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Construction of some special subsequences within a Farey sequence

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Abstract

Recently it has been found that some special subsequences within a Farey sequence play a crucial role in determining the ranges of coupling constant for which quantum soliton states can exist for an integrable derivative nonlinear Schrödinger model. In this article, we find a novel mapping which connects two such subsequences belonging to Farey sequences of different orders. By using this mapping, we construct an algorithm to generate all of these special subsequences within a Farey sequence. We also derive the continued fraction expansions for all the elements belonging to a subsequence and observe a close connection amongst the corresponding expansion coefficients.

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1 Introduction

The concept of Farey sequences appearing in number theory has recently found interesting applications in diverse subjects in physics like phase transitions in one dimensional statistical models with long-range interaction [1, 2, 3], fractal statistics [4], diffraction patterns of aperiodic crystals [5], 2+1 dimensional gravity [6] and quantum soliton states of an integrable 1+1 dimensional derivative nonlinear Schrödinger (DNLS) model [7]. For any positive integer N, the Farey sequence of order N (denoted by F_N) is defined to be the set of all the fractions a/b in increasing order such that (i) $0 \leq a \leq b \leq N$, and (ii) a and b are relatively prime (i.e., the greatest common divisor of a and b is 1) [8, 9]. The Farey sequences for the first few integers are given by

$$\begin{array}{llllllllllll} F_1 : & \frac{0}{1} & \frac{1}{1} \\ F_2 : & \frac{0}{1} & \frac{\mathbf{1}}{\mathbf{2}} & \frac{1}{1} \\ F_3 : & \frac{0}{1} & \frac{\mathbf{1}}{\mathbf{3}} & \frac{1}{2} & \frac{\mathbf{2}}{\mathbf{3}} & \frac{1}{1} \\ F_4 : & \frac{0}{1} & \frac{\mathbf{1}}{\mathbf{4}} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{\mathbf{3}}{\mathbf{4}} & \frac{1}{1} \\ F_5 : & \frac{0}{1} & \frac{\mathbf{1}}{\mathbf{5}} & \frac{1}{4} & \frac{1}{3} & \frac{\mathbf{2}}{\mathbf{5}} & \frac{1}{2} & \frac{\mathbf{3}}{\mathbf{5}} & \frac{2}{3} & \frac{3}{4} & \frac{\mathbf{4}}{\mathbf{5}} & \frac{1}{1} \; . \end{array} \tag{1.1}$$

These sequences enjoy many interesting properties [8, 9], of which we list the relevant ones below.

(i) If $a/b < a'/b'$ are two successive fractions in a Farey sequence F_N, then

$$a'b \; - \; ab' \; = \; 1 \; , \quad b \; , \; b' \; \leq \; N \; \text{ and } \quad b + b' > N \; . \tag{1.2}$$

In this case, it follows that both a and b' are relatively prime to a' and b.

(ii) If $a/b < a'/b' < a''/b''$ are three successive fractions in a Farey sequence F_N, then

$$\frac{a'}{b'} \; = \; \frac{a + a''}{b + b''} \; . \tag{1.3}$$

(iii) For $N \geq 2$, let n/N be a fraction appearing somewhere in the sequence F_N (such fractions are denoted by bold letters in Eq. (1.1)). Then the fractions a_1/b_1 and a_2/b_2 appearing immediately to the left and to the right respectively of n/N satisfy

$$\begin{aligned} a_1 \; , \; a_2 \; \leq \; n \; , \quad & \text{and} \quad a_1 \; + \; a_2 \; = \; n \; , \\ b_1 \; , \; b_2 \; < \; N \; , \quad & \text{and} \quad b_1 \; + \; b_2 \; = \; N \; . \end{aligned} \tag{1.4}$$

We have recently investigated the ranges of coupling constant (called bands) within which localized N-body soliton states can be constructed for a quantum integrable DNLS model [7]. Interestingly, it is found that such bands have a one-to-one correspondence with the fractions n/N, which appear within the sequence F_N. Consequently, we use the notation $B_{n,N}$ to denote such a band. Farey sequences also play a crucial role in finding the end points of these bands beyond which localized N-soliton states do not exist for the DNLS model. It is found that the fractions a_1/b_1 and a_2/b_2, appearing immediately to the left and to the right respectively of n/N within the Farey sequence F_N, determine the end points of the band $B_{n,N}$. Therefore, special subsequences like $a_1/b_1, n/N, a_2/b_2$ belonging to F_N play a key role in the context of quantum soliton states of DNLS model.

In this article, we focus our attention on such subsequences within Farey sequence and find a novel connection between these subsequences. Let us denote the subsequence of three successive fractions, $a_1/b_1, n/N, a_2/b_2$ in F_N, as $F_{n,N}$. It may be noted that the first subsequence in F_N, i.e., $F_{1,N}$, has a very simple form for all values of N. Namely, $F_{1,N}$ consists of $0/1, 1/N, 1/(N-1)$. However, higher subsequences in F_N (i.e., $F_{n,N}$ with $n > 1$) cannot be expressed in such a simple way. In Sec. 2 of this article, we find a mapping between two subsequences within two Farey sequences of different orders. Due to this mapping, any higher subsequence in F_N can be generated through the first subsequence in some F_M, where $M < N$. In this way, we find an algorithm to generate all $F_{n,N}$ with $n > 1$. In Sec. 3 we show that the above mentioned algorithm to generate the elements of $F_{n,N}$ can be expressed in an elegant way through continued fractions. In this section we also discuss a method for finding out the successive fraction of any given fraction within a Farey sequence. Sec. 4 is the concluding section.

2 Mapping between subsequences of Farey sequences

Following the standard convention [8], we call a/b an irreducible fraction when a and b are relative prime numbers. For any two irreducible positive fractions a/b and c/d, we define

$$\Delta(a/b, c/d) \equiv cb - ad. \tag{2.1}$$

For the case $a/b < c/d$, $\Delta(a/b, c/d)$ is a positive integer. Let us now consider the following theorem which may be regarded as the converse of relation (1.2).

Theorem 2.1 *If $0 \le a/b < a'/b' \le 1$ are two irreducible fractions which satisfy $\Delta(a/b, a'/b') = 1$, then they will appear as successive fractions in F_N for any value of N which lies in the range $max(b, b') \le N < b + b'$.*

Proof It is obvious that a/b and a'/b' are some fractions belonging to F_N. Let us now suppose that a/b and a'/b' are not successive fractions in F_N. Let c/d be a fraction which lies between them in F_N (there may be more than one such fraction; in that case, we choose any one of them). Then we have two relations like $\Delta(a/b, c/d) = n_1$, and $\Delta(c/d, a'/b') = n_2$, where $n_1, n_2 \ge 1$. Let us multiply the first relation by b', the second by b, and add these two relations. Thus we get $d\ \Delta(a/b, a'/b') = b'n_1 + bn_2$. Since $\Delta(a/b, a'/b') = 1$, we see that $d = b'n_1 + bn_2$. However, since $b + b' > N$, and $n_1, n_2 \geq 1$, we necessarily have $d = b'n_1 + bn_2 > N$. This proves by contradiction that a/b and a'/b' must be successive fractions in F_N.

Let us now consider another theorem which provides a novel mapping between two subsequences of Farey sequences with different orders.

Theorem 2.2 *If $a_1/b_1, a_2/b_2, a_3/b_3$ is a subsequence in F_{b_2}, then $b_3/(\rho b_3 + a_3), b_2/(\rho b_2 + a_2), b_1/(\rho b_1 + a_1)$ is a subsequence in $F_{\rho b_2 + a_2}$, where ρ is any positive integer.*

Proof Since $a_1/b_1, a_2/b_2, a_3/b_3$ is a subsequence in F_{b_2}, the pairs (a_i, b_i) are relative prime. By using this property, it is easy to show that the pairs $(b_i, \rho b_i + a_i)$ are also relative prime. Thus irreducible fractions associated with the pairs $(b_i, \rho b_i + a_i)$ are fit candidates to form a subsequence in $F_{\rho b_2 + a_2}$. Using the relation (2.1), we find that

$$\Delta\left(\frac{b_2}{\rho b_2 + a_2}, \frac{b_1}{\rho b_1 + a_1}\right) = \Delta\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) \ , \quad \Delta\left(\frac{b_3}{\rho b_3 + a_3}, \frac{b_2}{\rho b_2 + a_2}\right) = \Delta\left(\frac{a_2}{b_2}, \frac{a_3}{b_3}\right) \ . \quad (2.2)$$

Since $a_1/b_1, a_2/b_2, a_3/b_3$ is a subsequence in F_{b_2}, $\Delta(a_1/b_1, a_2/b_2) = \Delta(a_2/b_2, a_3/b_3) = 1$. So, from Eq. (2.2), we obtain

$$\Delta\left(\frac{b_2}{\rho b_2 + a_2}, \frac{b_1}{\rho b_1 + a_1}\right) = \Delta\left(\frac{b_3}{\rho b_3 + a_3}, \frac{b_2}{\rho b_2 + a_2}\right) = 1 \ . \quad (2.3)$$

Moreover, using property (1.4) for the subsequence $a_1/b_1, a_2/b_2, a_3/b_3$, one finds that $a_2 = a_1 + a_3$ and $b_2 = b_1 + b_3$. Due to these relations, it follows that $\rho b_2 + a_2 = (\rho b_1 + a_1) + (\rho b_3 + a_3)$ and

$$max(\rho b_2 + a_2, \rho b_1 + a_1) = max(\rho b_2 + a_2, \rho b_3 + a_3) = \rho b_2 + a_2 \ . \quad (2.4)$$

Consequently, by applying Theorem 2.1 along with relations (2.3) and (2.4), we find that $b_3/(\rho b_3 + a_3), b_2/(\rho b_2 + a_2), b_1/(\rho b_1 + a_1)$ is a subsequence within $F_{\rho b_2 + a_2}$. The converse of Theorem 2.2 is also valid and may be proved in a similar fashion.

Starting from any known subsequence in F_N, and applying Theorem 2.2 repeatedly, one can easily construct many other subsequences in F_{N_1}, where $N_1 > N$. We have already noted that $F_{1,N}$ has a very simple form, namely, $0/1, 1/N, 1/(N-1)$. Since such subsequences cannot be generated from some other subsequences by using Theorem 2.2, they may be classified as 'fundamental subsequences'. In the following, we shall show that all F_{n_1,N_1} with $n_1 > 1$ are 'composite subsequences', which can be constructed from the fundamental subsequences by using Theorem 2.2. To construct F_{n_1,N_1}, it is only necessary to find the first and last fraction appearing within this composite subsequence. We can always map the middle fraction of F_{n_1,N_1} to the middle fraction of some fundamental subsequence through successive steps given by

$$\frac{n_1}{N_1} \xrightarrow{\rho_1} \frac{n_2}{N_2} \cdots\cdots\cdots \frac{n_i}{N_i} \xrightarrow{\rho_i} \frac{n_{i+1}}{N_{i+1}} \cdots\cdots\cdots \frac{n_k}{N_k} \xrightarrow{\rho_k} \frac{1}{N_{k+1}} \ , \quad (2.5)$$

where $\rho_i = \left[\frac{N_i}{n_i}\right]$ (here the symbol $[q]$ denotes the integer part of q), $n_{i+1} = N_i - \rho_i n_i$, $N_{i+1} = n_i$, and it is assumed that $n_{k+1} = 1$. It may be noted that, within the fundamental subsequence $F_{1,N_{k+1}}$, $0/1$ and $1/(N_{k+1}-1)$ appear in the left and the right side of the fraction $1/N_{k+1}$ respectively. Applying the mappings of Eq. (2.5) in reverse order to the fractions $0/1$ and $1/(N_{k+1}-1)$, we obtain

$$\begin{aligned} \frac{0}{1} &\equiv \frac{n'_{k+1}}{N'_{k+1}} \xrightarrow{\rho_k} \frac{n'_k}{N'_k} \cdots\cdots\cdots \frac{n'_{i+1}}{N'_{i+1}} \xrightarrow{\rho_i} \frac{n'_i}{N'_i} \cdots\cdots\cdots \frac{n'_2}{N'_2} \xrightarrow{\rho_1} \frac{n'_1}{N'_1}, \\ \frac{1}{N_{k+1}-1} &\equiv \frac{n''_{k+1}}{N''_{k+1}} \xrightarrow{\rho_k} \frac{n''_k}{N''_k} \cdots\cdots\cdots \frac{n''_{i+1}}{N''_{i+1}} \xrightarrow{\rho_i} \frac{n''_i}{N''_i} \cdots\cdots\cdots \frac{n''_2}{N''_2} \xrightarrow{\rho_1} \frac{n''_1}{N''_1}, \end{aligned} \tag{2.6}$$

where $n'_i = N'_{i+1}, N'_i = n'_{i+1} + \rho_i N'_{i+1}$ and $n''_i = N''_{i+1}, N''_i = n''_{i+1} + \rho_i N''_{i+1}$. Since $n'_{k+1}/N'_{k+1}, n_{k+1}/N_{k+1}, n''_{k+1}/N''_{k+1}$ represents the fundamental subsequence in $F_{N_{k+1}}$, by applying Theorem 2.2 with $\rho = \rho_k$ one can easily show that $n''_k/N''_k, n_k/N_k, n'_k/N'_k$ is a subsequence in F_{N_k}. Applying Theorem 2.2 repeatedly in this fashion, we find that

$$\frac{n'_1}{N'_1}, \frac{n_1}{N_1}, \frac{n''_1}{N''_1}, \qquad \text{or,} \qquad \frac{n''_1}{N''_1}, \frac{n_1}{N_1}, \frac{n'_1}{N'_1}, \tag{2.7}$$

generates the subsequence F_{n_1,N_1} within F_{N_1}, when k is an even or odd integer respectively.

Thus we observe that all higher subsequences within F_{N_1} are in fact composite subsequences, which can be generated in the above mentioned way. As an example, let us try to find the composite subsequence $F_{5,39}$ which lies within F_{39}. By using (2.5), we find that

$$\frac{5}{39} \xrightarrow{\rho_1=7} \frac{4}{5} \xrightarrow{\rho_2=1} \frac{1}{4}.$$

Thus $1/4$ is the middle fraction of the corresponding fundamental subsequence $F_{1,4}$. The first and last fraction of this fundamental subsequence are given by $0/1$ and $1/3$ respectively. Applying the mapping (2.6) to these fractions, we obtain

$$\begin{aligned} \frac{0}{1} &\xrightarrow{\rho_2=1} \frac{1}{1} \xrightarrow{\rho_1=7} \frac{1}{8}, \\ \frac{1}{3} &\xrightarrow{\rho_2=1} \frac{3}{4} \xrightarrow{\rho_1=7} \frac{4}{31}. \end{aligned}$$

Since this is two step process ($k = 2$), by using (2.7) we find that $F_{5,39} = 1/8, 5/39, 4/31$.

3 Farey subsequences and continued fractions

The above mentioned procedure of generating a subsequence of Farey sequence can be described in an elegant way with the help of continued fractions. We first discuss the idea of a continued fraction [9]. Any positive real number x has a simple continued fraction expansion of the form

$$x \ = \ n_0 \ + \ \cfrac{1}{n_1 \ + \ \cfrac{1}{n_2 \ + \ \cfrac{1}{n_3 \ + \ \cdots}}} \ , \tag{3.1}$$

where the n_i's are integers satisfying $n_0 \geq 0$, and $n_i \geq 1$ for $i \geq 1$. Given a number x, the integers n_i can be found as follows. Let us first define $x_0 = x$ and $n_0 = [x_0]$. We then recursively define $x_{i+1} = 1/(x_i - n_i)$, and obtain $n_{i+1} = [x_{i+1}]$ for $i = 0, 1, 2, \cdots$. This expansion ends at a finite stage with a last integer n_k if x is rational. In that case, we can assume that the last integer satisfies $n_k \geq 2$ (if n_k is equal to 1, we can stop at the previous stage and increase n_{k-1} by 1). With this convention for n_k, the continued fraction expansion written in the form

$$x = [n_0, n_1, n_2, \cdots, n_{k-1}, n_k] \ ,$$

is unique for any rational number x. If x is an irrational number, the continued fraction expansion does not end, and it is unique.

As shown in Eq. (2.7), the subsequence F_{n_1,N_1} contains the fractions n_1'/N_1', n_1/N_1, and n_1''/N_1''. In Sec. 2 we have given an algorithm for finding out the fractions n_1'/N_1' and n_1''/N_1'' around the known middle fraction n_1/N_1. Let us now try to find the continued fraction expansion for all of these elements in F_{n_1,N_1}. To this end, we observe that the relation between fractions n_i/N_i and n_{i+1}/N_{i+1} appearing in Eq. (2.5) can be written in the form

$$\frac{n_i}{N_i} = \frac{1}{\rho_i + \frac{n_{i+1}}{N_{i+1}}} \ . \tag{3.2}$$

Since n_{i+1}/N_{i+1} is the middle term of a composite subsequence having value less than 1, the continued fraction expansion for n_{i+1}/N_{i+1} can be written as $[0, X]$, where $X \equiv n_1, n_2, \cdots n_k$. Then, due to Eq. (3.2), it follows that

$$\frac{n_{i+1}}{N_{i+1}} = [0, X] \Longrightarrow \frac{n_i}{N_i} = [0, \rho_i, X] \ . \tag{3.3}$$

Taking $n_{k+1}/N_{k+1} \equiv 1/N_{k+1} = [0, N_{k+1}]$ as the 'initial condition', and applying Eq. (3.3) repeatedly, we easily obtain

$$\frac{n_1}{N_1} = [0, \rho_1, \rho_2, \cdots, \rho_k, N_{k+1}] \ . \tag{3.4}$$

It may be noted that the fractions n_i'/N_i' and n_{i+1}'/N_{i+1}' (or, n_i''/N_i'' and n_{i+1}''/N_{i+1}'') appearing in Eq. (2.6) also satisfy relations exactly of the form (3.2) and (3.3). Taking $0/1 = [0]$ and $1/(N_{k+1} - 1) = [0, N_{k+1} - 1]$ as initial conditions, and applying relations of the form (3.3), we derive the continued fraction expansion for n_1'/N_1' and n_1''/N_1'' as

$$\begin{aligned} \frac{n_1'}{N_1'} &= [0, \rho_1, \rho_2, \cdots, \rho_k] \ , \\ \frac{n_1''}{N_1''} &= [0, \rho_1, \rho_2, \cdots, \rho_k, N_{k+1} - 1] \ . \end{aligned} \tag{3.5}$$

Comparison of Eqs. (3.4) and (3.5) reveals that the continued fraction expansions of three elements within subsequence F_{n_1,N_1} are closely connected with each other. Previously we have established such a connection through a completely a different route by using some properties of the continued fraction expansion [7]. By finding the continued fraction expansion of the middle term n_1/N_1 and applying Eq. (3.5), one can easily construct the subsequence F_{n_1,N_1}. As an explicit example, let us try to construct the subsequence $F_{9,25}$. The continued fraction expansion of its middle term is given by $9/25 = [0, 2, 1, 3, 2]$. Eq. (3.5) then gives $n_1'/N_1' = [0, 2, 1, 3] = 4/11$ and $n_1''/N_1'' = [0, 2, 1, 3, 1] = 5/14$. Since $k = 3$ for this case, using Eq. (2.7) we obtain $F_{9,25} = 5/14, 9/25, 4/11$.

Finally, it may be noted that our procedure of finding a subsequence of Farey sequence can also be used to find the successive term of any given fraction within a Farey sequence. Let us try to find the successive term on the right side of the fraction a/b within a Farey sequence F_N. By using the above mentioned procedure of continued fractions, we can easily find such successive term of a/b within the Farey sequence F_b. Let us denote this term as a_0/b_0. Since we have $\Delta(a/b, a_0/b_0) = 1$ and $b > b_0$, due to Theorem 2.1 it is clear that a/b and a_0/b_0 will be successive terms within any F_N with N lying in the range $b \le N < b + b_0$. Next, we want to find the successive term of a/b within F_N for the case $N \ge b + b_0$. It is easy to see that

$$\Delta\left(\frac{a}{b}, \frac{la + a_0}{lb + b_0}\right) = \Delta\left(\frac{a}{b}, \frac{a_0}{b_0}\right) = 1 \ ,$$

where l is any positive integer. Consequently, due to Theorem 2.1, a/b and $(la+a_0)/(lb+b_0)$ will be successive terms within any F_N with N lying in the range $lb + b_0 \le N < (l + 1)b + b_0$. Thus, for any given value of N, the successive term on the right side of the fraction a/b within the sequence F_N is obtained as

$$\frac{a_l}{b_l} = \delta_{l,0}\, \frac{a_0}{b_0} + (1 - \delta_{l,0})\, \frac{la + a_0}{lb + b_0} \ , \tag{3.6}$$

where $l = [(N - b_0)/b]$. As a concrete example, let us try to calculate the successive term on the right side of $9/25$ within the Farey sequence F_{100}. By using the method of continued fractions, we have already found in the preceding paragraph that $4/11$ appears in the right side of $9/25$ within the Farey sequence F_{25}. Since $a/b = 9/25$ and $a_0/b_0 = 4/11$ in this case, we have $l = [(100 - 11)/25] = 3$. Eq. (3.6) then gives the successive term of $9/25$ within F_{100} as $a_3/b_3 = 31/86$.

4 Concluding remarks

In this work, we have studied some special subsequences within a Farey sequence which appear naturally in the context of quantum soliton states for an integrable derivative nonlinear Schrödinger model. In particular, we have found a novel mapping (as stated in Theorem 2.2) through which one can generate many such subsequences within Farey sequences from the knowledge of any given subsequence. This mapping allows us to classify all of these subsequences into two types - the 'fundamental subsequences' and the 'composite subsequences'. We find that all composite subsequences can be constructed by mapping them to fundamental subsequences which are always expressed in a known simple form. In this way, we obtain an algorithm to generate all composite subsequences within a Farey sequence. We have also derived the continued fraction expansions (3.4) and (3.5) for all elements within a subsequence, and have found a close connection amongst the corresponding expansion coefficients. Consequently, our algorithm for generating all subsequences within a Farey sequence can be expressed in an elegant way through the continued fraction expansions.

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