-t)(x-4)} \text{Im}f_1^1(x) \\ &+ \frac{3}{2} (s-t) \frac{u^2}{\pi} \int_4^\infty \frac{dx}{x^2(x-u)(x-4)} \text{Im}f_1^1(x) \\ &\left. + \frac{1}{2} \left(\frac{t^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-t)} \text{Im}f_0^2(x) + \frac{u^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-u)} \text{Im}f_0^2(x) \right) \right), \end{aligned} \quad (11)$$

where

$$\alpha_0^I = a_0^I - \frac{4}{\pi} \int_4^\infty \frac{dx}{x(x-4)} \text{Im}f_0^I(x) + \frac{4}{\pi} \int_4^\infty \frac{dx}{x^2} \text{Im}f_0^I(x) \quad I = 0, 2$$

$$\begin{aligned}
\gamma_0^I &= \frac{1}{\pi} \int_4^\infty \frac{dx}{x^3} \text{Im} f_0^I(x) \quad I = 0, 2 \\
\beta_1^1 &= \frac{3}{\pi} \int_4^\infty \frac{dx}{x^2(x-4)} \text{Im} f_1^1(x) \\
\alpha_0^1 &= \gamma_0^1 = \beta_1^0 = \beta_1^2 = 0.
\end{aligned}
\tag{12}$$

Although the property of crossing symmetry places constraints on the absorptive parts of the amplitude in general, the presence of two subtractions in these dispersion relations implies that the S- and P-waves do not face any constraints [22]. It has been shown that the dispersive representation for the amplitude in the approximation that only S- and P- waves saturate the absorptive parts of the amplitudes lends itself to a straightforward comparison with chiral amplitudes [21]. We reproduce below some of the pertinent discussion.

Chiral perturbation theory at lowest order reproduces the current algebra result

$$A(s, t, u) = \frac{s-1}{F_\pi^2} \tag{13}$$

with only two free parameters F_π and m_π (note that $m_\pi \equiv 1$), leading to the prediction for $a_0^0 = 7/(32\pi F_\pi^2) \simeq 0.16$. Due to the non-renormalizability of the theory four additional low energy constants $\bar{l}_{1,2,3,4}$ enter the $\pi\pi$ scattering amplitude at order $O(p^4)$, where the presence of infrared singularities of the theory modifies the current algebra result for $A(s, t, u)$ substantially. Estimates for these quantities from disparate sources such as D- wave scattering lengths (alternatively from $\pi\pi$ phase information directly), $SU(3)$ mass relations and the ratio of the decay constants F_K/F_π gives a correction of about 25% to the leading order prediction, $a_0^0 = 0.20 \pm 0.01$ [3] (the experimental value for this number is quoted as 0.26 ± 0.05 [23]).

The scattering amplitudes in chiral perturbation theory are perturbatively unitarity; at one-loop order the loops contribute to the scattering amplitude terms that have the correct analytic structure corresponding to producing them from the tree level amplitude by elastic unitarity.

At next to leading order the representation for the function $A(s, t, u)$ at $O(p^4)$ reads [3]:

$$A(s, t, u) = A^{(2)}(s, t, u) + A^{(4)}(s, t, u) + O(p^6), \tag{14}$$

with

$$\begin{aligned}
A^{(2)}(s, t, u) &= \frac{s-1}{F_\pi^2}, \\
A^{(4)}(s, t, u) &= \frac{1}{6F_\pi^4} \left(3(s^2-1)\bar{J}(s) \right. \\
&\quad \left. + [t(t-u) - 2t + 4u - 2]\bar{J}(t) + (t \leftrightarrow u) \right) \\
&\quad + \frac{1}{96\pi^2 F_\pi^4} \{ 2(\bar{l}_1 - 4/3)(s-2)^2 + (\bar{l}_2 - 5/6)[s^2 + (t-u)^2] \\
&\quad + 12s(\bar{l}_4 - 1) - 3(\bar{l}_3 + 4\bar{l}_4 - 5) \} \\
\text{and } \bar{J}(z) &= -\frac{1}{16\pi^2} \int_0^1 dx \ln[1 - x(1-x)z], \quad \text{Im}\bar{J}(z) = \frac{\rho(s)}{16\pi} \Theta(z-4).
\end{aligned}$$

Note also that at $O(p^4)$ the imaginary parts of the partial waves above threshold ($s > 4$) computed from the amplitude above are

$$\begin{aligned}
\text{Im}f_0^0(s) &= \frac{\rho(s)}{1024\pi^2 F_\pi^4} (2s-1)^2 \\
\text{Im}f_1^1(s) &= \frac{\rho(s)}{9216\pi^2 F_\pi^4} (s-4)^2 \\
\text{Im}f_0^2(s) &= \frac{\rho(s)}{1024\pi^2 F_\pi^4} (s-2)^2 \\
\text{Im}f_l^I(s) &= 0, \quad l \geq 2,
\end{aligned} \tag{15}$$

(the chiral power counting enforces the property that the absorptive parts of the D- and higher waves arise only at $O(p^8)$) verifying the property of perturbative unitarity, viz., when the $O(p^2)$ predictions for the threshold parameters $a_0^0 = 7/(32\pi F_\pi^2)$, $a_0^2 = -1/(16\pi F_\pi^2)$, $b_0^0 = 1/(4\pi F_\pi^2)$, $b_0^2 = -1/(8\pi F_\pi^2)$ and $a_1^1 = 1/(24\pi F_\pi^2)$ are inserted into the pertinent form of the perturbative unitarity relations:

$$\begin{aligned}
\text{Im}f_0^I(s) &= \rho(s)(a_0^I + b_0^I(s-4)/4)^2, \quad I = 0, 2 \\
\text{Im}f_1^1(s) &= \rho(s)(a_1^1(s-4)/4)^2.
\end{aligned}$$

In order to carry out the comparison between the chiral expansion and the physical scattering data, we first recall that up to $O(p^6)$, it is possible to

decompose $A(s, t, u)$ into a sum of three functions of single variables as follows [24]:

$$A(s, t, u) = 32\pi \left[\frac{1}{3}W_0(s) + \frac{3}{2}(s-u)W_1(t) + \frac{3}{2}(s-t)W_1(u) + \frac{1}{2} \left(W_2(t) + W_2(u) - \frac{2}{3}W_2(s) \right) \right]. \quad (16)$$

One convenient decomposition of the chiral one-loop amplitude is:

$$W_0(s) = \frac{3}{32\pi} \left[\frac{s-1}{F_\pi^2} + \frac{2}{3F_\pi^4}(s-1/2)^2 \bar{J}(s) + \frac{1}{96\pi^2 F_\pi^4} (2(\bar{l}_1 - 4/3)(s-2)^2 + 4/3(\bar{l}_2 - 5/6)(s-2)^2 + 12s(\bar{l}_4 - 1) - 3(\bar{l}_3 + 4\bar{l}_4 - 5)) \right], \quad (17)$$

$$W_1(s) = \frac{1}{576\pi F_\pi^4} (s-4) \bar{J}(s), \quad (18)$$

$$W_2(s) = \frac{1}{16\pi} \left[\frac{1}{4F_\pi^4} (s-2)^2 \bar{J}(s) + \frac{1}{48\pi^2 F_\pi^4} (\bar{l}_2 - 5/6)(s-2)^2 \right], \quad (19)$$

where we note that this decomposition is not unique, with ambiguities in the real part only. We observe that the imaginary parts of these functions verify the relation:

$$\begin{aligned} \text{Im}W_I(x) &= \text{Im}f_0^I(x), \quad I = 0, 2 \\ \text{Im}W_1(x) &= \text{Im}f_1^1(x)/(x-4), \end{aligned} \quad (20)$$

which may be used to demonstrate the following dispersion relations:

$$W_0(s) = \frac{-1 + 72\bar{l}_1 + 48\bar{l}_2 - 27\bar{l}_3 - 108\bar{l}_4 - 864\pi^2 F_\pi^2}{9216\pi^3 F_\pi^4} + \frac{59 - 144\bar{l}_1 - 96\bar{l}_2 + 216\bar{l}_4 + 1728\pi^2 F_\pi^2}{18432\pi^3 F_\pi^4} s + \frac{-797 + 360\bar{l}_1 + 240\bar{l}_2}{184320\pi^3 F_\pi^4} s^2 + \frac{s^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-s)} \text{Im}f_0^0(x), \quad (21)$$

$$W_1(s) = \frac{-s}{13824\pi^3 F_\pi^4} + \frac{s^2}{\pi} \int_4^\infty \frac{dx}{x^2(x-4)(x-s)} \text{Im}f_1^1(x), \quad (22)$$

$$\begin{aligned}
W_2(s) = & \frac{6\bar{l}_2 - 5}{1152\pi^3 F_\pi^4} + \frac{23 - 24\bar{l}_2}{4608\pi^3 F_\pi^4} s + \frac{60\bar{l}_2 - 77}{46080\pi^3 F_\pi^4} s^2 \\
& + \frac{s^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-s)} \text{Im} f_0^2(x).
\end{aligned} \tag{23}$$

We now reconstruct $A(s, t, u)$ from this dispersive representation for the W 's to obtain:

$$\begin{aligned}
A(s, t, u) = & \frac{s-1}{F_\pi^2} + \frac{-540 + 480\bar{l}_1 + 960\bar{l}_2 - 180\bar{l}_3 - 720\bar{l}_4}{5760\pi^2 F_\pi^4} \\
& - \frac{110 + 480\bar{l}_1 - 720\bar{l}_4}{5760\pi^2 F_\pi^4} s - \frac{163 - 120\bar{l}_1}{5760\pi^2 F_\pi^4} s^2 \\
& + \frac{460 - 480\bar{l}_2}{5760\pi^2 F_\pi^4} (t+u) - \frac{20u(s-t) + 20t(s-u)}{5760\pi^2 F_\pi^4} \\
& - \frac{154 - 120\bar{l}_2}{5760\pi^2 F_\pi^4} (t^2 + u^2) \\
& + 32\pi \left(\frac{1}{3} \frac{s^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-s)} (\text{Im} f_0^0(x) - \text{Im} f_0^2(x)) \right. \\
& + \frac{3}{2} (s-u) \frac{t^2}{\pi} \int_4^\infty \frac{dx}{x^2(x-t)(x-4)} \text{Im} f_1^1(x) \\
& + \frac{3}{2} (s-t) \frac{u^2}{\pi} \int_4^\infty \frac{dx}{x^2(x-u)(x-4)} \text{Im} f_1^1(x) \\
& \left. + \frac{1}{2} \left(\frac{t^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-t)} \text{Im} f_0^2(x) + \frac{u^3}{\pi} \int_4^\infty \frac{dx}{x^3(x-u)} \text{Im} f_0^2(x) \right) \right).
\end{aligned} \tag{24}$$

This is seen to be the sum of a polynomial of second degree in s , t and u and a dispersive piece. The problem associated with the non-uniqueness of the real part of the decomposition into the W 's is eliminated by setting $u = 4 - s - t$ upon which we obtain a second degree polynomial in s and t :

$$\begin{aligned}
P = & \left(\frac{29}{120\pi^2 F_\pi^4} - \frac{\bar{l}_2}{6\pi^2 F_\pi^4} \right) \left(t - \frac{t^2}{4} - \frac{st}{4} \right) \\
& + \left(-\frac{33}{640\pi^2 F_\pi^4} + \frac{\bar{l}_1}{48\pi^2 F_\pi^4} + \frac{\bar{l}_2}{48\pi^2 F_\pi^4} \right) s^2 \\
& + \left(\frac{1}{F_\pi^2} + \frac{97}{960\pi^2 F_\pi^4} - \frac{\bar{l}_1}{12\pi^2 F_\pi^4} - \frac{\bar{l}_2}{12\pi^2 F_\pi^4} + \frac{\bar{l}_4}{8\pi^2 F_\pi^4} \right) s
\end{aligned} \tag{25}$$

$$+ \left(-\frac{1}{F_\pi^2} - \frac{97}{480\pi^2 F_\pi^4} + \frac{\bar{l}_1}{12\pi^2 F_\pi^4} + \frac{\bar{l}_2}{6\pi^2 F_\pi^4} - \frac{\bar{l}_3}{32\pi^2 F_\pi^4} - \frac{\bar{l}_4}{8\pi^2 F_\pi^4} \right).$$

We now wish to employ the knowledge that the low energy representation is fully determined by the two subtraction constants a_0^0 and a_0^2 and in terms of the three functions of single variables W_I with right hand cuts only and verifying eq.(20). Indeed, a Roy equation fit allows us to obtain a representation for the S- and P- wave absorptive parts, (with some effects of higher angular momentum states absorbed into the driving terms).

We are now in a position to compare the two representations for $A(s, t, u)$ namely the chiral representation eq.(24) and the axiomatic representation eq.(11). These are formally equivalent, with the dispersive integrals in the former described by chiral absorptive parts whereas in the latter by the physical S- and P-wave absorptive parts. For the chiral expansion to reproduce low energy physics accurately we now require the effective subtraction constants to match. Once more setting $u = 4 - s - t$ yields the polynomial piece of the representation eq.(11):

$$\begin{aligned} P = & -128\pi(\beta_1^1 + \gamma_0^2)\left(t - \frac{t^2}{4} - \frac{st}{4}\right) + \frac{16\pi}{3}(2\gamma_0^0 + \gamma_0^2 - 3\beta_1^1)s^2 \\ & + 8\pi\left(\frac{\alpha_0^0}{3} - \frac{5}{6}\alpha_0^2 + 8\beta_1^1 - 16\gamma_0^2\right)s + 16\pi(\alpha_0^2 + 16\gamma_0^2). \end{aligned} \quad (26)$$

A straightforward comparison of eq. (25) and eq. (26) yields explicit expressions for $\bar{l}_1, \bar{l}_2, \bar{l}_3$ and \bar{l}_4 . In particular we have for \bar{l}_1 and \bar{l}_2 :

$$\bar{l}_1 = 24\pi^2 F_\pi^4 \left(\frac{41}{960\pi^2 F_\pi^4} - \frac{64\pi}{3}(\gamma_0^2 - \gamma_0^0 + 3\beta_1^1) \right), \quad (27)$$

$$\bar{l}_2 = 24\pi^2 F_\pi^4 \left(\frac{29}{480\pi^2 F_\pi^4} + 32\pi(\beta_1^1 + \gamma_0^2) \right). \quad (28)$$

For the numerical values we find for \bar{l}_1 and \bar{l}_2 [21]:

$$\begin{aligned} \bar{l}_1 &= -1.7 \pm 0.15 \\ \bar{l}_2 &= 5.0. \end{aligned}$$

These have an interesting dependence on the actual physical phase shifts: one observes that in eq.(27), the presence of γ_0^0 . As a result we can anticipate \bar{l}_1 to be influenced by the input for a_0^0 . In contrast, \bar{l}_2 has no dependence

on γ_0^0 and depends almost totally on the P- wave contribution via β_1^1 , as a result of the weakness of the $I = 2$ channel which renders γ_0^2 negligible in comparison with β_1^1 (and γ_0^0). Since the P- wave happens to be the best determined experimental quantity, even the Roy equation fits to it are not strongly influenced by the input value of a_0^0 . Thus we expect a determination of \bar{l}_2 in this manner to be very stable.

The higher partial waves have to be treated with care. The problem must be accounted for when we saturate fixed-t dispersion relations with absorptive parts which are modeled theoretically (perhaps in terms of resonance propagators, Pomeron exchange, say the Veneziano model, etc.) A recent analysis of these inputs has been reported [25]. Alternatively, one may write down dispersion relations in terms of homogeneous variables [22] which manifestly enforce crossing symmetry; however there might be a dependence on parameters which parametrize the curves in the plane of the homogeneous variables on which the dispersion relations are written down. Recently this method has been exploited to extract the contributions of some resonances [26]. A discussion concerning the evaluation of higher threshold parameters of pion-pion scattering has been presented in some detail [27].

In particular, today the chiral amplitudes beyond one loop have been computed [28] and to two loops [29] (also see Ref. [30]). The work of [29] in standard chiral perturbation theory affords an accurate prediction for the parameter a_0^0 which is a soft quantity from the point of view of dispersion relations and work is in progress to this end. For some of the latest information on the experimental as well as theoretical aspects of the subject, see Ref. [31].

3 Further processes

πN scattering is a further well known process represented by

$$\pi^a(q_1) + N(p_1) \rightarrow \pi^b(q_2) + N(p_2). \quad (29)$$

The pion-nucleon system has been studied in detail with the methods of current algebra, PCAC, and early-days phenomenological Lagrangians. An excellent introduction and review of these topics may be found in Chapter 19 of Ref. [7]. For instance, we have the lowest order predictions for the iso-spin

3/2 and 1/2 scattering lengths of -0.075 and 0.15 which compare favorably with the data [23].

The πN amplitude may be written in terms of the four invariant amplitudes A^\pm, B^\pm :

$$T_{ab} = T^+ \delta_{ab} - T^- i \epsilon_{abc} \tau_c$$

$$T^\pm = \bar{u}(p_2) \left[A^\pm(\nu, t) + \frac{1}{2} \gamma_\mu (q_1^\mu + q_2^\mu) B^\pm(\nu, t) \right] u(p_1) \quad (30)$$

with $s = (p_1 + q_1)^2$, $t = (q_1 - q_2)^2$, $u = (p_1 - q_2)^2$, $\nu = (s - u)/(4m_N)$.

In chiral perturbation theory, this process has been first considered in a relativistic framework in [32]. In this work, the authors have extended the analysis of the Green's functions of QCD with an external nucleon. Furthermore the πN environment is an important one for the tests of chiral predictions [33, 34]. However, the fact that the nucleon has a non-vanishing mass in the chiral limit causes some difficulties: the presence of the chiral-limit nucleon mass in the lowest order $\pi - N$ Lagrangian makes it necessary to renormalize the the chiral-limit nucleon mass. This destroys the correspondence between the number of loops and the powers of the external momenta and makes the bookkeeping of the chiral expansion somewhat difficult. These difficulties are circumvented in heavy baryon chiral perturbation theory (HBChPT) [35], where the nucleon mass is taken to be heavy compared to the external momenta. This way, the nucleon mass does not show up in the lowest order Lagrangian restoring the chiral counting scheme at the price of losing the manifest Lorentz invariance⁴.

Both methods have in common that because of the non-renormalizability additional low energy constants have to be introduced at each order of the chiral expansion. Again, they can not be determined by the theory alone. Only a comparison with experimental data allows one to fix the low energy constants. As an example we consider HBChPT. There one usually expresses the amplitudes in terms of the isoscalar/isovector non-spin-flip amplitudes g^\pm and the isoscalar/isovector spin-flip amplitudes h^\pm :

$$A^\pm = C_1 g^\pm + C_2 h^\pm \quad (31)$$

$$B^\pm = C_3 g^\pm + C_4 h^\pm \quad (32)$$

⁴Note that in a recent work it is shown that both chiral counting scheme and Lorentz invariance can be preserved [36].

with suitably chosen coefficients C_i . To order $O(q^3)$ the amplitude g^\pm , e.g., reads [37]:

$$g_{tree}^+(\omega, t) = -\frac{g_A^2}{F_\pi^2} \frac{1}{16m\omega^2} \left[4M^4 + t^2 + 4\omega^2 t - 4M^2 t \right] \\ + \frac{g_A^2}{F_\pi^2} \frac{1}{32m^2\omega^3} \left[16\omega^2 M^4 + 5\omega^2 t^2 - 16M^6 - 8M^2 t^2 + 4\omega^4 t + 20M^4 t \right. \\ \left. + t^3 - 20\omega^2 M^2 t \right], \quad (33)$$

$$g_{ct}^+(\omega, t) = \frac{1}{F_\pi^2} \left[-4c_1 M^2 + 2c_2 \omega^2 + c_3 (2M^2 - t) \right] + \frac{c_2 \omega}{m F_\pi^2} \left[4\omega^2 - 4M^2 + t \right],$$

$$g_{loop}^+(\omega, t) = i \frac{\omega^2}{8\pi F_\pi^4} \sqrt{\omega^2 - M^2} + \\ \frac{g_A^2}{F_\pi^4} \frac{1}{32\pi} (M^2 - 2t) \left(M + \frac{2M^2 - t}{2\sqrt{-t}} \arctan \frac{\sqrt{-t}}{2M} \right)$$

where the LEC's c_1, c_2, c_3 show up in the counter term contribution g_{ct}^+ . ω is the cm energy of the pion and M and m are the pion and the nucleon mass, respectively. The above chiral calculation is most reliable inside the Mandelstam triangle, which is a unphysical part of the Mandelstam plane. The only way to extrapolate the experimental information to this region is to use dispersion relations. By using crossing symmetry, unitarity and analyticity the real part of A^\pm and B^\pm may be written as

$$\text{Re } A^\pm(s, t) = \frac{1}{\pi} P \int_{s_{th.}}^{\infty} ds' \text{Im } A^\pm(s', t) \left\{ \frac{1}{s' - s} \pm \frac{1}{s' - u} \right\} \quad (34)$$

$$\text{Re } B^\pm(s, t) = \frac{g_{\pi N}^2}{m^2 - s} \mp \frac{g_{\pi N}^2}{m^2 - u} \\ + \frac{1}{\pi} P \int_{s_{th.}}^{\infty} ds' \text{Im } B^\pm(s', t) \left\{ \frac{1}{s' - s} \mp \frac{1}{s' - u} \right\}, \quad (35)$$

where $g_{\pi N}$ is the pion-nucleon coupling constant and $s_{th.} = (m + M)^2$. The inclusion of all the aspects of dispersion relations can be easily found in the literature [15]. The advantage of the above relations is obvious: as the range of integration is restricted to the physical domain of s , the integrands of the dispersion integrals can be calculated from the available experimental data. On the other hand, the dispersion relations are valid for any s and t , i.e. by using the data in the *physical* domain one is able to extrapolate the

amplitudes $A^\pm(s, t)$ and $B^\pm(s, t)$ to the *unphysical* region. By inverting eq. (31,32) the results of a dispersive analysis may then be compared with chiral expressions for the amplitudes g^\pm and h^\pm [38], yielding numerical results for the LEC's.

The methods of chiral perturbation theory are also of great use for strong, semi-leptonic, electromagnetic, etc. decays. One example where all the methods of effective Lagrangian as well as those of dispersion relations are of utility are in the decay $\eta \rightarrow 3\pi$ [39].

4 Afterword

In this talk we have described in some detail the effective field theories describing a sector of the standard model where conventional perturbation theory fails. Questions still remain in the sense of a rigorous equivalence of solving the underlying field theory and its effective field theory. In a somewhat indirect context these questions have recently been raised [40]. The ideas and techniques used are manifold and present a great challenge to the student and avid reader which promises sharp quantitative and qualitative understanding of otherwise intractable phenomena.

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