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Computation of Certain Measures of Proximity Between Convex Polytopes: A Complexity Viewpoint

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Abstract

Four problems of proximity between two convex polytopes in R^s are considered. The convex polytopes are represented as convex hulls of finite sets of points. Let the total number of points in the two finite sets be n. We show that three of the proximity problems, viz., checking intersection, checking whether the polytopes are just touching and finding the distance between them, can be solved in $O(n)$ time for fixed s and in polynomial time for varying s. We also show that the fourth proximity problem of finding the intensity of collision for varying s is NP-complete.

1 Introduction

In applications such as robot motion planning and VLSI layout, which involve the relative motion or placement of several objects, the quantification of proximity between a pair of objects plays an important role. Let A and B be two objects (compact, connected sets in R^s). There are four important proximity problems whose solutions adequately describe the proximity between A and B. They are:

P_1: check whether A and B intersect;

P_2: check if A and B are just touching;

P_3: if A and B are non-intersecting, compute the minimum distance between them; and,

P_4: if A and B are intersecting, compute the intensity of collision between them.

The problems will be formally defined in the next section.

Let us briefly mention the usefulness of these problems in Robotics. Gilbert and Johnson [11], and, Gilbert and Hong [10] use the euclidean distance between the obstacle and the robot parts in collision-avoidance path planning and collision detection. Distance is used as a measure of how far a robot part is from collision with an obstacle. When the two objects intersect the distance between them is zero. This gives no information about the intensity of collision. The negative of the minimum euclidean distance by which the two objects must be relatively translated so as to separate is defined as the intensity of collision. Buckley [1] uses this measure in a penalty approach to collision-avoidance robot motion planning. Problems P_1 and P_3 are directly related to the distance and the intensity of collision. Hence they are also useful.

In this paper we discuss the complexity of solving the four problems. The representation for the objects, A and B plays an important role in deciding the complexities. In the literature the two objects are usually taken to be convex. If an object is non-convex it can be usually expressed as the union of a finite number of convex parts. Then, given two objects, the solution of the four problems for each pair of convex parts of the two objects usually yields a solution of the problems for the given pair of objects. If A and B have a smooth boundary then the problems may not be solvable using a finite amount of computation [13]. In this paper we take A and B to be convex polytopes, a representation popularly used in Robotics. A polytope can be described in two ways: (1) the point description, in which the polytope is given to be the convex hull of a finite number of points; and (2) the facial description, in which the polytope is given as the set of points satisfying a finite number of linear inequalities. Most of the results in this paper concern the point description. (A few comments will be made at the end of the paper about complexities associated with the facial description.) Let m and n be integers satisfying $1 < m < n < \infty$,

$$A = \mathrm{Co}(A_p) \text{ and } B = \mathrm{Co}(B_p), \qquad (1.1)$$

where $\mathrm{card}(A_p)=m$, $\mathrm{card}(B_p)=n-m$, and, given $X \subset R^s$, $\mathrm{Co}(X)$ and $\mathrm{card}(X)$ denote the convex hull of X and the cardinality of X, respectively.

We will analyze algorithms for the four problems in terms of two complexity types, described as follows.

- *Type 1 Complexity:* s is assumed to be fixed and, complexity is viewed as a function of n and is measured in terms of the number of arithmetic operations required.
- *Type 2 Complexity:* s is taken as a variable, all data (i.e., A_p and B_p) are assumed to be rational and, complexity is viewed as a function of L, the total data size in number of bits, and measured in terms of the number of bit operations required.

Type 1 complexity is appropriate in applications such as Robotics where s is small, say $s \leq 3$. Type 2 com-

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plexity is useful in other applications which involve larger values for s.

There is a wealth of literature on the problems, P_1 and P_3. There are two different approaches. The first approach, with its roots in Computational Geometry, aims to design algorithms with a low type **1** complexity. For $s = 2$, with A_p and B_p given such that their members are the vertices of A and B given in order, Chazelle and Dobkin [4] have given an $O(\log n)$ algorithm for P_1 and, Chin and Wang [5] and Edelsbrunner [8] have independently proposed an $O(\log n)$ algorithm for P_3. For $s = 3$, with the complete vertex-edge-face descriptions of A and B given (note that Euler's formula for three dimensional polytopes implies that the size of this description is $O(n)$), Chazelle and Dobkin [4] have given an $O(n)$ algorithm for P_1, and, Dobkin and Kirkpatrick [7] have proposed an $O(n)$ algorithm for P_3. Megiddo [17, 9] has given an $O(n)$ algorithm for P_1 assuming only point descriptions for A and B. His algorithm does not restrict itself to $s = 3$. Thus Megiddo's result on the complexity of P_1 is a clear improvement of Chazelle and Dobkin's result. Furthermore, Megiddo's ideas can be used to show that P_1 has a polynomial type **2** complexity.

The second approach to the solution of P_1 and P_3, considered by Gilbert et.al., [12], has its roots in Mathematical Programming. The main aim of this approach is to design practically efficient algorithms. The algorithm suggested in [12] solves P_1 and P_3 simultaneously. Although it has a severe worst case complexity, detailed computational tests have shown that its average type **1** complexity is $O(n)$.

Let us now come to the main contributions of this paper. We propose a new approach to the solution of P_2 and P_3 which can be viewed as a nontrivial extension of Megiddo's approach to P_1. Let A and B be given by (1.1). We first reduce P_3 to a nice quadratic programming problem. This reduction is combined with two recent quadratic programming algorithms [16, 17, 19] to give: (1) an algorithm for P_3 whose type **1** complexity is $O(n)$; and (2) an algorithm for P_3 whose type **2** complexity is polynomial. The former algorithm implies a complexity result which is a clear improvement of Dobkin and Kirkpatrick's result for P_3 since: (1) our algorithm is not restricted to $s = 3$; and (2) even for $s = 3$, our algorithm does not require the vertex-edge-face descriptions of A and B. (Note that, for $s = 3$, the determination of the vertex-edge-face descriptions of A and B from the point descriptions of A and B requires $O(n \log n)$ effort.) These improvements are similar to those achieved by Megiddo's algorithm over Chazelle and Dobkin's algorithm for solving P_1.

Reducing P_2 to a problem of checking the feasibility of linear inequalities, we also derive two algorithms for P_2 similar to those mentioned above for P_3. It appears that, for P_1, P_2 and P_3, $O(n)$ is the optimal type 1 complexity. As already remarked, the type **2** complexity of these problems is polynomial. We briefly remark, towards the end of the paper that, if facial description of A and B are given and n denotes the total number of linear inequalities, then algorithms can be given to solve P_1, P_2 and P_3 with $O(n)$ type **1** complexity and polynomial type **2** complexity.

The fourth proximity problem of computing the intensity of collision has not received much attention. The only algorithms known for this problem are those of Buckley and Leifer [2], Cameron and Culley [3] and Keerthi and Sridharan [15]. For $s = 3$ these algorithms have an $O(n^2 \log n)$ type **1** complexity and for variable s they have an exponential type *2* complexity. For P_4 our main contribution is the following. We reduce P_4 to a problem of maximizing a convex quadratic function subject to linear inequalities and show that, in terms of type *2* complexity, P_4 is NP-complete.

The following notations will be used. For $x \in R^s$, x^i = ith component of x and $\|x\| = l_2$-norm of x. $x \cdot y$ will denote the inner product of $x, y \in R^s$. If $x \in R^m$ and $y \in R^n$, (x, y) will denote the joint vector, $(x', y')'$ where prime denotes transpose. $A - B$ is the Caratheodory-Minkowski set difference between A and B, i.e., $A - B = \{z : z = x - y \text{ for some } x \in A, y \in B\}$. We will denote the boundary of set A by $\mathrm{bd}(A)$.

2 Preliminaries

In this section we give formal definitions of the proximity measures between convex polytopes and also define the four proximity problems formally. We will assume the two objects A and B to be convex polytopes for further discussions.

Definition **2.1.** The (euclidean) distance between A and B, D_+ is given by

$$D_+ = \min \{\|x - y\| : x \in A, y \in B\}.$$

Note that when $A \cap B \neq \phi$, $D_+ = 0$. Under such circumstances, it is useful to talk in terms of the intensity of collision. When $A \cap B = \phi$, there is no collision and hence the intensity of collision is taken as zero. A good measure of the intensity of collision should reflect on how close A and B are from separation, i.e., the least relative translation needed for separating A and B. To describe relative motion, it is sufficient to consider one of the objects (say, A) to be static and other (B) to be moving. Let us first concentrate on translation of B by $z \in R^s$. Given $z \in R^s$, let

$$B(z) = B + \{z\} = \{x + z : x \in B\}.$$

Definition **2.2.** The intensity of collision between A and B, D_- is given by

$$D_- = -\inf \{\|z\| : A \cap B(z) = \phi\}.$$

Note that these defintions of D_+ and D_- are applicable also to the case when A and B are general convex objects (not necessarily convex polytopes). Let us now define the concept of A and B just touching each other.

Definition **2.3.** A and B are said to be just touching if $A \cap B \neq \phi$ and $\exists\, \mu \in R^s$ such that $\forall\, \epsilon > 0$ $A \cap B(\mu\epsilon) = \phi$.

The above definition is also applicable when A and B are general convex objects. For nonconvex objects only Definition **2.1** is useful as a proximity measure. The following lemma contains useful results about

D_+, D_- and the just touching property. We omit its proof as it is easy to establish.

Lemma 2.1. (i) $D_+ \geq 0$, $D_- \leq 0$; (ii) $A \cap B \neq \phi$ iff $D_+ = 0$; (iii) A and B are just touching iff $D_+ = D_- = 0$; and, (iv) $D_- = -\sup\{r : S(r) \subset A - B\}$, where $S(r) = \{x : \|x\| \leq r\}$, the ball of radius r centred at origin.

Problems P_1–P_4 can now be formally defined as

P_1: check if $A \cap B = \phi$;

P_2: check if A and B are just touching;

P_3: if $A \cap B = \phi$, find D_+; and,

P_4: if $A \cap B \neq \phi$ find D_-.

We define the support functions of the objects, $h_1 : R^s \to R$ and $h_2 : R^s \to R$ as

$$h_1(\eta) = \min\{\eta \cdot x : x \in A\}, \tag{2.1}$$

and,

$$h_2(\eta) = \max\{\eta \cdot x : x \in B\}. \tag{2.2}$$

Define $h : R^s \to R$ as,

$$h(\eta) = h_1(\eta) - h_2(\eta). \tag{2.3}$$

The following lemma contains a few useful properties of h_1, h_2 and h. We omit its proof as it is easy to establish.

Lemma 2.2. **(i) h_1 is concave, h_2 is convex and h is concave; (ii) $h(\eta) = \min\{\eta \cdot z : z \in A - B\}$; (iii) $h_1(\alpha\eta) = \alpha h_1(\eta)$, $h_2(\alpha\eta) = \alpha h_2(\eta)$, and $h(\alpha\eta) = \alpha h(\eta)$ $\forall\ \alpha \geq 0$; (iv) furthermore, if A and B are given by (1.1) then $h_1(\eta) = \min\{\eta \cdot v : v \in A_p\}$ and $h_2(\eta) = \max\{\eta \cdot v : v \in B_p\}$.**

3 Solutions of P_1, P_2 and P_3

In this section we reduce each of the problems, P_1 and P_2, to a problem of checking the feasibility of linear inequalities, and, problem P_3 to a convex quadratic programming problem (QP). Then, the application of well known efficient algorithms for feasibility and QP yields efficient algorithms for P_1–P_3. Throughout this section and the next we assume the following:

$$A_p = \{v_1, v_2, \ldots, v_m\}; B_p = \{v_{m+1}, v_{m+2}, \ldots, v_n\};$$

$$A = \mathrm{Co}(A_p); B = \mathrm{Co}(B_p);$$

$$E = \{(\eta, c) \in R^{s+1} : \eta \cdot v_i - c \geq 1\ \forall i = 1, \ldots, m,\ \eta \cdot v_i - c \leq -1\ \forall i = m+1, \ldots, n\};$$

and,

$$F = \{(\eta, c) \in R^{s+1} : \eta \neq 0, \eta \cdot v_i \geq c\ \forall i = 1, \ldots, m,\ \eta \cdot v_i \leq c\ \forall i = m+1, \ldots, n\}.$$

The reductions of problems P_1–P_3 are described by the following three lemmas. The ideas for P_1 are somewhat different from those given by Megiddo [17, 9].

Lemma 3.1. $A \cap B = \phi \Leftrightarrow E \neq \phi$.

Proof. By the separating hyperplane theorem $A \cap B = \phi$ iff there exists a hyperplane, $H = \{x : \beta \cdot x = \alpha\}$ strictly separating A and B, where $\alpha \in R$, $\beta \in R^s$ and $\beta \neq 0$. Without loss of generality we can assume that $\beta \cdot x \geq \alpha\ \forall x \in A$ and $\beta \cdot x < \alpha\ \forall x \in B$. Let

$$\gamma = \max\{\beta \cdot v_i : m+1 \leq i \leq n\}.$$

Clearly $\gamma < \alpha$ and,

$$\beta \cdot x \geq \alpha\ \forall x \in A \text{ and } \beta \cdot x \leq \gamma\ \forall x \in B. \tag{3.1}$$

Since $A = \mathrm{Co}(A_p)$ and $B = \mathrm{Co}(B_p)$, (3.1) is equivalent to

$$\begin{array}{lcll} \beta \cdot v_i & \geq & \alpha, & i = 1, \ldots, m, \\ \beta \cdot v_i & \leq & \gamma, & i = m+1, \ldots, n. \end{array} \tag{3.2}$$

Define $\rho = (\alpha + \gamma)/2$. We can rewrite (3.2) as,

$$\begin{array}{lcll} \beta \cdot v_i - \rho & \geq & (\alpha - \gamma)/2, & i = 1, \ldots, m \\ \beta \cdot v_i - \rho & \leq & -(\alpha - \gamma)/2, & i = m+1, \ldots, n. \end{array} \tag{3.3}$$

Define: $\eta = 2\beta/(\alpha - \gamma)$; and $c = 2\rho/(\alpha - \gamma)$. Dividing both sides of each of the inequalities in (3.3) by the positive scalar, $2/(\alpha - \gamma)$ it is easy to see that (3.2) is equivalent to

$$\begin{array}{lcll} \eta \cdot v_i - c & \geq & 1, & i = 1, \ldots, m \\ \eta \cdot v_i - c & \leq & -1, & i = m+1, \ldots, n. \end{array} \tag{3.4}$$

This proves the lemma. ∎

Lemma 3.2. Suppose $A \cap B \neq \phi$. Then A and B are just touching iff $F \neq \phi$.

Proof. Let A and B be just touching. Let $\mu \in R^s$ be as in Definition 2.3. Take a sequence of hyperplanes, $\{H_i\}$, where $H_i = \{x : \eta_i \cdot x = c_i\}$, $\|\eta_i\| = 1$, $A \subset \{x : \eta_i \cdot x \geq c_i\}$ and H_i strictly separates A and $B(\mu/i)$. We get a bounded sequence $\{(\eta_i, c_i)\}$ in R^{s+1} which admits a convergent subsequence. Let the subsequence converge to (η, c). It is easy to verify that $(\eta, c) \in F$. Thus, $F \neq \phi$.

Now assume $F \neq \phi$. Let $\eta \in F$. It is easy to verify that $\forall\ \epsilon > 0$, $A \cap B(-\eta\epsilon) = \phi$, so that A and B are just touching. ∎

Lemma 3.3. Suppose $A \cap B = \phi$. Then:

(i) $D_+ = \max\{h(\eta) : \|\eta\| = 1, h(\eta) > 0\}$; and,

(ii) $D_+ = 2/\min\{\|\eta\| : h_1(\eta) - c \geq 1$ and $h_2(\eta) - c \leq -1$ for some $c \in R\}$.

Proof. By definition,

$$\begin{array}{lcl} D_+ & = & \min\{\|x - y\| : x \in A, y \in B\} \\ & = & \min\{\|z\| : z \in A - B\}. \end{array}$$

Let $\bar{z} \in A - B$ solve the minimization problem above. (Existence of $\bar{z}$ follows from the compactness of $A - B$ and Weirstrass' theorem.) Define $\bar{\eta} = \bar{z}/\|\bar{z}\|$. (Note that $\|\bar{z}\| = D_+ > 0$.) Optimality of $\bar{z}$ implies that the hyperplane, $\{x : \bar{\eta} \cdot x = D_+\}$ contains $\bar{z}$ and strictly separates $A - B$ from the origin. In other words,

$$\|\bar{z}\| = D_+ = h(\bar{\eta}). \tag{3.5}$$

Now choose any η satisfying $\|\eta\| = 1$. By part (ii) of Lemma 2.2 and compactness of $A - B$ there exists $w \in A - B$ that satisfies $h(\eta) = \eta \cdot w$. Then, by part (ii) of Lemma 2.2, Schwartz inequality and **(3.5)** we get

$$h(\eta) = \eta \cdot w \le \eta \cdot \bar{z} \le \|\bar{z}\| = D_+ = h(\bar{\eta}).$$

Thus $D_+ = \max\{h(\eta) : \|\eta\| = 1\}$. Since $D_+ > 0$, part (i) follows.

Using part (iii) of Lemma 2.2 and part (i) just proved, it is not difficult to see that

$$D_+ = \max\{h(\eta)/\|\eta\| : h(\eta) > 0\}.$$

For a particular η, $h(\eta)/\|\eta\|$ and $h(\alpha\eta)/\|\alpha\eta\|$ are same $\forall\, \alpha > 0$. Therefore, we can fix a positive value of $h(\eta)$, say $h(\eta) = 2$. Thus,

$$\begin{aligned} D_+ &= \max\{2/\|\eta\| : h(\eta) = 2\} \\ &= 2/\min\{\|\eta\| : h(\eta) = 2\}. \end{aligned}$$

Suppose $h(\tilde{\eta}) > 2$ for some $\tilde{\eta}$. Then $\eta = \alpha\tilde{\eta}$, with $\alpha = 2/h(\tilde{\eta})$, satisfies $h(\eta) = 2$ and $\|\eta\| < \|\tilde{\eta}\|$. Thus,

$$D_+ = 2/\min\{\|\eta\| : h(\eta) \ge 2\}. \tag{3.6}$$

Also,

$$\begin{aligned} h(\eta) \ge 2 \;&\Leftrightarrow\; h_1(\eta) - h_2(\eta) \ge 2 \\ &\Leftrightarrow\; h_1(\eta) - 1 \ge h_2(\eta) + 1 \\ &\Leftrightarrow\; \exists\, c \in R \text{ such that } h_1(\eta) - 1 \ge c \\ &\;\; \text{and } h_2(\eta) + 1 \le c. \end{aligned}$$

Using this in **(3.6)** proves the lemma. ∎

The above three lemmas yield the reductions of P_1–P_3 to standard problems. This is stated in the following theorem. Its proof is omitted since it is nearly a restatement of Lemmas **3.1–3.3.**

Theorem 3.4.

(i) $A \cap B = \phi$ iff $\exists\, \eta \in R^s$ and $c \in R$ such that

$$(\eta, c) \in E. \tag{3.7}$$

(ii) A and B are just touching iff $A \cap B \ne \phi$ and $\exists\, \eta \in R^s$, $c \in R$, such that

$$(\eta, c) \in F. \tag{3.8}$$

(iii) If $A \cap B = \phi$,

$$D_+ = 2/\min\{\|\eta\| : (\eta, c) \in E\}. \tag{3.9}$$

Remark 3.1. In order to solve P_1, we have to check the feasibility of **(3.7).** For this we can use any dummy linear objective function (say, η^1) and solve the resulting linear program(LP). This LP is infeasible iff $A \cap B \ne \phi$. Thus to solve P_1, we have to solve one LP involving $(s+1)$ variables and n inequality constraints.

Remark 3.2. Theorem **3.4** shows that P_2 also reduces to a feasibilty problem. But, we have a non-standard constraint $\eta \ne 0$. We breifly indicate how to solve this problem using a sequence of at most 2s feasibilty problems. Consider the system,

$$\begin{array}{lll} \eta \cdot v_i - c & \le & 0, \quad i = 1, \ldots, m, \\ \eta \cdot v_i - c & \ge & 0, \quad i = m+1, \ldots, n. \end{array} \tag{3.10}$$

The forlowing steps solve P_2

0. Set $k = 1$.
1. Fix $\eta^j = 0\ \forall\, j \ni 1 \le j < k$ in **(3.10)** and, let L_+^k and L_-^k denote the systems obtained by further fixing $\eta^k = 1$ and $\eta^k = -1$, respectively. (Essentially, L_+^k and L_-^k are systems involving $s-k$ variables and n inequalities.) Check the feasibilties of L_+^k and L_-^k. If one of them is feasible, conclude that A and B are just touching and stop; else, go to step 2.
2. If $k = s$, conclude that A and B are not just touching and stop; else, reset $k = k+1$ and go back to step **1.**

Remark 3.3. Solving P_3 requires the solution of a convex quadratic programming problem(QP) involving $(s+1)$ variables and n inequality constraints.

Let us now condense some well-known results associated with the solution of the standard problems mentioned in Remarks **3.1–3.3.** Consider the QP

$$\min \quad y \cdot Ty + p \cdot y \tag{3.11}$$

$$\text{s.t.} \quad q_i \cdot y \le r_i, \quad i = 1, \ldots, n, \tag{3.12}$$

where $y \in R^d$, $d \le (s+1)$, and T is a symmetric positive semidefinite matrix. The feasibilty problem, i.e., checking the feasibility of (3.12) forms a part of QP and therefore it is a special case of QP. LP is a special case of QP and it corresponds to setting $T = 0$ in **(3.11).** It is easy to see that each of the problems mentioned in Remarks **3.1–3.3** is a special case of QP. The following results concerning the complexity of solving QP are well-known.

Lemma 3.5. There exists an algorithm with $O(n)$ type **1** complexity that solves QP.

Lemma 3.6. There exists an algorithm with polynomial type 2 complexity that solves QP.

For LP, Lemma **3.5** is proved by Megiddo in [17], where he gives a detailed algorithm with $O(n)$ type **1** complexity and mentions that a similar algorithm can be derived for QP. In [18] Megiddo gives the details of his QP algorithm for $d \le 3$. This algorithm requires somewhat messy modifications of his LP ideas. In [19] we have given an elegant algorithm for QP with $O(n)$ type **1** complexity which is directly along the lines of Megiddo's LP algorithm. For LP, Lemma **3.6** is due to Khachiyan [14] and, for QP it is due to Kozlov et.al. [16].

Combining the reductions given in Theorem **3.4** with Remarks **3.1–3.3** and using Lemmas **3.5–3.6** we get the following result on the complexities of P_1–P_3.

Theorem 3.7. For each of the problems, P_1–P_3, there exist algorithms with $O(n)$ type **1** complexity and polynomial type **2** complexity.

As far as the worst case complexity analysis is concerned, the above result is promising. We have not implemented any of the algorithms which lead to the complexities mentioned in Theorem **3.7.** We expect that, for small s (which is typically the case in Robotics), low type **1** complexity algorithms based on those in [17, 19] will be useful. However, practical usefulness of these algorithms is yet to be demonstrated. The algorithms need to be implemented and compared against practically efficient algorithms such as those in [12].

4 Complexity of P_4

In this section we discuss some complexity issues associated with problem P_4. As in section **3,** we assume that A and B are given by point descriptions. Our main result is that the intensity of collision computation can be reduced to a concave quadratic programming problem. This reduction follows from the following lemma.

Lemma 4.1. Suppose: $A \cap B \neq \phi$; A and B are not just touching each other; and, h_1, h_2 and h are defined as in **(2.1)**–**(2.3)**. Then:

(i) $D_- = \max\{h(\eta) : \|\eta\| = 1, h(\eta) < 0\}$; and,

(ii) $D_- = -2/\max\{\|\eta\| : h_1(\eta) - c \geq -1$ and $h_2(\eta) - c \leq 1$ for some $c \in R\}$.

Proof. From parts (iii) and (iv) of Lemma **2.1,**

$$0 > D_- = -\sup\{r : S(r) \subset A - B\}.$$

Let $\bar{r} = -D_-$. Compactness of $A - B$ implies that $S(\bar{r}) \subset A - B$ and that there exists $\bar{z} \in \mathrm{bd}(A - B) \cap S(\bar{r})$. Clearly, $\|\bar{z}\| = \bar{r} = -D_- > 0$. Define $\bar{\eta} = -\bar{z}/\|\bar{z}\|$. Since $S(\bar{r}) \subset A - B$, any hyperplane which supports $A - B$ at $\bar{z}$ also supports $S(\bar{r})$. But, there is a unique hyperplane, $H = \{x : \bar{\eta} \cdot x = D_-\}$, which supports $S(\bar{r})$ at $\bar{z}$. Thus, H also supports $A - B$. This implies that $h(\bar{\eta}) = D_-$. Now choose any η such that $\|\eta\| = 1$. Since $S(\bar{r}) \subset A - B$, $-\bar{r}\eta \in A - B$. Hence,

$$h(\eta) \leq \eta \cdot (-\bar{r}\eta) = -\bar{r} = D_- = h(\bar{\eta}).$$

Thus $D_- = \max\{h(\eta) : \|\eta\| = 1\}$. Since $\|\eta\| = 1$ implies $h(\eta) < 0$, this condition is superfluous and part (i) follows.

Using part (iii) of Lemma **2.2** and part (i) just proved, it is not difficult to see that

$$D_- = \max\{h(\eta)/\|\eta\| : h(\eta) < 0\}.$$

For a particular η, $h(\eta)/\|\eta\|$ and $h(\alpha\eta)/\|\alpha\eta\|$ are same $\forall\, \alpha > 0$. Therefore, we can fix a negative value of $h(\eta)$, say $h(\eta) = -2$. Thus,

$$\begin{aligned} D_- &= \max\{-2/\|\eta\| : h(\eta) = -2\} \\ &= -2/\max\{\|\eta\| : h(\eta) = -2\}. \end{aligned}$$

It is not difficult to see that even if we allow $h(\eta) \geq -2$ in the maximization problem of the above equation, then maximum will be attained at $h(\eta) = -2$. Hence,

$$D_- = -2/\max\{\|\eta\| : h(\eta) \geq -2\}. \qquad (4.1)$$

Now,

$$\begin{aligned} h(\eta) \geq -2 \quad &\Leftrightarrow \quad h_1(\eta) - h_2(\eta) \geq -2 \\ &\Leftrightarrow \quad h_1(\eta) + 1 \geq h_2(\eta) - 1 \\ &\Leftrightarrow \quad \exists\, c \in R \text{ such that } h_1(\eta) - c \geq -1 \\ &\qquad \text{and } h_2(\eta) - c \leq 1. \end{aligned}$$

Using this in **(4.1)** proves the lemma. ■

Combining the above lemma with part (iv) of Lemma **2.2** we get $D_- = -2/\delta$, where δ is the solution of the following convex maximization problem.

$$\begin{aligned} \max \quad & \|\eta\| \\ \text{s.t.} \quad & \eta \cdot v_i - c \geq -1, \quad i = 1, \ldots, m, \\ & \eta \cdot v_i - c \leq 1, \quad i = m+1, \ldots, n. \end{aligned} \qquad (4.2)$$

Vavasis [20] has shown that **(4.2)** belongs to NP. To show that **(4.2)** is NP-complete we convert the conjunctive normal SAT problem [6] to an instance of **(4.2).** Consider a formula with s logical variables $\chi^1, \ldots \chi^s$, and M clauses. Let η^j denote a real variable corresponding to χ^j, which takes a value of **1** if χ^j is true. and -1 if χ^j is false. The SAT problem has a solution iff the maximum value of $\|\eta\|$ subject to $\eta^j \leq 1$, $-\eta^j \leq 1$ $j = 1, \ldots, s$, and the clause constraints is s. Each clause constraint can be equivalently written as $\eta \cdot w_i \leq 1$ where w_i is appropriately defined. For example, the clause, $\chi^1 \vee \neg\chi^2 \vee \neg\chi^3$ will translate to the inequality, $(1 + \eta^1) + (1 - \eta^2) + (1 - \eta^3) \geq 0.5$, i.e., $0.4\eta^1 - 0.4\eta^2 - 0.4\eta^3 \leq 1$. Thus, after a multiplication of each of the inequalities by a factor of **2,** SAT reduces to an instance of

$$\max \ \|\eta\| \quad \text{s.t.} \quad \eta \cdot v_i \leq 2, \ i = 2, \ldots, n, \qquad (4.3)$$

where $n = 2s + M + 1$ and the v_i's are appropriately defined. If we define $m = 1, v_1 = 0$, the origin in R^s, then it is easy to conclude from **(4.2)** and **(4.3)** (by eliminating c from **(4.2)**) that the maximum value of the objective function in **(4.3)** is nothing but $-2/D^-$. Thus we have the following result.

Theorem 4.2. P_3 **is** NP-complete in the sense of type **2** complexity.

5 Complexity for Facial Representations

In sections **3** and **4** we have considered only point descriptions of A and B. In this section we consider facial representations of A and B. Let

$$\begin{aligned} A &= \{x \in R^s : a_i \cdot x \leq b_i, \ i = 1, \ldots, m\}, \\ B &= \{x \in R^s : a_i \cdot x \leq b_i, \ i = m+1, \ldots, n\}. \end{aligned}$$

Theorem **3.7** holds for this modified representation too. The reasoning is as follows. First, P_1 is equivalent to checking the feasibilty of the system,

$$\begin{aligned} a_i \cdot x &\leq b_i \quad i = 1, \ldots, m, \\ a_i \cdot x &\leq b_i \quad i = m+1, \ldots, n. \end{aligned} \qquad (5.1)$$

$A \cap B = \phi$ iff the above system is infeasible. Similar to Remark 3.2, P_2 can also be solved using a sequence of feasibility problems. The details can be easily worked out. P_3 is equivalent to the solution of the QP

$$\min \|x - y\| \text{ s.t. } (5.1).$$

Whether or not a polynomial type 2 complexity algorithm exists for P_4 is an interesting open problem.

6 Conclusion

In this paper we have analyzed two types of complexity for four proximity problems associated with a pair of objects whose point descriptions are given. We have shown that P_1–P_3 can be solved in linear type 1 and polynomial type 2 complexities. We have also shown that P_3 is NP-complete.

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