

Novel multi-band quantum soliton states for a derivative nonlinear Schrödinger model

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Abstract

We show that localized N -body soliton states exist for a quantum integrable derivative nonlinear Schrödinger model for several non-overlapping ranges (called bands) of the coupling constant η . The number of such distinct bands is given by Euler's ϕ -function which appears in the context of number theory. The ranges of η within each band can also be determined completely using concepts from number theory such as Farey sequences and continued fractions. We observe that N -body soliton states appearing within each band can have both positive and negative momentum. Moreover, for all bands lying in the region $\eta > 0$, soliton states with positive momentum have positive binding energy (called bound states), while the states with negative momentum have negative binding energy (anti-bound states).

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1 Introduction

Soliton states in integrable quantum field theory models in 1+1 dimensions have been studied extensively for many years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The quantum soliton states are usually constructed by using either the coordinate Bethe ansatz or the algebraic Bethe ansatz. For an integrable nonrelativistic Hamiltonian, the coordinate Bethe ansatz can yield the exact eigenfunctions in the coordinate representation. If such an eigenfunction decays sufficiently fast when any of the particle coordinates tends towards infinity (keeping the center of mass coordinate fixed), we call such a localized square-integrable eigenfunction a quantum soliton state. It is also possible to construct quantum soliton states using the algebraic Bethe ansatz, by choosing appropriate distributions of the spectral parameter in the complex plane [2, 3, 4]. The stability of quantum soliton states, in the presence of small external perturbations, can be determined by calculating their binding energy. It is usually found that localized quantum soliton states of various integrable models, including the well known nonlinear Schrödinger model (NLS) and the sine-Gordon model, have positive binding energy [1, 2, 3, 4, 5].

In this paper, we will study the quantum soliton states of an integrable derivative nonlinear Schrödinger (DNLS) model [6, 7, 8, 9]. Classical and quantum versions of the DNLS model have found applications in different areas of physics like circularly polarized nonlinear Alfvén waves in a plasma [11, 12], quantum properties of solitons in optical fibers [13], and some chiral Luttinger liquids which are obtained by dimensional reduction of a Chern-Simons model defined in two dimensions [14, 15]. It is known that the classical DNLS model can have solitons with momenta in only one direction, where the direction depends on the sign of the coupling constant; namely, the ratio of the momentum to the coupling constant is positive [16, 17, 18]. Here we want to investigate whether this chirality property of the classical solitons is preserved at the quantum level. The Lagrangian and Hamiltonian of the quantum DNLS model in its second quantized form are given by [6]

$$\begin{aligned} L &= i \int_{-\infty}^{\infty} dx \psi^\dagger \psi_t - H, \\ H &= \int_{-\infty}^{+\infty} dx \left[\hbar \psi_x^\dagger \psi_x + i\eta \{ \psi^{\dagger 2} \psi \psi_x - \psi_x^\dagger \psi^\dagger \psi^2 \} \right], \end{aligned} \quad (1.1)$$

where the subscripts t and x denote partial derivatives with respect to time and space respectively, η ($\neq 0$) is the real coupling constant, and we have set the particle mass $m = 1/2$. The field operators $\psi(x, t)$, $\psi^\dagger(x, t)$ obey the equal time commutation relations, $[\psi(x, t), \psi(y, t)] = [\psi^\dagger(x, t), \psi^\dagger(y, t)] = 0$, and $[\psi(x, t), \psi^\dagger(y, t)] = \hbar \delta(x - y)$. The quantum soliton states of this DNLS model have been constructed through both the algebraic Bethe ansatz [7, 8, 9] and the coordinate Bethe ansatz [6]. By applying the coordinate Bethe ansatz, it was found that quantum N -body soliton (henceforth called N -soliton) states exist for this DNLS model provided that η lies in the range $0 < |\eta| < \tan\left(\frac{\pi}{N}\right)$. Moreover it was observed that, similar to the classical case, such N -soliton states can have only positive values of P/η , where P is the momentum [6]. However, it was found recently

that soliton states can exist even with $P/\eta < 0$ provided that $\tan\left(\frac{\pi}{N}\right) < |\eta| < \tan\left(\frac{\pi}{N-1}\right)$ [10]. These soliton states have the surprising feature that their binding energy is negative. This naturally leads to the question of whether there are additional ranges of values of η for which there are N -solitons in the quantum DNLS model, and if so, what the values of momentum and binding energy of those solitons are.

In this paper, we re-investigate the ranges of values of η for which localized quantum N -soliton states exist in the DNLS model. In Sec. 2, we apply the coordinate Bethe ansatz and find the condition which the Bethe momenta have to satisfy in order that a quantum N -soliton state should exist. In Sec. 3, we find that there are certain nonoverlapping ranges of η , called bands, in which solitons can exist. The N -solitons of DNLS model found earlier [6,10] are contained only within the lowest band. Thus the existence of solitons in higher bands is a novel feature of DNLS model which is revealed through our present investigation. We also apply the idea of Farey sequences in number theory to completely determine the ranges of all bands for which N -soliton states exist for a given value of N . In Sec. 4, we show that the solitons appearing within each band can have both positive and negative values of P/η ; these have positive and negative binding energies respectively, and we call them bound and anti-bound states. In Sec. 5, we will use another concept from number theory, that of continued fractions, to give an explicit expression for the end points of the bands. We will also address the inverse problem of finding the values of N for which N -soliton states exist for a given value of η . We make some concluding remarks in Sec. 6.

2 Conditions for quantum N -soliton states in DNLS model

To apply the coordinate Bethe ansatz, we separate the full bosonic Fock space associated with the Hamiltonian (1.1) into disjoint N -particle subspaces $|S_N\rangle$. We want to solve the eigenvalue equation $H|S_N\rangle = E|S_N\rangle$. The coordinate representation of this equation is given by

$$H_N \tau_N(x_1, x_2, \dots, x_N) = E \tau_N(x_1, x_2, \dots, x_N), \quad (2.1)$$

where the N -particle symmetric wave function $\tau_N(x_1, x_2, \dots, x_N)$ is defined as

$$\tau_N(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{n!}} \langle 0 | \psi(x_1) \cdots \psi(x_N) | S_N \rangle, \quad (2.2)$$

and H_N , the projection of the second-quantized Hamiltonian H (1.1) on to the N particle sector, is given by

$$H_N = -\hbar^2 \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2i\hbar^2\eta \sum_{l < m} \delta(x_l - x_m) \left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_m} \right). \quad (2.3)$$

It is evident that H_N commutes with the total momentum operator in the N -particle sector, which is defined as

$$P_N = -i\hbar \sum_{j=1}^N \frac{\partial}{\partial x_j} . \quad (2.4)$$

Note that the Hamiltonian and momentum enjoy the scaling property $H_N \rightarrow \lambda^2 H_N$ and $P_N \rightarrow \lambda P_N$ if all the coordinates $x_i \rightarrow x_i/\lambda$. Given any one eigenfunction of H_N and P_N , therefore, we can find a one-parameter family of eigenfunctions by scaling all the x_i . We also observe that H_N remains invariant while P_N changes sign if we change the sign of η and transform all the $x_i \rightarrow -x_i$ at the same time; let us call this the parity transformation. Hence it is sufficient to study the problem for one particular sign of η , say, $\eta > 0$. The eigenfunctions for $\eta < 0$ can then be obtained by changing $x_i \rightarrow -x_i$; this leaves the energy invariant but reverses the momentum.

Next, we divide the coordinate space $R^N \equiv \{x_1, x_2, \dots, x_N\}$ into various N -dimensional sectors defined through inequalities like $x_{\omega(1)} < x_{\omega(2)} < \dots < x_{\omega(N)}$, where $\omega(1), \omega(2), \dots, \omega(N)$ represents a permutation of the integers $1, 2, \dots, N$. Within each such sector, the interaction part of the Hamiltonian (2.3) is zero, and the resulting eigenfunction is just a superposition of the free particle wave functions. The coefficients associated with these free particle wave functions can be obtained from the interaction part of the Hamiltonian (2.3), which is nontrivial at the boundary of two adjacent sectors. It is known that all such necessary coefficients for the Bethe ansatz solution of a N -particle system can be obtained by solving the corresponding two-particle problem [19]. Let us therefore construct the eigenfunctions of the Hamiltonian (2.3) for the two-particle case, without imposing any symmetry property on $\tau_2(x_1, x_2)$ under the exchange of the particle coordinates. For the region $x_1 < x_2$, we may take the eigenfunction to be

$$\tau_2(x_1, x_2) = \exp \{i(k_1 x_1 + k_2 x_2)\} , \quad (2.5)$$

where k_1 and k_2 are two distinct wave numbers. Using Eq. (2.1) for $N = 2$, we find that this two-particle wave function takes the following form in the region $x_1 > x_2$:

$$\tau_2(x_1, x_2) = A(k_1, k_2) \exp \{i(k_1 x_1 + k_2 x_2)\} + B(k_1, k_2) \exp \{i(k_2 x_1 + k_1 x_2)\} , \quad (2.6)$$

where the ‘matching coefficients’ $A(k_1, k_2)$ and $B(k_1, k_2)$ are given by

$$A(k_1, k_2) = \frac{k_1 - k_2 + i\eta(k_1 + k_2)}{k_1 - k_2} , \quad B(k_1, k_2) = 1 - A(k_1, k_2) . \quad (2.7)$$

By using these matching coefficients, we can construct completely symmetric N -particle eigenfunctions for the Hamiltonian (2.3). In the region $x_1 < x_2 < \dots < x_N$, these eigenfunctions are given by [6, 19]

$$\tau_N(x_1, x_2, \dots, x_N) = \sum_{\omega} \left(\prod_{l < m} \frac{A(k_{\omega(m)}, k_{\omega(l)})}{A(k_m, k_l)} \right) \rho_{\omega(1), \omega(2), \dots, \omega(N)}(x_1, x_2, \dots, x_N) , \quad (2.8)$$

where

$$\rho_{\omega(1),\omega(2),\dots,\omega(N)}(x_1, x_2, \dots, x_N) = \exp \{i(k_{\omega(1)}x_1 + \dots + k_{\omega(N)}x_N)\} . \quad (2.9)$$

In the expression (2.8), the k_n 's are all distinct wave numbers, and \sum_{ω} implies summing over all permutations of the integers $(1, 2, \dots, N)$. The eigenvalues of the momentum (2.4) and Hamiltonian (2.3) operators, corresponding to the eigenfunctions $\tau_N(x_1, x_2, \dots, x_N)$, are given by

$$P_N \tau_N(x_1, x_2, \dots, x_N) = \hbar \left(\sum_{j=1}^N k_j \right) \tau_N(x_1, x_2, \dots, x_N) , \quad (2.10a)$$

$$H_N \tau_N(x_1, x_2, \dots, x_N) = \hbar^2 \left(\sum_{j=1}^N k_j^2 \right) \tau_N(x_1, x_2, \dots, x_N) . \quad (2.10b)$$

The wave function in (2.8) will represent a localized soliton state if it decays when any of the relative coordinates measuring the distance between a pair of particles tends towards infinity. To obtain the condition for the existence of such a localized soliton state, let us consider the following wave function in the region $x_1 < x_2 < \dots < x_N$:

$$\rho_{1,2,\dots,N}(x_1, x_2, \dots, x_N) = \exp \left(i \sum_{j=1}^N k_j x_j \right) . \quad (2.11)$$

As before, the momentum eigenvalue corresponding to this wave function is given by $\hbar \sum_{j=1}^N k_j$. Since this must be a real quantity, we obtain the condition

$$\sum_{j=1}^N q_j = 0 , \quad (2.12)$$

where q_j denotes the imaginary part of k_j . The probability density corresponding to the wave function $\rho_{1,2,\dots,N}(x_1, x_2, \dots, x_N)$ in (2.11) can be expressed as

$$|\rho_{1,2,\dots,N}(x_1, x_2, \dots, x_N)|^2 = \exp \left\{ 2 \sum_{r=1}^{N-1} \left(\sum_{j=1}^r q_j \right) y_r \right\} , \quad (2.13)$$

where the y_r 's are the $N - 1$ relative coordinates: $y_r \equiv x_{r+1} - x_r$, and we have used Eq. (2.12). It is evident that the probability density in (2.13) decays exponentially in the limit $y_r \rightarrow \infty$ for one or more values of r , provided that all the following conditions are satisfied:

$$q_1 < 0 , \quad q_1 + q_2 < 0 , \quad \dots \dots \dots , \quad \sum_{j=1}^{N-1} q_j < 0 . \quad (2.14)$$

Note that the wave function (2.11) is obtained by taking ω as the identity permutation in (2.9). However, the full wave function (2.8) also contains terms like (2.9) with ω representing all possible nontrivial permutations. The conditions which ensure the decay of such

a term with a nontrivial permutation will, in general, contradict the conditions (2.14). Consequently, in order to have an overall decaying wave function (2.8), the coefficients of all terms $\rho_{\omega(1),\omega(2),\dots,\omega(N)}(x_1, x_2, \dots, x_N)$ with nontrivial permutations must be required to vanish. This leads to a set of relations like

$$A(k_1, k_2) = 0, \quad A(k_2, k_3) = 0, \quad \dots \dots, \quad A(k_{N-1}, k_N) = 0. \quad (2.15)$$

Thus the simultaneous validity of the conditions (2.12), (2.14) and (2.15) ensures that the full wave function $\tau_N(x_1, x_2, \dots, x_N)$ (2.8) represents a localized soliton state. Using the conditions (2.12) and (2.15), we obtain an expression for the complex k_n 's in the form

$$k_n = \chi e^{-i(N+1-2n)\phi}, \quad (2.16)$$

where χ is a real parameter, and ϕ is related to the coupling constant as

$$\phi = \tan^{-1}(\eta). \quad (2.17)$$

To obtain an unique value of ϕ from the above equation, we restrict it to the fundamental region $-\frac{\pi}{2} < \phi (\neq 0) < \frac{\pi}{2}$. [Note that η and ϕ have the same sign. Due to the parity symmetry mentioned above, we can restrict our attention to the range $0 < \phi < \frac{\pi}{2}$].

In this context, it may be mentioned that a relation equivalent to (2.16) can also be obtained through the method of the algebraic Bethe ansatz, when quantum soliton states of DNLS model are considered [7, 8, 9].

Now, let us verify whether the k_n 's in (2.16) satisfy the conditions (2.14). Summing over the imaginary parts of these k_n 's, we can express the conditions (2.14) in the form

$$\chi \frac{\sin(l\phi)}{\sin\phi} \sin[(N-l)\phi] > 0 \quad \text{for } l = 1, 2, \dots, N-1. \quad (2.18)$$

Thus, for some given values of ϕ , N and χ , a soliton state will exist when all the above inequalities are simultaneously satisfied. By using Eqs. (2.16) and (2.10), we obtain the momentum eigenvalue of such soliton state to be

$$P = \hbar\chi \frac{\sin(N\phi)}{\sin\phi}, \quad (2.19)$$

and the energy to be

$$E = \frac{\hbar^2\chi^2 \sin(2N\phi)}{\sin(2\phi)}. \quad (2.20)$$

The next section of our paper will be devoted to finding the ranges of values of ϕ where all the inequalities (2.18) are simultaneously satisfied for a given value of the particle number N .

3 Finding the values of ϕ where N -soliton states exist

In this section, we will study the values of ϕ where N -soliton states exist for different values of N . For the simplest case $N = 2$, the condition (2.18) is satisfied when ϕ lies in the range $0 < \phi < \frac{\pi}{2}$ ($-\frac{\pi}{2} < \phi < 0$) for the choice $\chi > 0$ ($\chi < 0$). Thus any nonzero value of ϕ within its fundamental region can generate a 2-soliton state. Looking at the momentum P in (2.19), we see that the ratio $P/\phi > 0$ since ϕ and χ have the same sign. Thus the chirality property of the classical solitons is preserved in the quantum theory for $N = 2$.

We will now consider the more interesting case with $N \geq 3$. Due to the parity symmetry of the Hamiltonian in (2.3), we will henceforth assume that $\phi > 0$. The inequalities in Eq. (2.18) can therefore be rewritten as

$$\chi \sin(l\phi) \sin[(N-l)\phi] > 0 \quad \text{for } l = 1, 2, \dots, N-1, \quad (3.1)$$

or

$$\chi [\cos[(N-2l)\phi] - \cos(N\phi)] > 0 \quad \text{for } l = 1, 2, \dots, N-1. \quad (3.2)$$

It is now convenient to consider the cases $\chi > 0$ and $\chi < 0$ separately.

For $\chi > 0$, Eq. (3.2) takes the form

$$\cos[(N-2l)\phi] > \cos(N\phi) \quad \text{for } l = 1, 2, \dots, N-1. \quad (3.3)$$

Let us now consider a value of ϕ of the form

$$\phi_{N,n} \equiv \frac{\pi n}{N}, \quad (3.4)$$

where n is an integer satisfying $1 \leq n < N/2$. Now, if n is odd, $\cos(N\phi_{N,n}) = -1$ and it is possible that all the inequalities in (3.3) will be satisfied since -1 is the minimum possible value of the cosine function. However, a closer look reveals that all the inequalities in (3.3) are satisfied only if N and n are relatively prime, i.e., if the greatest common divisor of N and n is 1. If N and n are not relatively prime, then let p be an integer (greater than 1) which divides both of them. Consider the integers $N' = N/p$ and $n' = n/p$. Since n is odd, p must be odd and therefore n' is also odd. Similarly, $N' = N/p$ is even (odd) if N is even (odd), and therefore $N - N'$ is an even integer which is less than N . We then see that there is an integer l for which $N - 2l = N'$, and therefore $\cos[(N-2l)\phi_{N,n}] = \cos(\pi n') = -1$; hence that particular inequality in Eq. (3.3) will be violated. We therefore conclude that all the inequalities in (3.3) will be satisfied for $\phi = \phi_{N,n}$ and n odd, if and only if N and n are relatively prime.

Similarly, we can consider the case $\chi < 0$. Eq. (3.2) then takes the form

$$\cos[(N-2l)\phi] < \cos(N\phi) \quad \text{for } l = 1, 2, \dots, N-1. \quad (3.5)$$

It is clear that all of these cannot be satisfied if N is even; in particular the inequality with $l = N/2$ will fail. We therefore assume that N is odd. Once again, we consider a value

of ϕ of the form given in (3.4) where n is now even. Now $\cos(N\phi_{N,n}) = 1$ and there is a chance that all the inequalities in (3.5) will be satisfied since 1 is the maximum possible value of the cosine function. However, we again find that this is true only if N and n are relatively prime. If N and n are not relatively prime, then let p be an integer (greater than 1) which divides both of them. Consider the integers $N' = N/p$ and $n' = n/p$. Since N is odd, p must be odd and therefore N' is also odd; hence $N - N'$ is an even integer which is less than N . Similarly, $n' = n/p$ is even since n is even. Thus there is an integer l for which $N - 2l = N'$, and then $\cos[(N - 2l)\phi_{N,n}] = \cos(\pi n') = 1$; hence that particular inequality is violated in Eq. (3.5). We therefore conclude that all the inequalities in (3.5) will be satisfied for $\phi = \phi_{N,n}$ and n even, if and only if N and n are relatively prime.

Putting these statements together, we see that all the inequalities in (3.2) are satisfied for $\phi = \phi_{N,n}$, if and only if N and n are relatively prime (with n odd for $\chi > 0$, and n even for $\chi < 0$). By continuity, it then follows that all the inequalities will hold in a neighborhood of $\phi_{N,n}$ extending from a value $\phi_{N,n,-}$ to a value $\phi_{N,n,+}$, such that $\phi_{N,n,-} < \phi_{N,n} < \phi_{N,n,+}$. We will call the region

$$\phi_{N,n,-} < \phi < \phi_{N,n,+} \quad (3.6)$$

as the band $B_{N,n}$. In this band, there is a soliton state with N particles.

For a given value of N , the number of bands in which soliton states exist is equal to the number of integers n which are relatively prime to N and satisfy $1 \leq n < N/2$. This is equal to half the number of integers which are relatively prime to N and satisfy $1 \leq n < N$. The latter number is called Euler's ϕ -function or totient function $\Phi(N)$ [20]. If N has the prime factorization

$$N = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} , \quad (3.7)$$

where p_1, p_2, \dots, p_k are all prime numbers, then the totient function is given by

$$\Phi(N) = N \prod_{i=1}^k \left(1 - \frac{1}{p_i} \right) . \quad (3.8)$$

Thus the number of bands lying within the region $\eta > 0$ coincides with $\Phi(N)/2$. From the parity symmetry of Hamiltonian (2.3), it follows that the total number of bands lying within the full range of η is given by $\Phi(N)$.

We now have to determine the end points $\phi_{N,n,-}$ and $\phi_{N,n,+}$ of the band $B_{N,n}$. The original inequalities in (3.1) show that the end points are given by ϕ of the form

$$\phi = \frac{\pi j}{l} , \quad (3.9)$$

where j and l are integers satisfying

$$1 \leq l < N , \quad \text{and} \quad j < \frac{l}{2} \quad (3.10)$$

(since $\phi < \pi/2$). In general, these j and l may not be relative prime numbers. However, one can always choose two relative primes j' and l' satisfying the constraint (3.10) such that $j'/l' = j/l$. Thus the end points of the band $B_{N,n}$ are given by two rational numbers ϕ/π of the form j/l (where j, l are relative primes satisfying the conditions in (3.10)) which lie *closest* to (and on either side of) the point $\phi_{N,n}/\pi = n/N$. The solution to this problem is well known in number theory and is described by the Farey sequences [20].

For a positive integer N , the Farey sequence is defined to be the set of all the fractions a/b in increasing order such that (i) $0 \leq a \leq b \leq N$, and (ii) a and b are relatively prime. The Farey sequences for the first few integers are given by

$$\begin{aligned}
F_1 : & \quad \frac{0}{1} \quad \frac{1}{1} \\
F_2 : & \quad \frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1} \\
F_3 : & \quad \frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1} \\
F_4 : & \quad \frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1} \\
F_5 : & \quad \frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1}
\end{aligned} \tag{3.11}$$

These sequences enjoy several properties, of which we list the relevant ones below.

(i) $a/b < a'/b'$ are two successive fractions in a Farey sequence F_N , if and only if

$$a'b - ab' = 1, \quad \text{and} \quad b, b' \leq N. \tag{3.12}$$

It then follows that both a and b' are relatively prime to a' and b .

(ii) For $N \geq 2$, if n/N is a fraction appearing somewhere in the sequence F_N (this implies that N and n are relatively prime according to the definition of F_N), then the fractions a_1/b_1 and a_2/b_2 appearing immediately to the left and to the right respectively of n/N satisfy

$$\begin{aligned}
a_1, a_2 & \leq n, \quad \text{and} \quad a_1 + a_2 = n, \\
b_1, b_2 & < N, \quad \text{and} \quad b_1 + b_2 = N.
\end{aligned} \tag{3.13}$$

To return to our problem, we now see that the points $\phi_{N,n}$ in (3.4) (which lie in the bands $B_{N,n}$) have a one-to-one correspondence with the fractions n/N , which appear on the left side of $1/2$ within the sequence F_N . Due to Eqs. (3.9) and (3.10), the end points of the band $B_{N,n}$ are given by

$$\begin{aligned}
\phi_{N,n,-} & = \frac{\pi a_1}{b_1}, \\
\phi_{N,n,+} & = \frac{\pi a_2}{b_2},
\end{aligned} \tag{3.14}$$

where a_1/b_1 and a_2/b_2 are the fractions lying immediately to the left and right of n/N in the Farey sequence F_N . These are the two unique fractions which lie closest to (and on either side) of n/N whose denominators are less than N (due to (3.13)). Therefore, as we move away from $\phi = \phi_{N,n}$, one of the inequalities in (3.1) will be violated for the first time at these two points. (The property $b_1 + b_2 = N$ shows that the same inequality in (3.1) is violated at the two end points of a given band $B_{N,n}$).

The end points of a band given in Eq. (3.14) satisfy the property

$$\begin{aligned} nb_1 - Na_1 &= 1, \\ nb_2 - Na_2 &= -1, \end{aligned} \tag{3.15}$$

due to the property (3.12) of Farey sequences. Given two integers N and n satisfying the conditions given above, one can numerically find the integers a_i and b_i by searching for solutions of Eqs. (3.15) within the limits given in (3.13). For the case of lowest band ($n = 1$), these equations can also be solved analytically to yield: $a_1 = 0$, $b_1 = 1$, $a_2 = 1$, $b_2 = N - 1$. Thus, for every $N \geq 3$, the range of lowest band is given by $0 < \phi/\pi < 1/(N-1)$ which agrees with the result obtained in Ref. [10]. It turns out that there is a way to analytically determine the integers a_i and b_i for the case of a general n by using the idea of continued fractions; this will be described in Sec. 5.

In Table 1, we show the ranges of values of ϕ for which solitons exist for $N = 2$ to 9. In Fig. 1, we present the same information pictorially for N going up to 20.

N	n	Range of values of ϕ/π
2	1	$0 < \phi/\pi < 1/2$
3	1	$0 < \phi/\pi < 1/2$
4	1	$0 < \phi/\pi < 1/3$
5	1	$0 < \phi/\pi < 1/4$
5	2	$1/3 < \phi/\pi < 1/2$
6	1	$0 < \phi/\pi < 1/5$
7	1	$0 < \phi/\pi < 1/6$
7	2	$1/4 < \phi/\pi < 1/3$
7	3	$2/5 < \phi/\pi < 1/2$
8	1	$0 < \phi/\pi < 1/7$
8	3	$1/3 < \phi/\pi < 2/5$
9	1	$0 < \phi/\pi < 1/8$
9	2	$1/5 < \phi/\pi < 1/4$
9	4	$3/7 < \phi/\pi < 1/2$

Table 1. The range of values of ϕ/π for which solitons exist for various values of N .

Due to the relations in (3.15), we see that the width of the right side of the band $B_{N,n}$ from $\phi_{N,n}$ to $\phi_{N,n,+}$ is $\pi/(Nb_2)$, while the width of the left side from $\phi_{N,n,-}$ to $\phi_{N,n}$ is

$\pi/(Nb_1)$. For later use, we note that each of these widths is strictly larger than π/N^2 , since $b_1, b_2 < N$. The total width $W_{N,n}$ of the band $B_{N,n}$ is given by

$$W_{N,n} = \frac{\pi}{Nb_1} + \frac{\pi}{Nb_2} . \quad (3.16)$$

For a given value of N , we will now find an expression for the total width of all the bands.

Consider two different bands $B_{N,n}$ and $B_{N,n'}$ and their end points given by

$$\begin{aligned} \frac{a_1}{b_1} &< \frac{n}{N} < \frac{a_2}{b_2} , \\ \frac{a'_1}{b'_1} &< \frac{n'}{N} < \frac{a'_2}{b'_2} , \end{aligned} \quad (3.17)$$

where we know from (3.12) that $n, n', b_1, b_2, b'_1, b'_2$ are all relatively prime to N . We also know that $n, n' < N/2$; we will assume that $n' < n$. We will now show that b'_1 and b'_2 are not equal to either b_1 or b_2 . Eq. (3.12) implies that

$$\begin{aligned} Na_1 - nb_1 &= -1 , \\ Na_2 - nb_2 &= 1 , \\ Na'_1 - n'b'_1 &= -1 . \end{aligned} \quad (3.18)$$

We then see that

$$\begin{aligned} N(a_1 - a'_1) &= nb_1 - n'b'_1 , \\ N(a_2 + a'_1) &= nb_2 + n'b'_1 . \end{aligned} \quad (3.19)$$

If b'_1 was equal to b_1 , we would get $N(a_1 - a'_1) = b'_1(n - n')$, i.e., N is a factor of $b'_1(n - n')$. Since b'_1 is relatively prime to N , this would mean that N must be a factor of $n - n'$ [20]. But this is not possible since $n - n' < N$. Similarly, if b'_1 was equal to b_2 , we would get $N(a_2 + a'_1) = b'_1(n + n')$, i.e., N is a factor of $n + n'$. But this is also not possible since $n + n' < N$. We therefore conclude that b'_1 is not equal to b_1 or b_2 . Similarly, we can show that b'_2 is not equal to b_1 or b_2 .

We thus have the result that for any value of N , the denominators of the end points of the various bands $B_{N,n}$ are all different from each other; we also know that they are all smaller than and relatively prime to N . Since the numbers of bands is equal to half the number of integers less than and relatively prime to N , and each band has two end points, the set of denominators of the end points of all the bands must be exactly the same as the set of integers less than and relatively prime to N . Eq. (3.16) now implies that the total width of all the bands is given by

$$W_N = \frac{\pi}{N} \sum_l \frac{1}{l} , \quad (3.20)$$

where the sum runs over all values of l which are relatively prime to N and satisfy $1 \leq l \leq N - 1$. If N is prime, we obtain

$$W_N = \frac{\pi}{N} \sum_{l=1}^{N-1} \frac{1}{l} \simeq \frac{\pi}{N} [\ln(N-1) + \gamma] \quad \text{as } N \rightarrow \infty, \quad (3.21)$$

where $\gamma = 0.57721 \dots$ is Euler's constant. We have numerically studied the behaviors of W_N and the sum $I_N \equiv \sum_{n=2}^N W_N$ for values of N up to 10^4 . We find that although W_N is a non-monotonic function of N , on the average it grows with N as $\ln N/N$; consequently, I_N grows as $(\ln N)^2$ as N becomes large. [In the range $N = 10^2$ to 10^4 , we find that $NW_N/[\pi(\ln(N-1) + \gamma)]$ fluctuates between 0.255 and 1.001, while $W_N/(\ln N)^2 = 1.179$ for $N = 10^4$].

We thus see that the allowed range of values of ϕ for which solitons exist goes to zero generically as $\ln N/N$ as $N \rightarrow \infty$. In contrast to this, the lowest band runs from 0 to $\pi/(N-1)$ and therefore has a width of $\pi/(N-1)$. This implies that the width of the lowest band becomes an insignificant fraction of the total width as N becomes large.

4 Momentum and binding energy of a N -soliton state

In this section, we will calculate the momentum and binding energy for the N -soliton states described above. We first look at the momentum of the solitons in a particular band $B_{N,n}$ using Eq.(2.19). The form of the end points given in (3.14) shows that $\sin(N\phi) = 0$ at only one point in the band $B_{N,n}$, namely, at $\phi = \phi_{N,n}$. In the right part of the band (i.e., from $\phi_{N,n}$ to $\phi_{N,n,+}$), the sign of $\sin(N\phi)$ is $(-1)^n$. In the left part of the band (i.e., from $\phi_{N,n,-}$ to $\phi_{N,n}$), the sign of $\sin(N\phi)$ is $(-1)^{n+1}$. Now, the analysis given above showed that χ has the same sign as $(-1)^{n+1}$. Hence the momentum given in (2.19) is positive in the left part of the band, negative in the right part of the band, and zero at $\phi = \phi_{N,n}$.

Next, we look at the energy using Eq. (2.20). To calculate the binding energy, we consider a reference state in which the momentum P of the N -soliton state given in (2.19) is equally distributed among the N single-particle scattering states. The real wave number associated with each of these single-particle states is denoted by k_0 . From Eqs. (2.10a) and (2.19), we obtain

$$k_0 = \frac{\chi \sin(N\phi)}{N \sin \phi}. \quad (4.1)$$

Using Eq. (2.10b), we can calculate the total energy for the N single-particle scattering states as

$$E_s = \hbar^2 N k_0^2 = \frac{\hbar^2 \chi^2 \sin^2(N\phi)}{N \sin^2 \phi}. \quad (4.2)$$

Subtracting E in (2.20) from E_s in (4.2), we obtain the binding energy of the N -soliton state as

$$E_B(\phi, N) \equiv E_s - E = \frac{\hbar^2 \chi^2 \sin(N\phi)}{\sin \phi} \left\{ \frac{\sin(N\phi)}{N \sin \phi} - \frac{\cos(N\phi)}{\cos \phi} \right\}. \quad (4.3)$$

It may be noted that the above expression of binding energy remains invariant under the transformation $\phi \rightarrow -\phi$.

Substituting $N = 2$ in Eq. (4.3), we obtain $E_B(\phi, 2) = 2\hbar^2 \chi^2 \sin^2 \phi$. Thus we get $E_B(\phi, 2) > 0$ for any nonzero value of ϕ . Let us now look at the case $N \geq 3$. We can rewrite Eq. (4.3) in the form

$$\begin{aligned} E_B(\phi, N) &= \frac{\hbar^2 \chi \sin(N\phi)}{N \sin^2 \phi \cos \phi} f(\phi, N), \\ f(\phi, N) &= \chi [\sin(N\phi) \cos \phi - N \cos(N\phi) \sin \phi]. \end{aligned} \quad (4.4)$$

We will now prove that the function $f(\phi, N)$ is positive in all the bands $B_{N,n}$ for all values of N and n . To show this, we add up all the inequalities given in (3.2), and use the identities

$$\begin{aligned} \sum_{l=1}^{N-1} \cos[(N-2l)\phi] &= \frac{\sin[(N-1)\phi]}{\sin \phi}, \\ \sin[(N-1)\phi] &= \sin(N\phi) \cos \phi - \cos(N\phi) \sin \phi. \end{aligned} \quad (4.5)$$

We then find that $f(\phi, N) > 0$. Hence, E_B given in (4.4) has the same sign as $\chi \sin(N\phi)$. Following arguments similar to that of the momentum, we find that the binding energy is positive in the left part of each band (called bound states), negative in the right part (called anti-bound states), and zero at the point $\phi = \phi_{N,n}$.

We thus see that for $\phi > 0$, the momentum and the binding energy are both positive in the left part of each band, and they are both negative in the right part. This is to be contrasted with the classical solitons in the classical DNLS model which always have positive momentum. If $\phi < 0$, we can similarly show that solitons with positive values of P/ϕ have positive binding energy, and solitons with negative values of P/ϕ have negative binding energy.

In Fig. 2, we show the binding energy E_B in (4.3) as a function of ϕ/π for three different values of N . (We have set $\hbar^2 \chi^2 = 1$ in that figure). We see that E_B is indeed positive (negative) in the left (right) part of each band.

5 Continued fractions

In this section, we will apply the technique of continued fractions to our problem. This will lead us to an explicit expression for the end points of the band $B_{N,n}$ for any value

of N and n . We will also address the problem of determining the values of N for which N -soliton states exist for a given value of ϕ . It will turn out that continued fractions provide a way of finding some (but not necessarily all) values of N for which solitons exist for a given ϕ .

We first discuss the idea of a continued fraction [20]. Any positive real number x has a simple continued fraction expansion of the form

$$x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}} , \quad (5.1)$$

where the n_i 's are integers satisfying $n_0 \geq 0$, and $n_i \geq 1$ for $i \geq 1$. The expansion ends at a finite stage with a last integer n_k if x is rational. In that case, we can assume that the last integer satisfies $n_k \geq 2$. (If n_k is equal to 1, we can stop at the previous stage and increase n_{k-1} by 1). With this convention for n_k , the continued fraction expansion is unique for any rational number x . If x is an irrational number, the continued fraction expansion does not end, and it is unique.

We will be concerned below with the continued fraction expansion for $x = \phi/\pi$ which lies between 0 and 1/2; hence, $n_0 = 0$. We will use the notation $x = [0, n_1, n_2, n_3, \dots]$ for such an expansion.

Given a number x , the integers n_i can be found as follows. We define $x_0 = x$. Then $n_0 = [x_0]$, where $[y]$ denotes the integer part of a non-negative number y . We then recursively define $x_{i+1} = 1/(x_i - n_i)$, and obtain $n_{i+1} = [x_{i+1}]$ for $i = 0, 1, 2, \dots$. If we stop at the k^{th} stage, we obtain a rational number $r_k = [n_0, n_1, n_2, \dots, n_k]$ which is an approximation to the number x . If we write $r_k = p_k/q_k$, where p_k and q_k are relatively prime, then we have the following three properties for all values of $k \geq 1$ [20],

$$\begin{aligned} q_k &< q_{k+1} , \\ p_{k+1}q_k - p_kq_{k+1} &= (-1)^k , \\ |x - \frac{p_k}{q_k}| &< \frac{1}{q_k^2} . \end{aligned} \quad (5.2)$$

Using these properties, we can now find the end points $\phi_{N,n,\pm}$ of the band $B_{N,n}$. In Sec. 3, we saw that this is equivalent to finding the fractions a_1/b_1 and a_2/b_2 which lie to the immediate left and right of the fraction n/N in the Farey sequence F_N . Let us suppose that n/N has a continued fraction expansion given by

$$\begin{aligned} \frac{n}{N} &= [0, n_1, n_2, \dots, n_{k-1}, n_k] \\ &= [0, n_1, n_2, \dots, n_{k-1}, n_k - 1, 1] , \end{aligned} \quad (5.3)$$

where the expression in the second equation can be seen to be equivalent to the one in the first equation. Now consider the two fractions

$$\begin{aligned} \frac{a}{b} &= [0, n_1, n_2, \dots, n_{k-1}] , \\ \frac{c}{d} &= [0, n_1, n_2, \dots, n_{k-1}, n_k - 1] . \end{aligned} \quad (5.4)$$

On comparing these expressions to those given in (5.3) and using the properties of continued fractions in (5.2), we see that

$$\begin{aligned} b, d &< N, \\ nb - Na &= (-1)^{k-1}, \\ nd - Nc &= (-1)^k. \end{aligned} \tag{5.5}$$

We now see from Eq. (3.12) that if k is odd, then a/b and c/d are to the immediate left and right respectively of n/N in the Farey sequence F_N , while if k is even, then a/b and c/d are to the immediate right and left respectively of n/N in the sequence F_N . Hence the end points $\phi_{N,n,-}/\pi$ and $\phi_{N,n,+}/\pi$ are given by a/b and c/d respectively (in Eq. (5.4)) if k is odd, and vice versa if k is even. Thus the continued fraction expansion gives a convenient way of determining $\phi_{N,n,\pm}$ if N and n are given.

As an explicit example, consider the value $\phi/\pi = 3/8$. The continued fractions expansion of this is given by $\phi/\pi = [0, 2, 1, 2] = [0, 2, 1, 1, 1]$. Eq. (5.4) then gives the two neighboring fractions in the Farey sequence F_8 to be $a/b = [0, 2, 1] = 1/3$ and $c/d = [0, 2, 1, 1] = 2/5$. The band $B_{8,3}$ therefore lies in the range $1/3 < \phi/\pi < 2/5$ as shown in Table 1.

Next, we study the question of determining the values of N for which N -body solitons exist if a value of ϕ is given. Suppose that we know the continued fraction expansion

$$\frac{\phi}{\pi} = [0, n_1, n_2, \dots]. \tag{5.6}$$

Suppose that we stop at any any point in this expansion, say, at the k^{th} stage, and we get

$$\frac{p_k}{q_k} = [0, n_1, n_2, \dots, n_k]. \tag{5.7}$$

Then we know from (5.2) that

$$\left| \frac{\phi}{\pi} - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}. \tag{5.8}$$

We now recall from Sec. 3 that both the right and the left part of the band B_{q_k, p_k} have widths which are larger than $1/q_k^2$. Eq. (5.8) therefore implies that ϕ/π must lie within the band B_{q_k, p_k} . We have thus found a value of $N = q_k$ for which we know that a N -body soliton state exists for the given value of ϕ . We can generate several such values of N by stopping at different stages k in the expansion given in (5.6).

If ϕ/π is rational, the continued fraction expansion stops at a finite stage, so we only obtain a finite number of values of N in this way. This can also be seen directly from Eq. (3.1). If $\phi/\pi = p/q$ is rational, then at least one of the inequalities in (3.1) will be violated if $N > q$. Moreover, one gets $k_n = k_{n+q}$ from Eq.(2.16) by putting $\phi/\pi = p/q$. Thus at least two wave numbers would coincide when $N > q$. As a result, the eigenfunction (2.8) becomes trivial in this case. We thus conclude that if ϕ/π is rational, there is only a finite

number of values of N for which a N -body soliton state exists. On the other hand, if ϕ/π is irrational, then the expansion in (5.6) does not end, and we can use the procedure described above to find an infinite number of possible values of N for which a N -body soliton exists.

Given a value of ϕ , the procedure described above will find values of N for which we can prove that a N -body soliton exists. This does not rule out the possibility that there may be other values of N for which such solitons exist. [To give a simple example, consider the value $\phi = \pi/10$. Using Eq. (3.1), we see that there are bound states for all values of N from 2 to 10. However, the continued fraction technique only gives the value of $N = 10$ for which a soliton exists]. The continued fraction technique therefore provides a sufficient but not necessary condition for finding the desired values of N .

6 Conclusion

By applying the coordinate Bethe ansatz, we have investigated the range of the coupling constant η for which localized quantum N -soliton states exist in the DNLS model. Using the ideas of Farey sequences and continued fractions, we have given explicit expressions for all the allowed ranges (bands) of η for which N -body solitons exist. The continued fraction also provides a way of finding some values of N for which soliton states exist for a given value of η . If $\phi = \tan^{-1}(\eta)$ is rational (irrational), the number of values of N for which soliton states exist is finite (infinite).

We find that for $N \geq 3$, the N -body solitons can have both positive and negative momentum. Thus the chirality property of classical DNLS solitons is generally not preserved at the quantum level. We also calculated the binding energy for the soliton states. We find that solitons with positive values of P/η form bound states with positive binding energy. Solitons with negative values of P/η have negative binding energy and hence form anti-bound states. The anti-bound solitons would be expected to be unstable in the presence of external perturbations. As a future study, it may be interesting to calculate the decay rate of such quantum solitons by introducing small perturbations in the DNLS Hamiltonian.

In the continuum version of the DNLS model (this can be obtained from the N -body Schrödinger model by taking the limit $N \rightarrow \infty$), it is known that solitons exist only if $0 < N|\phi| < \pi$ [6, 18]; this corresponds to the lowest band. One might ask whether there is a continuum version of the solitons in the higher bands which we have found in this paper. This may be an interesting problem for future study. It is possible that only the lowest band has a counterpart in the continuum theory. Note that if ϕ lies in the lowest band, then the phases of k_n in Eq. (2.16) vary slowly with n ; the spacing between two neighboring phases is of order $1/N$. This may be necessary to ensure that the continuum limit exists. For the generic band, where ϕ is of order 1 rather than of order $1/N$, the

phases of neighboring values of k_n vary rapidly; this may make it difficult to take the continuum limit.

To conclude, we have shown that the soliton structure in the DNLS model is much richer than in the well known NLS model. In the latter model, the interaction part of the Hamiltonian is given by $\mu \sum_{l < m} \delta(x_l - x_m)$ in the coordinate representation. This model has N -soliton states for any value of $N \geq 2$ and any value of $\mu < 0$; all the solitons have positive binding energy. In contrast to this, the DNLS model allows soliton states in only certain bands of the coupling constant for $N \geq 3$, and these bands form intricate number theoretic patterns.

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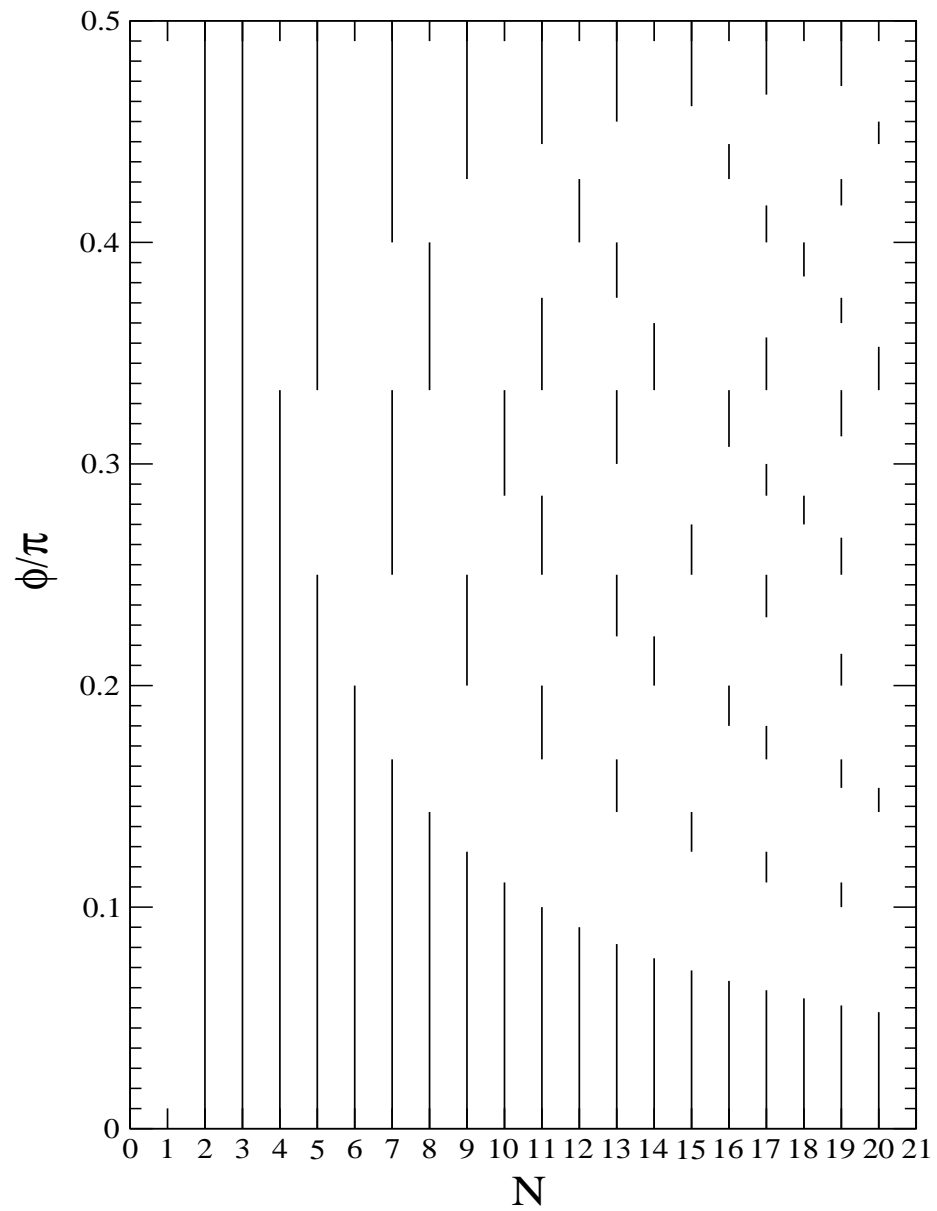


Figure 1: The values of ϕ/π for which N -body soliton states exist for various values of N .

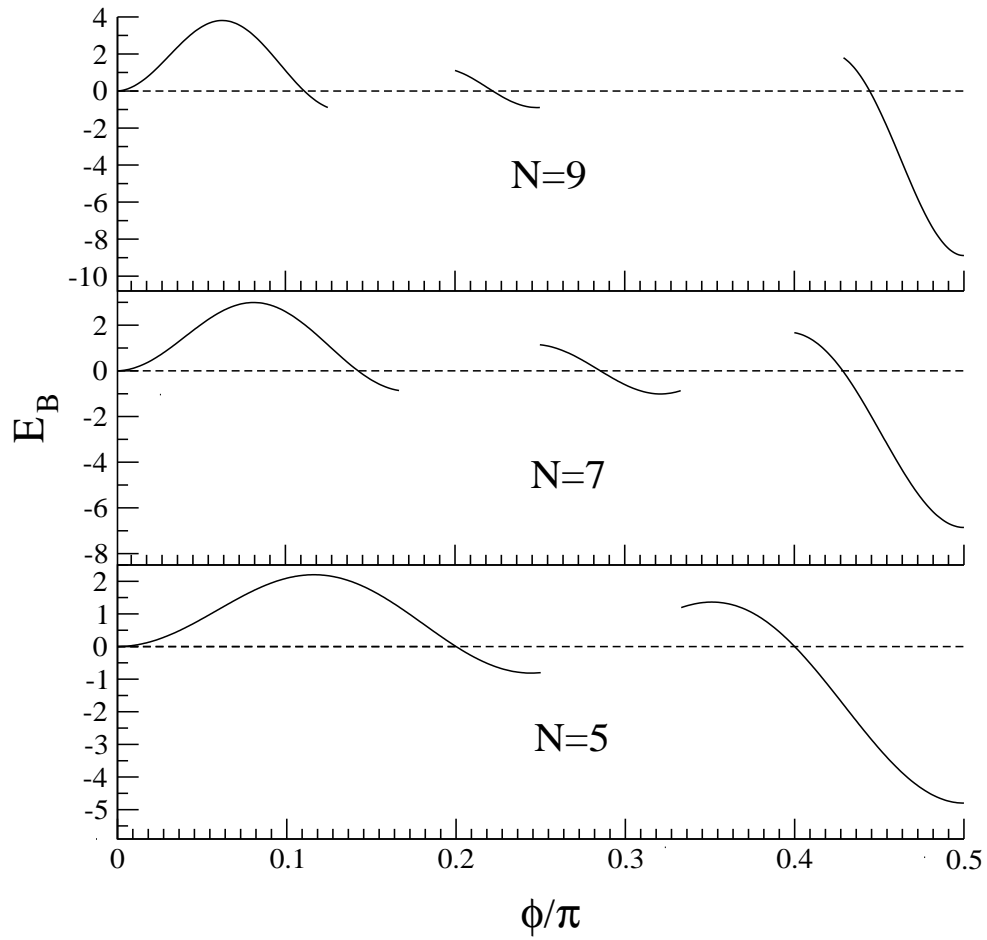


Figure 2: The binding energy E_B of the N -body soliton states as a function of ϕ/π for three different values of N . Positive and negative values of E_B denote bound and anti-bound states respectively.