Discrete Mathematics 345 (2022) 112829

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

Upper bounding rainbow connection number by forest number

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ARTICLE INFO

Article history:
Received 30 September 2020
Received in revised form 6 January 2022
Accepted 28 January 2022
Available online xxxx

Keywords:
Rainbow connection
Forest number
Upper bound

ABSTRACT

A path in an edge-colored graph is *rainbow* if no two edges of it are colored the same, and the graph is *rainbow-connected* if there is a rainbow path between each pair of its vertices. The minimum number of colors needed to rainbow-connect a graph G is the *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$.
A simple way to rainbow-connect a graph G is to color the edges of a *spanning tree* with distinct colors and then re-use any of these colors to color the remaining edges of G. This proves that $\mathrm{rc}(G) \le |V(G)| - 1$. We ask whether there is a stronger connection between tree-like structures and rainbow coloring than that is implied by the above trivial argument. For instance, is it possible to find an upper bound of $t(G) - 1$ for $\mathrm{rc}(G)$, where $t(G)$ is the number of vertices in the largest induced tree of G? The answer turns out to be negative, as there are counter-examples that show that even $c \cdot t(G)$ is not an upper bound for $\mathrm{rc}(G)$ for any given constant c.
In this work we show that if we consider the *forest number* $\mathrm{f}(G)$, the number of vertices in a maximum induced forest of G, instead of $t(G)$, then surprisingly we do get an upper bound. More specifically, we prove that $\mathrm{rc}(G) \le \mathrm{f}(G) + 2$. Our result indicates a stronger connection between rainbow connection and tree-like structures than that was suggested by the simple spanning tree based upper bound.

1. Introduction

Let G be a connected, simple and finite graph. Consider any edge-coloring of G. A path in G is said to be *rainbow* if no two edges of it are colored the same. The graph G is **rainbow-connected** if there is a rainbow path between each pair of its vertices. If there is a rainbow *shortest* path between every pair of its vertices, we say that G is **strongly rainbow-connected**. The minimum number of colors required to rainbow-connect G is known as the **rainbow connection number** of G, and denoted as $\mathrm{rc}(G)$. Similarly, the minimum number of colors needed to strongly rainbow-connect G is the **strong rainbow**

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[1] The major part of the work was done when this author was on a long-term research visit at Max Planck Institute for Informatics, Saarbrücken, Germany. The visit was funded by the Alexander von Humboldt Foundation fellowship.
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https://doi.org/10.1016/j.disc.2022.112829

connection number of G, denoted as $\mathrm{src}(G)$. These measures of rainbow connectivity were introduced by Chartrand et al. [5] in 2008. The concept has gathered significant attention from both combinatorial and algorithmic perspectives. Indeed, the work of Chartrand et al. [5] has already amassed more than 400 citations. In addition to being a theoretically interesting way of strengthening the usual notion of connectivity, rainbow connectivity has potential applications in networking [3], layered encryption [7], and broadcast scheduling [9].

While introducing the parameters, Chartrand et al. [5] established basic bounds along with exact values of the parameters for some structured graphs. To repeat their results, recall that the *diameter* of G, denoted by $\mathrm{diam}(G)$, is the length of a longest shortest path in G. Now, it is straightforward to verify that $\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq \mathrm{src}(G) \leq m$, where m is the number of edges of G. In other words, both $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ are always sandwiched between one and m. The extremal cases are not difficult to see: $\mathrm{rc}(G) = \mathrm{src}(G) = 1$ if and only if G is complete; $\mathrm{rc}(G) = \mathrm{src}(G) = m$ if and only if G is a tree. The authors also determined the exact rainbow connection numbers for cycle graphs, wheel graphs, and complete multipartite graphs.

Much of the research on rainbow connectivity has focused on finding bounds on the parameters, either in terms of the number of vertices n or some other well-known parameters. It follows that $\mathrm{rc}(G) \leq n - 1$ by taking a spanning tree and coloring its edges with distinct colors, and repeating an already used color for the other edges. For 2-connected graphs, Ekstein et al. [8] showed that $\mathrm{rc}(G) \leq \lceil n/2 \rceil$, and this is tight as witnessed by e.g., odd cycles. Further, it has turned out that domination is a useful concept when deriving upper bounds on $\mathrm{rc}(G)$ (see e.g., [2,11,4]). Specifically, Krivelevich and Yuster [11] showed that $\mathrm{rc}(G) \leq \frac{20n}{\delta}$, later improved by Chandran et al. [4] to $\mathrm{rc}(G) \leq \frac{3n}{\delta+1} + 3$, where δ denotes the minimum degree of G. Moreover, the latter authors derived that when $\delta \geq 2$, then $\mathrm{rc}(G) \leq \gamma_c(G) + 2$, where $\gamma_c(G)$ is the connected domination number. For some structured graph classes, this leads to upper bounds of the form $\mathrm{rc}(G) \leq \mathrm{diam}(G) + c$, where c is a small constant. For instance, it follows that $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 1$ when G is an interval graph and $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 3$, when G is an AT-free graph, both bounds holding when $\delta \geq 2$. Basavaraju et al. [1] show that for every bridgeless graph G with radius r, $\mathrm{rc}(G) \leq r(r+2)$, and for a bridgeless graph with radius r and chordality (length of a largest induced cycle) k, $\mathrm{rc}(G) \leq rk$.

In addition to domination, various authors (see e.g., [2]) have noted trees to be useful in bounding $\mathrm{rc}(G)$. As mentioned earlier, $\mathrm{rc}(G) \leq n - 1$ follows by coloring the edges of a spanning tree of G with distinct colors. Moreover, Kamčev et al. [10] proved that $\mathrm{rc}(G) \leq \mathrm{diam}(G_1) + \mathrm{diam}(G_2) + c$, where $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are connected spanning subgraphs of G and $c \leq |E_1 \cap E_2|$. For a more comprehensive treatment, we refer the curious reader to the books [6,14] and the surveys [13,15] on rainbow connectivity.

In light of the above results, it makes sense to search for bounds on $\mathrm{rc}(G)$ in terms of other graph parameters, that possibly arise from "tree-related" and "dominating" graph structures. Intuitively, a graph structure that has both characteristics is a maximum induced forest of a graph. Hence, the question arises whether one can bound $\mathrm{rc}(G)$ in terms of its **forest number** $\mathrm{f}(G)$, the number of vertices in the largest induced forest in the graph. We answer this in the affirmative by proving the following theorem.

Theorem 1. *A connected graph G with forest number $\mathrm{f}(G)$ has $\mathrm{rc}(G) \leq \mathrm{f}(G) + 2$.*

Observe that the bound is tight up to an additive factor of 3 due to trees that have $\mathrm{rc}(G) = n - 1 = \mathrm{f}(G) - 1$. Our bound improves the upper bound of $n - 1$ obtained by coloring the edges of a spanning tree in distinct colors, except when $\mathrm{f}(G) \geq n - 2$. We leave as an open problem the question of whether the stronger upper bound of $\mathrm{f}(G) - 1$ is true.

One might be tempted to conjecture a strengthening of our bound, namely that $\mathrm{rc}(G)$ is at most $t(G)$, the number of vertices in the largest induced *tree* in the graph. However, this turns out to be not true. To see this, one can consider a graph G obtained by taking a K_k for any $k \geq 3$ with a pendant vertex attached to each of its vertices. Then, we have that $\mathrm{rc}(G) = k$ whereas $t(G) = 4$.

Finally, we note that the complement of an induced forest is a *feedback vertex set*. The *feedback vertex set number* is the size of the smallest set of vertices in a graph whose removal leaves an induced forest. Hence, Theorem 1 directly implies the following.

Corollary 1. *A connected graph G with feedback vertex set number $\mathrm{fvs}(G)$ has $\mathrm{rc}(G) \leq |V(G)| - \mathrm{fvs}(G) + 2$.*

1.1. Overview of our techniques

Here, we give a summary of the ideas used for proving Theorem 1. We first fix a maximum induced forest $\mathcal{F}$ of G and define H to be the graph obtained from G by contracting each connected component of $\mathcal{F}$, each of which is a tree, into a single vertex. Thus H consists of *tree vertices* and *non-tree vertices*. An edge from a non-tree-vertex u to a tree-vertex x_T is classified as a 2-edge if u has at least two edges to the tree T (the tree that was contracted into the tree-vertex x_T), and as a 1-edge otherwise. We fix a carefully chosen spanning tree of H, root it at some (contracted) tree-vertex, and direct all the edges towards root. We call this the *skeleton* B. The *inner skeleton* B_1 is defined to be B minus the leaves of B that are non-tree vertices.

We color all the edges of the forest $\mathcal{F}$ with distinct colors (call them forest colors), then associate with each tree of $\mathcal{F}$, an additional color called its *surplus color*, and also keep aside two *global surplus colors*. Note that this makes the total number of colors $f(G)+2$ as required. The idea is to color the edges of G that appear in the inner skeleton B_1 using the surplus colors in such a way that between any pair of vertices in B_1 there is a rainbow path using edges of B_1 and $\mathcal{F}$. However, this turned out to be not always possible; in some cases we had to use some special edges outside B_1 that we call shortcut edges. After rainbow-connecting vertex-pairs in inner skeleton B_1, we connect the vertices outside of the inner skeleton to the inner skeleton using the two global surplus colors. The hard part of the proof is in making the coloring of B_1 work with one surplus color per tree. For this, we do a case analysis to color the edges around a vertex in B_1. The cases are differentiated mainly on the basis of the number of edges and the number of 2-edges incident on a vertex.

1.2. Preliminaries

For a graph G, a subgraph G_1 of G, and any $E' \subseteq E(G)$, we use $E'(G_1)$ to denote $E' \cap E(G_1)$. For a vertex v of (di)graph G, we use $\deg_G(v)$ to denote the degree of v in G. We use $\mathsf{dist}_G(u,v)$ to denote number of vertices in any shortest path between u and v in G. For graph G and $S \subseteq V(G)$, we define $G \setminus S := G[V(G) \setminus S]$. We use uv for an edge between u and v and for a directed edge from u to v, we use $\overrightarrow{uv}$. For the latter, we may omit the arrow, when the direction is not relevant. For a directed graph G, we denote by $\tilde{G}$, the underlying undirected graph of it. Since, for a forest $\mathcal{F}$, each connected component is a tree, we will use the phrases "tree of $\mathcal{F}$" and "connected component of $\mathcal{F}$" analogously. An *in-arborescence* is a directed graph with a special root vertex such that all vertices have a unique directed path to the root vertex. For a tree T and vertices u and v in T, we use T_{uv} to denote the unique path in T between u and v. The following is a general easy-to-see observation about trees that we use in the proof.

Observation 1. *Let v_1, v_2, and v_3 be three vertices in any tree T, and let e be an edge in $T_{v_2v_3}$. Then either $T_{v_1v_2}$ or $T_{v_1v_3}$ does not contain the edge e.*

2. Proof of Theorem 1

Let $G=(V,E)$ be a connected graph. Our goal is to prove that $\mathrm{rc}(G) \le f(G)+2$.

Let $\mathcal{F}$ be a maximum induced forest of G that has the smallest number of connected components (trees) out of all the maximum induced forests of G. Let $F = V(\mathcal{F})$ be the set of vertices in $\mathcal{F}$. Let $\mathcal{T}$ be the set of connected components (trees) of $\mathcal{F}$ and let $t = |\mathcal{T}|$. Let $S := V \setminus F$. Also, let $f = |V(F)| = f(G)$. We call an edge uv of G a **tree-edge** if both u and v belong to the same tree in $\mathcal{T}$; otherwise, the edge is called a **non-tree edge**.

Let H be the graph obtained from G by contracting each connected component of $\mathcal{F}$ to a single vertex (see Fig. 1). Formally, we define H as:

$$V(H) := V_{\mathcal{T}} \cup S, \text{ where}$$
$$V_{\mathcal{T}} := \{x_T : T \in \mathcal{T}\} \text{ and}$$
$$E(H) := E(G[S]) \cup \{ux_T : u \in S,\ T \in \mathcal{T},\ u \text{ has at least one edge to } V(T) \text{ in } G\}.$$

We call the vertices in $V_{\mathcal{T}}$ the **tree vertices** and the vertices in S the **non-tree vertices** of H. Notice that $V_{\mathcal{T}}$ is an independent set in H, because there are no edges in G between any two distinct connected components of $\mathcal{F}$. We partition the edges of H into the following two sets:

$$E_1 := E(G[S]) \cup \{ux_T : u \in S,\ T \in \mathcal{T},\ u \text{ has exactly one edge to } V(T) \text{ in } G\} \text{ and}$$
$$E_2 := \{ux_T : u \in S,\ T \in \mathcal{T},\ u \text{ has at least two edges to } V(T) \text{ in } G\}.$$

The edges in E_1 are called **1-edges** while those in E_2 are **2-edges.** See Fig. 1 for an illustration of the above definitions. We define a function $f_{\mathcal{T}} : V_{\mathcal{T}} \to \mathcal{T}$ that maps a tree-vertex to its corresponding tree, i.e., $f_{\mathcal{T}}(x_T) = T$. For each edge in H, we define its **representatives** in G as follows. Consider first a 2-edge e between $u \in S$ and $x_T \in V_{\mathcal{T}}$. By definition of a 2-edge, u has at least two edges to $V(T)$ in G. We arbitrarily choose two of these edges as the representatives in G of the 2-edge e and denote them by $(e)_1$ and $(e)_2$. For a 1-edge e between $u \in S$ and $x_T \in V_{\mathcal{T}}$, there is a unique edge between u and $V(T)$ in G, by the definition of a 1-edge. We call this edge the representative of ux_T in G, and denote it $(e)_1$. For a 1-edge e between $u \in S$ and $v \in S$, we call uv its own representative in G. For simplicity, we might simply say *representatives* instead of *representatives in G*. Whenever we say a representative, it is implicitly assumed that we are talking about an edge in G. For a 2-edge ux_T with representatives uv_1 and uv_2, we call the vertices v_1 and v_2, the **foots** of ux_T. The unique path between v_1 and v_2 in T is called the **foot-path** of ux_T. For a 1-edge ux_T (whose one endpoint is a tree-vertex) with representative uv, we call the vertex v, the foot of ux_T.

A **skeleton** is an in-arborescence obtained by taking a spanning tree of H with an arbitrary node of $V_{\mathcal{T}}$ fixed as its root with all edges directed towards the root. Given a skeleton B with root r, we define the **level** of each node v, denoted by $\ell_B(v)$, as its distance (in terms of number of vertices) to r in B. Note that $\ell_B(r) = 1$ per this definition. For a skeleton B, we define its **configuration vector** as the following vector:

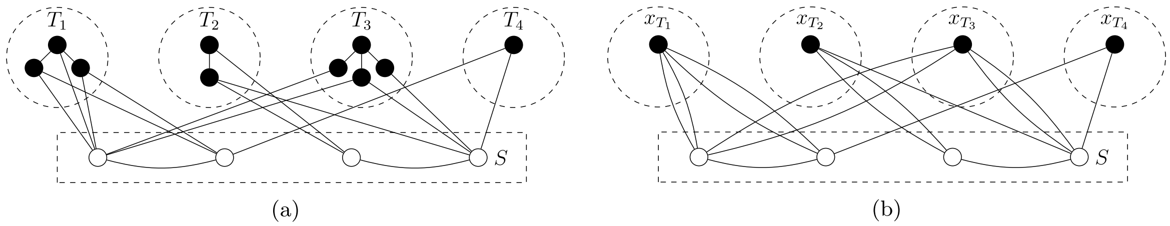

Fig. 1. (a) A graph G is partitioned into a maximum induced forest $\mathcal{F}$ and $S = V(G) \setminus V(\mathcal{F})$. The connected components (trees) of $\mathcal{F}$ are T_1, T_2, T_3 and T_4. The edges between two black vertices, corresponding to vertices in $V(\mathcal{F})$, are tree-edges. **(b)** The graph H obtained after contracting the connected components of $\mathcal{F}$. We draw a 2-edge with 2 lines and a 1-edge with a single line.

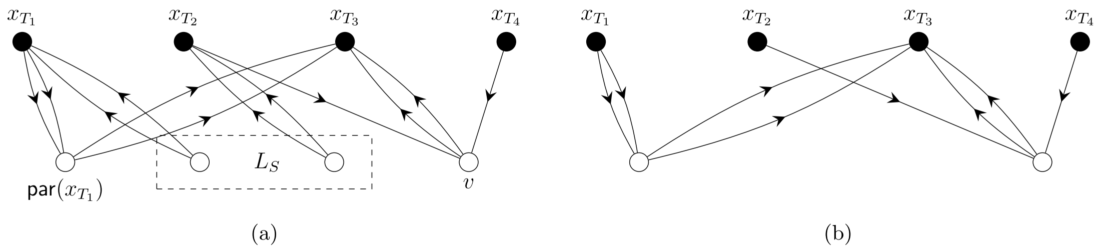

Fig. 2. (a) A skeleton B, where x_{T_3} is the root. For vertex v, x_{T_2} and x_{T_4} are the children. **(b)** The inner skeleton $B_1 := B[V(B) \setminus L_S]$.

$$\langle\ |E_2(B)|, n_2, n_3, \ldots, n_{|V|}\ \rangle,$$

where n_i denotes the number of vertices in level i in B.

We now fix a skeleton B such that it has the lexicographically highest configuration vector out of all possible skeletons. The **parent** of a non-root vertex v in B, denoted by par(v), is the unique out-neighbor of v in B. The **children** of v are the in-neighbors of v in B. Whenever we say the parent (or child), we mean the parent (or child) in B, even if B is not mentioned explicitly. We call a directed edge $\overrightarrow{uv}$ in B, a 1-edge (or 2-edge respectively), if uv is a 1-edge (or 2-edge respectively) in H. Let L_S be the set of vertices of S that are leaves of B and let B_1 be the sub-arborescence of B defined as $B_1 := B[V(B) \setminus L_S]$. We call B_1 the **inner skeleton**. Let $\tilde{B_1}$ be the underlying undirected tree of B_1. These concepts are illustrated in Fig. 2.

We now prove a lemma and three corollaries that are useful for our coloring procedure.

Lemma 1. *Every vertex in S has at least one 2-edge incident on it in B.*

Proof. Suppose for the sake of contradiction that v is a vertex in S that has only 1-edges incident on it in B. There exists a $T \in \mathcal{T}$ such that v has at least two edges to T in G, because otherwise, $G[F \cup \{v\}]$ is a forest, contradicting the maximality of $\mathcal{F}$. Therefore, vx_T is a 2-edge in H. Let C be the connected component of $B \setminus v$ that contains the vertex x_T. Let e be the unique edge in B between v and C. Note that e is a 1-edge by assumption. Removing e from B and adding 2-edge vx_T gives a skeleton with higher number of 2-edges than B. This is a contradiction to the choice of B. □

The above lemma has the following corollaries.

Corollary 2. *For every vertex in L_S, the unique edge incident on it in B is a 2-edge.*

Corollary 3. *Every leaf of B_1 is a tree-vertex.*

Proof. Suppose for the sake of contradiction that there is a leaf v of B_1 that is a non-tree-vertex. Clearly, $v \notin L_S$ by the definition of B_1. Hence, v is not a leaf of B. Then, there must be a vertex u in L_S that has an edge to v in B. Since both u and v are in S, the edge uv is a 1-edge. This is a contradiction to Corollary 2. □

Corollary 4. *For each 1-edge $\overrightarrow{uv}$ in B_1, either u is a tree-vertex, or a child u' of u in B_1 is a tree-vertex with $u'u$ being a 2-edge.*

Proof. Suppose that u is not a tree-vertex. Then there is an incoming 2-edge on u in B, because its outgoing edge is a 1-edge and there has to be at least one 2-edge incident on it due to Lemma 1. Let the other endpoint of this edge be u'. Since at least one of the endpoints of a 2-edge has to be a tree-vertex, u' is a tree-vertex. Since u' is a tree-vertex, it has to be in B_1. □

We define a mapping h from G to H as follows. For a vertex v in $V(G)$, if $v \in S$ then define $h(v) := v$, otherwise (i.e., if $v \in F$) define $h(v) := x_T$, where $T \in \mathcal{T}$ is the tree containing v. For a non-tree edge $e = uv$ in G, we define $h(e)$ to be the edge $h(u)h(v)$. For a vertex subset U of $V(G)$, we define $h(U)$ to be $\bigcup_{a \in U} h(a)$. For an edge subset E' of $E(G)$, we define $h(E')$ to be $\{h(e) : e \in E' \text{ and } e \text{ is a non-tree edge}\}$. For a subgraph G' of G, we define $h(G')$ as the subgraph of H with vertex set $h(V(G'))$ and edge set $h(E(G'))$.

Let the palette of colors be $\{1, 2, \ldots, f+2\}$. We call colors $f+1$ and $f+2$ the **global surplus colors**, and denote them by g_1 and g_2 respectively. We reserve g_1 and g_2 to color the edges incident on L_S. We will first give a coloring of some edges of G using colors $\{1, 2, \ldots, f\}$ such that there is a rainbow path between every pair of vertices in $V(G) \setminus L_S$. Then we will extend the coloring to L_S using the global surplus colors. We give our coloring procedure as a list of **coloring rules**.

For $a, b \in V(G) \setminus L_S$, let Q_{ab} denote the unique path in the inner skeleton B_1 between $h(a)$ and $h(b)$. For each such pair of vertices (a, b), we will maintain a subgraph P_{ab} of G. Each P_{ab} is initialized to $\emptyset$. After the application of each coloring rule, we will apply a **path rule** for each pair (a, b), which (possibly) adds some newly colored edges to P_{ab}. We say that an edge in B_1 is colored if its representatives in G are colored (we will make sure that for a 2-edge, either both representatives are colored or both are uncolored at any point of time). Whenever an edge in B_1 gets colored by a coloring rule and if it is in Q_{ab}, we make sure that we add exactly one of its representatives to P_{ab} in the subsequent path rule. Whenever it happens during a path rule that two edges u_1v_1 and u_2v_2 are in P_{ab} such that both v_1 and u_2 are in some $T \in \mathcal{T}$, but u_1 and v_2 are not in T, then we add the path $T_{v_2u_1}$ to P_{ab} (if it is not already included). Similarly, if it happens that there is an edge uv in P_{ab} such that $v, a \in V(T)$ ($v, b \in V(T)$ resp.) but $u \notin V(T)$, we add the path T_{va} (T_{vb} resp.) to P_{ab}. Also, if both a and b are in the same tree T, then we add the path T_{ab} to P_{ab} (during Path Rule 1 below). Thus, when all the coloring rules and path rules have been applied, we will have that for all $a, b \in V(G) \setminus L_S$, it holds that P_{ab} is a path between a and b. We will prove that P_{ab} is also a rainbow path. For this, we will maintain the following invariant.

Invariant 1. *For each pair $a, b \in V(G) \setminus L_S$, no two edges in P_{ab} have the same color.*

We will prove that the invariant still holds after each path rule. Since new edges are added to P_{ab} only during path rules, this means that the invariant always holds. We also maintain the following three auxiliary invariants. But they are rather straightforward to check from the coloring and path rules and hence we will not explicitly prove them.

Invariant 2. *For any 2-edge in B, either both representatives of it are colored or both are uncolored.*

A vertex in B_1 is said to be **completed** if all the incident edges on it in B_1 are colored and is said to be **incomplete** otherwise.

Invariant 3. *For an incomplete tree-vertex x_T, the colors of $E(T)$ are disjoint from the colors of the rest of the graph G.*

Invariant 4. *A nonempty subset of internal edges of a tree T is contained in P_{ab} only if a representative of each edge in Q_{ab} that is incident on x_T (there can be at most two of such edges as Q_{ab} is a path) is in P_{ab}.*

Now, we start with the coloring and path rules.

Coloring Rule 1. *Color all the edges in $\mathcal{F}$ with distinct colors $1, 2, \ldots, f-t$.*

Path Rule 1. *For each $a, b \in V(G) \setminus L_S$, if a and b are in the same tree T for some $T \in \mathcal{T}$, then add the path T_{ab} to P_{ab}.*

It is easy to see that Invariant 1 is satisfied after the above Path rule as the color of each edge is distinct so far.

For each tree $T \in \mathcal{T}$, we designate a color in $[f-t+1, f]$ as its **surplus color**, denoted by $s(T)$. More specifically, the surplus color of i^{th} tree in $\mathcal{T}$ is defined as the color $f-t+i$. Also, the colors of the edges of T (colored by Coloring Rule 1) are called the **internal colors** of T.

Coloring Rule 2. *For each 1-edge $\overrightarrow{uv}$ in B: if u is a tree-vertex, then color $(uv)_1$ with $s(f_{\mathcal{T}}(u))$; otherwise, i.e., if u is not a tree-vertex, by Corollary 4, there is at least one child of u in B_1 that is a tree-vertex; pick one such tree-vertex x_T and color $(uv)_1$ with color $s(T)$.*

Note that after Coloring Rule 2, any tree-vertex x_T such that T is just a single vertex, is completed.

Path Rule 2. *Do the following for each $a, b \in V(G) \setminus L_S$. For each 1-edge e in Q_{ab}, add $(e)_1$ to P_{ab}. Next, we add edges inside trees as follows.*

- *If for some tree T it holds that $a \in V(T)$ and there is a 1-edge ux_T in Q_{ab}, then add the path T_{wa} to P_{ab}, where w is the foot of the edge ux_T in T.*

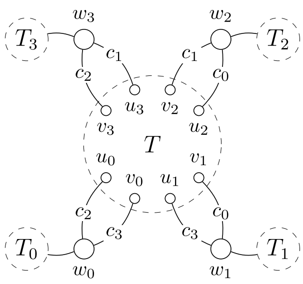

Fig. 3. Illustration of Coloring Rule 3 applied on a tree-vertex x_T with 2-edge degree 4. Here, $c_i = s(T_i)$.

- *If for some tree T it holds that $b \in V(T)$ and there is a 1-edge ux_T in Q_{ab}, then add the path T_{wb} to P_{ab}, where w is the foot of the edge ux_T in T.*
- *If for some tree T there are two 1-edges ux_T and vx_T in Q_{ab}, add the path T_{wz} to P_{ab}, where w is the foot of the edge ux_T in T and z the foot of the edge vx_T in T.*

Lemma 2. *Invariant 1 is satisfied so far. Moreover, each color is used at most for one edge in G.*

Proof. It is clear that during Coloring Rule 1, all edges are colored distinct. In Coloring Rule 2, we use the surplus colors, which are disjoint from the colors used in Coloring Rule 1. It is also not difficult to see that during Coloring Rule 2, the surplus color of a tree-vertex is used for only one 1-edge. □

For a colored edge $e \in E(G)$, we define $c(e)$ to be the color of e. For a subgraph G' of G, we define $c(G')$ to be the set of colors used in $E(G')$. We call the number of 2-edges of B_1 incident on a vertex, the *2-edge degree* of it. For any two vertices u and v, the connected component of $B_1 \setminus u$ containing v is denoted by $\mathtt{ST}(u, v)$. Note that $\mathtt{ST}(u, v)$ is a subtree of B_1. We fix a closest (breaking ties arbitrarily) tree-vertex to v in $\mathtt{ST}(u, v)$ in $\tilde{B_1}$ and denote it by $\mathtt{CT}(u, v)$. Note that at least one tree-vertex exists in $\mathtt{ST}(u, v)$ because all leaves of B_1 are tree vertices by Corollary 3. Also note that if v is a tree-vertex, then $\mathtt{CT}(u, v) = v$.

Coloring Rule 3. *For each tree-vertex x_T with 2-edge degree at least 4 (see Fig. 3 for an illustration). Let q be the 2-edge degree of x_T. Let $w_0, w_1, w_2, \ldots, w_{q-1}$ be the other endpoints of the 2-edges incident on x_T. For $i \in [0, q-1]$, let $x_{T_i} := \mathtt{CT}(x_T, w_i)$ and let $c_i := s(T_i)$. For each $i \in [0, q-1]$, color the edge $(x_T w_i)_1$ with $c_{((i+2) \bmod q)}$ and the edge $(x_T w_i)_2$ with $c_{((i+3) \bmod q)}$.*

The following lemma follows from the way in which we have colored the edges incident on x_T in Coloring Rule 3.

Lemma 3. *For each tree-vertex x_T on which Coloring Rule 3 has been applied as above, for all distinct $i, j \in [0, q-1]$, there is a rainbow path from w_i to w_j in G that uses only the colors from $\left(\left\{c_0, c_1, \ldots, c_{q-1}\right\} \setminus \left\{c_i, c_j\right\}\right) \cup c(T)$. Moreover, for any $i \in [q-1]$ and some $u \in V(T)$, there is a rainbow path in G from u to w_i that uses only colors from $\left(\left\{c_0, c_1, \ldots, c_{q-1}\right\} \setminus \{c_i\}\right) \cup c(T)$.*

Proof. Let u_i and v_i be the endpoints in T of $(x_T w_i)_1$ and $(x_T w_i)_2$ respectively, for each $i \in \{0, 1, \ldots, q-1\}$. First, we prove that there is a rainbow path from w_i to w_j with the required colors as claimed by the lemma. Suppose for the sake of contradiction that there was no such path. Consider the following three paths between w_i and w_j: $P := w_i u_i T_{u_i u_j} u_j w_j$, $P' := w_i v_i T_{v_i v_j} v_j w_j$, and $P'' := w_i v_i T_{v_i u_j} u_j w_j$. By our assumption, each of these paths, is either not a rainbow path, or uses a color that is not in $\left(\left\{c_0, c_1, \ldots, c_{q-1}\right\} \setminus \left\{c_i, c_j\right\}\right) \cup c(T)$. Also, from Coloring Rules 1 and 3, we know that the only colors that are not in $\left(\left\{c_0, c_1, \ldots, c_{q-1}\right\} \setminus \left\{c_i, c_j\right\}\right) \cup c(T)$ that any of these three paths can use are c_i and c_j. Thus, each of P, P' and P'' is either not a rainbow path or uses c_i or c_j. However, we know that the paths $T_{u_i u_j}$, $T_{v_i v_j}$, and $T_{v_i u_j}$ are all rainbow paths due to Coloring Rule 1, and moreover the colors used by them are disjoint from $\left\{c_0, \ldots, c_{q-1}\right\}$. For the path P, this means that either $c(w_i u_i) = c(u_j w_j)$ or $\left\{c(w_i u_i), c(u_j w_j)\right\} \cap \left\{c_i, c_j\right\} \neq \emptyset$. That is, either $c_{(i+2) \bmod q} = c_{(j+2) \bmod q}$ or $\left\{c_{(i+2) \bmod q}, c_{(j+2) \bmod q}\right\} \cap \left\{c_i, c_j\right\} \neq \emptyset$. That is, either $i = j$ or $\{(i+2) \bmod q, (j+2) \bmod q\} \cap \{i, j\} \neq \emptyset$. But we know that $(i+2) \bmod q \neq i$ and that $(j+2) \bmod q \neq j$. Therefore, either $(i+2) \bmod q = j$ or $(j+2) \bmod q = i$. Without loss of generality assume that $(i+2) \bmod q = j$.

By using the same reasoning as above for path P', we derive that either $(i+3) \bmod q = j$ or $(j+3) \bmod q = i$. Since we already have that $(i+2) \bmod q = j$, it should be the latter case, i.e, $(j+3) \bmod q = i$.

Now consider the third path P''. We have that either $c(w_i v_i) = c(u_j w_j)$ or $\left\{c(w_i v_i), c(u_j w_j)\right\} \cap \left\{c_i, c_j\right\} \neq \emptyset$. That is, either $c_{(i+3) \bmod q} = c_{(j+2) \bmod q}$ or $\left\{c_{(i+3) \bmod q}, c_{(j+2) \bmod q}\right\} \cap \left\{c_i, c_j\right\} \neq \emptyset$. That is, either $(i+3) \bmod q = (j+2) \bmod q$ or $\{(i+3) \bmod q, (j+2) \bmod q\} \cap \{i, j\} \neq \emptyset$. Substituting that $(i+3) \bmod q = (((i+2) \bmod q) + 1) \bmod q = (j+1) \bmod q$ and that $i = (j+3) \bmod q$, we get that either $(j+1) \bmod q = (j+2) \bmod q$ or $\{(j+1) \bmod q, (j+2) \bmod q\} \cap$

$\{(j+3) \bmod q, j\} \neq \emptyset$. Since $j, (j+1) \bmod q, (j+2) \bmod q$, and $(j+3) \bmod q$ are distinct for $q \geq 4$, we have a contradiction.

Next, we prove the second part of the lemma, i.e., we prove that there is a rainbow path from u to w_i with the colors claimed by the lemma. Suppose for the sake of contradiction that there was no such path. Consider the path $P''' := w_i u_i T_{u_i u}$. We know that the path $T_{u_i u}$ uses only colors from $c(T)$ and is rainbow, and that the edge $w_i u_i$ is colored $c_{(i+2) \bmod q}$. Also, $c_{(i+2) \bmod q} \neq c_i$ as $(i+2) \bmod q \neq i$. Thus P is a rainbow path and uses only the colors in $\left(\{c_0, c_1, \ldots, c_{q-1}\} \setminus \{c_i\}\right) \cup c(T)$. □

Path Rule 3. *For each x_T on which Coloring Rule 3 has been applied as above and for each $a, b \in V(G) \setminus L_S$ such that Q_{ab} contains x_T (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following.*

Case 1: *There are two 2-edges incident on x_T in Q_{ab}.*

Let w_i and w_j be the neighbors of x_T in Q_{ab}. Add to P_{ab} the rainbow path from w_i to w_j as given by Lemma 3.

Case 2: *There is one 2-edge and one 1-edge incident on x_T in Q_{ab}.*

Let $x_T w_i$ be the 2-edge. Let u be the endpoint in T of the representative of the 1-edge. There is a rainbow path from w_i to u as given by Lemma 3. Add this path to Q_{ab}. (Note that the representative of the 1-edge has been already added to P_{ab} during Path Rule 2).

Case 3: *x_T is an endpoint of Q_{ab} and the only edge incident on x_T in Q_{ab} is a 2-edge.*

Let w_i be the neighbor of x_T in Q_{ab}. We know one of a or b is in T. From this vertex (a or b whichever is in T) to w_i, there is a rainbow path as given by Lemma 3. Add this path to P_{ab}.

The following lemma follows from Lemma 3 and Path Rule 3.

Lemma 4. *Suppose for some $a, b \in V(G) \setminus L_S$ and for some tree $T' \in \mathcal{T}$, P_{ab} contains an edge e that was colored with $s(T')$ during the application of Coloring Rule 3 on some tree-vertex x_T. Then, $T' \neq T$ and Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T'})$.*

Proof. Since $s(T')$ was used during the application of Coloring Rule 3 on x_T, the vertex $x_{T'}$ should have been taken as x_{T_i} (in Coloring Rule 3) for some i and $s(T')$ was taken as c_i (in Coloring Rule 3). Since $T_i \neq T$, it is clear that $T' \neq T$. Suppose Q_{ab} intersects $\mathtt{ST}(x_T, x_{T'})$ for the sake of contradiction. That is, Q_{ab} intersects $\mathtt{ST}(x_T, x_{T_i})$. Then the color c_i was not used in Path Rule 3 according to Lemma 3. That means e was not colored with c_i, which is a contradiction. □

Lemma 5. *Invariant 1 is not violated during Path Rule 3.*

Proof. Suppose Invariant 1 is violated during the application of Path Rule 3 on the pair (x_T, P_{ab}). Then there exist edges e and e' in P_{ab} having the same color after the application of the path rule. We can assume without loss of generality that e was added during the application of Path Rule 3 on (x_T, P_{ab}). That means e was colored during the application of Coloring Rule 3 on x_T. Then either $e \in E(T)$ or $h(e) = w_i x_T$ for some $i \in [0, q-1]$. Since each color in $c(T)$ has been used only in one edge in G, we have that $h(e) = w_i x_T$ for some $i \in [0, q-1]$ and hence $c(e) = s(T_j)$ for some $j \in [0, q-1] \setminus i$. Also Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T_j})$ by Lemma 4. Since the application of Path Rule 3 on (x_T, P_{ab}) added a rainbow path to P_{ab}, the edge e' was not added during this application. Since each color in $c(F)$ has been used for only one edge in G so far, we know that e' was not added during Path Rule 1. Hence, the following two cases are exhaustive and in both cases we derive a contradiction.

Case 1: e' was added during the application of Path Rule 3 on $(x_{T'}, P_{ab})$ for some tree $T' \neq T$.

Since P_{ab} contains e', we have that Q_{ab} contains $h(e')$. Since e' was added during the application of Path Rule 3 on $(x_{T'}, P_{ab})$, either $e' \in E(T')$ or $h(e')$ is incident on $x_{T'}$. In either case, $x_{T'}$ is in Q_{ab}. Since Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T_j})$, we have that $x_{T'}$ is not in $\mathtt{ST}(x_T, x_{T_j})$. This implies that $\mathsf{dist}_{\vec{B}_1}(x_{T'}, x_T) < \mathsf{dist}_{\vec{B}_1}(x_{T'}, x_{T_j})$. But then during the application of Coloring Rule 3 on $x_{T'}$, the color $s(T_j)$ would never be used as $x_{T_j} \neq \mathtt{CT}(x_{T'}, v)$ for any vertex v. Thus, the color of e' is not $s(T_j)$. But we know that $c(e') = c(e) = s(T_j)$, a contradiction.

Case 2: e' was added during the application of Path Rule 2 on P_{ab}.

This means e' is the representative of a 1-edge and was colored during Coloring Rule 2. Since e' is colored with $s(T_j)$, we have that $h(e')$ should either be the outgoing edge of x_{T_j} or the outgoing edge of the parent of x_{T_j}, from Coloring Rule 2. This implies that $h(e')$ is in $\mathtt{ST}(x_T, x_{T_j})$, as the parent of x_{T_j} is a non-tree-vertex. But then Q_{ab} does not contain $h(e')$ as Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T_j})$. Thus P_{ab} does not contain e', which is a contradiction. □

Coloring Rule 4. *For each tree-vertex x_T with 2-edge degree exactly 3 (see Fig. 4), let w_1, w_2, and w_3 be the other endpoints of the three 2-edges incident on x_T. Further, for $i \in \{1, 2, 3\}$, let $x_{T_i} = \mathtt{CT}(x_T, w_i)$, let u_i and v_i be the foots of $x_T w_i$ in T, let $P_i := T_{u_i v_i}$, and let $c_i := s(T_i)$.*

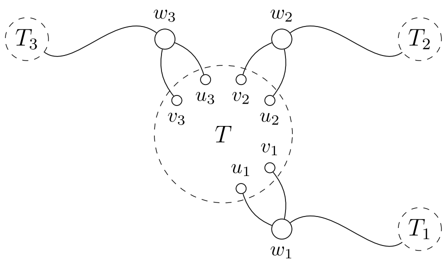

Fig. 4. A scenario in which Coloring Rule 4 is applicable on x_T.

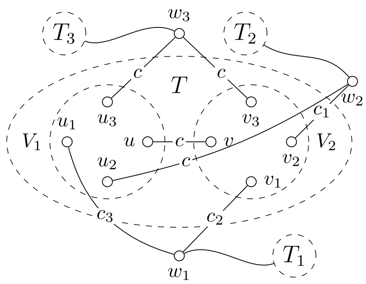

Fig. 5. Case 1 of Coloring Rule 4.

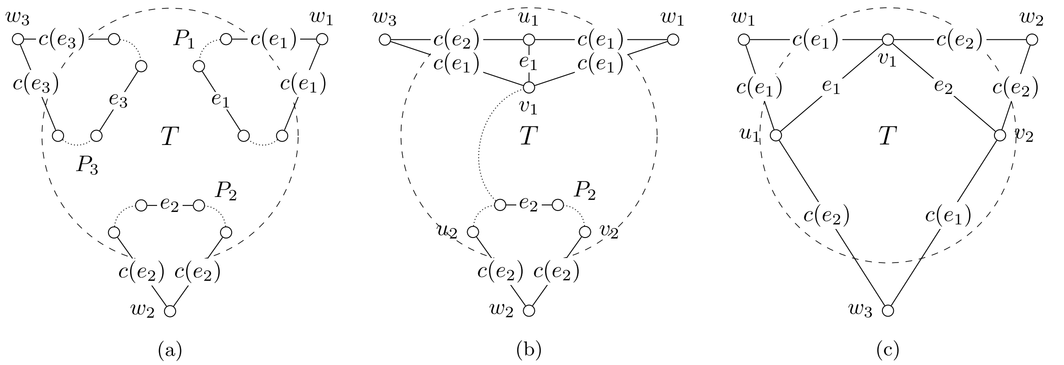

Fig. 6. Cases 2 and 3 of Coloring Rule 4. **(a)** Case 2. Note that P_1, P_2, and P_3 are not necessarily disjoint. **(b)** Case 3, scenario 1. Note that $u_1 = u_3$ and $v_1 = v_3$. **(c)** Case 3, scenario 2. Note that $u_1 = u_3$, $v_1 = u_2$ and $v_3 = v_2$.

Case 1: *There exists an edge uv in T such that the cut (V_1, V_2) induced by uv in T is such that for all $i \in \{1, 2, 3\}$, $|V_1 \cap \{u_i, v_i\}| = 1$ and $|V_2 \cap \{u_i, v_i\}| = 1$. (For an illustration, see Fig. 5).*

Without loss of generality, let u_i and v_i be the foots of $x_T w_i$ in V_1 and V_2 respectively for each $i \in \{1, 2, 3\}$. Let c be the color of uv. Color $u_1 w_1$ with c_3, $v_1 w_1$ with c_2, $u_2 w_2$ with c, $v_2 w_2$ with c_1, $u_3 w_3$ with c, and $v_3 w_3$ with c, as shown in Fig. 5.

Case 2: *There exist distinct edges e_1, e_2, e_3 such that $e_i \in E(P_i)$ for each $i \in \{1, 2, 3\}$. (For an illustration, see Fig. 6 (a)).*

Color both the representatives of $x_T w_i$ with the color of e_i for each $i \in \{1, 2, 3\}$.

Case 3: *Case 1 and 2 do not apply.*

Because Case 1 and 2 do not apply, there exist $i, j \in \{1, 2, 3\}$ such that $E(P_i) \cap E(P_j) = \emptyset$, because otherwise $E(P_1) \cap E(P_2) \cap E(P_3) \neq \emptyset$ using the Helly property of trees[4] and then any edge in this intersection qualifies as uv of Case 1. So, without loss of generality assume that $E(P_1) \cap E(P_2) = \emptyset$. Also, note that $E(P_3) \subseteq E(P_1) \cup E(P_2)$ because otherwise Case 2 applies. So, without loss of generality assume that $E(P_3) \cap E(P_1) \neq \emptyset$. But then $E(P_3) \cap E(P_1) = E(P_1)$ and P_1 consists of a single edge so that Case 2 does not apply. Let this edge be e_1. Note that $e_1 = u_1 v_1$. Furthermore, at least one of the end-vertices of P_1 and P_3 coincide so that Case 2 does not apply. Thus, assume without loss of generality that $u_1 = u_3$. Let e_2 be any edge in P_2. Without loss of generality assume that

[4] We use the following Helly property of trees: if $T_1, T_2, \ldots, T_k$ are subtrees of a tree T that pairwise intersect each other on at least one edge, then there is an edge of T that is common to all of $T_1, T_2, \ldots, T_k$.

v_1 is the closer vertex among u_1 and v_1 to path P_2 in T. The two possible scenarios in this case are shown in Fig. 6 (b) and (c). Color w_1u_1 and w_1v_1 with $c(e_1)$, w_2u_2 and w_2v_2 with $c(e_2)$, w_3u_3 with $c(e_2)$ and w_3v_3 with $c(e_1)$.

The following lemma follows from the way in which we have colored the edges incident on x_T in Coloring Rule 4. The lemma is easy to verify (with the help of Figs. 5 and 6, and using Observation 1) and hence we state it without proof.

Lemma 6. *For each tree-vertex x_T on which Coloring Rule 4 has been applied as above, for distinct $i, j \in \{1, 2, 3\}$, there is a rainbow path from w_i to w_j in G that uses only the colors from $(\{c_1, c_2, c_3\} \setminus \{c_i, c_j\}) \cup c(T)$. Also, for any $i \in \{1, 2, 3\}$, and any $z \in V(T)$, there is a rainbow path from z to w_i, that uses only the colors from $(\{c_1, c_2, c_3\} \setminus \{c_i\}) \cup c(T)$.*

Path Rule 4. *For each x_T on which Coloring Rule 4 has been applied as above and for each P_{ab} such that Q_{ab} contains x_T (we say that the rule is being applied on the pair (x_T, P_{ab})), do the following.*

Case 1: *x_T has two 2-edges incident in Q_{ab}.*

Let w_i and w_j be the neighbors of x_T in Q_{ab}. Add to P_{ab} the rainbow path from w_i to w_j as given by Lemma 6.

Case 2: *x_T has exactly one 2-edge and exactly one 1-edge incident in Q_{ab}.*

Let x_Tw_i be the 2-edge and let z be the endpoint in T of the 1-edge. Add to P_{ab} the rainbow path from w_i to z as given by Lemma 6.

Case 3: *x_T is an endpoint of Q_{ab} and has one 2-edge incident in Q_{ab}.*

Let w_i be the neighbor of x_T in Q_{ab}. We know one of a or b is in T. From this vertex (a or b, whichever is in T) to w_i, there is a rainbow path as given by Lemma 6. Add this path to P_{ab}.

The following lemma follows from Lemma 6 and Path Rule 4. The proof is similar to that of Lemma 4 and is omitted.

Lemma 7. *Suppose for some $a, b \in V(G) \setminus L_S$ and for some tree $T' \in \mathcal{T}$, P_{ab} contains an edge e that was colored with $s(T')$ during the application of Coloring Rule 4 on some tree-vertex x_T. Then, $T' \neq T$ and Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T'})$.*

Lemma 8. *Invariant 1 is not violated during Path Rule 4.*

Proof. Suppose for the sake of contradiction that Invariant 1 is violated during the application of Path Rule 4 on the pair (x_T, P_{ab}) as above. Then there exist edges e and e' in P_{ab} having the same color. We can assume without loss of generality that e was colored during the application of Coloring Rule 4 on x_T. This means $e \in E' := E(T) \cup R$, where R is defined as the set of representatives of w_1x_T, w_2x_T, and w_3x_T. Since the application of Path Rule 4 on (x_T, P_{ab}) added a rainbow path to P_{ab}, the edge e' was not added during this application and hence $e' \notin E'$. Each color in $c(T)$ have been used only in E' so far. That means $c(e) = c(e') \notin c(T)$. Hence $e \in E' \setminus E(T) = R$. Without loss of generality assume that e is a representative of w_1x_T. Now, $c(e) = s(T_j)$ where $j \in \{2, 3\}$. Without loss of generality assume that $c(e) = s(T_2)$. This also means $c(e') = s(T_2)$. That means e' was colored during Coloring Rules 2, 3 or 4. Hence the following two cases are exhaustive and in each case we prove a contradiction.

Case 1: e' was colored during the application of Coloring Rules 3 or 4 on $x_{T'}$, for some tree $T' \neq T$.

Since P_{ab} contains e', we have that Q_{ab} contains $h(e')$. Since e' was colored during the application of Coloring Rules 3 or 4 on $x_{T'}$, either $e' \in E(T')$ or $h(e')$ is incident on $x_{T'}$, and hence $x_{T'}$ is in Q_{ab}. Since Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T_2})$ by Lemmas 4 and 7, we have that $x_{T'}$ is not in $\mathtt{ST}(x_T, x_{T_2})$. Then $\mathsf{dist}_{\tilde{B}_1}(x_{T'}, x_T) < \mathsf{dist}_{\tilde{B}_1}(x_{T'}, x_{T_2})$. But then during the application of Coloring Rule 3 or 4 on $x_{T'}$, the color $s(T_2)$ would never be used as $x_{T_2} \neq \mathtt{CT}(x_{T'}, v)$ for any vertex v. Thus, the color of e' is not $s(T_2)$. But we know that $c(e') = c(e) = s(T_2)$, a contradiction.

Case 2: e' was colored during the application of Coloring Rule 2.

This means e' is the representative of a 1-edge. Since e' is colored with $s(T_2)$, we have that $h(e')$ should either be the outgoing edge of x_{T_2} or the outgoing edge of the parent of x_{T_2}, from Coloring Rule 2. This implies that $h(e')$ is in $\mathtt{ST}(x_T, x_{T_2})$, as the parent of x_{T_2} is a non-tree edge. But then Q_{ab} does not contain $h(e')$ as Q_{ab} does not intersect $\mathtt{ST}(x_T, x_{T_2})$, by Lemmas 4 and 7. Thus P_{ab} does not contain e', which is a contradiction. □

Coloring Rule 5. *For each non-tree-vertex u with degree at least 3 in B_1 (see Fig. 7), let q be the number of children of u (note that $q \geq 2$ as degree of u is at least 3), let $u_1, u_2, \ldots, u_q$ be the children of u and let x_{T_i} be $\mathtt{CT}(u, u_i)$. Let $\overrightarrow{uv}$ be the outgoing edge from u in B_1. If uv is a 1-edge, due to Coloring Rule 2, we know that there exists an $i \in [q]$ such that u_i is a tree-vertex (and hence $T_i = f_{\mathcal{T}}(u_i)$), and uv is colored with $s(T_i)$. Hence, if uv is a 1-edge, assume without loss of generality that u_1 is a tree-vertex (and hence $x_{T_1} = u_1$) and that uv is colored with $s(T_1)$.*

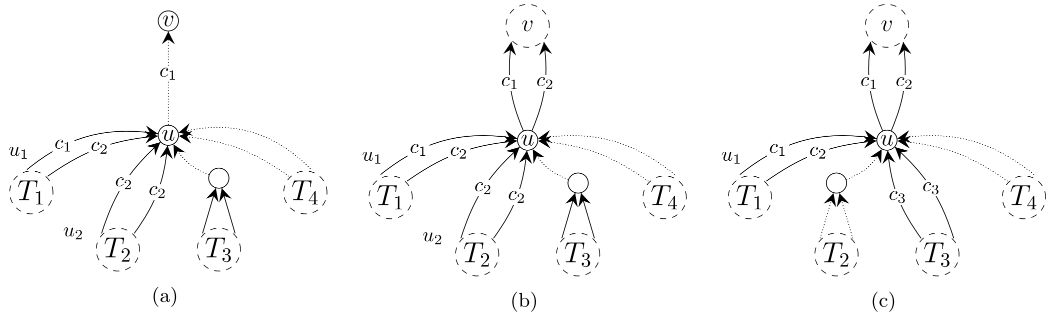

Fig. 7. Three examples of Coloring Rule 5. Here $c_i = s(T_i)$. The edges that were colored before the application of the rule are drawn as densely dotted lines.

- *If u_1u is uncolored (then u_1u is a 2-edge due to Coloring Rule 2, implying that u_1 is a tree-vertex and hence $T_1 = f_{\mathcal{T}}(u_1)$), then color $(u_1u)_1$ with $s(T_1)$ and $(u_1u)_2$ with $s(T_2)$.*
- *For each $2 \le i \le q$, if u_iu is uncolored (then u_iu is a 2-edge due to Coloring Rule 2, implying that u_i is a tree-vertex and hence $T_i = f_{\mathcal{T}}(u_i)$), then color both its representatives with $s(T_i)$.*
- *If uv is uncolored (in which case it is a 2-edge due to Coloring Rule 2) then color $(uv)_1$ with $s(T_1)$ and $(uv)_2$ with $s(T_2)$.*

Path Rule 5. *For each non-tree-vertex u on which Coloring Rule 5 has been applied as above and for each P_{ab} such that Q_{ab} contains u (we say that the rule is being applied on the pair (u, P_{ab})), execute the following two parts (in the mentioned order).*

Part 1

- *If Q_{ab} contains edge u_1u and u_1u is colored during the application of Coloring Rule 5 on u, do the following. If the other neighbor (if any) of u in Q_{ab} is u_2, then add $(u_1u)_1$ (which has color $s(T_1)$) to P_{ab}. Otherwise, add $(u_1u)_2$ (which has color $s(T_2)$) to P_{ab}.*
- *For each $i \in [2, q]$, if Q_{ab} contains edge u_iu and u_iu is colored during the application of Coloring Rule 5 on u, add $(u_iu)_1$ (which has color $s(T_i)$) to P_{ab}.*
- *If Q_{ab} contains edge uv and uv is colored during the application of Coloring Rule 5 on u: if the other neighbor (if any) of u in Q_{ab} is u_1 and u_1u is a 1-edge, then add $(uv)_2$ (which has color $s(T_2)$) to P_{ab}; otherwise add $(uv)_1$ (which has color $s(T_1)$) to P_{ab}.*

Part 2

- *For each tree-vertex x_T such that the degree of x_T in $h(P_{ab})$ became 2 during the addition of above edges in Part 1, let x and y be the endpoints in T of the two edges of P_{ab} incident on T. Add T_{xy} to P_{ab}.*
- *For each tree-vertex $x_T \in \{h(a), h(b)\}$ such that the degree of x_T in $h(P_{ab})$ became 1 during the addition of above edges in Part 1, let x be the endpoint in T of the edge of P_{ab} incident on T. If $x_T = h(a)$, add T_{ax} to P_{ab}; otherwise (i.e., if $x_T = h(b)$), add T_{bx} to P_{ab}.*

Lemma 9. *Invariant 1 is not violated during Path Rule 5.*

Proof. Suppose Invariant 1 is violated during the application of Path Rule 5 on the pair (u, P_{ab}) as above. Then there exist edges e and e' in P_{ab} having the same color. We can assume without loss of generality that e was colored during the application of Coloring Rule 5 on u. Suppose e was added during Part 2 of Path Rule 5. Observe that if we add a path inside a tree T in Part 2, then x_T was *incomplete* before the application of Coloring Rule 5. By Invariant 3, this implies that the internal colors of T were not used anywhere else so far. Thus, the color of e is unique, in particular $c(e') \neq c(e)$, a contradiction. Thus, the edge e was not added during Part 2. Then e was added during Part 1 and hence $c(e) = c(e') = s(T_i)$ for some $i \in [q]$. Then e' was colored during one of Coloring Rules 5, 4, 3, or 2.

Case 1: e' was colored during the Coloring Rule 5.

Note that during the application of Path Rule 5 on (u, P_{ab}), we have added at most two edges to P_{ab}. And, if we have added two edges, they are of different colors. Thus e' was not added to P_{ab} during the application of Path Rule 5 on (u, P_{ab}) and hence was not colored during the application of Coloring Rule 5 on u. So, e' was colored during the application of Coloring Rule 5 on some non-tree-vertex $u' \neq u$. Notice that for any tree $T \in \mathcal{T}$, $s(T)$ is used during the application of Coloring Rule 5 only when the rule is applied to an ancestor of x_T in B_1. Hence, both u and u' are ancestors of x_{T_i}. Without loss of generality, assume that u' is closer than u to x_{T_i}. Then, u cannot have any tree vertices as children because otherwise $x_{T_i} \neq \mathtt{CT}(u, u_i)$. Then, the only edges colored during the application of Coloring Rule 5 on u, are the representatives of uv. Thus $h(e) = uv$.

Case 1.1 $e = (uv)_2$.

We know that $e=(uv)_2$ is colored with $s(T_2)$ by Coloring Rule 5. Thus, $c(e')=c(e)=s(T_2)$ and $T_i=T_2$. Since the edge $(uv)_2$ is added during application of Path Rule 5 on (u, P_{ab}), the neighbors of u in Q_{ab} are v and u_1, by Path Rule 5. Since $u' \in Q_{ab}$, we have that u' is a descendant of u_1 and not u_2 in B_1. This implies $T_i=T_1 \neq T_2$, a contradiction.

Case 1.2 $e=(uv)_1$.

Since the edge $(uv)_1$ is added during application of Path Rule 5 on (u, P_{ab}), either u_1 is not a neighbor of u in Q_{ab}, or uu_1 is a 2-edge, by Path Rule 5. But uu_1 cannot be a 2-edge as both u and u_1 are non-tree vertices. (Recall that we said all children of u are non-tree vertices in Case 1). Hence u_1 is not a neighbor of u in Q_{ab}. Since $u' \in Q_{ab}$, this implies that $T_i \neq T_1$, and hence $c(e)=c(e')=s(T_i) \neq s(T_1)$. But we know that $e=(uv)_1$ is colored with $s(T_1)$, by Coloring Rule 5. Thus, we have a contradiction.

Case 2: e' was colored during the Coloring Rules 4 or 3.

Let T' be the tree on which e' is incident. Then e' was colored with $s(T_i)$ during the application of Coloring Rules 4 or 3 on $x_{T'}$. Then Q_{ab} does not intersect $\mathtt{ST}(T', T_i)$ due to Lemmas 7 and 4. Since $x_{T_i}=\mathtt{CT}(u, u_i)$, there is no other tree-vertex in the path from u to x_{T_i}. Thus, u is in $\mathtt{ST}(T', T_i)$. Hence, we have that u is not in Q_{ab}. We know that e is adjacent on u as every edge colored during the application of Coloring Rule 5 on u is incident on u. But then $e \notin P_{ab}$ as u is not in Q_{ab}. This is a contradiction.

Case 3: e' was colored during the Coloring Rule 2.

This means that e' is a 1-edge.

Case 3.1 $h(e')=uv$.

In the case when uv is a 1-edge, we selected u_1 during Coloring Rule 5 in such a way that $c(uv)=s(T_1)$. Thus $c(e)=c(e'=uv)=s(T_1)$. The only edges that can be potentially colored with $s(T_1)$ during the application of Coloring Rule 5 on u are $(uv)_1$ and $(uu_1)_1$. Since e and e' are distinct we have $e=(u_1u)_1$. But since uv is in Q_{ab}, we would have added $(u_1u)_2$ and not $(u_1u)_1$ to P_{ab} during Path Rule 5. Thus we have a contradiction.

Case 3.2. $h(e') \neq uv$.

Then e' is on the path between x_{T_i} and u. Also, x_{T_i} is not a child of u. Then, the only possibility for e to have color $s(T_i)$ is if $i=2$ and $e=(u_1u)_2$. Then Q_{ab} contains both u_1 and u_2. In that case, we would have added $(u_1u)_1$ and not $(u_1u)_2$ to P_{ab} during Path Rule 5. Hence, $e \neq (u_1u)_2$, a contradiction. □

Coloring Rule 6. *For each incomplete tree-vertex x_T having 2-edge degree exactly 1: let e be the only 2-edge incident on x_T, pick an edge e_1 in the foot-path of e, color the representatives of e with the color of e_1.*

Path Rule 6. *For each tree-vertex x_T on which Coloring Rule 6 has been applied as above and for each P_{ab} such that Q_{ab} contains $h(e)$ (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following.*

We pick vertex w as follows. If $a \in V(T)$, let $w := a$, and if $b \in V(T)$ let $w := b$. (Note that both a and b cannot be in T as Q_{ab} contains $h(e)$). If $a, b \notin V(T)$ then there is an edge $e_2 \neq e$ of Q_{ab} incident on x_T. Furthermore, since e is the only 2-edge incident on x_T, the edge e_2 is a 1-edge. In this case, let w be the endpoint of $(e_2)_1$ in T.

By Observation 1, there is a path in T that excludes e_1, from w to one of the foots of e. Let this foot be z. Add the path in T between w and z to P_{ab}. Also add to P_{ab} the representative of e having z as its endpoint in T.

Lemma 10. *Invariant 1 is not violated during Path Rule 6.*

Proof. Let E_N be the set of new edges added to P_{ab} during the application of Path Rule 6 to (x_T, P_{ab}) and let E_O be the set of already included edges in P_{ab} before this application. Suppose Invariant 1 is violated for the sake of contradiction. Then either there are two edges in E_N with the same color or $c(E_N) \cap c(E_O) \neq \emptyset$. Recall that E_N consists of $E(T_{wz})$ and a representative of e, say $(e)_j$. Note that $c((e)_j)=c(e_1)$ by Coloring Rule 6. So, all the edges in E_N are colored from $c(T)$, the internal colors of T. Recall that e_1 is not in T_{wz} by our choice of z. Thus the edges in E_N all have distinct colors. So, it has to be the case that $c(E_O) \cap c(E_N) \neq \emptyset$. Since $c(E_N) \subseteq c(T)$, this implies that $c(E_O) \cap c(T) \neq \emptyset$. Let d be an edge in E_O with color in $c(T)$. Since the representative of at least one edge of Q_{ab} incident on x_T (namely e) was not added to P_{ab} before the application of Path Rule 6, we have that $E(T) \cap E_O = \emptyset$ by Invariant 4. Thus $d \notin E(T)$ but has color in $c(E(T))$ and $d \in E_O$. By Invariant 3, this implies that x_T was complete before the application of Coloring Rule 6, making the rule not applicable on x_T, a contradiction. □

Coloring Rule 7. *For each tree-vertex x_T such that the 2-edge degree of x_T is exactly 2 and $E(T)$ contains at least 2 edges, let e_1 and e_2 be the 2-edges incident on x_T, let w and z be the other endpoints of e_1 and e_2, respectively, and let P_1 and P_2 be the foot-paths of e_1 and e_2, respectively.*

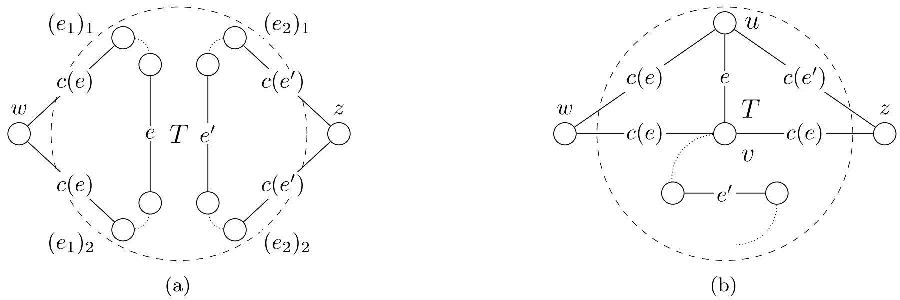

Fig. 8. Coloring Rule 7. **(a)** Case 1 **(b)** Case 2.

Case 1: $|E(P_1 \cup P_2)| \geq 2$. *(See Fig. 8 (a)).*

Pick distinct edges e and e' from P_1 and P_2 respectively. If $(e_1)_1$ and $(e_1)_2$ are uncolored, color them with color of e and if $(e_2)_1$ and $(e_2)_2$ are uncolored, color them with color of e'.

Case 2: *Case 1 does not hold. (See Fig. 8 (b)).*

Clearly, P_1 and P_2 both are a single edge and they are the same edge. Let this edge be $e = uv$. Pick any other edge e' in T (such an edge exists because we said that the rule is applicable only if $E(T)$ contains at least two edges). Without loss of generality, assume that e' is closer to v than u in T. If uw and vw are uncolored, color them with color of e. If uz and vz are uncolored, color them with colors of e' and e, respectively.

Lemma 11. *Consider a tree-vertex x_T on which Coloring Rule 7 has been applied as above. There is a rainbow path in G from w to z using only the colors in $c(T)$. Also, from any vertex $x \in V(T)$, there is a rainbow path to both w and z using only the colors in $c(T)$.*

Proof. If Case 2 of Coloring Rule 7 has been applied then this is rather easy to see as follows. The path wuz is a rainbow path from w to z. Also for the second statement, note that at least one of T_{xu} or T_{xv} avoids e. Then at least one of the paths T_{xu} followed by uw or uz, or T_{xv} followed by vw or vz, is a rainbow path (it avoids e and if it contains e', then it is the second case and $c(e')$ is used only once).

So, it only remains to prove the lemma when Case 1 of Coloring Rule 7 is applied. Let w_1, w_2 be the foots of e_1 and z_1, z_2 be the foots of e_2. To prove the first statement, it is sufficient to prove that at least one of the four paths $T_{w_1z_1}, T_{w_1z_2}, T_{w_2z_1}$ and $T_{w_2z_2}$ contains neither e nor e'. Given this, it is easy to show the necessary rainbow path from w to z: if the path $T_{w_iz_j}$ contains neither e nor e' then the path $ww_iT_{w_iz_j}z_jz$ is the required path. So for the sake of contradiction assume that each $T_{w_iz_j}$ contains either e or e'. Without loss of generality assume that $T_{w_1z_1}$ contains e. Let y be the last vertex on $T_{w_1z_1}$ that is in P_1 (while going from w_1 to z_1). Now, T_{yz_1} and T_{yw_2} do not contain e and hence $T_{z_1w_2} = T_{yz_1} \cup T_{yw_2}$ does not contain e. This implies that $T_{z_1w_2}$ contains e'. Let y' be the last vertex on $T_{z_1w_2}$ that is in P_2 (while going from z_1 to w_2). Now, $T_{y'z_2}$ and $T_{w_2y'}$ contains neither e nor e' and hence $T_{w_2z_2} = T_{y'z_2} \cup T_{w_2y'}$ contains neither e nor e'.

To prove the second statement, observe that there is a rainbow path from x to either w_1 or w_2 not containing e, and a rainbow path from x to either z_1 or z_2 not containing e', due to Observation 1. □

Path Rule 7. *For each tree-vertex x_T on which Coloring Rule 7 has been applied as above and for each P_{ab} such that Q_{ab} contains at least one of e_1 and e_2 (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following.*

If Q_{ab} contains both e_1 and e_2 then let $y_1 := w$ and $y_2 := z$. If Q_{ab} contains only e_1 and not e_2 then let $y_1 := w$. If Q_{ab} contains only e_2 and not e_1 then let $y_1 := z$. If $a \in V(T)$, let $y_2 := a$, and if $b \in V(T)$, let $y_2 := b$. (Note that both a and b cannot be in T as Q_{ab} contains e_1 or e_2). If $a, b \notin V(T)$ and only one of e_1, e_2 is in Q_{ab}, then there is an edge $e'' \notin \{e_1, e_2\}$ incident on x_T in Q_{ab}; and since e_1 and e_2 are the only 2-edges incident on x_T, the edge e'' is a 1-edge; let y_2 be the endpoint of $(e'')_1$ in T. Add to P_{ab} the path between y_1 and y_2 given by Lemma 11.

Lemma 12. *Invariant 1 is not violated during Path Rule 7.*

Proof. Let E_N be the set of new edges added to P_{ab} during the application of Path Rule 7 to (x_T, P_{ab}) and let E_O be the set of already included edges in P_{ab} before this application. Suppose Invariant 1 is violated for the sake of contradiction. Then either there are two edges in E_N with the same color or $c(E_N) \cap c(E_O) \neq \emptyset$. By Lemma 11, all edges in E_N are colored from $c(T)$ and have distinct colors. So, it has to be the case that $c(E_O) \cap c(E_N) \neq \emptyset$. Since $c(E_N) \subseteq c(T)$, this implies that $c(E_O) \cap c(T) \neq \emptyset$. Let d be an edge in E_O with color in $c(T)$. The representative of at least one edge of Q_{ab} incident on x_T was not in P_{ab} before the application of Path Rule 7 on (x_T, P_{ab}), because otherwise the path rule is not applicable on (x_T, P_{ab}). Then, by Invariant 4, we have that $E(T) \cap E_O = \emptyset$. Thus $d \notin E(T)$ but has color in $c(E(T))$ and $d \in E_O$. But then by Invariant 3, we have that x_T was completed before the application of Coloring Rule 7, thereby making the rule not applicable on x_T, which is a contradiction. □

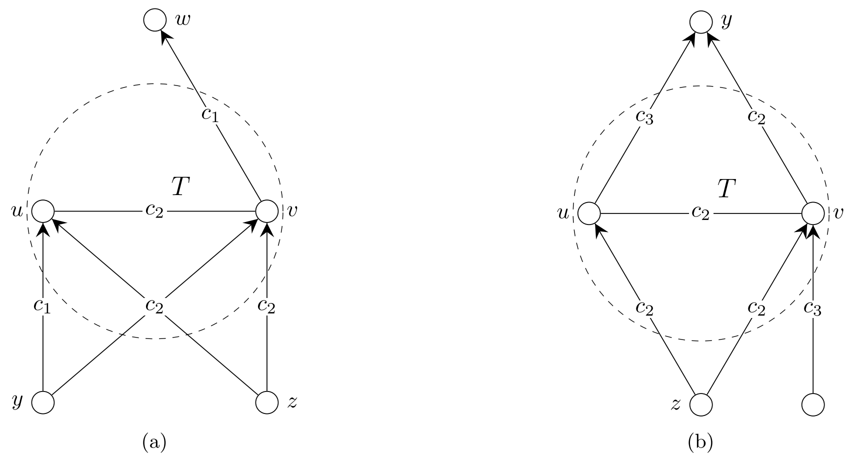

Fig. 9. Coloring Rule 8. **(a)** Case 1 where $c_1 = s(T)$ and $c_2 = c(uv)$. **(b)** Case 2 where $c_2 = c(uv)$ and c_3 is the color of the representative of an arbitrarily chosen 1-edge incident on x_T.

Coloring Rule 8. *For each incomplete tree-vertex x_T having degree at least 3 in B_1: We can assume that Coloring Rules 3, 4, 6, 7 are not applicable on x_T as otherwise x_T would have been completed. If x_T has at least three 2-edges incident on it, then Coloring Rule 3 or 4 would have been applicable on x_T. If it has 2-edge degree 1, then Coloring Rule 6 would have been applicable on x_T. If it has 2-edge degree 0, then it would have been completed after Coloring Rule 2. Hence, we can assume that x_T has 2-edge degree exactly 2. Now, if $|E(T)| \geq 2$, Coloring Rule 7 becomes applicable on x_T. Hence, we can assume that the tree T is an edge. Let uv be this edge. Let the two 2-edges incident on x_T be yx_T and zx_T.*

Case 1: *Both yx_T and zx_T are incoming to x_T (see Fig. 9 (a)).*

Then the outgoing edge of x_T in B_1 (if any) is a 1-edge, say $\overrightarrow{x_T w}$. Assume without loss of generality that its representative is vw. Let $c_1 = s(T)$ and c_2 be the color of uv. Note that vw is colored with c_1 due to Coloring Rule 2. If yu and yv are uncolored, color them with c_1 and c_2 respectively. If zu and zv are uncolored, color both of them with c_2.

Case 2: *One of the 2-edges, say yx_T, is outgoing from x_T (see Fig. 9 (b)).*

Let c_2 be the color of uv and c_3 be the color of representative of any 1-edge incoming on x_T. Note that at least one such 1-edge exists as the degree of x_T is at least 3. If yu and yv are uncolored, color them with c_3 and c_2 respectively. If zu and zv are uncolored, color both of them with c_2.

Path Rule 8. *For each tree-vertex x_T on which Coloring Rule 8 has been applied as above and for each P_{ab} such that Q_{ab} contains at least one of yx_T and zx_T (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following:*

- *If Q_{ab} contains both yx_T and zx_T then add yu and uz to P_{ab}.*
- *If Q_{ab} contains only yx_T and not zx_T then let $y_1 := y$. If Q_{ab} contains only zx_T and not yx_T then let $y_1 := z$. If $a \in V(T)$, let $y_2 := a$, and if $b \in V(T)$ let $y_2 := b$. (Note that both a and b cannot be in T as Q_{ab} contains yx_T or zx_T). If $a, b \notin V(T)$ and only one of yx_T, zx_T is in Q_{ab} then there is an edge $e'' \notin \{yx_T, zx_T\}$ incident on x_T in Q_{ab}. Further, since yx_T and zx_T are the only 2-edges incident on x_T, the edge e'' is a 1-edge. Note that $(e'')_1$ is already added to P_{ab} in Path Rule 2. Let y_2 be the endpoint of $(e'')_1$ in T.*
  Note that in all cases $y_2 \in \{u, v\}$ and $y_1 \in \{y, z\}$. Hence, the edge y_1y_2 exists. Add the edge y_1y_2 to P_{ab}.

Lemma 13. *Invariant 1 is not violated during Path Rule 8.*

Proof. Suppose the invariant is violated. Then there exist edges e and e' in P_{ab} having the same color. We can assume without loss of generality that e was colored during the application of Coloring Rule 8 on (x_T, P_{ab}). We added at most two edges during the application of Path Rule 8 on x_T and if we added two edges we have made sure they have distinct colors. Thus e' was not added during the application of Path Rule 8 on (x_T, P_{ab}). The colors that are possible for e are c_1, c_2 and c_3 according to Coloring Rule 8.

Case 1: $c(e) = c(e') = c_2$.

Recall $c_2 = c(uv)$. If $e' = uv$, then by Invariant 4, the representatives of all the edges of Q_{ab} incident on x_T are in P_{ab} even before the application of Path Rule 8 on (x_T, P_{ab}). Then Path Rule 8 is not applicable on (x_T, P_{ab}). Thus $e' \notin E(T)$ but $c(e') \in c(E(T))$. Then by Invariant 3, x_T was completed before the application of Coloring Rule 8, making the rule not applicable on x_T. Thus, such an e' does not exist.

Case 2: $c(e) = c(e') = c_1 = s(T)$.

This means $e = yu$ and that Case 1 of Coloring Rule 8 (see Fig. 9 (a)) was applied on x_T. The only coloring rules so far that use surplus colors are Coloring Rules 8, 5, 4, 3, and 2.

Case 2.1 e' was colored during Coloring Rule 2.

Note that this means e' is a 1-edge and the only way e' can have color $s(T)$ is if $e' = vw$. But, in Path Rule 8, we add $yu = e$ to P_{ab} only when vw is not in Q_{ab}. Thus we have a contradiction.

Case 2.2 e' was colored during Coloring Rules 4 or 3.

Let T' be the tree on which e' is incident. Since e' was colored with $s(T)$ during Coloring Rules 4 or 3, we know that Q_{ab} does not intersect $\mathrm{ST}(x_{T'}, x_T)$ due to Lemmas 7 and 4. Then Q_{ab} does not contain x_T and hence P_{ab} does not contain e, which is a contradiction.

Case 2.3 e' was colored during Coloring Rule 5.

Then $h(e')$ is a 2-edge in the path from x_T to root of B_1. Since e' is in Q_{ab}, this means that $x_T w$ is in Q_{ab}. But then by Path Rule 8, we would have added yv instead of $yu = e$ to P_{ab}, a contradiction.

Case 2.4 e' was colored during Coloring Rule 8.

The only application of Coloring Rule 8 that uses $s(T)$ is the application on x_T. But since e' was not colored during this application, we have a contradiction.

Case 3: $c(e) = c(e') = c_3$.

This means $e = uy$ and that Case 2 of Coloring Rule 8 was applied on x_T. Let x be the neighbor of x_T such that xx_T is the 1-edge incident on x_T whose representative is colored with c_3. By Coloring Rule 2, there exists a tree T' that is a descendant of x such that $s(T') = c_3$. The only coloring rules so far that use surplus colors are Coloring Rules 8, 5, 4, 3, and 2.

Case 3.1 e' was colored during Coloring Rule 2.

This means e' is a 1-edge. Since xx_T is the only 1-edge with color $s(T')$ by Lemma 2, we have that $e' = xx_T$. Hence, xx_T is in Q_{ab}. But if xx_T is in Q_{ab}, we would have added vy and not uy in Path Rule 8. This is a contradiction to $e = uy$.

Case 3.2 e' was colored during Coloring Rules 4 or 3.

Let T'' be such that e' is adjacent on $x_{T''}$. Since e' was colored with $s(T')$ during Coloring Rules 4 or 3, we have that Q_{ab} does not intersect $\mathrm{ST}(x_{T''}, x_{T'})$ due to Lemmas 7 and 4. Since P_{ab} contains e that is incident on x_T, we have that Q_{ab} contains x_T. This implies that $x_T \notin \mathrm{ST}(x_{T''}, x_{T'})$ implying that $x_{T''}$ is on the path between x_T and $x_{T'}$. But then $s(T'')$ and not $s(T')$ would have been used to color xx_T, a contradiction.

Case 3.3 e' was colored during Coloring Rule 5 or 8.

Since e' is colored with $s(T')$, by Coloring Rule 5 and 8 this implies e' is in $\mathrm{ST}(x_T, x_{T'}) = \mathrm{ST}(x_T, x)$, implying that Q_{ab} contains x. But then we would have added vy and not $uy = e$ to P_{ab} according to Path Rule 8, a contradiction. □

The following Lemma follows from the previous coloring rules.

Lemma 14. *Consider an edge e in B_1 that remains uncolored after the application of Coloring Rules 1 through 8. Let x_T and v be the endpoints of e (Note that due to Coloring Rule 2, e is a 2-edge, and hence one of its endpoints is a tree-vertex). Then, both x_T and v have degree exactly 2 in B_1, both edges incident on x_T are 2-edges, and T consists of a single edge.*

Proof. Suppose $u \in \{v, x_T\}$ has degree not equal to 2 in B_1. First, suppose the degree was greater than 2. Then Coloring Rule 8 or 5 would have been applicable on u, and hence u would have been completed. Therefore, u has degree 1 in B_1. By Corollary 3, every leaf of B_1 is a tree-vertex. Hence, u is a tree-vertex and $u = x_T$. But then Coloring Rule 6 would have been applicable on x_T, and x_T would have been completed. Thus, e is already colored, which is a contradiction. Hence, x_T and v have degree 2 in B_1.

Now, suppose x_T has only one 2-edge incident in B_1. Then, Coloring Rule 6 would have been applied on x_T and x_T would have been completed. Thus, both edges incident on x_T in B_1 are 2-edges. If T contained at least two edges, Coloring Rule 7 would have been applied on x_T and x_T would have been completed. Hence, T contains only one edge. □

Coloring Rule 9. *For each tree-vertex x_T with exactly one uncolored 2-edge e incident on it: note that it follows by Lemma 14 that the tree T comprises of a single edge e'. Let $e = vx_T$ and $e' := u_1u_2$. Color $(e)_1 = vu_1$ and $(e)_2 = vu_2$ with the color of e'.*

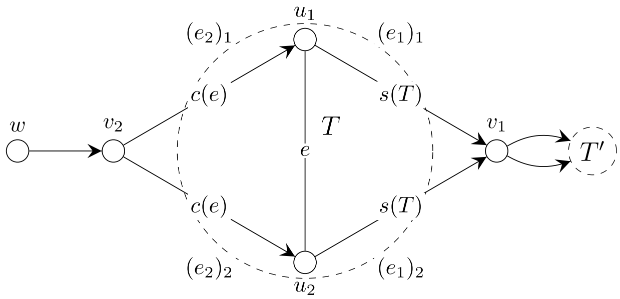

Fig. 10. Coloring Rule 10.

Path Rule 9. *For each tree-vertex x_T on which Coloring Rule 9 has been applied as above and for each P_{ab} such that Q_{ab} contains e (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following:*

First we pick vertex $w \in V(T) = \{u_1, u_2\}$ as follows: if $a \in V(T)$, let $w := a$; if $b \in V(T)$ let $w := b$; (note that both a and b cannot be in T as Q_{ab} contains e); if $a, b \notin V(T)$ then there is an edge $e_2 \neq e$ of Q_{ab} incident on x_T; furthermore, since e is the only uncolored edge incident on x_T, a representative of the edge e_2 is already in P_{ab}; let w be the endpoint in T of this representative of e_2.
Add vw to P_{ab}.

Lemma 15. *Invariant 1 is not violated during Path Rule 9.*

Proof. The edge added to P_{ab} during the application of Path Rule 9 to (x_T, P_{ab}) has color $c(e') \in c(T)$. If the invariant is violated, then there was an edge e'' in P_{ab} already with color $c(e')$. By Invariant 4 e' was not already in P_{ab} as the edge e incident on x_T is in Q_{ab} and the representative of e was not added to P_{ab} before. Thus $e'' \neq e'$ but $c(e'') = c(e')$. Since x_T was incomplete before the application of current coloring rule, by Invariant 3, none of the colors in $c(T)$ were used before anywhere outside of T. So, such an e'' does not exist, a contradiction. □

Lemma 16. *Consider a 2-edge e incident on tree-vertex x_T that remains uncolored after the application of Rules 1 to 9. Then, x_T has degree exactly 2 in B_1, T contains only one edge, and the other edge incident on x_T is an uncolored 2-edge.*

Proof. By Lemma 14 it follows that x_T has degree exactly 2 in B_1, T contains only one edge, and the other edge incident on x_T is a 2-edge. If this other 2-edge is colored, then Coloring Rule 9 would have been applied on x_T and x_T would have been completed. □

Coloring Rule 10. *For each incomplete tree-vertex x_T whose parent's outgoing edge is a 2-edge: (See Fig. 10 for an Illustration). Let v_1 be the parent of x_T. From Lemma 16, it follows that x_T has degree exactly 2 in B_1, has one incoming and one outgoing 2-edge incident on it, both the 2-edges are uncolored, and the tree T is just a single edge. Let e_1 and e_2 respectively be the outgoing and incoming 2-edges of x_T. Let e be the only edge in T. Let v_2 be the other endpoint of e_2. Let u_1 be the endpoint of $(e_1)_1$ and $(e_2)_1$ in T. Let u_2 be the endpoint of $(e_1)_2$ and $(e_2)_2$ in T. From Lemma 14, we know that v_1 and v_2 have degree exactly 2. Let $\overrightarrow{v_1 x_{T'}}$ be the outgoing 2-edge from v_1 and let $\overrightarrow{w v_2}$ be the incoming edge on v_2 in B_1.*
Color $(e_2)_1$ and $(e_2)_2$ with the color of e, and color $(e_1)_1$ and $(e_1)_2$ with $s(T)$.

Path Rule 10. *For each tree-vertex x_T on which Coloring Rule 10 has been applied as above and for each P_{ab} such that Q_{ab} contains e_1 or e_2 (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following.*

- *If Q_{ab} contains both e_1 and e_2, add $v_1 u_1$ and $u_1 v_2$ to P_{ab}.*
- *If Q_{ab} contains exactly one edge among e_1 and e_2, then either a or b is in $V(T)$. Also both of them cannot be in $V(T)$. Let z be the one among a or b that is in $V(T)$. If Q_{ab} contains e_1, add $v_1 z$ to P_{ab}; otherwise, i.e., if Q_{ab} contains e_2, add $v_2 z$ to P_{ab}.*

Lemma 17. *Invariant 1 is not violated during Path Rule 10.*

Proof. Suppose for the sake of contradiction that the invariant is violated. Then there exist distinct edges d_1 and d_2 in P_{ab} having the same color. We can assume without loss of generality that d_1 was colored during the application of Coloring Rule 10 on x_T. We added at most two edges during the application of Path Rule 10 on x_T, and in the cases where we added two edges, the two edges have distinct colors. Thus, d_2 was not added during the application of Path Rule 10 on x_T and hence was not colored during the application of Coloring Rule 10 on x_T.

The colors that are possible for d_1 are $s(T)$ and $c(e)$.

Case 1: $c(d_1) = c(d_2) = c(e)$.

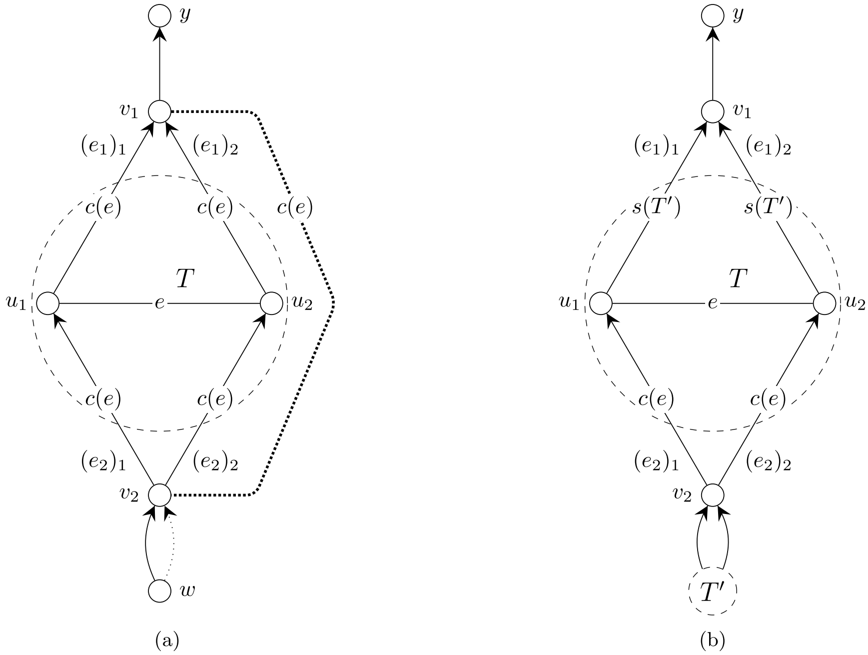

Fig. 11. Coloring Rule 11. **(a)** Case 1; here the edge v_1v_2 is drawn as a thick dotted line to highlight that it is not in B_1, and the edge wv_2 is drawn with one solid line and one dotted line to denote that it could be a 1-edge or a 2-edge **(b)** Case 2; here $w = x_{T'}$.

This is not possible since the color of e has not been used to color any other edges so far by Invariant 3, and e is not in P_{ab} by Invariant 4.

Case 2: $c(d_1) = c(d_2) = s(T)$.

This means $h(d_1) = v_1x_T$. The only coloring rules so far that use the surplus colors of trees are Coloring Rules 2, 3, 4, 5, 8, and 10. Hence, d_2 was colored with $s(T)$ during one of them.

Case 2.1 d_2 was colored during Coloring Rule 2.

This means that d_2 is a 1-edge. According to Coloring Rule 2, the only 1-edge that can be colored with $s(T)$ is either the outgoing edge of x_T or the outgoing edge of the parent of x_T. However, both of them are 2-edges and hence we have a contradiction.

Case 2.2 d_2 was colored during Coloring Rules 4 or 3.

Let T'' be such that d_2 is adjacent on $x_{T''}$. Then, by Lemmas 7 and 4, we know that Q_{ab} does not intersect $\mathtt{ST}(x_{T''}, x_T)$, in particular Q_{ab} does not contain x_T. Since $h(d_1) = v_1x_T$, this implies P_{ab} does not contain d_1, which is a contradiction.

Case 2.3 d_2 was colored during application of Coloring Rule 5.

From Coloring Rule 5, this implies that d_2 was colored during application of Coloring Rule 5 on some ancestor v' of x_T such that there are no other tree vertices in the path from x_T to v'. Then, the only possibility for v' is v_1 as the parent of v_1 is a tree-vertex. However, we know that v_1 has degree 2 in B_1, and hence Coloring Rule 5 could not have been applied on v_1. Thus, we have a contradiction.

Case 2.4 d_2 was colored during application of Coloring Rule 8.

Since d_2 is colored with $s(T)$ during Coloring Rule 8, Case 1 of the rule (see Coloring Rule 8) was applied on x_T and hence the outgoing edge from x_T is a 1-edge. However, this is a 2-edge and hence we have a contradiction.

Case 2.5 d_2 was colored during Coloring Rule 10.

Since d_2 was not colored during the application of Coloring Rule 10 on x_T, we have that d_2 was colored during the application of Coloring Rule 10 on some $x_{T''} \neq x_T$. But then d_2 is not colored with $s(T)$, a contradiction. □

Coloring Rule 11. *For each incomplete tree-vertex x_T: from Lemma 16, it follows that x_T has degree exactly 2 in B_1, has one incoming and one outgoing 2-edge incident on it, both the 2-edges are uncolored, and the tree T is just a single edge. Let e_1 and e_2 respectively be the outgoing and incoming 2-edges of x_T. Let v_1 be the other endpoint of e_1 and v_2 be the other endpoint of e_2. Let $e = u_1u_2$ be the* *only edge in T. Without loss of generality, u_1 be the endpoint of $(e_1)_1$ and $(e_2)_1$ in T, and u_2 be the endpoint of $(e_1)_2$ and $(e_2)_2$ in T. From Lemma 14, we know that v_1 and v_2 have degree exactly 2. Let $\overrightarrow{v_1 y}$ be the outgoing edge from v_1 and $\overrightarrow{w v_2}$ be the incoming edge on v_2 in B_1. We have that $v_1 y$ is a 1-edge because otherwise Coloring Rule 10 would have been applicable on x_T, and x_T would have been already completed.*

Case 1: *There is an edge between v_1 and v_2 in G. (See Fig. 11 (a) for an illustration).*

Color the representatives of e_1 and e_2 with $c(e)$. Color $v_1 v_2$ with $c(e)$. We say that $v_1 v_2$ is a ***shortcut edge****. Note that shortcut edges are the only colored edges in G that are not representatives of edges in B.*

Case 2: *Case 1 does not apply. (See Fig. 11 (b) for an illustration).*

We will prove in Lemma 24 that wv_2 is a 2-edge in this case. Let $T' = f_{\mathcal{T}}(w)$. Color $(e_1)_1$ and $(e_1)_2$ with $s(T')$ and color $(e_2)_1$ and $(e_2)_2$ with color of e.

Path Rule 11. *For each tree-vertex x_T on which Coloring Rule 11 has been applied as above and for each P_{ab} such that Q_{ab} contains e_1 or e_2 (we say that the path rule is being applied on the pair (x_T, P_{ab})), do the following.*

- *If Q_{ab} contains both e_1 and e_2: if $v_1 v_2 \in E(G)$, add $v_1 v_2$ to P_{ab}; otherwise add $v_1 u_1$ and $v_2 u_1$ to P_{ab}.*
- *If Q_{ab} contains exactly one edge among e_1 and e_2, then either a or b is in $V(T)$. Also, both of them cannot be in $V(T)$. Let z be the one among a or b that is in $V(T)$. If Q_{ab} contains e_1, add $v_1 z$ to P_{ab}. If Q_{ab} contains e_2, add $v_2 z$ to P_{ab}.*

Lemma 18. *Invariant 1 is not violated during Path Rule 11.*

Proof. Suppose for the sake of contradiction that the invariant is violated. Then there exist distinct edges d_1 and d_2 in P_{ab} having the same color. We can assume without loss of generality that d_1 was colored during the application of Coloring Rule 11 on x_T. We added at most two edges during the application of Path Rule 11 on x_T, and in the cases where we added two edges, the two edges have distinct colors. Thus d_2 was not added during the application of Path Rule 11 on x_T and hence was not colored during Coloring Rule 11 on x_T.

The colors that are possible for d_1 are $c(e)$ and $s(T')$.

Case 1: $c(d_1) = c(d_2) = c(e)$.

This is not possible since the color of e has not been used to color any other edges so far by Invariant 3, and e is not in P_{ab} by Invariant 4.

Case 2: $c(d_1) = c(d_2) = s(T')$.

This means $h(d_1) = e_1$ and that Case 2 of Coloring Rule 11 was applied on x_T. The only coloring rules so far that use the surplus colors of trees are Coloring Rules 2, 3, 4, 5, 8, 10, and 11. Hence, d_2 was colored with $s(T')$ during one of them.

Case 2.1 d_2 was colored during Coloring Rule 2.

This means d_2 is a 1-edge and $h(d_2)$ is either $x_{T'} v_2$ or $v_2 x_T$. But since both $x_{T'} v_2$ and $v_2 x_T$ are 2-edges (since Case 2 of Coloring Rule 11 was applied on x_T), this is not possible.

Case 2.2 d_2 was colored during Coloring Rules 4 or 3.

Let T'' be the tree such that d_2 is adjacent to $x_{T''}$. By Lemmas 7 and 4, we know that Q_{ab} does not intersect $\mathtt{ST}(x_{T''}, x_{T'})$. Then x_T is not in $\mathtt{ST}(x_{T''}, x_{T'})$. This implies $x_{T''}$ is in the path from x_T to $x_{T'}$. But the only vertex in the path from x_T to $x_{T'}$ is v_2, a non-tree-vertex. Thus, we have a contradiction.

Case 2.3 d_2 was colored during application of Coloring Rule 5.

From Coloring Rule 5, this implies that d_2 was colored during application of Coloring Rule 5 on some ancestor v' of $x_{T'}$ such that there are no other tree vertices in the path from $x_{T'}$ to v'. Then, the only possibility for v' is v_2 as the parent of v_2 is a tree-vertex. However, we know that v_2 has degree 2 in B_1, and hence Coloring Rule 5 could not have been applied on v_2. Thus, we have a contradiction.

Case 2.4 d_2 was colored during application of Coloring Rule 8.

Since d_2 is colored with $s(T')$ during Coloring Rule 8, Case 1 of the rule (see Coloring Rule 8) was applied on $x_{T'}$ and hence the outgoing edge from $x_{T'}$ is a 1-edge. However, this is a 2-edge and hence we have a contradiction.

Case 2.5 d_2 was colored during application of Coloring Rule 10.

Since $c(d_2) = s(T')$, from Coloring Rule 10 we get that d_2 was colored during the application of Coloring Rule 10 on $x_{T'}$. In Lemma 25, we will prove that Coloring Rule 10 was not applied on $x_{T'}$. Thus, we have a contradiction.

Fig. 12. An illustration of the proof of Lemma 19. The densely dotted edges denote the edges of H that are not in B. **(a)** The scenario given by the precondition of the lemma, and **(b)** the transformation to new skeleton B' as described in the proof.

Fig. 13. An illustration of the transformation in the proof of Lemma 20. The densely dotted edges denote the edges of H that are not in B. **(a)** The scenario given by the precondition of the lemma, and **(b)** the transformation to the new skeleton B' as described in the proof.

Case 2.6 d_2 was colored during application of Coloring Rule 11.

Since $c(d_2) = s(T')$, from Coloring Rule 11, we get that d_2 was colored during the application of Coloring Rule 11 on x_T. Moreover, $h(d_2) = e_1$. Recall that we have $h(d_1) = e_1$ too. Since we picked only one representative of e_1 into P_{ab} by Path Rule 11, we have that $d_1 = d_2$. This is a contradiction to the fact that d_1 and d_2 are distinct. $\square$

Now we proceed towards proving Lemmas 24 and 25 that were used above. For this we need to prove some auxiliary lemmas first.

Lemma 19. *Let v be a non-tree-vertex and x_{T_1} be a tree-vertex that is a descendant of v in B_1. If vx_{T_1} is a 2-edge in H then x_{T_1} is a child of v in B_1.*

Proof. See Fig. 12 for an illustration of the proof. Suppose x_{T_1} is not a child of v in B_1. Let $\overrightarrow{x_{T_1}z}$ be the outgoing edge of x_{T_1} in B_1. Let B' be the *skeleton* obtained by deleting $\overrightarrow{x_{T_1}z}$ from B and adding $\overrightarrow{x_{T_1}v}$. Going from B to B', the number of 2-edges is non-decreasing, the level of all vertices in the subtree rooted at x_{T_1} decrease by 1 and the level of all other vertices remain the same. Thus B' has a lexicographically higher configuration vector than B. Thus we have a contradiction to the choice of B. $\square$

Lemma 20. *Let x_T be a vertex on which Coloring Rule 11 is being applied. Let v_1 be as defined in Coloring Rule 11. The vertex v_1 has no 2-edge in H to any vertex except x_T.*

Proof. See Fig. 13 for an illustration of the proof. Suppose for the sake of contradiction that v_1 has a 2-edge in H to a tree-vertex $x_{T_1} \in V(H) \setminus \{x_T\}$. The edge v_1y is a 1-edge as otherwise Coloring Rule 10 would have been applicable on x_T and x_T would have been completed already. Thus $x_{T_1} \neq y$. Then the edge $v_1x_{T_1}$ is not in B_1 as y and x_T are the only neighbors of v_1 in B_1. Since x_{T_1} is a tree-vertex, it is not in L_S (recall that L_S is the set of non-tree leaves of H). Thus, since the edge $v_1x_{T_1}$ is not in B_1, it is not in B also (recall $B_1 = B \setminus L_S$). Thus, $v_1x_{T_1} \in E(H) \setminus E(B)$. Also x_{T_1} is not a descendant of v_1 due to Lemma 19. Thus, $x_{T_1} \in \mathrm{ST}(v_1, y)$. Then, by deleting the 1-edge $\overrightarrow{v_1y}$ from B and adding the 2-edge $\overrightarrow{v_1x_{T_1}}$, we get a skeleton B' that has a higher number of 2-edges than B and hence has a lexicographically higher configuration vector. Thus we have a contradiction to the choice of B. $\square$

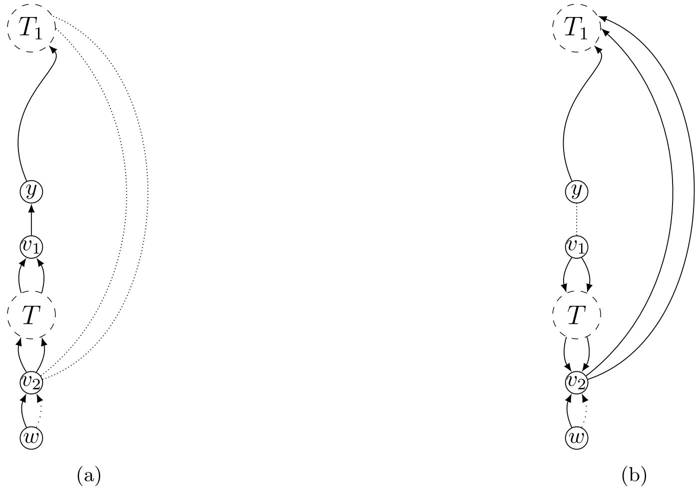

Fig. 14. An illustration of the transformation in the proof of Lemma 21. The densely dotted edges denote the edges of H that are not in B and the edge wv_2 is drawn with 1 solid line and 1 dotted line to denote that it could be a 1-edge or a 2-edge. **(a)** The scenario given by the precondition of the lemma, and **(b)** the transformation to the new skeleton B' as described in the proof.

Fig. 15. An illustration of the transformation in the proof of Lemma 22. The densely dotted edges denote the edges of H that are not in B. **(a)** The scenario before the transformation, and **(b)** the transformation to the new forest $\mathcal{F}'$ as described in the proof where T is removed and a new tree including v, u_2 and the vertices of T_1 are added.

Lemma 21. *Let x_T be a vertex on which Coloring Rule 11 is being applied. Let v_2, w be as defined in Coloring Rule 11. Then, v_2 has no 2-edge in H to any vertex in $V(H) \setminus \{w, x_T\}$.*

Proof. See Fig. 14 for an illustration of the proof. Suppose for the sake of contradiction that v_2 has a 2-edge in H to a tree-vertex $x_{T_1} \in V(H) \setminus \{x_T, w\}$. Then the edge $v_2x_{T_1}$ is not in B_1 as the only neighbors of v_2 in B_1 are x_T and w. Since x_{T_1} is a tree-vertex, it is not in L_S (recall that L_S is the set of non-tree leaves of H). Thus, since the edge $v_2x_{T_1}$ is not in B_1, it is not in B also (recall $B_1 = B \setminus L_S$). Thus, $v_2x_{T_1} \in E(H) \setminus E(B_1)$. Also x_{T_1} is not a descendant of v_2 due to Lemma 19. Clearly, then $x_{T_1} \in \mathtt{ST}(v_2, x_T)$. Since $x_{T_1} \neq x_T$, and the degree of x_T and v_1 in B_1 is 2, we have that the edge v_1y is on the path from v_2 to x_{T_1} in B_1. Then, by deleting the 1-edge v_1y from B and adding the 2-edge $v_2x_{T_1}$, we get a skeleton B' that has a higher number of 2-edges and hence has a lexicographically higher configuration vector. Thus we have a contradiction to the choice of B. □

Lemma 22. *Let v be a non-tree-vertex having degree 2 in B_1 such that the neighbors of v in B_1 are a tree-vertex x_T and a vertex x', and the tree T consists of a single edge u_1u_2. If v does not have a 2-edge to any tree-vertex except x_T in H, then v does not have edges to any tree-vertex in H except x_T.*

Proof. See Fig. 15 for an illustration. Suppose v has an edge in H to a tree-vertex $x_{T_1} \neq x_T$. Note that vx_{T_1} is not a 2-edge by assumption. Thus vx_{T_1} is a 1-edge. We define the forest $\mathcal{F}'$ as $F' := (F \setminus \{u_1\}) \cup \{v\}$ and $\mathcal{F}' := G[F']$. We claim that $\mathcal{F}'$ is a forest with fewer trees than $\mathcal{F}$, which is a contradiction to the choice of $\mathcal{F}$. (Recall that out of all maximum induced forests, we picked $\mathcal{F}$ to be one having the fewest number of trees). Since v has at most one edge to any tree in $G[F \setminus \{u_1\}]$, it is clear that $\mathcal{F}'$ is indeed a forest. The number of trees in $\mathcal{F}'$ is at least one smaller than that of $\mathcal{F}$ because $\{u_2, v\} \cup V(T_1)$ induces a single tree in $\mathcal{F}'$. □

Lemma 23. *Let x_T be a vertex on which Coloring Rule 11 is being applied. Let v_1, v_2, y, w be as defined in Coloring Rule 11.*

1. *v_1 has no edge in H to any tree-vertex except x_T (which also implies that y is a non-tree-vertex), and*

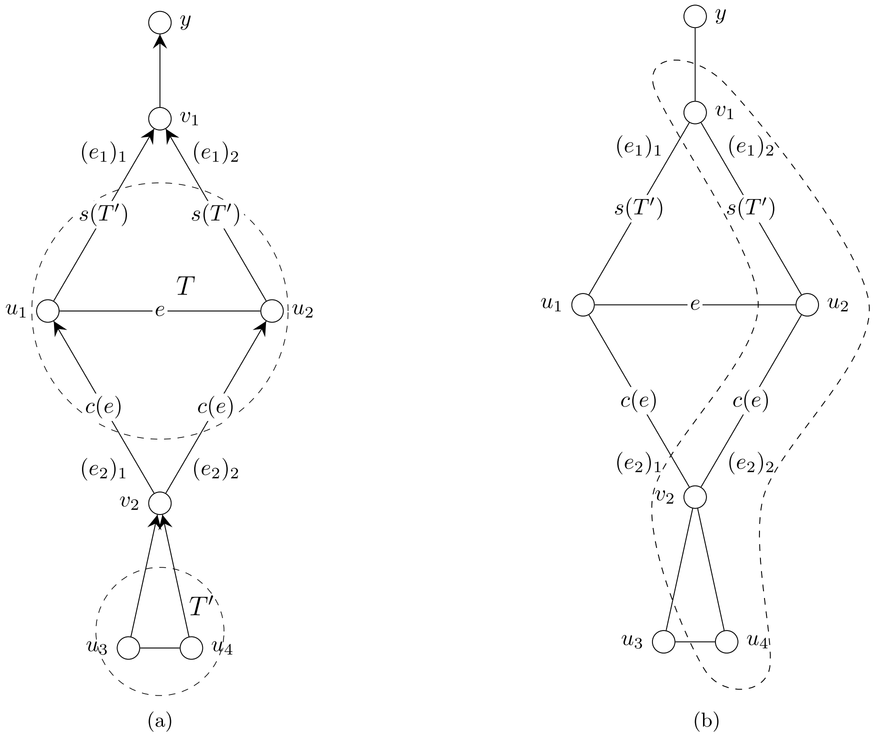

Fig. 16. An illustration of the transformation in the proof of Lemma 25. **(a)** The initial scenario before transformation, and **(b)** the transformation to the new forest $\mathcal{F}'$ as described in the proof where T and T' are removed and a new tree including u_4, u_2, v_1, v_2 is added, thereby reducing the number of trees.

2. *if wv_2 is a 1-edge, vertex v_2 has no edge in H to any tree-vertex except x_T (which also implies that w is a non-tree-vertex in this case).*

Proof. The first statement follows from Lemmas 20, and 22 and the fact that v_1y is a 1-edge, and the second statement follows from Lemmas 21 and 22. □

Lemma 24. *Let x_T be a vertex on which Coloring Rule 11 is being applied and suppose the precondition of Case* 1 *of the rule is* not *satisfied. Let v_2 and w be as defined in Coloring Rule 11. Then, wv_2 is a 2-edge.*

Proof. Suppose for the sake of contradiction that wv_2 is a 1-edge. Let v_1, u_1, u_2 be also as given in Coloring Rule 11 (see Fig. 11). Let $F' = (F \setminus \{u_1\} \cup \{v_1, v_2\})$. Let $\mathcal{F}' = G[F']$. To see that $\mathcal{F}'$ is a forest, observe that by Lemma 23, both v_1 and v_2 are not adjacent in H to any tree-vertex except x_T. Then since $|F'| > |F|$, we have that $\mathcal{F}$ is not a maximum induced forest, a contradiction. □

Lemma 25. *Let x_T be a vertex on which Coloring Rule 11 is being applied and suppose the precondition of Case* 1 *of the rule is not satisfied. Let T' be as defined in Coloring Rule 11 (see Fig. 11b). Then, $x_{T'}$ is not a vertex on which Coloring Rule 10 was applied.*

Proof. Suppose for the sake of contradiction that Coloring Rule 10 was applied on $x_{T'}$. Then $x_{T'}$ has degree 2 in B_1 and T' consists of a single edge. Let this edge be u_3u_4 (see Fig. 16a). Observe that the representatives of the outgoing edge of $x_{T'}$ are u_3v_2 and u_4v_2. Let $F' := (F \setminus \{u_1, u_3\} \cup \{v_1, v_2\})$ and $\mathcal{F}' := G[F']$. Note that $|F'| = |F|$. We claim that $\mathcal{F}'$ is a forest with fewer trees than $\mathcal{F}$, thereby showing a contradiction to the choice of $\mathcal{F}$. (Recall that out of all maximum induced forests, we picked $\mathcal{F}$ to be one having the fewest number of trees). To see that $\mathcal{F}'$ is indeed a forest, observe that v_1 does not have an edge to any tree-vertex in H except x_T (by Lemma 23), v_1 and v_2 are not adjacent in G (since the precondition of Case 1 of Rule 11 is not satisfied), and that v_2 does not have an edge in H to any tree-vertex in B_1 except x_T and $x_{T'}$ (by Lemma 21). It is clear from construction that $\mathcal{F}'$ has at least one tree less than $\mathcal{F}$, which concludes the proof. □

Thus we have proved the Lemmas that we used in Coloring Rule 11.

By the end of Coloring Rule 11, we have colored the representatives of all edges in B_1. We may have also colored some additional edges of G that are not in B_1, namely the shortcut edges (during Coloring Rule 11). We next show that the vertices in B_1 are now rainbow connected through these colored edges.

Lemma 26. *For any pair of vertices $v_1, v_2 \in V(G) \setminus L_S$, P_{ab} is a rainbow path between v_1 and v_2 in G and uses only colors in $[f]$.*

Proof. There are no more incomplete tree-vertices because Coloring Rule 11 is applicable on each incomplete tree-vertex and each tree-vertex on which the rule is applied is completed during the rule. This means there are no uncolored 2-edges in B_1. Also, Coloring Rule 2 colors all 1-edges in B_1. Thus, each edge in B_1, and hence their representatives in G, have been colored.

Whenever an edge in B_1 is colored by a coloring rule and if it is in Q_{ab}, we have added exactly one of its representatives to P_{ab} in the proceeding path rule, except possibly Path Rule 11 where we might have added a shortcut edge instead. In the case when a shortcut edge is added, the shortcut edge shortcuts the two consecutive edges in Q_{ab} whose representatives were not added to P_{ab} and hence the path is not broken.

Also, whenever a tree T has two edges of P_{ab} incident on it, we have added the path between the endpoints of the edges in the tree to P_{ab}. And, whenever a tree T with $a \in V(T)$ has one edge of P_{ab} incident on it, we have added the path between the endpoints of the edge and a in the tree to P_{ab}. Similarly, whenever a tree T with $b \in V(T)$ has one edge of P_{ab} incident on it, we have added the path between the endpoints of the edge and b in the tree to P_{ab}. If there is a tree T with $a, b \in V(T)$ we added the path between the endpoints of a and b in the tree to P_{ab} during Path Rule 1. It follows that P_{ab} is indeed a path between a and b in G. Since Invariant 1 holds, we know that P_{ab} is a rainbow path. Since we have used only the colors from 1 to f so far, the lemma follows. □

So, now we only need to worry about how to rainbow connect vertices in L_S between themselves and to the other vertices. For this, we give the following coloring rule.

Coloring Rule 12. *For each $v \in L_S$, let e be the unique 2-edge incident on v which exists by Lemma 1. Color $(e)_1$ with $g_1 = f + 1$ and $(e)_2$ with $g_2 = f + 2$. (Recall that g_1 and g_2 are the global surplus colors).*

Now, we complete the proof of the main theorem.

Proof of Theorem 1. Consider any pair of vertices $a_1, a_2 \in V(G)$. If $a_1 \in L_S$, let e_1 be the edge incident on a_1 that is colored with g_1, and let a be the other end of e_1. If $a_1 \notin L_S$, let $a = a_1$. If $a_2 \in L_S$, let e_2 be the edge incident on a_2 that is colored with g_2, and let b be the other end of e_2. If $a_2 \notin L_S$, let $b = a_2$. We know there is a rainbow path P_{ab} from a to b that uses only colors in $[f]$ due to Lemma 26. We define path P as follows. If $a_1, a_2 \in L_S$, then $P := a_1 a P_{ab} b a_2$. If $a_1 \in L_S$ but $a_2 \notin L_S$, then $P := a_1 a P_{ab}$. If $a_2 \in L_S$ but $a_1 \notin L_S$, then $P := P_{ab} b a_2$. If $a_1, a_2 \notin L_S$, then $P := P_{ab}$. It is clear from the construction that P is a path between a_1 and a_2. Since edge $a_1 a$ is colored with $g_1 = f + 1$, edge $b a_2$ is colored with $g_2 = f + 2$, and path P_{ab} uses only colors in $[f]$, the path P is indeed a rainbow path. □

3. Conclusions

We gave an upper bound of $\mathrm{f}(G) + 2$ on $\mathrm{rc}(G)$, strengthening the intuition that tree-like and dominating structures are helpful in rainbow-connecting graphs. Our bound is tight up to an additive factor of 3 as shown by any tree. The question remains whether the bound can be improved to $f(G) - 1$ so that the bound is tight even with respect to additive factors. Also, then the bound would be a strict improvement over the bound $n - 1$ obtained by coloring the edges of a spanning tree in distinct colors. From our insight developed during the current work, we conjecture such a slightly stronger bound.

Conjecture 1. *A connected graph G has* $\mathrm{rc}(G) \leq \mathrm{f}(G) - 1$.

We expect that proving this conjecture requires further extensive case analysis. Further, we note that Lauri [12] proposed the following stronger version of the above conjecture, discovered using the automated conjecture-making software GraPHedron [16].

Conjecture 2 *([12]). A connected graph G has* $\mathrm{src}(G) \leq \mathrm{f}(G) - 1$.

Another interesting direction is to discover other graph parameters that yield tight bounds on the rainbow connection number of general graphs or of particular graph classes.

Declaration of competing interest

None declared.

Acknowledgement

We thank Kurt Mehlhorn for funding the research visit of Erik Jan van Leeuwen and Juho Lauri to Max Planck Institute for Informatics, Saarbrücken, which led to this work.

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