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Path Planning : An Approach Based on Connecting All the Minimizers and Maximizers of a Potential Function

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Abstract

An improved potential-based method for robot path planning is developed by connecting all the local minima and local maxima of the potential function defined in the configuration space of the robot. The method is based on the stability theory of dynamical systems. The usefulness of the method is demonstrated on a two dimensional piano mover's problem with three degrees of freedom.

1 Introduction

The problem of computing a geometric (kinematic) robot path which is free from any collision with obstacles has been a topic of active research during the past decade [2, 8, 12]. In this paper we concentrate on the potential function approach for path planning. The robot is represented as a point in the configuration space under the influence of a potential field whose local variations are expected to reflect the structure of the free space. Let Y be a smooth connected manifold with boundary, denoting the free configuration space of the robot. Suppose $y_F \in Y$ is a fixed final destination to be reached and it is desired to bring any other point in Y to y_F using feedback control. The problem will be solved if we are able to find a smooth function(potential) $f : Y \to R$, with the following properties:

1. All the critical points of f in Y are isolated.
2. f has a unique local minimum at $y_F \in Y$.
3. All the trajectories of $\dot{y} = -\nabla f$ starting from $y_I \in Y$, completely lie in Y, where ∇f = gradient of f with respect to g.

Properties (1) and (2) together imply that all the critical points of f in Y other than y_F are either saddles or local maxima. Then for all except a set of measure zero starting points in Y, y_F is the unique globally asymptotic stable point of the system $\dot{y} = -\nabla f$. The existence of a potential function f, satisfying properties (1)–(3) is guaranteed by the theory of Morse and Smale for manifolds with boundary [9, 7]. However, constructing such a potenial function is extremely hard. Khatib [6] was the first to propose the above method of potentials for obstacle-avoidance. However the heuristic for f suggested by him fails to satisfy property (2) mentioned, above i.e., his potential function has local minima other than the destination point y_F. In [7], Koditchek and Rimon have shown how to construct f for a highly specialized configuration space of sphere worlds, where Y consists of a sphere with several distinct spherical holes in it. In [11], they have extended the ideas to slightly more general situations. Their ideas are yet to be made practically useful.

Hence for the general path planning problem, the functions suggested as potentials may have several local minima other than y_F and they are an important source of inefficiency for potential function methods. This is the major issue one has to face in designing a planner based on this approach.

Canny [2] solves the problem by the roadmap algorithm, which can connect all the critical points of a polynomial potential function over a semi algebraic set. Apart from its resriction to problems with algebraic description, Canny's method is difficult to implement and is not proven to be practically useful. Barraquand and Latombe [1] have developed a stochastic technique for escaping the region of attraction of one local minimum, followed by a gradient motion that follows the negated gradient of the potential function to reach another local minimum. The escape from a local minimum is done by a series of bounded random motions which are known to converge towards Brownian motions when the steps of the walks tend towards zero. The method when implemented yields good results but lacks elegance.

In this paper, we formulate the motion planning problem as the motion of a point on a compact manifold and develop a new method for determining and connecting all the local minima and maxima of the potential function. The determining and connecting processes are simultaneous. We construct an adjacency graph of the local minima and maxima of the potential function. An edge connecting a local minimum and a local maximum has an associated pair of perturbations which gives a way of moving between them. Section 2 describes the formulation of the problem and section 3 gives the theory of the new method together with some important heuristics associated with the implementation. In section 4, we describe in detail a motion planning example of a three degrees of freedom mobile robot moving in a maze. Section 5 gives the conclusions and mentions the future work that can be

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undertaken.

2 Problem Formulation

Let the robot operate in a physical space $S \subset R^s$ (usually $s = 2$ or 3). The robot is made up of links, $L_1, \ldots, L_r$, $(r \geq 1)$ which are rigid bodies in R^s. A configuration or placement is a k-tuple, $x = (x_1, \ldots, x_k) \in R^k$, denoting the k degrees of freedom of the robot. The workspace boundary and the static obstacles are specified as $O_j \subset R^s$, $j = 0, 1, \ldots, l$, where O_0 is the region outside S and, for $j \geq 1$, O_j is the region occupied by the jth obstacle. We will assume each O_j, $j = 0, 1, \ldots, l$ to be a closed set. Let $d_{ij}(y)$ denote the euclidean distance between $L_i(y)$ and O_j, where $L_i(y)$ is the region occupied by the ith link when the robot is in configuration y. It can be shown [4] that if L_i and O_j have sufficiently smooth boundaries, then d_{ij} will be a sufficiently smooth function of y. The configuration space obstacle corresponding to link i and obstacle j is defined as

$$C_{ij} = \{y : d_{ij}(y) \leq \epsilon\} \tag{2.1}$$

where $\epsilon > 0$ is a given clearance which is the distance to be maintained between the robot and the obstacles. The free configuration space, Y is defined as

$$Y = \{y \in R^r : d_{ij}(y) - \epsilon \geq 0, \forall i, j\}. \tag{2.2}$$

Collecting the negative distance functions $\{-d_{ij}(y) + c)$ into a single vector functibn $g : R^k \rightarrow R^m$ $(m = r(l+1))$, we can compactly write

$$Y = \{y \in R^k : g(y) \leq 0). \tag{2.3}$$

For various reasons, it may be more convenient to deal with equalities than inequalities. To describe Y by equalities, we introduce an additional vector $z = (z_1, \ldots, z_m)$, set $x = (y, z)$, $n = k + m$, and define $h : R^n \rightarrow R^m$ by

$$h_i(x) = g_i(y) + z_i^2, \quad i = 1, \ldots, m. \tag{2.4}$$

Then, let

$$\tilde{\Omega} = \{x : h(x) = 0\}. \tag{2.5}$$

Clearly Y is the projection of $\tilde{\Omega}$ on to the y-space, i.e.,

$$Y = \{y : (y, z) \in \tilde{\Omega} \text{ for some } z\}.$$

Let C^2 denote the set of functions from $R^n \rightarrow R$ whose derivatives upto second order are continuous. Let us make the following assumption.

Assumption 2.1.

1. $h_i \in C^2 \; \forall \; i = 1, \ldots, m$.
2. $h_x(x)$, the $m \times n$ Jacobian of h with respect to x, has full row rank $\forall x \in R$.
3. $\tilde{\Omega}$ is compact.

This assumption implies that $\tilde{\Omega}$ is a compact manifold of dimension k. We have already reasoned condition 1 of assumption 2.1. Condition 2 is generic. Condition 3 generally holds if the physical space S is bounded and a proper cyclic representation is used for joint angles.

Let y_F be a fixed destination point in Y. Define $z_i = \sqrt{-g_i(y_F)}, i = 1, \ldots, m$ and set $x_F = (y_F, z_1, \ldots, z_m)$. Formulate a smooth potential function $f : R^n \rightarrow R$ such that f attains its global minimum at x_F. One such choice is $f(x) = \|x - x_F\|^2$. For good performance the choice of the potential function should be carefully done for each given problem. We consider the problem of connecting all the local minima and local maxima of f on the set $\Omega \subset \tilde{\Omega}$ defined by

$$\Omega = \text{the connected component of } \tilde{\Omega} \text{ containing } x_F. \tag{2.6}$$

3 Our approach

We begin this section with some definitions. Define $L : R^n \times R^m \rightarrow R$, the Lagrangian of f over Ω as

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x),$$

where λ_i $i = 1, \ldots, m$ are the Lagrange multipliers. A point $\bar{x} \in \Omega$ is said to be a *critical point* of *over* Ω if $\exists \bar{\lambda} \in R^m$ such that

$$\nabla_x L(\bar{x}, \bar{\lambda}) = \left(\frac{\partial L}{\partial x_1}(\bar{x}, \bar{\lambda}), \cdots, \frac{\partial L}{\partial x_n}(\bar{x}, \bar{\lambda}) \right)^T = 0,$$

where T denotes transpose. By condition 2 of assumption 2.1, the $\bar{\lambda}$ associated with $\bar{x}$ is unique.

Let: Z denote the set of all critical points of over Ω; and, MIN and MAX represent the set of all local minima and local maxima of over Ω respectively. Condition 2 of assumption 2.1 allows Langrange multiplier rule to be used and so we get $MIN \cup MAX \subset Z$.

In our approach, we form a directed graph G with all the local minima and local maxima as its vertices. G is a bipartite graph with MIN and MAX as partitions. An edge connecting x_i to x_j has an associated weight vector which encodes a way of connecting x_i to x_j via a continuous path on R. For $x \in R$, denote the tangent space of Ω at x by $T_x(\Omega)$. Consider x^*, a critical point of over Ω. Let: λ^* denote the Lagrangian multiplier vector associated with x^*; $\nabla_x^2 L(x^*, \lambda^*)$ denote the Hessian of L with respect to x at x^*; B be a $n \times (n - m)$ matrix whose columns form an orthogonal basis for $T_{x^*}(\Omega)$; $H(x^*) = B^T \nabla_x^2 L(x^*, \lambda^*) B$; and $\alpha_1, \ldots, \alpha_{n-m}$ be the eigen values of $H(x^*)$. We say that x^* is a *hyperbolic critical point* if $\alpha_i \neq 0$, $i = 1, \ldots, (n - m)$. Let us make the following generic assumption [10].

Asumption 3.1. All critical points of over Ω are hyperbolic.

By compactness of Ω, there will only be a finite number of hyperbolic critical points off over Ω [5].

Let $\dot{x} = dx/dt$ and consider the following differential systems:

$$\begin{aligned} \dot{x} &= -\nabla_x L(x,\lambda) \\ <\nabla h_i, \dot{x}> &= 0, \ i = 1,\ldots,m; \end{aligned} \tag{3.1}$$

$$\begin{aligned} \dot{x} &= \nabla_x L(x,\lambda) \\ <\nabla h_i, \dot{x}> &= 0, \ i = 1,\ldots,m. \end{aligned} \tag{3.2}$$

These gradient systems denote the vector fields in which $\dot{x}$ is the projection of $-\nabla f(x)$ and $\nabla f(x)$ onto $T_x(Q)$. The existence and uniqueness of the solution of (3.1) and (3.2) can be easily established using the compactness of Ω and the fact that $f, h_i \in C^2$. Given $x_0 \in \Omega$, let $\Phi^-(t,x_0)$ and $\Phi^+(t,x_0)$ denote the solutions of (3.1) and (3.2) respectively, with $x(0) = x_0$. Allsolutions of (3.1) and (3.2) which start in Ω will remain in Ω and asymptotically reach one of the critical points off over Ω.

Let $\bar{x}$ be a critical point off over Ω. Define,

$$W^-(\bar{x}) = \{x \in \Omega : \lim_{t\to\infty} \Phi^-(t,x) = \bar{x}\},$$

and,

$$W^+(\bar{x}) = \{x \in \Omega : \lim_{t\to\infty} \Phi^+(t,x) = \bar{x}\}.$$

If $\bar{x} \in MIN$ then $W^-(\bar{x})$ denotes the region of attraction of $\bar{x}$. Similarly, if $\bar{x} \in MAX$, then $W^+(\bar{x})$ is the region of attraction of $\bar{x}$. Let $\partial W^-(\bar{x})$ and $\partial W^+(\bar{x})$ respectively represent the boundaries of $W^-(\bar{x})$ and $W^+(\bar{x})$. We make the following generic assumption [3].

Assumption 3.2. $W^+(\bar{x})$ and $W^-(\bar{y})$ intersect transversally for all $\bar{x}, \bar{y} \in Z$.

Given $x \in R^n$ and $\epsilon > 0$, let $B_\epsilon(x) = \{z : \|z - x\| < \epsilon\}$, where $\|.\|$ denotes the euclidean norm in R^n. We define a local maximum y to be *adjacent* to a local minimum x if $\forall \epsilon > 0, \exists z \in B_\epsilon(x)$ such that $\lim_{t\to\infty} \Phi^+(t,z) = y$. Similarly a local minimum x is said to be *adjacent* to a local maximum y if $\forall \epsilon > 0, \exists z \in B_\epsilon(y)$ such that $\lim_{t\to\infty} \Phi^-(t,z) = x$. With x and y as given above, it is easy to see that y is adjacent to x iff x is adjacent to y. Hence we can simply talk in terms of x and y being adjacent. As stated before, our aim is to connect all the local maxima and local minima off over Ω. We form a directed graph G as follows. The set of vertices of G is $MZN \cup MAX$. An edge is connected from vertex x to vertex y if and y are adjacent. We state a theorem on the connectedness of G. The proof of the theorem is given in the detailed version of this paper[13]. It requires some results on dynamical system theory which are taken from[3].

Theorem 3.1. Suppose Assumptions 2.1, 3.1 and 3.2 hold. Then G is connected.

Consider the construction of C. Let x be a local minimum and $y_1, \ldots, y_l$ be the local maxima adjacent to x. We need an algorithm A^+ which takes x as input, gives $\{y_i\}$ as output and for any given $\epsilon > 0$, determines perturbations $\{p_i\}$ such that $\|p_i\| \leq \epsilon$ and $\lim_{t\to\infty} \Phi^+(t, x+p_i) = y_i$, $i = 1,\ldots,l$. In other words A^+ connects a local minimum to all its adjacent local maxima. We also need an algorithm A^- which takes a local maximum as input and outputs all its adjacent local minima with the corresponding perturbations. An exact realization of A^+ and A^- is hard. For practical implementation, A^+ and A^- are replaced by heuristic approximations. We shall shortly describe some powerful heuristics for the implementation of A^+ and A−.

The construction of G is given by the following algorithm.

ALGORITHM FORM_GRAPH(x^*) ($x^* \in \Omega$ is given)

1: Set $V_1 = \phi$, $V_2 = \phi$, $E = \phi$, $V = \phi$, where ϕ denotes the null set. Choose a small $\epsilon > 0$ for use in A^+ and A^-.

2: Integrate (3.1) to find $\bar{x} = \lim_{t\to\infty} \Phi^-(t,x^*)$.
Set: $V_1 = \{\bar{x}\}$, $V = \{\bar{x}\}$

3: While ($V \neq \phi$) do
Begin
Pick $x \in V$. Set $V = V \backslash \{x\}$.
If $x \in MZN$, apply A^+ to find all the adjacent local maxima $y_1, \ldots, y_l$ and corresponding perturbations $p_1, \ldots, p_l$ such that $\lim_{t\to\infty} \Phi^+(t, x + p_i) = y_i$, $i = 1,\ldots,l$.
Set: $V = V \cup [\{y_1,\ldots,y_l\} \backslash V_2]$;
$V_2 = V_2 \cup \{y_1,\ldots,y_l\}$; and,
$E = E \cup \{(x,y_1,p_1,1), \ldots, (x,y_l,p_l,1)\}$.
Else
If $x \in MAX$, apply A to find all the adjacent local minima $z_1, \ldots, z_r$ and corresponding perturbations $q_1, \ldots, q_r$ such that $\lim_{t\to\infty} \Phi^-(t, x + q_i) = z_i$, $i = 1,\ldots,r$.
Set: $V = V \cup [\{z_1,\ldots,z_r\} \backslash V_1]$;
$V_1 = V_1 \cup \{z_1,\ldots,z_r\}$; and,
$E = E \cup \{(x,z_1,q_1,-1), \ldots, (x,z_r,q_r,-1)\}$.
End.

On completion of the above procedure we et two partitions of vertices, V_1 and V_2 and an edge-fist, E. With probability one we can say that the $\bar{x}$ determined in step 2 is a local minimum. By Theorem 3.1, the graph determined by V_1, V_2 and E is the same as G. Thus V_1 is the set of local minima and V_2 is the set of local maxima of f over Ω. Each $e \in E$ is of the form $e = (x,y,p,s)$. $s = 1$ implies that x is a local minimum, y is a local maximum and $\lim_{t\to\infty} \Phi^+(t, x + p) = y$. Similarly $s = -1$ implies that x is a local maximum, y is a local minimum, and $\lim_{t\to\infty} \Phi^-(t, x + p) = y$.

Now we address the solution of the path find problem. Determine the graph G using form_graph(x_F). Given any $x \in \tilde{\Omega}$, find the critical point $\bar{x}$ by integrating either (3.1) or (3.2) with $x(0) = x$. With probability one we can say that, $x \in \Omega$ iff $\bar{x}$ is a vertex of G.

Therfore, if $\bar{x}$ is not a vertex of G, we can conclude, with probability one, that $x \notin \Omega$. If $\bar{x}$ is a vertex of G, we can use a suitable graph search algorithm on G to connect $\bar{x}$ and x_F. This yields an overall path from x to x_F.

Now we briefly describe three heuristics for the implementation of A^+ and $A-$. We will describe the details only for $A-$. The ideas for A^+ are similar. Let x^* be a local maximum. Consider a solution of (3.2) that approaches x^*. (With t reversed, this is a solution of (3.1).) From the theory of dynamical systems, it is clear that the solution will asymptotically approach x^* along an eigen direction corresponding to one of the eigen vectors of $H(x^*)$, which was defined in section 2 as the projection of the Lagrangian on the tangent space of Ω at x^*. Since $H(x^*)$ is symmetric, all its eigen values are real. Generically, $\alpha_1, \ldots, \alpha_{n-m}$, the eigen values of $H(x^*)$ will be distinct. Without loss of generality, assume that $\alpha_1 < \alpha_2 \ldots < \alpha_{n-m}$. Let $v_1, v_2, \ldots v_{n-m}$ be the corresponding eigen directions in the tangent space of Ω at x^*. Since $H(x^*)$ is symmetric these eigen directions are mutually orthogonal.

Eigen-axis heuristic. In the Eigen-axis heuristic we choose one perturbation vector along each eigen direction. In other words, $P(x^*)$, the set of perturbation vectors at x^*, is taken as

$$P(x^*) = \{\beta v_i \ : \beta = \pm 1, \forall i = 1, \ldots, n-m\}.$$

Choose a small $\epsilon > 0$ and find adjacent minimizers as $\lim_{t\to\infty} \Phi^+(t, x^* + \epsilon p) \ \forall p \in P(x^*)$.

Practical implementation shows that many of these perturbations lead to saddle points. This is understandable, because perturbations along the eigen direction from x^* could lie on the stability boundary of a local minimum, which is invariant under (3.1) and contains saddles. A heuristic, which avoids this problem and is found to work very well in practice is described below.

Eigen-orthant heuristic. The eigen directions, $v_1, \ldots, v_{n-m}$ form an orthogonal coordinate system for $T_{x^*}(\Omega)$. There are 2^{n-m} orthants of this coordinate system. In this heuristic, we choose one perturbation vector from each orthant. In other words $P(x^*)$, the set of perturbation vectors at x^* is given by

$$P(x^*) = \{\sum_{i=1}^{n-m} \beta_i v_i : \beta_i = \pm\delta, \forall i = 1, \ldots, n-m.\}$$

where $\delta = 1/\sqrt{n-m}$. Choose a small $\epsilon > 0$ and find all the adjacent minima as $\lim_{t\to\infty} \Phi^+(t, x^* + \epsilon p) \ \forall p \in P(x^*)$. This heuristic tries to choose the perturbation vectors in such a way as to maintain a maximum angle from each of the eigen directions. For path planning applications $n-m$ denotes the number of degrees of freedom and is usually small. Hence 2^{n-m}, the cardinality of $P(x^*)$, is not a bothersome factor. Numerical experiments suggest that the eigen-orthant heuristic works extremely well when the number of local minima and local maxima are approximately equal.

Dominating Eigen-axis heuristic. The eigen-orthant heuristic basically assumes that every local maximum has at most 2^{n-m} adjacent local minima and vice versa. But this may not always be the case. In [13] we construct situations in which there are more adjacent local minima. Although our experience with a number of problems shows that such situations are rare, the fact is that the eigen-orthant heuristic will fail in such situations. The dominating eigen-axis heuristic, described next, is meant mainly to handle these situations.

Let $x^* \in MAX$. As already mentioned, all the solutions of (3.1) except a set of measure zero approach x^* along v_{n-m}. We shall call v_{n-m} as the dominating eigen-axis and concentrate the choice of perturbation vectors in a region around v_{n-m}. Let

$$C(+1, x^*, \theta) = \{x \in T_{x^*}(\Omega) : -\theta \leq \arccos\left(\frac{x^T v_{n-m}}{\|x\|\|v_{n-m}\|}\right) \leq \theta\},$$

and,

$$C(-1, x^*, \theta) = \{x \in T_{x^*}(\Omega) : -\theta \leq \arccos\left(\frac{x^T v_{n-m}}{\|x\|\|v_{n-m}\|}\right) \leq \theta\}.$$

$C(+1, x^*, \theta)$ and $C(-1, x^*, \theta)$ denote the cones, whose apex is at x^*, with apex angle=θ. v_{n-m} and $-v_{n-m}$ are the axes of the two cones respectively. Let $NTOL$ be a chosen positive integer, say $NTOL = 3$. (Larger the $NTOL$, better the heuristic, but more is the computational effort.) Choose the perturbation vectors from the above cones randomly with a uniform distribution until $NTOL$ consecutive perturbations fail to give a new minimizer. We have found this heuristic to work extremely well in practice. Detailed numerical examples using the above heuristics have been worked out in [13]. In this paper we describe only one example.

4 A Path Planning Example

The algorithm Form-Graph, using the heuristics for A^+ and $A-$, has been implemented and run successfully on several path planning examples. Here we describe one example.

Fig. 1 shows a workspace which is in the form of a maze. The mobile robot is an ellipse with three degrees of freedom namely X-translation, Y-translation and rotation. The origin for (X, Y) and the placement of the ellipse corresponding to y_F, the destination configuration are shown in Fig. 1. The rotation, θ is measured as the angle made by one of the major axis directions with the positive X axis. A cyclic representation was used for θ by taking $c = \cos\theta$ and $s = \sin\theta$ as variables and including the constraint, $c^2 + s^2 = 1$. Hence $y = (X, Y, c, s)$. The potential function used was $\|y - y_F\|^2$.

The positions of the local minima and local maxima and their adjacency graph obtained by the heuristics are adequately detailed in Table 1 and Fig. 2 respectively. Fig. 1 shows the X and Y co-ordinates of the local minima and maxima. Local minima are numbered 1-9 and maxima are marked as a-o. Fig. 3 and

4 show a typical path determined using the adjacency graph. The complete path to v_F has been represented in two parts in Fig. 3 and 4 for the sake of clarity. The path is certainly not optimal in any sense. If optimality according to some objective function is important, then path optimization techniques can be used to improve the path generated by our approach.

5 Conclusions

A new method of connecting the local minima and maxima of a potential function has been developed. Given a work space, the adjacency graph describing this connection can be developed offline. Then the method requires just a graph search to be done online and can serve as a very good feedback law. We are currently working on some more powerful heuristics for the implementation of A^+ and $A-$, the procedures needed to determine the adjacency graph.

Vertex	X	Y	$\cos\theta$	$\sin\theta$
1	-3.0	2.0	1.0	0.0
2	-3.0	0.489	1.0	0.0
3	-2.998	3.51	1.0	0.0
4	-2.988	-2.51	1.0	0.0
5	1.527	1.998	0.1522	0.9883
6	1.527	1.998	0.1522	-0.9883
7	3.019	2.001	0.1128	0.9936
8	3.3479	-1.677	0.7764	0.6302
9	2.858	-3.372	-0.5853	-0.8108
a	-3.99	1.51	-1.0	0.0
b	4.481	3.994	-0.1048	0.9945
c	4.481	3.994	-0.1048	-0.9945
d	-0.001	-3.48	-1.0	0.0
e	2.444	4.048	-0.3878	0.9217
f	-3.9899	2.4899	-1.0	0.0
g	3.989	-4.4899	-1.0	0.0
h	4.47	-3.999	-0.1599	0.98716
i	4.47	-3.999	-0.1599	-0.98716
j	-3.9899	4.4899	1.0	0.0
k	2.277	4.484	0.9962	0.0868
l	-2.988	-3.4899	1.0	0.0
m	-4.4866	-2.274	0.0671	0.9975
n	-3.995	-1.988	-0.9934	0.1146
o	-4.49	2.723	0.0	-1.0

Table 1. Description of the local minima and maxima

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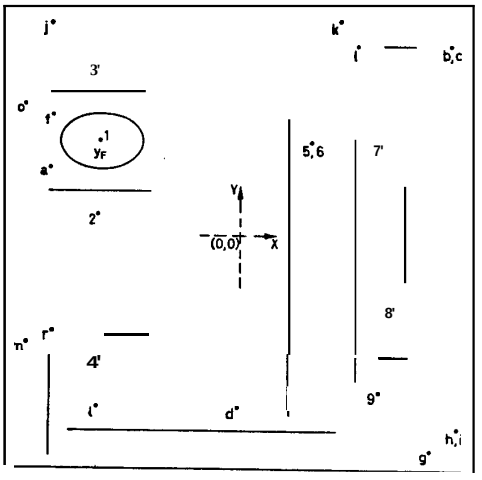

Fig. 1: The workspace with the X–Y co-ordinates of local minima and maxima.

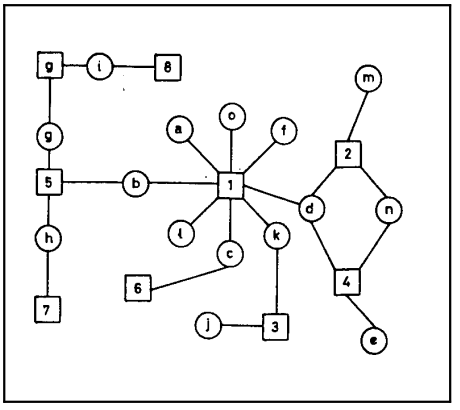

Fig. 2: The adjacency graph of the local minima and maxima.

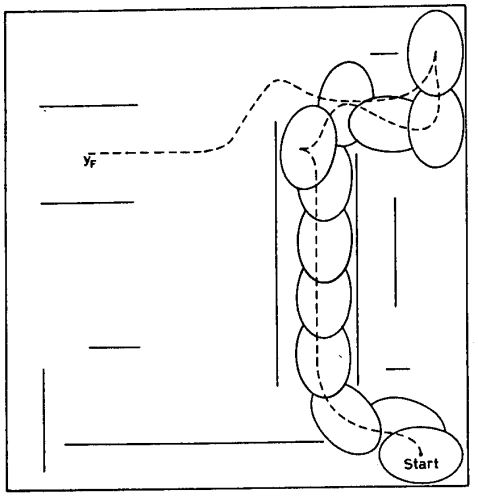

Fig. 3: A part of the obtained path to y_F.

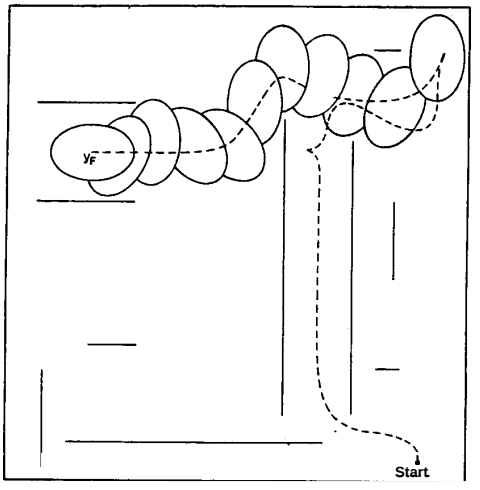

Fig. 4: The remaining part of the path to y_F.