Technical Notes and Correspondence

Regional Pole Placement of Multivariable Systems Under Control Structure Constraints

S. Sathiya Keerthi and Makarand S. Phatak

***Abstract*— Many controller realizations are structurally constrained. Some typical examples are static output feedback, constant gain feedback for multiple operating points of a system, two-controller feedback, and decentralized feedback. A general class of problems of regional pole placement of multivariable systems with such control structure constraints is considered and a unified numerical method is given to solve them. First a problem in this class is converted to a problem of solving a system of equalities and inequalities. This system is then solved by using a modified homotopy method.**

I. Introduction

Consider structurally constrained controllers, such as static output feedback, constant gain feedback for multiple operating points of a system, two-controller feedback, and decentralized feedback. With such constrained controllers it is in general not possible to place the eigenvalues arbitrarily in the complex plane. For satisfactory dynamical behavior of a system, it usually suffices to place the eigenvalues in some desired stability region, S in the complex plane, i.e., to S-stabilize the system. With structurally constrained controllers S-stabilization is quite possible. In this paper we consider a general class of problems of S-stabilization (or regional pole placement) of multivariable systems with control structure constraints and give a numerical method to solve them. This work is a nontrivial extension of the authors' earlier work [6] to the multivariable case. While doing the extension the possible overlap between [6] and this note is kept at a minimum by omitting the most repetitive details.

The problems mentioned here have also been considered in many earlier papers from which we choose references somewhat arbitrarily and point to [3] for the case of static output feedback, [8] for the case of constant gain feedback for multiple operating points of a system, [9] for the case of two-controller feedback, and [5] for the case of decentralized feedback. Our method is fundamentally different from the above methods and treats different problems under a single framework.

In our method we formulate (Section II) each of the regional eigenvalue placement problems as a problem of solving a system of equalities and inequalities, which we solve (Section III) by employing a modified homotopy framework. An example is presented (Section IV) to show the efficacy of our method.

II. Problem Formulation

Consider the multivariable dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{2.1}$$

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S. S. Keerthi is with the Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India.

M. S. Phatak was with the Department of Computer Science and Automation, Indian Institute of Science and is now with Aerospace Systems Private Limited, Bangalore, India.

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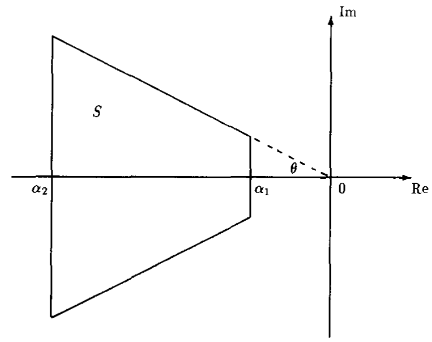

Fig. 1. An example of a popular stability region S in the complex plane.

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and A, B, C are constant matrices of appropriate dimensions. We will assume the following throughout: system (2.1) is controllable and observable, rank $B = m$ and rank $C = p$. The feedback we consider is of the form

$$u(t) = -Kx(t). \tag{2.2}$$

Our objective is to S-stabilize, i.e., to place the eigenvalues of $(A - BK)$ in some desired stability region, S in the complex plane under given control structure constraints. S comes from specifications of the desirable dynamical behavior of the closed-loop system. In particular we focus our attention on the popular stability region, S shown in Fig. 1. Control structure constraints are imposed by the way feedback is realized. For example, the in case of static output feedback the constraint is that rows of K must belong to the range space of rows of C so that state feedback (2.2) can be realized as static output feedback. To solve this constrained problem we formulate a system of inequalities and equalities

$$g(v) \leq 0 \tag{2.3}$$

$$h(v) = 0 \tag{2.4}$$

where v is some set of intermediate variables. It is worth mentioning here that a neat transformation of our control problem to an instance of (2.3)–(2.4) is by no means trivial; so our way of doing it is interesting in its own right. Once a v satisfying (2.3)–(2.4) is found the gain matrix required for stabilization can easily be found. In Section II-A we present details of how v is chosen and how to formulate the inequalities. The equalities will be derived in Section II-B.

A. Choice of Variables and Formation of Inequalities

Given a controllable pair (A, B) the feedback matrix K which gives a specified characteristic polynomial for $(A - BK)$ is not unique. K, however, is determined uniquely by a specified matrix polynomial $P(s)$ [1]. $P(s)$ is defined by

$$P(s) = \overline{P}(s) + \Gamma V(s) \tag{2.5}$$

where Γ is a constant (i.e., independent of s) $m \times n$ matrix

$$\overline{P}(s) = \text{diag}\,[s^{\nu_1}, \cdots, s^{\nu_m}],$$

$$V(s) = \text{block diag}\,[(1\, s \cdots s^{n_1 - 1})^T, \cdots, (1\, s \cdots s^{n_m - 1})^T]$$

and $n_1, \cdots, n_m$ are the controllability indexes of the pair (A, B). The K which satisfies

$$\det(sI - A + BK) = \det(P(s)) \tag{2.6}$$

is given by

$$K = U^{-1}(\overline{K} + \Gamma T) \tag{2.7}$$

where the matrices U (nonsingular and upper triangular), $\overline{K}$, and T (nonsingular) depend on (A, B); see [1] for details of the dependency. Since the relationship between K and Γ is affine, as given by (2.7) Γ is an attractive choice for the set of variables, v. However the problem of finding conditions on the elements of Γ so that the set of eigenvalues, $\Lambda = \lambda(A - BK) \subset S$ is very hard. On the other hand, suppose we denote

$$\det(P(s)) = \det(\overline{P}(s) + \Gamma V(s)) = \left[\prod_{i=1}^{l}(s^2 + a_i s + b_i)\right](s - \lambda) \tag{2.8}$$

where $l = \lfloor n/2 \rfloor$, the integer part of $n/2$ and the last term $(s - \lambda)$ occurs when n is odd. Then the following proposition shows that the constraint $\lambda(A - BK) \subset S$ can be easily reformulated in terms of the elements of γ, where

$$\gamma = [a_1 b_1 \cdots a_l b_l \lambda]^T. \tag{2.9}$$

Proposition 2.1: Let S be as shown in Fig. 1, then $\lambda(A - BK) \subset S$ is equivalent to

$$\alpha_2 \le -a_i/2 \le \alpha_1, \qquad b_i \le (1 + \tan^2\theta)(a_i/2)^2$$

$$\alpha_1^2 + \alpha_1 a_i + b_i \ge 0, \qquad \alpha_2^2 + \alpha_2 a_i + b_i \ge 0, \qquad \forall i \in \{1, \cdots, l\}$$

$$\alpha_2 \le \lambda \le \alpha_1. \tag{2.10}$$

By using the results on representation of general stability regions in terms of inequalities [1, p. 298–299] Proposition 2.1 can be easily proved. We omit the details for the sake of brevity.

The use of quadratic factors in (2.8) avoids the need to work with complex numbers. We define the set of variables, v by

$$v = \{\Gamma, \gamma\}. \tag{2.11}$$

It is clear that the elements of v are not independent. The reason for choosing such an overdetermined set, v, is given later in Remark 2.2. In Section II-B we develop equality constraints which must be satisfied by the elements of Γ and γ to ensure (2.6) and (2.8). The system of inequalities (2.10) define $g(v) \le 0$ (truely $g(\gamma) \le 0$) in (2.3).

B. Equalities

We first formulate the equalities which are directly associated with the definition of the set of variables, v in (2.11).

Assumption 2.1: Let $p_i(s) = s^2 + a_i s + b_i$, $i = 1, \cdots, l$, and $p_{l+1}(s) = s - \lambda$. For each $i, j \in \{1, \cdots, l+1\}$, $i \ne j$, $p_i(s)$ and $p_j(s)$ are relatively prime.

Proposition 2.2: Suppose that Assumption 2.1 holds. Then (2.8) is equivalent to the system of n equalities given by

$$e_1[\text{block}\det(\overline{P}(C_i) + \Gamma V(C_i))] = 0, \qquad \forall i \in \{1, \cdots l\},$$
$$\det(\overline{P}(\lambda) + \Gamma V(\lambda)) = 0 \tag{2.12}$$

where

$$e_1 = [1 \quad 0], \qquad C_i = \begin{bmatrix} 0 & 1 \\ -b_i & -a_i \end{bmatrix} \quad \text{and}$$

$$\text{block}\det(P(C_i)) = \det(P(s))|_{s=C_i}.$$

Proof: We can eliminate s in (2.8) by substituting it by C_i, $i = 1, 2, \cdots l$, and λ so that the right-hand side of (2.8) is zero by Cayley–Hamilton's theorem. Then we get an intermediate system of equations which is the same as (2.12) except that the term e_1 is absent. This intermediate system of equations contains a total of $4l + 1 = (n + 2l)$ equations. Out of these $2l$ are redundant and when they are eliminated we get (2.12). Thus (2.8) implies (2.12).

We now show, using Assumption 2.1, that (2.12) implies (2.8). Let $p_i(s)$ be as in Assumption 2.1. If we show that, for each $i = 1, \cdots, l+1$

$$p_i(s) \text{ divides } \det(\overline{P}(s) + \Gamma V(s)) \tag{2.13}$$

then we are done because of Assumption 2.1 and the fact that the degree of $\det(\overline{P}(s) + \Gamma V(s))$ is equal to n. By the second-level equation in (2.12) it directly follows that (2.13) holds for $i = l + 1$. Now take $i \in \{1, \cdots, l\}$. To show (2.13) we need to consider two cases: (1) $\lambda_1 = \lambda_2$; (2) $\lambda_1 \ne \lambda_2$, where λ_1 and λ_2 are the roots of $s^2 + a_i s + b_i = 0$.

In case 1, $s^2 + a_i s + b_i = 0$ has a root λ_1 of multiplicity two. We have

$$C_i \begin{bmatrix} 1 & 0 \\ \lambda_1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \lambda_1 & 1 \end{bmatrix}^{-1}$$

and

$$\begin{aligned} &\text{block}\det(\overline{P}(C_i) + \Gamma V(C_i)) \\ &= \begin{bmatrix} 1 & 0 \\ \lambda_1 & 1 \end{bmatrix} \\ &\cdot \begin{bmatrix} \det(\overline{P}(\lambda_1) + \Gamma V(\lambda_1)) & \frac{d}{d\lambda_1}\det(\overline{P}(\lambda_1)) + \Gamma V(\lambda_1) \\ 0 & \det(\overline{P}(\lambda_1) + \Gamma V(\lambda_1)) \end{bmatrix} \\ &\cdot \begin{bmatrix} 1 & 0 \\ \lambda_1 & 1 \end{bmatrix}^{-1}. \end{aligned} \tag{2.14}$$

From (2.12) and (2.14) it follows that

$$\det(\overline{P}(\lambda_1) + \Gamma V(\lambda_1)) = 0, \qquad \frac{d}{d\lambda_1}\det(\overline{P}(\lambda_1)) + \Gamma V(\lambda_1) = 0$$

and hence (2.13) holds. A similar (actually easier) proof holds for case 2. This completes the proof of the proposition. ■

Remark 2.1: Equations in (2.12) are independent. The proof of this is implicit in Lemma 3.2 which is stated and proved in Section III.

Remark 2.2: In the single-input case K is determined uniquely by γ and can be expressed directly in terms of the elements of γ as [using factorization of the scalar polynomial $P(s)$ in (2.8)] $K = q_n[\prod_{i=1}^{l}(A^2 + a_i A + b_i I)](A - \lambda I)$, where q_n is as defined in [1]. So (2.12) is not required in the single-input case [6]. In the multi-input case K is determined uniquely by Γ and cannot be expressed directly in terms of the elements of γ and some other variables required for its unique determination. This is because of the lack of a factorized parameterization of the matrix polynomial $P(s)$. So we define the overdetermined set v in (2.11) and the constraining equations (2.12) for the multi-input case.

The block determinant in (2.12) and its partial derivatives with respect to the elements of v, which are required in the numerical solution described in Section III, can be obtained by using a transformation of a polynomial matrix to Hermite form [2]. This transformation uses elementary operations to reduce a square polynomial matrix to triangular form. First $\det(P(s))$ is obtained from the Hermite form of $P(s)$ as $\det(P(s)) = s^n + \beta_1 s^{n-1} + \cdots + \beta_n$. Then block $\det P(C)$ is obtained by evaluating the above polynomial at C, i.e., (subscript i in C_i, a_i, and b_i is dropped in the following expressions for notational simplicity) block $\det P(C) = \det(P(s))|_{s=C} = C^n + \beta_1 C^{n-1} + \cdots + \beta_n I_2$. Partial derivatives of block determinant with respect to elements of Γ are obtained as follows. From $P(s)$ equal to the expression found at the bottom of the page where γ_i^{jk}'s are the elements of Γ; the partial derivative of block $\det P(C)$ with respect to, say, γ_1^{11}, is nothing but $C^{n_1-1}[\operatorname{adj} P(s)]_{11}|_{s=C}$, where $[\operatorname{adj} P(s)]_{11}$ is the $(1, 1)$ element of the adjoint of $P(s)$. The partial derivatives with respect to other $\gamma_i^{jk'}$s are similarly obtained. Partial derivatives of block determinant with respect to elements of C are obtained from the partial derivatives of power of C which are given below

$$\frac{\partial}{\partial a} C^r = \sum_{j=1}^{r} C^{j-1} \frac{\partial}{\partial a}(C) C^{r-j},$$

$$\frac{\partial}{\partial b} C^r = \sum_{j=1}^{r} C^{j-1} \frac{\partial}{\partial b}(C) C^{r-j}, \qquad 1 \le r \le n$$

where

$$\frac{\partial}{\partial a}(C) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \frac{\partial}{\partial b}(C) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Remark 2.3: Assumption 2.1 can be enforced by simple inequalities, using 2×2 McDuffe resultants [2]. These inequalities along with (2.10) actually define $g(v) \le 0$ in (2.3). In the actual implementation, however, we have preferred to omit these extra inequalities arising from enforcing Assumption 2.1. We have also observed, on all the examples tried, that this omission does not lead to any difficulties in obtaining the final solution. We will discuss more about this in Section III (Remark 3.1).

Remark 2.4: An important observation is worth mentioning here. Our algorithm for solving the constrained stabilization problems, which will be discussed later, starts from an initial guess point and varies a_i and b_i continuously, while satisfying (2.12), so as to reach a solution. Since (2.12) allows a pair of repeated roots, a continuous change between a pair of real roots and a pair of complex conjugate roots is possible.

Now we consider the equalities which arise from structural constraints on state feedback. By appropriately appending (2.12) to these equalities we get (2.4). We demonstrate the setting up of the equalities only for the control structure of decentralized state feedback. Details for the other control structures mentioned earlier are straightforward and can be found in [6]. The K given by (2.7) is function of Γ, a part of v, and so we denote K by $K(v)$. The control structure constraints on K are treated as constraints on v.

Decentralized State Feedback

Let r be the number of subsystems of a large scale system for which a decentralized feedback is to be realized. The state feedback gain matrix, K typically takes the block diagonal form: $K = \text{block diag}[K_1, \cdots, K_r]$. The control structure constraints corresponding to this form can be expressed as

$$E_i^L K(v) E_i^R = 0, \qquad i = 1, 2, \cdots, r \tag{2.15}$$

where the matrices E_i^L, E_i^R are constant matrices of appropriate dimensions which select all blocks in the ith block row except the ith one. Sometimes there exists an overlap between the state variables of the subsystems. In such a case K cannot be expressed in block diagonal form. The structural constraints can still be represented as in (2.15), however, with an appropriate choice of E_i^L and E_i^R, $i = 1, \cdots, r$. Together (2.15) along with (2.12) define (2.4).

Remark 2.5: The equations in (2.12) involve characteristic polynomials and determinants, quantities which are known to have bad sensitivity properties. In other words, in the presence of finite precision arithmetic, the difference between the set of eigenvalues of $(A - BK)$ and the set of roots of (2.8) can be large even when the errors in the satisfaction of (2.12) are small. Therefore, when a K is determined using the numerical method proposed in this paper, it is necessary to evaluate the eigenvalues of $(A - BK)$ and check whether they belong to S. If this check fails then it indicates that our method has suffered from numerical instability. If computations are done with a high precision this would not happen. See for instance, the example of Section IV.

III. A Modified Homotopy Method

We solve the system of equalities and inequalities obtained in the previous section by using a modified homotopy method. Homotopy methods are popularly used to find zeros of a square system of nonlinear equations. Our main reason for preferring homotopy methods over the usual iterative methods is that it has been observed in practice that the domain of attraction of a solution point for iterative methods is usually much smaller than that for homotopy methods. A good survey of homotopy (or, continuation) methods together with guiding references on theory, numerical methods and applications is given in [7]. We devise a modification of a homotopy method to solve the system of equalities (2.4) (which is typically underdetermined) along with the inequalities (2.3).

Let $H(v, t)$ be a function with the following properties: a) when $t = 0$, the system of equations $H(v, 0) = 0, g(v) \le 0$ is trivial in the sense that it is easy to find a $\bar{v}$ satisfying them, and b) when $t = 1$, $H(v, 1) = h(v)$. Thus when $t = 0$ we know that $\bar{v}$ is a solution to $H(v, 0) = 0$, $g(v) \le 0$. In a homotopy method one starts from this known solution $(\bar{v}, 0)$, and moves on the solution hypersurface defined by $\{(v, t)\colon H(v, t) = 0,\ g(v) \le 0,\ 0 \le t \le 1\}$ in an attempt to reach a solution v^*, if it exists. Our choice of the H function is

$$H(v, t) = \begin{bmatrix} h^1(\Gamma) + (t-1) h^1(\bar{\Gamma}) \\ h^2(\Gamma, \gamma) \end{bmatrix} \tag{3.1}$$

where: the h^1 function represents the given control structure constraint, see for example, (2.15) for the case of decentralized state

$$\begin{bmatrix} s^{n_1} + \gamma_1^{11} s^{n_1-1} + \cdots + \gamma_{n_1}^{11} & \gamma_1^{12} s^{n_2-1} + \cdots + \gamma_{n_2}^{12} & \cdots & \gamma_1^{1m} s^{n_m-1} + \cdots + \gamma_{n_m}^{1m} \\ \gamma_1^{21} s^{n_1-1} + \cdots + \gamma_{n_1}^{21} & s^{n_2} + \gamma_1^{22} s^{n_2-1} + \cdots + \gamma_{n_2}^{22} & \cdots & \gamma_1^{2m} s^{n_m-1} + \cdots + \gamma_{n_m}^{2m} \\ \vdots & \vdots & & \vdots \\ \gamma_1^{m1} s^{n_1-1} + \cdots + \gamma_{n_1}^{m1} & \gamma_1^{m2} s^{n_2-1} + \cdots + \gamma_{n_2}^{m2} & \cdots & s^{n_m} + \gamma_1^{mm} s^{n_m-1} + \cdots + \gamma_{n_m}^{mm} \end{bmatrix}$$

feedback; the h^2 function is the left-hand side of (2.12); and $\overline{\Gamma}$ is chosen as explained below. Recall that $v = \{\Gamma, \gamma\}$. When $t = 0$, we obtain a solution $\overline{v}$ to $H(v, 0) = 0$ as follows. Choose a $\overline{\gamma}$ so that $g(\overline{\gamma}) < 0$. Note that it is easy to get such a $\overline{\gamma}$ because of the decoupled nature of the inequalities; see (2.10). Once a $\overline{\gamma}$ is chosen, choose a $\overline{\Gamma}$ so that $h^2(\overline{\Gamma}, \overline{\gamma}) = 0$. Again, it is easy to choose such a $\overline{\Gamma}$; for example one possibility is to choose $\overline{\Gamma}$ which corresponds to a diagonal matrix $P(s)$. Then, $\overline{v}$ is nothing but $\{\overline{\Gamma}, \overline{\gamma}\}$.

To move on the solution hypersurface from $t = 0$ to $t = 1$ we devise a numerical method by using an optimization set-up. Here our description of the optimization set-up will be very brief; see [6] for its detailed discussion and the algorithm based on it.

Since $t = 1$ has to be reached for the homotopy method to be successful, and $t = 1$ has to be reached before crossing over to $t > 1$, $(1 - t)$ is a good choice for the objective function to be minimized. So we define the following optimization problem

$$\min (1 - t) \text{ subject to } H(v, t) = 0, \qquad g(v) \leq 0. \tag{3.2}$$

It is important to note that, with $(1 - t)$ as the cost function, the aim is not to solve the optimization problem (3.2). It is formulated only to drive the motion from $t = 0$ to $t = 1$. It is easy to maintain feasibility with respect to the inequalities $g(v) \leq 0$ simply by using a barrier method [4] and replacing (3.2) by

$$\min (1 - t) + c_k B(v) \text{ subject to } H(v, t) = 0 \tag{3.3}$$

where $B(v)$ is a barrier function which is smooth and which tends to ∞, i.e., establishes a barrier, as v approaches the boundary of $\{v\colon g(v) \leq 0\}$ and c_k is positive. Problem (3.3) is repeatedly solved for a sequence of c_k values (which monotonically decrease to zero). This solution process is terminated as soon as $t = 1$ is reached and there is no need to go for an actual minimization of (3.2). Therefore, unlike usual barrier methods, our method requires that the solution of (3.3) is repeated only for a few c_k values. In our numerical implementation we have used a logarithmic barrier function [4]. See [6] for the rationale used to choose the decreasing sequence $\{c_k\}$. In all the test examples it was observed that the initial choice $c_1 = 0.1$ worked well in that it was sufficient to solve (3.3) only once, i.e., $t = 1$ was reached while solving (3.3) with c_1.

Suppose that the S-stabilization problem does not have a solution. In that case the method will reach a stage where the solution trajectory comes close to making one or more of the inequalities active and, as a result t gets stuck at a value less than one. In that case the region S can be expanded by appropriately changing the active inequalities and the procedure can be restarted.

Problem (3.3) is only an equality constrained problem. We solve (3.3) by a continuous realization of the gradient projection method [4, 6]. Let

$$Z = \{(v, t)\colon H(v, t) = 0\} \tag{3.4}$$

and $\nabla f_p(v, t)$ be the projection of the gradient of $f = (1 - t) + c_k B(v)$ onto the tangent space of Z at (v, t). In the gradient projection method one moves on Z along the trajectory of the vector field defined by $-\nabla f_p$. Careful tracking of the solution manifold Z using the vector field $-\nabla f_p$ can be done by using a simple modification of any state-of-the-art ODE solving package [10]. Such a tracking allows continuous check on any deviation from the solution manifold caused by the propagation and accumulation of numerical errors and applies a correction to get back to the solution manifold whenever necessary.

Let $\mathcal{C}$ denote the curve on Z starting from $(\overline{v}, 0)$ and defined using the above process. The tracking of $\mathcal{C}$ requires Z to be smooth. The smoothness of H does not automatically guarantee smoothness of Z. The following result describes the smoothness of Z.

Theorem 3.1: Let Z be as defined in (3.4) and $V = \{(v, t)\colon \text{Assumption 2.1 holds}\} \subset R^N$. Suppose that h^1_Γ, the Jacobian of h^1 with respect to Γ, has full row rank for all $(v, t) \in Z \cap V$. Then $Z \cap V$ is a differentiable manifold.

Remark 3.1: Typically $\mathcal{C}$ cuts ∂V, the boundary of V, transversally and so there is really no need to worry about any difficulties imposed by any crossing of ∂V (if at all it occurs) during the tracking of $\mathcal{C}$. In doing all the numerical examples we have simply ignored Assumption 2.1 and yet did not face any difficulties in reaching a solution $(v^*, 1)$.

Remark 3.2: The assumption that h^1_Γ has full row rank usually comes directly from the way the control structure constraints are formulated.

To prove Theorem 3.1 we use the following lemma.

Lemma 3.2: Suppose v is such that Assumption 2.1 holds and $h^2(v) = h^2(\Gamma, \gamma) = 0$, where h^2 is as in (3.1). Then $h^2_\gamma(\Gamma, \gamma)$, the Jacobian of h^2 with respect to γ, is nonsingular.

Proof: Let $\delta(s) = \det(\overline{P}(s) + \Gamma V(s))$. With straightforward algebra it is easy to see that $h^2_\gamma(\Gamma, \gamma)$ has block diagonal structure with ith $(i \in \{1, \cdots, l\})$ block given by

$$\begin{bmatrix} \frac{\partial e_1 \delta(C_i)}{\partial a_i} \\ \frac{\partial e_1 \delta(C_i)}{\partial b_i} \end{bmatrix} = \begin{bmatrix} (0 & -1) \\ (-1 & 0) \end{bmatrix} \prod_{j=1,\, j\neq i}^{l+1} G_j(C_i),$$

$$G_j(C_i) = C_i^2 + a_j C_i + b_j I_2, \qquad j = 1, \cdots l$$

$$G_{l+1}(C_i) = C_i - \lambda I_2.$$

The last $((l+1)\text{th})$ block is $\partial\delta(\lambda)/\partial\lambda = \prod_{i=1}^{l}(\lambda^2 + a_i\lambda + b_i)$. If Assumption 2.1 holds then for each i, j, $G_j(C_i)$ (which is the McDuffe resultant associated with $p_i(s)$ and $p_j(s)$) is nonsingular [2] and $\lambda^2 + a_i\lambda + b_i$ is nonzero. Hence Lemma 3.2 follows. ∎

Proof of Theorem 3.1: To prove Theorem 3.1 it is sufficient to show that the Jacobian J of H with respect (Γ, γ, t) has full row rank for every $(v, t) = (\Gamma, \gamma, t) \in Z \cap V$. We have

$$J = \begin{bmatrix} h^1_\Gamma & 0 & h^1(\overline{\Gamma}) \\ h^2_\Gamma(\Gamma, \gamma) & h^2_\gamma(\Gamma, \gamma) & h^2(\overline{\Gamma}, \overline{\gamma}) \end{bmatrix}. \tag{3.5}$$

Then from (3.5) and Lemma 3.2, Theorem 3.1 follows directly. ∎

It is very important that $\mathcal{C}$ connects the points $(\overline{v}, 0)$ and $(v^*, 1)$ (*if* v^* exists). This connectivity condition depends on the choice of H. For any of the S-stabilization problems which are considered in this paper choosing an H such that the connectivity condition is satisfied is a very hard problem. We believe that it can only be solved when complete system theoretic solutions to these problems become available. Our choice of H given in (3.1) is not guaranteed to satisfy the connectivity condition. Tests on several problems have shown, however, that the homotopy method based on the H as in (3.1) and the optimization formulation (3.2) has good empirical success. On all the problems tried we were always successful in reaching $t = 1$ without encountering any local minima of (3.2) or any unboundedness on the elements of Γ.

IV. Example

We carried out numerical tests on several constrained S-stabilization problems using the VAX-88 computer. Here we present only one example to demonstrate the effectiveness of our approach.

Power System: We consider the example of seven-state, two-input model of load frequency control of a two-area power system. The numerical data for the A, B matrices are taken from [5]. Here we also take into consideration integral control action which is employed to reject step disturbances due to changes in load. Thus we append the state vector by two more variables which represent the integrals of the area control errors defined as $\Delta p_{\text{tie}} + \Delta f_1$ and $-\Delta p_{\text{tie}} + \Delta f_2$;

TABLE I
SOLUTION FOR POWER SYSTEM

Initial values ($t = 0$)	
Λ	$\{ -2 \pm j1, \ -3 \pm j1, \ -4 \pm j1, \ -5 \pm j1, \ -2 \}$
$P(s)$	$p_{11}(s) = s^5 + 12s^4 + 59s^3 + 148s^2 + 190s + 100$ $p_{12}(s) = 0$ $p_{21}(s) = 0$ $p_{22}(s) = s^4 + 18s^3 + 123s^2 + 378s + 442$
$E_i^L K E_i^R$	$E_1^L K E_1^R = [0.984 \ 1.127 \ 0.480 \ 0.517]$ $E_2^L K E_2^R = [0.234 \ 0.078 \ 0.000 \ 0.884]$

$\Downarrow$
Number of integration steps = 1954, cpu time = 736.15 sec
$\Downarrow$

Feasible solution values ($t = 1$)	
Λ	$\{-1.47087 \pm j0.691058, \ -2.72493 \pm j0.992018, \ -4.42945 \pm j1.49412, \ -6.27487 \pm j3.05654, \ -1.99674\}$
$P(s)$	$p_{11}(s) = s^5 + 13.9234s^4 + 82.3513s^3 + 271.316s^2 + 338.132s + 234.420$ $p_{12}(s) = -3.95005s^3 - 40.9614s^2 - 101.740s - 108.530$ $p_{21}(s) = -3.27041s^3 - 58.2824s^2 - 10.7900s - 146.782$ $p_{22}(s) = s^4 + 17.8735s^3 + 116.771s^2 + 269.408s + 269.346$
$E_i^L K E_i^R$	$E_1^L K E_1^R = [0\ 0\ 0\ 0]$ $E_2^L K E_2^R = [0\ 0\ 0\ 0]$
K	$(k_1, k_{15}) = [0.238848 \ 0.959565 \ 1.84323 \ 0.643260 \ -0.580120]$ $(k_{25}, k_2) = [2.70128 \ 0.637658 \ 1.621895 \ 2.159241 \ 1.07738]$

see [5] for the physical meanings of $-\Delta p_{\text{tie}}$ and Δf_i, $i = 1, 2$ and the other state variables. The feedback is decentralized with some overlap. The 2×9 gain matrix K has the following structure

$$K = \begin{bmatrix} \times & \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} k_1 & k_{15} & 0 \\ 0 & k_{25} & k_2 \end{bmatrix}$$

where the $\times$'s represent the free gain elements. Results obtained by our method on this example are tabulated in Table I. We used the stability region of Fig. 1 with $\alpha_1 = -1$, $\alpha_2 = -10$ and $\theta = 45°$. For doing homotopy curve tracking we used relative tolerance of 10^{-8}. In view of Remark 2.5 we compared the eigenvalues of $(A - BK)$ and the roots of (2.8). They matched up to three decimal digits.

V. CONCLUSION

In this note we have given a numerical method for solving a general class of multivariable regional pole placement problems with control constraints, which has worked well on many examples. Given that for general control structure constraints the problem is very hard, such a method should prove very useful.

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Continuous Robust Control Design for Nonlinear Uncertain Systems Without *a Priori* Knowledge of Control Direction

J. Kaloust and Z. Qu

***Abstract*—In this paper, a robust control scheme is proposed for a class of nonlinear systems that have not only additive nonlinear uncertainties but also unknown multiplicative signs. These signs are called control directions since they represent effectively the direction of motion under any given control. Except for the unknown control directions, the class of systems satisfy the generalized matching conditions. Nonlinear robust control is designed to identify on-line unknown control directions and to guarantee global stability of uniform ultimate boundedness without the knowledge of nonlinear dynamics except their size bounding functions. It is also shown that the proposed robust control can be made continuous through utilizing the so-called shifting laws that change smoothly and accordingly the signs of robust controls and that, no matter what time constants and gains of the shifting laws are, global stability is always ensured. The analysis and design is done using Lyapunov's direct method.**

I. INTRODUCTION

Robust control of nonlinear systems in the presence of nonlinear uncertainties has been studied extensively. The important classes of stabilizable uncertain systems and their robust control laws can be found in [1], [3], [6], [9]-[11]. The uncertainties in those systems can be general nonlinear functions and input-related and/or input-unrelated. The input-related uncertainties studied so far in the previous results, however, are not only sign-invariant but also have known signs. These signs, called control directions, represent motion directions of the system under any control, and knowledge of these signs makes robust control design much easier. The objective of this paper is to develop a robust control design procedure based on Lyapunov's direct method that achieve global stability but does not require a priori knowledge of control directions. For practical implementation, designs of both continuous and discontinuous robust controls are considered and compared. It is shown that continuity of robust control can be achieved by designing the so-called shifting laws that change the signs of the robust control in a continuous fashion. The shifting laws are based on the results of on-line identification of unknown control directions.

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The authors are with the Department of Electrical Engineering, University of Central Florida, Orlando, FL 32826 USA.

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