Parametrizing the mixing matrix : A unified approach

S. Chaturvedi*
School of Physics, University of Hyderabad, Hyderabad 500 046, India

N. Mukunda†
Centre for Theoretical Studies and Department of Physics, Indian Institute of Science, Bangalore 560 012, India, and
Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur, Bangalore 560 064, India
(March 29, 2004)

A unified approach to parametrization of the mixing matrix for N generations is developed. This approach not only has a clear geometrical underpinning but also has the advantage of being economical and recursive and leads in a natural way to the known phenomenologically useful parametrizations of the mixing matrix.

arXiv:hep-ph/0004219 v1 25 Apr 2000

I. INTRODUCTION

In the standard $SU(3) \times SU(2) \times U(1)$ model of strong , weak and electromagnetic interactions, all aspects of the charged weak interactions among quarks can be described in terms of a 3×3 unitary matrix

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1.1}$$

specified by four real parameters: three generalized Cabibbo angles and one Kobayashi-Maskawa phase. After the pioneering work of Kobayashi and Maskawa [1], this matrix, which describes the mixing between quark mass eigenstates and the charged weak current eigenstates, has been parametrized in a number of phenomenologically useful ways [2-6]. Generalizations to $N \geq 3$ generations of quarks, where the mixing matrix is characterized by $N(N-1)/2$ angles and $(N-1)(N-2)/2$ phases, have also been proposed [6-14]. Analogues of the mixing matrix also arise in the lepton sector if the neutrinos are taken as massive Dirac particles. In most of the parametrizations hitherto proposed, the mixing matrix is expressed as an ordered product of $N(N-1)/2$ factors each of which carries an angle. Of these $N(N-1)/2$ factors, a prescribed set of $(N-1)(N-2)/2$ factors carry phases as well. Different parametrizations differ from each other in the ordering prescription and the location of the phase factors within the matrices carrying them. In this work, we present a parametrization of the mixing matrix based on a decomposition, involving, in the $N=3$ case, just two factors. This parametrization, apart from having a clear geometrical picture underlying it, also enables us to recover and relate other parametrizations and to generate new ones in a unified manner.

II. PARAMETRIZING SU(N) ELEMENTS BY A SEQUENCE OF COMPLEX UNIT VECTORS

The proposed parametrization of the mixing matrix is based on the observation that a generic matrix $g \in SU(N)$ can be parametrized by a sequence of complex unit vectors $\boldsymbol{\xi}, \cdots, \boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha}$ of dimensions $n, n-1, \cdots, 3, 2$. This can be seen as follows. Let

$$\Sigma_n = \{\boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_n \end{pmatrix} \mid \boldsymbol{\psi}^\dagger \boldsymbol{\psi} = 1\} \tag{2.1}$$

denote the set of unit vectors in complex n-dimensional Hilbert space i.e. a set of real dimension $(2n-1)$. Any $\boldsymbol{\psi} \in \Sigma_n$ can be mapped to the vector

*e-mail: scsp@uohyd.ernet.in
†email: nmukunda@cts.iisc.ernet.in

$$\mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \tag{2.2}$$

via a suitable $SU(n)$ element. (Note that we are really using $SU(n)$ here , not $U(n)$) Therefore, $SU(n)$ acts transitively on Σ_n. The subgroup of $SU(n)$ that leaves $\mathbf{e}_n$ invariant is $SU(n-1)$ on the first $(n-1)$ dimensions and hence

$$\Sigma_n \simeq \textit{ coset space } SU(n)/SU(n-1) \tag{2.3}$$

Therefore, we expect that, apart from global matching problems or ambiguities on a subset of measure zero, any element in $SU(n)$ is uniquely specified by a pair consisting of an element in $SU(n-1)$ and a unit vector $\boldsymbol{\psi} \in \Sigma_n$. Therefore, recursively, we see that an element $g \in SU(n)$ can be parametrized as $g = g(\boldsymbol{\xi}, \cdots, \boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$ by a string of complex unit vectors $\boldsymbol{\xi}, \cdots, \boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha}$ of dimensions $n, n-1, \cdots, 4, 3, 2$.

As a convention, we will let the above unit vectors stand for the last column. in the relevant $SU(n)$ matrices. This is because when a matrix of $SU(n)$ is multiplied on the right by a matrix of $SU(n-1)$ (leaving $\mathbf{e}_n$ invariant), it is the last column in the former matrix that remains unchanged. For elements of $SU(2)$ we will thus write

$$A \in SU(2) \quad ; \quad A = A(\boldsymbol{\alpha}) = \begin{pmatrix} \alpha_2^* & \alpha_1 \\ -\alpha_1^* & \alpha_2 \end{pmatrix} \quad ; \quad \boldsymbol{\alpha}^\dagger \boldsymbol{\alpha} = 1 \tag{2.4}$$

This is globally well defined.

III. SU(3) AND THE KOBAYASHI-MASKAWA PHASE

Let $\boldsymbol{\beta}$ denote a three component complex unit vector, $\boldsymbol{\beta}^\dagger \boldsymbol{\beta} = 1$. Then for $|\beta_1| < 1$, the matrix

$$B(\boldsymbol{\beta}) = \begin{pmatrix} P^{-1} & 0 & \beta_1 \\ -P\beta_1^*\beta_2 & P\beta_3^* & \beta_2 \\ -P\beta_1^*\beta_3 & -P\beta_2^* & \beta_3 \end{pmatrix} \quad , \quad P = (1-|\beta_1|^2)^{-1/2} \tag{3.1}$$

is in $SU(3)$. the unit vector $\boldsymbol{\beta}$ is a label for right $SU(2)$ cosets in $SU(3)$, and $B(\boldsymbol{\beta})$ is a coset representative. So any $B \in SU(3)$, $|B_{13}| < 1$, can be uniquely written as

$$B = B(\boldsymbol{\beta}, \boldsymbol{\alpha}) = B(\boldsymbol{\beta}) \cdot \begin{pmatrix} A(\boldsymbol{\alpha}) & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

On multiplying out the two matrices on the rhs one obtains

$$B(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \begin{pmatrix} P^{-1}\alpha_2^* & P^{-1}\alpha_1 & \beta_1 \\ -P\beta_1^*\beta_2\alpha_2^* - P\beta_3^*\alpha_1^* & -P\beta_1^*\beta_2\alpha_1 + P\beta_3^*\alpha_2 & \beta_2 \\ -P\beta_1^*\beta_3\alpha_2^* + P\beta_2^*\alpha_1^* & -P\beta_1^*\beta_3\alpha_1 - P\beta_2^*\alpha_2 & \beta_3 \end{pmatrix} \tag{3.3}$$

Now we examine how $B(\boldsymbol{\beta}, \boldsymbol{\alpha})$ transforms under rephasing transformations i.e. we ask how $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ change when we multiply $B(\boldsymbol{\beta}, \boldsymbol{\alpha})$ on the left and on the right by independent diagonal elements of $SU(3)$:

$$B' = D(\theta') \; B \; D(\theta) \tag{3.4}$$

where $D(\theta) = diag(e^{i\theta_1 + i\theta_2}, e^{-i\theta_1 + i\theta_2}, e^{-2i\theta_2})$ and $D(\theta')$ is defined similarly. Then we find

$$D(\theta')B(\boldsymbol{\beta}, \boldsymbol{\alpha})D(\theta) = B(\boldsymbol{\beta}', \boldsymbol{\alpha}') \tag{3.5}$$

where

$$\alpha_1' = \alpha_1 e^{i(\theta_1' + \theta_2' - \theta_1 + \theta_2)} \quad ; \quad \alpha_2' = \alpha_2 e^{-i(\theta_1' + \theta_2' + \theta_1 + \theta_2)} \tag{3.6}$$

$$\beta_1' = \beta_1 e^{i(\theta_1' + \theta_2' - 2\theta_2)} \; ; \; \beta_2' = \beta_2 e^{i(-\theta_1' + \theta_2' - 2\theta_2)} \; ; \; \beta_3' = \beta_3 e^{-2i(\theta_2' + \theta_2)} \tag{3.7}$$

These transformation laws can easily be written down from the locations of $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ in the matrix (3.3).

As the dimension of $SU(3)$ is eight and we have four independent phases here, there should be four independent real invariants. Three of them, essentially the generalized Cabibbo angles may be chosen to be, say, $|\alpha_1|, |\beta_1|, |\beta_2|$. The fourth one can be found systematically as follows.

The phase θ_1 does not occur in $\boldsymbol{\beta}'$. So we form the combination $\alpha_1\alpha_2^*$ whose transformation law is θ_1 independent:

$$\alpha_1'\alpha_2'^* = \alpha_1\alpha_2^* e^{2i(\theta_1'+\theta_2'+\theta_2)} \tag{3.8}$$

Among the β's, θ_1' occurs only in β_1' and β_2', so we form a combination which can cancel $e^{2i\theta_1'}$ on the rhs of (3.8)

$$\beta_1^*\beta_2 \rightarrow \beta_1^*\beta_2 e^{-2i\theta_1'} \tag{3.9}$$

From (3.8) and (3.9) we find

$$\alpha_1\alpha_2^*\beta_1^*\beta_2 \rightarrow \alpha_1\alpha_2^*\beta_1^*\beta_2 e^{2i(\theta_2'+\theta_2)} \tag{3.10}$$

Comparing this with β_3' we see that $arg(\alpha_1\alpha_2^*\beta_1^*\beta_2\beta_3)$ is invariant under rephasing.

IV. COMPARISON WITH SOME WELL KNOWN PARAMETRIZATIONS OF THE MIXING MATRIX FOR N=3

Before we show how some well known parametrizations of the mixing matrix can be recovered from the considerations given above, it is useful to note that from the standard form(3.3) we can generate others by permutation of rows and columns and by taking transpose. The expressions for the invariants remain unchanged under these opertations as will become clear in section VII. This being the case, various parametrizations of the mixing matrix can be generated by choosing any one from $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ which appear in the invariant $arg(\alpha_1\alpha_2^*\beta_1^*\beta_2\beta_3)$ to be complex and all others real in the matrix (3.3) or in the matrices obtained by permuting rows and columns or by taking transpose. Thus, for instance, choosing β_2 to be complex, all others real, and putting

$$\alpha_1 = S_\theta, \alpha_2 = C_\theta, \beta_1 = S_\beta, \beta_2 = S_\gamma C_\beta e^{i\delta}, \beta_3 = C_\gamma C_\beta \tag{4.1}$$

in (3.3) one obtains the Maiani parametrization [2]

$$\begin{pmatrix} C_\beta C_\theta & C_\beta S_\theta & S_\beta \\ -S_\gamma C_\theta S_\theta e^{i\delta} - S_\theta C_\gamma & C_\gamma C_\theta - S_\gamma S_\beta S_\theta e^{i\delta} & S_\gamma C_\beta e^{i\delta} \\ -S_\beta C_\gamma C_\theta + S_\gamma S_\theta e^{i\delta} & -C_\gamma S_\beta S_\theta - S_\gamma C_\theta e^{-i\delta} & C_\gamma C_\beta \end{pmatrix} \tag{4.2}$$

The Chau-Keung parametrization [4] corresponds to choosing β_1 complex.

$$\alpha_1 = S_{12}, \alpha_2 = C_{12}, \beta_1 = S_{13}e^{-i\delta}, \beta_2 = S_{23}C_{13}, \beta_3 = C_{23}C_{13} \tag{4.3}$$

The mixing matrix, for this choice, is given by

$$\begin{pmatrix} C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta_{13}} \\ -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta_{13}} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_{13}} & S_{23}C_{13} \\ S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta_{13}} & -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{i\delta_{13}} & C_{23}C_{13} \end{pmatrix} \tag{4.4}$$

The Kobayashi-Maskawa form corresponds to taking β_2 complex and putting

$$\alpha_1 = -S_3, \alpha_2 = -C_3, \beta_1 = C_1, \beta_2 = -S_1C_2e^{-i\delta}, \beta_3 = S_1S_2 \tag{4.5}$$

in

$$B(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \begin{pmatrix} \beta_1 & P^{-1}\alpha_2^* & P^{-1}\alpha_1 \\ \beta_2 & -P\beta_1^*\beta_2\alpha_2^* - P\beta_3^*\alpha_1^* & -P\beta_1^*\beta_2\alpha_1 + P\beta_3^*\alpha_2 \\ \beta_3 & -P\beta_1^*\beta_3\alpha_2^* + P\beta_2^*\alpha_1^* & -P\beta_1^*\beta_3\alpha_1 - P\beta_2^*\alpha_2 \end{pmatrix} \tag{4.6}$$

and leads to

$$\begin{pmatrix} C_1 & -S_1C_3 & -S_1S_3 \\ -S_1C_2e^{-i\delta} & -C_1C_2C_3e^{-i\delta} + S_2S_3 & -C_1C_2S_3e^{-i\delta} - S_2C_3 \\ S_1S_2 & C_1S_2C_3 + C_2S_3e^{i\delta} & C_1S_2S_3 - C_2C_3e^{i\delta} \end{pmatrix} \tag{4.7}$$

which, on multiplying the second row by the phase factor $(e^{-i\delta})$ gives precisely the mixing matrix originally given by Kobayashi and Maskawa.

Similarly, taking β_2 to be complex and putting

$$\alpha_1 = C_{13}, \alpha_2 = -S_{13}, \beta_1 = -S_{12}, \beta_2 = C_{12}C_{23}e^{-i\alpha}, \beta_3 = C_{12}S_{23} \tag{4.8}$$

in

$$B(\boldsymbol{\beta},\boldsymbol{\alpha}) = \begin{pmatrix} P^{-1}\alpha_1 & \beta_1 & P^{-1}\alpha_2^* \\ -P\beta_1^*\beta_2\alpha_1 + P\beta_3^*\alpha_2 & \beta_2 & -P\beta_1^*\beta_2\alpha_2^* - P\beta_3^*\alpha_1^* \\ -P\beta_1^*\beta_3\alpha_1 - P\beta_2^*\alpha_2 & \beta_3 & -P\beta_1^*\beta_3\alpha_2^* + P\beta_2^*\alpha_1^* \end{pmatrix} \tag{4.9}$$

yields the parametrization due to Anselm et al [7]:

$$\begin{pmatrix} C_{12}C_{13} & -S_{12} & -C_{12}S_{13} \\ S_{12}C_{13}C_{23}e^{-i\alpha} - S_{13}S_{23} & C_{12}C_{23}e^{-i\alpha} & -S_{12}S_{13}C_{23}e^{-i\alpha} - C_{13}S_{23} \\ S_{12}C_{13}S_{23} + S_{13}C_{23}e^{i\alpha} & C_{12}S_{23} & -S_{12}S_{13}S_{23} + C_{13}C_{23}e^{i\alpha} \end{pmatrix} \tag{4.10}$$

V. SU(4) AND THREE KOBAYASHI-MASKAWA PHASES

We now consider the $N = 4$ case. Let $\boldsymbol{\gamma}$ denote a four dimensional complex unit vector. Then, for $|\gamma_1|^2 + |\gamma_2|^2 < 1$,

$$C(\boldsymbol{\gamma}) = \begin{pmatrix} Q^{-1} & 0 & 0 & \gamma_1 \\ -Q\gamma_1^*\gamma_2 & QR^{-1} & 0 & \gamma_2 \\ -Q\gamma_1^*\gamma_3 & -QR\gamma_2^*\gamma_3 & R\gamma_4^* & \gamma_3 \\ -Q\gamma_1^*\gamma_4 & -QR\gamma_2^*\gamma_4 & -R\gamma_3^* & \gamma_4 \end{pmatrix} \tag{5.1}$$

where $Q = (1-|\gamma_1|^2)^{-1/2}$, $R = (1-|\gamma_1|^2-|\gamma_2|^2)^{-1/2}$, is in $SU(4)$. The unit vector $\boldsymbol{\gamma}$ is a label for right $SU(3)$ cosets in $SU(4)$ and $C(\boldsymbol{\gamma})$ is a coset representative. So, except on a subset of measure zero, for a $C \in SU(4), |C_{14}|^2+|C_{24}|^2 < 1$, there is a unique sequence of complex unit vectors $\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\alpha}$ of dimensions $4, 3, 2$ respectively, such that

$$C = C(\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\alpha}) = C(\boldsymbol{\gamma}) \cdot \begin{pmatrix} B(\boldsymbol{\beta},\boldsymbol{\alpha}) & 0 \\ 0 & 1 \end{pmatrix} \tag{5.2}$$

Now we multiply C on the left and right by independent diagonal $SU(4)$ matrices, get the transformation laws for $\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\alpha}$, and then construct the invariants.

$$C' = D(\theta')\ C(\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\alpha})\ D(\theta) = C(\boldsymbol{\gamma}',\boldsymbol{\beta}',\boldsymbol{\alpha}') \tag{5.3}$$

where $D(\theta) = diag(e^{i\theta_1+i\theta_2+i\theta_3}, e^{-i\theta_1+i\theta_2+i\theta_3}, e^{-2i\theta_2+i\theta_3}, e^{-3i\theta_3})$ and $D(\theta')$ is defined similarly. For simplicity, let $B(\boldsymbol{\beta},\boldsymbol{\alpha})$ also denote the 4×4 matrix obtained by an appropriate bordering. Then, because of the way we parametrized $D(\theta)$ and $D(\theta')$, we find

$$\begin{aligned} C' &= C(\boldsymbol{\gamma}')B(\boldsymbol{\beta}',\boldsymbol{\alpha}') \\ &= D(\theta')C(\boldsymbol{\gamma})B(\boldsymbol{\beta},\boldsymbol{\alpha})D(\theta) \\ &= D(\theta')C(\boldsymbol{\gamma})\ diag(e^{i\theta_3}, e^{i\theta_3}, e^{i\theta_3}, e^{-3i\theta_3}) \\ &\quad \times B(\boldsymbol{\beta},\boldsymbol{\alpha})\ diag(e^{i\theta_1+i\theta_2}, e^{-i\theta_1+i\theta_2}, e^{-2i\theta_2}, 1) \end{aligned} \tag{5.4}$$

The expressions for $\boldsymbol{\gamma}'$ are easy to read off

$$\begin{aligned} \gamma_1' &= \gamma_1 e^{i(\theta_1'+\theta_2'+\theta_3'-3\theta_3)} \quad ; \quad \gamma_2' = \gamma_2 e^{i(-\theta_1'+\theta_2'+\theta_3'-3\theta_3)} \\ \gamma_3' &= \gamma_3 e^{i(-2\theta_2'+\theta_3'-3\theta_2)}\ ;\ \gamma_4' = \gamma_4 e^{-3i(\theta_3'+\theta_3)} \end{aligned} \tag{5.5}$$

A little algebra shows that

$$\begin{aligned} &D(\theta')C(\boldsymbol{\gamma})\ diag(e^{i\theta_3}, e^{i\theta_3}, e^{i\theta_3}, e^{-3i\theta_3}) \\ &= C(\boldsymbol{\gamma}')\ diag(e^{i(\theta_1'+\theta_2'+\theta_3'+\theta_3)}, e^{i(-\theta_1'+\theta_2'+\theta_3'+\theta_3)}, e^{-2i(\theta_2'+\theta_3'+\theta_3)}, 1) \end{aligned} \tag{5.6}$$

so that the rest reduces to an $SU(3)$ problem in 3×3 matrix form

$$\begin{aligned}
&B(\boldsymbol{\beta}', \boldsymbol{\alpha}') = \\
&diag(e^{i(\theta_1'+\theta_2'+\theta_3'+\theta_3)}, e^{i(-\theta_1'+\theta_2'+\theta_3'+\theta_3)}, e^{-2i(\theta_2'+\theta_3'+\theta_3)}) \\
&\times B(\boldsymbol{\beta}, \boldsymbol{\alpha}) diag(e^{i\theta_1+i\theta_2}, e^{-i\theta_1+i\theta_2}, e^{-2i\theta_2})
\end{aligned} \tag{5.7}$$

which is just the same as in (3.5) with the replacements $\theta_1 \to \theta_1, \theta_2 \to \theta_2, \theta_1' \to \theta_1', \theta_2' \to \theta_2' + \theta_3' + \theta_3$. Making these changes in (3.6) and (3.7) we see that for the $SU(4)$ problem to accompany (5.5), we have,

$$\alpha_1' = \alpha_1 e^{i(\theta_1'+\theta_2'+\theta_3'-\theta_1+\theta_2+\theta_3)} \quad ; \quad \alpha_2' = \alpha_2 e^{-i(\theta_1'+\theta_2'+\theta_3'+\theta_1+\theta_2+\theta_3)} \tag{5.8}$$

$$\begin{aligned}
&\beta_1' = \beta_1 e^{i(\theta_1'+\theta_2'+\theta_3'-2\theta_2+\theta_3)} \; ; \; \beta_2' = \beta_2 e^{i(-\theta_1'+\theta_2'+\theta_3'-2\theta_2+\theta_3)} \\
&\beta_3' = \beta_3 e^{-2i(\theta_2'+\theta_3'+\theta_2+\theta_3)}
\end{aligned} \tag{5.9}$$

From (5.5), (5.8) and (5.9) we need to construct the invariants. The six 'Cabibbo' angles may be taken to be given by $|\alpha_1|, |\beta_1|, |\beta_2|, |\gamma_1|, |\gamma_2|, |\gamma_3|$. The three KM phases can be obtained systematically as follows. Since θ_1 is involved only in $\boldsymbol{\alpha}'$, not in $\boldsymbol{\beta}'$ and γ', we see that the α's must enter only in the form $\alpha_1\alpha_2^*$ which obeys

$$\alpha_1\alpha_2^* \to \alpha_1\alpha_2^* e^{2i(\theta_1'+\theta_2'+\theta_3'+\theta_2+\theta_3)} \tag{5.10}$$

Next, we see that θ_2 is not involved in the γ''s at all, so we form independent expressions in $\alpha_1\alpha_2^*$ and the β's in which θ_2 goes away.

$$\begin{aligned}
\alpha_1\alpha_2^*\beta_1 &\to \alpha_1\alpha_2^*\beta_1 e^{3i(\theta_1'+\theta_2'+\theta_3'+\theta_3)} \\
\alpha_1\alpha_2^*\beta_2 &\to \alpha_1\alpha_2^*\beta_2 e^{i\theta_1'+3i(\theta_2'+\theta_3'+\theta_3)} \\
\alpha_1\alpha_2^*\beta_3 &\to \alpha_1\alpha_2^*\beta_3 e^{2i\theta_1'}
\end{aligned} \tag{5.11}$$

These three quantities and the four γ's involve $\theta_1', \theta_2', \theta_3', \theta_3$. We now form independent combinations in which θ_3 drops out. They are

$$\begin{aligned}
\gamma_1\gamma_2^* &\to \gamma_1\gamma_2^* e^{2i\theta_1'} \\
\gamma_1\gamma_3^* &\to \gamma_1\gamma_3^* e^{i(\theta_1'+3\theta_2')} \\
\gamma_1\gamma_4^* &\to \gamma_1\gamma_4^* e^{i(\theta_1'+\theta_2'+4\theta_3')} \\
\alpha_1\alpha_2^*\beta_1\gamma_4 &\to \alpha_1\alpha_2^*\beta_1\gamma_4 e^{3i(\theta_1'+\theta_2')} \\
\alpha_1\alpha_2^*\beta_2\gamma_4 &\to \alpha_1\alpha_2^*\beta_2\gamma_4 e^{i(\theta_1'+3\theta_2')} \\
\alpha_1\alpha_2^*\beta_3 &\to \alpha_1\alpha_2^*\beta_3 e^{2i\theta_1'}
\end{aligned} \tag{5.12}$$

Here θ_3' appears only in the rule for $\gamma_1\gamma_4^*$ so we just drop it. Then we quickly find a choice of three independent invariants:

$$arg(\alpha_1\alpha_2^*\beta_1^*\beta_2\beta_3) \;\; ; \;\; arg(\beta_1\beta_2^*\gamma_1^*\gamma_2) \;\; ; \;\; arg(\beta_2\beta_3^*\gamma_2^*\gamma_3\gamma_4) \;\; ; \tag{5.13}$$

The first of the three $SU(4)$ invariants is the same as the single $SU(3)$ invariant. This is explained by the observation that after the γ''s were determined in (5.5), the determination of the β''s and α''s was reduced to the $SU(3)$ level problem - the $SU(4)$ expressions for the β''s and α''s arise from those for $SU(3)$ in (3.6) and (3.7) by the replacements $\theta_1 \to \theta_1, \theta_2 \to \theta_2, \theta_1' \to \theta_1', \theta_2' \to \theta_2' + \theta_3' + \theta_3$.

The recursive procedure given above can easily be extended to N generations.

VI. COMPARISON WITH SOME EXISTING PARAMETRIZATIONS OF THE MIXING MATRIX FOR N=4

The parametrization due to Barger et al [8] and Oakes [9] correponds to choosing $\beta_2, \gamma_2, \gamma_3$ complex and all others real. Thus, on putting

$$\gamma_1 = C_1 \;\; ; \;\; \gamma_2 = -S_1C_2e^{-i(\delta_1+\delta_3)} \;\; ; \;\; \gamma_3 = -S_1S_2C_4e^{-i\delta_2} \;\; ; \;\; \gamma_4 = -S_1S_2S_4$$
$$\beta_1 = C_3 \;\; ; \;\; \beta_2 = S_3C_6e^{-i\delta_3} \;\; ; \;\; \beta_3 = S_3S_6$$
$$\alpha_1 = C_5 \;\; ; \;\; \alpha_2 = S_5 \tag{6.1}$$

in (5.2) and interchanging the first and the fourth columns and the second and the third we recover the parametrization in [8] and [9] after multiplying the second and the third row by phase factors $e^{i(\delta_1+\delta_3)}$ and $e^{i\delta_2}$ respectively.

The parametrization of the mixing matrix for $N = 4$ due to Anselm et al [7] is less economical. It corresponds to distributing the three phases four quantitities $\beta_3, \gamma_2, \gamma_3, \gamma_4$ with all others real:

$$\gamma_1 = -S_{12} \; ; \; \gamma_2 = C_{12}C_{23}C_{24}e^{-i\alpha} \; ; \; \gamma_3 = C_{12}(S_{23}C_{24}C_{34} - S_{24}S_{34})e^{i\gamma} \; ; \; \gamma_4 = C_{12}(S_{23}C_{24}S_{34}e^{-i(\beta+\gamma)} + S_{24}C_{34})e^{-i\beta}$$
$$\beta_1 = -S_{13} \; ; \; \beta_2 = C_{13}S_{23}/\sqrt{(1 - C_{23}^2C_{24}^2)} \; ; \; \beta_3 = C_{13}C_{23}S_{24}e^{i(\alpha-\beta)}/\sqrt{(1 - C_{23}^2C_{24}^2)}$$
$$\alpha_1 = C_{14} \; ; \; \alpha_2 = -S_{14} \tag{6.2}$$

Substituting these expressions in (5.2) one obtains the results of Anselm et al after suitable permutation of the columns and multiplication of second and fourth row by factors $e^{i\alpha}$ and $e^{i(\beta+\gamma)}$.

The parametrization due to Harari and Leurer [14] corresponds to choosing $\beta_1, \gamma_1, \gamma_2$ complex with all others real. Thus on putting

$$\alpha_1 = S_{12}, \alpha_2 = C_{12}, \beta_1 = S_{13}e^{-i\delta}, \beta_2 = S_{23}C_{13}, \beta_3 = C_{23}C_{13} \tag{6.3}$$

and

$$\gamma_1 = S_{14}e^{-i\delta_{14}} \; ; \; \gamma_2 = C_{14}S_{24}e^{-i\delta_{24}} \; \gamma_3 = C_{14}C_{24}S_{34} \; ; \; \gamma_4 = C_{14}C_{24}C_{34} \tag{6.4}$$

in (5.2) we recover their results. In fact in the Harari-Leurer parametrization, to go from 3 (*or* $N-1$) generations to 4 (*or* N) generations, one needs to multiply the (appropriately augmented) mixing matrix at 3 (*or* $N-1$) generations, on the left, by a matrix consisting of 3 (*or* $N-1$) factors. In the present case, it is easily seen that the three matrices when multiplied out have precisely the same structure as in (5.1). In general, the $N-1$ factors when multiplied out precisely correspond to the coset representative of $SU(N)/SU(N-1)$ characterized by an N-dimensional complex unit vector with its first $N-2$ components complex and the rest real.

VII. PHASES IN THE MIXING MATRIX AND THE BARGMANN INVARIANTS

It is known that, under rephasing, apart from the obvious invariants $|V_{\alpha i}|$, the magnitudes of the matrix elements of the mixing matrix, the following quantities, quartic in V's,

$$t_{\alpha i\beta j} \equiv V_{\alpha i}V_{\beta j}V_{\alpha j}^*V_{\beta i}^* \tag{7.1}$$

are invariant under the rephasing transformations

$$V_{\alpha i} \rightarrow e^{i\theta'_\alpha}V_{\alpha i}e^{i\theta_i} \tag{7.2}$$

It is also evident that this set of invariants remains unchanged under row and column permutations.

In the present context, these invariants were first discussed by Jarlskog [15] and by Greenberg [16] for the case of three generations (for which there is only one independent invariant) and were later generalized to N-generations by Nieves and Pal [17] who showed that of these the following $(N-1)(N-2)/2$ quantities can be taken as independent

$$t_{\alpha i1N} \;\; ; \;\; \alpha \leq i, \alpha \neq 1, i \neq N \tag{7.3}$$

It can easily be verified by explicit calculations that the invariant phases given earlier for $N = 3, 4$ precisely coincide with $arg(t_{\alpha i1N}) \;\; ; \;\; \alpha \leq i, \alpha \neq 1,$.

We would like to bring out the connection between these and the Bargmann invariants introduced by Bargmann in the context of Wigner's unitary-antiunitary theorem. If $\psi_1, \psi_2 \cdots, \psi_n$ are any n vectors in a Hilbert space, with no two consecutive ones being orthogonal, the n-vertex Bargmann invariant is

$$\Delta_n(\psi_1, \psi_2, \cdots, \psi_n) = <\psi_1|\psi_2><\psi_2|\psi_3> \cdots <\psi_n|\psi_1> \tag{7.4}$$

It is easily seen that, under a common unitary transformation applied to all the ψ's, and also, under independent phase changes of the ψ's, Δ_n remains unchanged. As an aside, we would like to remark here that there exists a deep connection between Bargmann invariants and the geometric phase as has been lucidly brought out by Mukunda and Simon [19].

To see the relevance of Bargmann invariants in the present context, notice that V being a unitary matrix can be thought of as effecting a change of basis from one set of orthonormal vectors $|f_i>$ to another $|e_\alpha>$ so that $V_{\alpha i} = <f_i|e_\alpha>$ and one can express $V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}$ as a Bargmann invariant

$$V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i} = <e_\alpha|f_i><f_i|e_\beta><e_\beta|f_j><f_j|e_\alpha> \tag{7.5}$$

VIII. SUMMARY

To summarize, the parametrization proposed here has the following special features:

- Introduction of N^{th} generation requires one new $N \times N$ matrix determined by one N-dimensional complex unit vector, a $SU(N)/SU(N-1)$ coset representative, multiplying the complete matrix at previous generation level after augmenting its dimension by one through bordering the last column and row suitably.
- All the invariants for $N-1$ generations remain invariants for N-generations as well.
- One matrix of ours determined by an N-dimensional unit vector corresponds to a product of $N-1$ factors of Harari and Leurer.
- The existing parametrizations are easily read off from our general expressions.
- Opens up new possibilities for alternatives parametrizations which may be phenomenologically useful, particularly for $N \geq 4$.
- The connection between the rephasing invariants and the Bargmann invariants is brought out.

We hope that the unified approach to parametrization of the mixing matrix developed here will prove to be phenomenologically useful as well. In particular, the connection between the phases and the Bargmann invariants brought out here may provide a new perspective on their origin.

One of us (SC) is grateful to Prof V. Gupta for asking a question which initiated this work. We are also grateful to Prof R. Simon for numerous discussions.

---

[1] M. Kobayashi and K. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
[2] L. Maiani Phys. Lett. B **62**, 183 (1976).
[3] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1984).
[4] L. L. Chau and W. Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984).
[5] H. Fritzsch, Phys. Rev. D **32**, 3085 1985.
[6] J. Schechter and J. W. F. Valle, Phys. Rev D **21**, 309 1980 ; D **22**, 2227 (1980) ; M. Gronau, R. Johnson and J. Schechter, **32** 3062 (1985) ; M. Gronau and J. Schechter, Phys. Rev. D **31**, 2773 (1985).
[7] A. A. Anselm, J. L. Chakreuli, N. G. Uraltsev and T. Zhukoskaya, Phys. Lett. B **156**, 102 (1985).
[8] V. Barger, K. Whisnant and R. J. N. Phillips, D **23**, 2773 (1981).
[9] R. J. Oakes, Phys. Rev. D **26**, 128 (1982).
[10] X.-G. He and S. Pakvasa, Phys. Lett. B **156**, 236 (1985).
[11] I.I. Bigi, Z. Phys. C **27**, 303 (1985).
[12] G. Kramer and I. Montvay, Z. Phys. C **11**, 1128 (1981).
[13] U. Türke, E. A. Paschos, H. Usler and R. Decker, Nucl. Phys. B **258**, 313 (1985).
[14] H. Harari and M. Leurer, Phys. Lett. B **181**, 123 (1986).
[15] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[16] O. W. Greenberg, Phys. Rev. D **32**, 1841 (1985) ; L. Dunietz, O. W. Greenberg and D. Wu, Phys. Rev. Lett. **55**, 2935 (1985).
[17] J. F. Nieves and P. B. Pal, Phys. Rev. D **36**, 315 (1987).
[18] V. Bargmann, J. Math. Phys. **5**, 862 (1964).
[19] N. Mukunda and R. Simon Ann. Phys. (NY) **228** , 205 (1993); **228** , 269 (1993)