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A derivation of Regge trajectories in large−N transverse lattice QCD*

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Large-N QCD is analysed in light-front coordinates with a transverse lattice at strong coupling. The general formalism can be looked up on as a $d+n$ expansion with a stack of $d-$dimensional hyperplanes uniformly spaced in n transverse dimensions. It can arise by application of the renormalisation group transformations only in the transverse directions. At leading order in strong coupling, the gauge field dynamics reduces to the constraint that only colour singlet states can jump between the hyperplanes. With $d=2$, $n=2$ and large−N, the leading order strong coupling results are simple renormalisations of those for the 't Hooft model. The meson spectrum lies on a set of parallel trajectories labeled by spin. This is the first derivation of the widely anticipated Regge trajectories in a regulated systematic expansion in QCD.

1. THE CORNERSTONES

The systematic study of strong interactions started with the discovery of lots of hadrons and the $S-$matrix analysis of their interactions. An important ingredient in this analysis was the observation that the hadron masses, when plotted in the $M^2 - J$ plane, lay on a set of approximately linear trajectories—the Regge trajectories—labeled by various quantum numbers. This phenomenological feature subsequently led to dual resonance models and string theory. Later on quark models and QCD replaced all the bootstrap ideas, but still it is generally accepted that QCD will reproduce an effective string description of the strong interactions at long distances.

Wilson's introduction of a non-perturbative lattice regulator and strong coupling expansion provided a qualitative realisation of this expectation [1]. In brief [2]:

• Area law for a large Wilson loop demonstrates linear confinement.

• In the strong coupling limit, $g \to \infty$, the gauge field action vanishes and the gauge field becomes completely random.

• The strong coupling expansion in inverse powers of g has a non-zero radius of convergence. It is the analogue of the high temperature expansion used in statistical mechanics.

• The strong coupling limit can be understood as the extreme stage of renormalisation group evolution. The regulating cutoff is lowered as much as possible. All the excited states are integrated out, leaving behind only the lowest state in each quantum number sector of the theory.

• As $g \to \infty$, the gauge field dynamics is reduced to the constraint that at every space-time point all the fields must combine into a gauge singlet.

• The strong coupling limit is not universal, and results depend on the type of lattice discretisation used. In any case, only subgroups of the continuum Lorentz and chiral symmetries remain exact.

• As $g \to \infty$, the string tension diverges and glueballs disappear from the theory. But hadrons containing quarks retain finite mass.

• Despite non-universality, the lightest hadron masses follow a pattern qualitatively in agreement with their experimental values. Quantitative fits can be obtained with about 30% accuracy.

• With only one surviving state in each quantum number sector, form factors show pole dominance. In coordinate space, the tails are exponential (and not gaussian as is often assumed in phenomenological analyses).

• The strong coupling analysis can be carried out with any gauge group in any number of dimensions. In 4-dimensional QCD, numerical simulations show that the strong coupling region is analytically connected to the weak coupling region.

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't Hooft introduced the expansion in inverse powers of the number of colours as another method of simplifying the non-abelian gauge field dynamics. The method could obtain the meson spectrum in 2-dimensions for any value of the gauge coupling [3]. The important features of this solution are [4]:
∘ The large−N limit has only planar diagrams.
∘ Choosing an axial gauge in two dimensions eliminates the non-linear self-coupling of the gauge fields.
∘ The dynamics of the gauge field is replaced by a linearly confining potential.
∘ The ultraviolet regulator can be removed, while the infrared divergence is controlled by the principle value prescription.
∘ Use of light-front coordinates simplifies the Lorentz index structure of the theory and eliminates non-dynamical variables, making Lorentz invariance manifest.
∘ The light-front wavefunctions are just the parton structure functions appropriate for analysis of deep inelastic scattering.
∘ Spin does not exist in the 2-dimensional theory, but parity is a good quantum number.
∘ The meson spectrum is determined by the eigenvalues of an integral equation. It lies on an approximately linear trajectory in the M^2-n plane, where n is the radial excitation quantum number.

In the following sections, a framework is constructed that combines these two approaches to simplifying the gauge field dynamics. The result is an analytically tractable systematic expansion in QCD which is closer to observed phenomenology than any previous attempt.

2. INCREASING DIMENSIONS OF A THEORY

When exact results for a $d-$dimensional theory are known, they can be used to make predictions for the same theory in neighbouring number of dimensions. A well-known method used with weak coupling perturbation theory is the $d+\epsilon$ expansion, where physical properties are evaluated as asymptotic series in the continuous variable ϵ. A method that uses strong coupling expansions can be constructed too, introducing the explicit regulator as a transverse lattice. The space-time geometry can be thought of as a stack of $d-$dimensional hyperplanes, uniformly spaced along n transverse dimensions. Such a geometry is illustrated in Fig.1. It can arise from the $(d+n)-$dimensional theory by anisotropic renormalisation group evolution, where the cutoff is lowered only in n transverse dimensions.

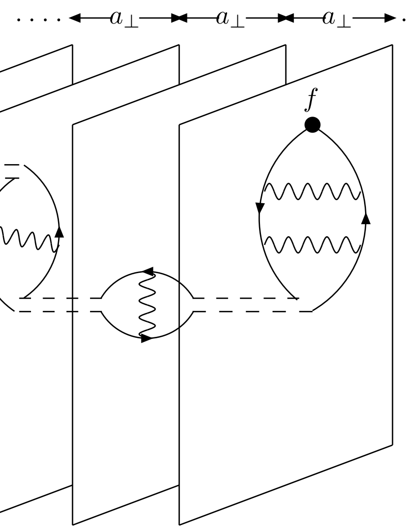

Figure 1. A schematic representation of the meson propagator on a transverse lattice at strong coupling. The jumps between hyperplanes are shown by dashed lines.

When the interaction between the hyperplanes becomes weak, we can use the exact results of the $d-$dimensional theory, to find the properties of the anisotropic $(d+n)$ dimensional theory. The expansion we need then is an expansion in the number of interactions between the hyperplanes[2]. In models of statistical mechanics, as $T_\perp \to \infty$, the hyperplanes decouple and the theory reduces to multiple copies of the $d-$dimensional theory. In gauge theories, as $g_\perp \to \infty$, charged states cannot propagate between hyperplanes, but neutral states can (see Fig.1). The resultant theory is then not the $d-$dimensional gauge theory, but related to it. Provided we can work out this relation, we can use the exact properties of the $d-$dimensinal theory to predict the properties of

[2]The subscript $\perp$ will be used to denote variables corresponding to the transverse directions.

the $(d+n)-$dimensional theory.

Conventionally, strong coupling (or high temperature) expansions have been constructed treating all directions on an equal footing (i.e. $d=0$). With a non-trivial start using the $d \neq 0$ exact solution, it is reasonable to expect that the anisotropic expansion would provide a better idea of the weak coupling behaviour of the $(d+n)-$dimensional theory. It is important to note that this strategy can be used only in situations when the $d-$ and the $(d+n)-$dimensional theory are in the same phase. (For example, the connection between the $2-$dimensional Schwinger model to $4-$dimensional QED is established in the strong coupling phase with a massive photon, and not in the weak coupling phase with a massless photon.)

The formalism of $(d+n)-$expansion is general enough to be applied to many other physical problems, e.g., layered high-temperature superconductors where Cooper pairs can hop between layers but individual electrons cannot, or higher-dimensional extensions of the standard model where the extra dimensions are accessible to only certain neutral particles. Here we use it to connect 4-dimensional large$-N$ QCD to the 2-dimensional 't Hooft model.

3. STRONG COUPLING TRANSVERSE LATTICE LARGE$-N$ QCD

With a transverse lattice, the coordinate space of QCD is reduced from R^4 to $R^2 \times Z^2$. It can also be thought of compactifying the momentum space of QCD to $R^2 \times T^2$. Transverse lattice formulation of QCD was introduced by Bardeen and Pearson long back [5]. Its strong coupling limit, however, has not been thoroughly investigated.

The transverse gauge field components are represented by the unitary link matrices, $U_\perp(x) = \exp(ia_\perp A_\perp(x))$, while the hyperplane gauge field components are the usual A_μ. Taking the strong coupling limit in the transverse directions replaces the dynamics of $U_\perp$ to constraints, i.e., only states contracted to colour singlets can propagate in the transverse direction. Choosing the light-front axial gauge $A^+ = 0$, the hyperplane gauge field is reduced to linear confining potential as in the 't Hooft model. Thus all the gauge dynamics is simplified, and it is easy to study the bound states of quarks[3]. The $\gamma-$matrices are not eliminated. But they are essential for defining the spin, and they can be handled in the same manner as in conventional strong coupling QCD.

The parameters of QCD are the gauge coupling and the quark masses. In lattice calculations, they are often traded off for the lattice cutoff and the hopping parameters, respectively. The anisotropic geometry increases the number of parameters; it is convenient to choose them as m and g for the $2-$dimensional hyperplanes and $a_\perp$ as the transverse lattice spacing. Thus the action can be written as (g is held fixed as $N \to \infty$)

$$\begin{aligned} S &= \frac{a_\perp^2 N}{g^2} \sum_{x_\perp} \int d^2x \Big[-\frac{1}{4} F^a_{\mu\nu b}(x) F^{\mu\nu b}_a(x) \\ &+ \overline{\psi}(x)(i\gamma^\mu \partial_\mu - \gamma^\mu A_\mu - m)\psi(x) \\ &+ \frac{i}{2a_\perp}\big(\overline{\psi}(x)\gamma^\perp U_\perp(x)\psi(x+a_\perp) \\ &- \overline{\psi}(x)\gamma^\perp U_\perp^\dagger(x-a_\perp)\psi(x-a_\perp)\big)\Big] \quad , \qquad (1) \end{aligned}$$

together with the constraint that in any correlation function products of $U_\perp(x)$ must contract to a colour singlet at each space-time point. The functional form of the lattice discretisation is not unique, and here I have chosen the naive fermion prescription for simplicity.

A mixed representation is suitable for explicit calculations—momentum space representation for the hyperplane variables (so they can be dealt with in the same way as in the 't Hooft model) and coordinate space representation for the transverse variables (so their colour singlet constraint can be enforced easily). For example, the hyperplane gauge field propagator becomes

$$D_{\mu\nu}(k; x_\perp, x'_\perp) = i\delta_{\mu+}\delta_{\nu+}\delta_{x_\perp,x'_\perp} \mathrm{P}\frac{1}{(k_-)^2} \quad . \qquad (2)$$

The transverse lattice spacing, $a_\perp$ converts the dimensionless QCD coupling g to the dimensionful coupling of the 't Hooft model.

[3]Numerical simulations of transverse lattice QCD have mostly concentrated on the pure gauge sector [6], which is more complicated in the strong coupling limit than the quark sector.

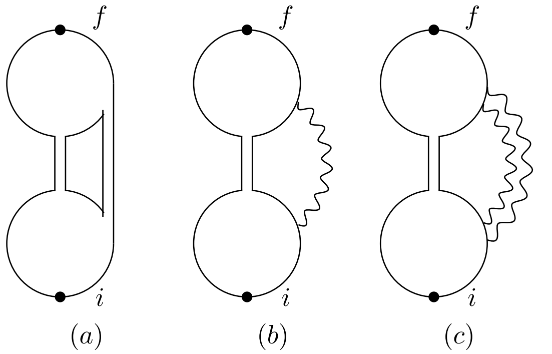

Figure 2. Corrections to the strong coupling transverse lattice large$-N$ QCD: (a) a quark loop is suppressed by N^{-1}, (b) a gluon jump between hyperplanes is suppressed by $N^{-1}g_\perp^{-2}$, (c) a glueball jump is suppressed by N^{-2}.

As illustrated in Fig.1, while propagating along the hyperplanes, the quarks spread out and exchange gluons. But for $g_\perp \to \infty$, they must come together and form colour singlets while jumping from one hyperplane to another. The corrections to this leading behaviour arise from multiple connections amongst parts of propagators on different hyperplanes, as shown in Fig.2. All these corrections involve loops and are suppressed at least by N^{-1}. The leading behaviour of the hadron propagators is thus "tree-like".

4. THE MESON SPECTRUM

It is easy to work out the explicit formulae for the propagators. The quark propagator receives gluon renormalisations as in the 't Hooft model, but it also gets corrections from hops back and forth between the hyperplanes (see Fig.3a). These additional corrections can be calculated in a closed form. It is sufficient to note, however, that they are tadpoles and cannot carry away any momentum or colour. Their only effect is to renormalise the quark mass, and since quark masses are ultimately fit parameters to reproduce the hadron masses, we do not need the explicit form for the tadpoles. Let $\tilde{m}$ and $\tilde{\kappa}_\perp$ be the renormalised mass and hopping parameters, after including all tadpole corrections. It is worthwhile to observe that the same tadpoles contribute to the chiral condensate $\langle\overline{\psi}\psi\rangle$, and so this mass renormalisation can be understood as generation of the constituent quark mass due to hops in the extra transverse directions.

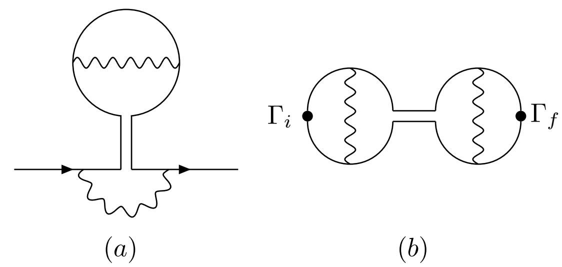

Figure 3. Modifications of the 't Hooft model results due to transverse lattice at strong coupling: (a) tadpole correction to the quark propagator, (b) jump correction to the meson propagator.

In order to evaluate the meson propagators, we have to keep track of the $\gamma-$matrices, unlike the 't Hooft model case. $\gamma-$matrices appear in the quark propagators as well as in the operators creating and destroying mesons (see Fig.3b). The anticommutation relations of $\gamma-$matrices are sufficient to evaluate the spinor traces, and the techniques for their evaluations are well-known in conventional strong coupling expansions.

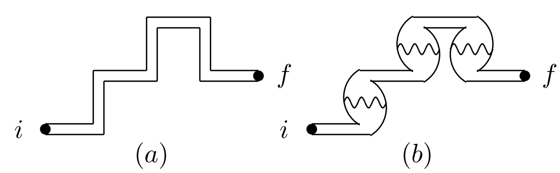

Figure 4. The strong coupling behaviour of the meson propagator, when (a) all the dimensions are latticised, (b) only the transverse dimensions are latticised.

As shown in Fig.4, the structure of the meson propagator in the transverse lattice QCD does not differ very much from that in the conventional strong coupling QCD. The graphs are "random walks" and can be conveniently summed using generating function techniques. The final result is simple: the complete inverse meson propagator is the sum of inverse meson propagators corresponding to propagation in individual directions. Compared to the 't Hooft model result, the poles of the meson propagator are thus translated due to propagation in the transverse directions.

It is known from conventional strong coupling expansions that this shift in the pole position depends on the $\gamma-$matrices used to create and destroy the meson, and is largely independent of the quark mass. Its explicit form depends on the lattice discretisation used for fermions in the transverse direction. For example, for naive fermions, the exact identity

$$G(x_i, x_f; \Gamma_i = \Gamma_f = \gamma_\perp)$$
$$= (-1)^{x_f - x_i} G(x_i, x_f; \Gamma_i = \Gamma_f = \gamma_5) \quad (3)$$

fixes the vector meson propagator to be the same as the pseudoscalar meson propagator shifted in momentum by $\pi/a_\perp$. The net result is that the meson spectrum lies on a set of parallel trajectories labeled by the $\gamma-$matrix used to create and destroy the meson, i.e. the spin. The separation between trajectories depends on $a_\perp$, which can therefore be determined using the experimental value as an input.

The fact that the meson spectrum consists of a tower of states in each quantum number channel is not unexpected. A successful combination of strong coupling and 't Hooft model results is bound to produce such a spectrum. The surprising part is that the results, though extracted in an extreme limit of QCD, fit the experimental data remarkably well. For example, $M_V^2 - M_P^2 \approx const.$, is observed to hold all the way from $\rho - \pi$ to $B^* - B$.

Thus we have arrived at the elusive Regge trajectories (and their daughthers) in a straightforward framework based on QCD. Because of the inherent transverse lattice structure, the theory has only discrete rotational symmetries. Consequently, the trajectories are labeled by the spin and not the total angular momentum J. Nevertheless, this is the closest we have got to the real world of strong interactions in a long time. A more rotationally symmetric regulator (instead of the transverse lattice), higher order calculations and/or numerical simulations should take us even closer to the real world.

5. FUTURE DIRECTIONS

The above results indicate that strong coupling transverse lattice large$-N$ QCD will be a highly useful phenomenological description of QCD. After all, it is QCD in an analytically tractable limit, and not an adhoc model. Some future investigations can be easily pointed out [7]:

- It is always possible to choose the reference frame such that the external legs of 2$-$point and 3$-$point correlation functions lie on a hyperplane. Such a choice will facilitate computation of many form factors, deep inelastic scattering amplitudes and decay matrix elements.
- Parameters of effective theories of the strong interactions, such as the chiral perturbation theory and the heavy quark effective theory, can be calculated without additional assumptions.
- With the light-front coordinates, the calculations automatically incorporate the Minkowski metric. So strong interaction phase-shifts are in principle calculable.
- The leading contribution of the sea quarks arises from loop graphs suppressed by N^{-1} (e.g., Fig.2a). This is calculable at the next order without giving up the $g_\perp \to \infty$ limit.
- Baryons can be included as solitons in this large$-N$ framework.
- As mentioned before, the $(d+n)-$expansion can be used to extend results of many exactly solved low-dimensional theories to higher dimensions.

REFERENCES

1. K. Wilson, Phys. Rev. D10 (1974) 2445.
2. See for example, M. Creutz "Quarks, Gluons and Lattices", Cambridge University Press, Cambridge, 1983.
3. G. 't Hooft, Nucl. Phys. B75 (1974) 461.
4. See for example, S. Coleman, in "Aspects of Symmetry: Selected Erice Lectures", Cambridge University Press, Cambridge, 1985, p.351.
5. W.A. Bardeen and R.B. Pearson, Phys. Rev. D14 (1976) 547; W.A. Bardeen, R.B. Pearson and E. Rabinovici, Phys. Rev. D21 (1980) 1037.
6. S. Dalley and B. van de Sande, Nucl. Phys. B (Proc. Suppl.) 83-84 (2000) 116; hep-lat/0010082.
7. A.D. Patel, in progress.