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The p-norm of circulant matrices via Fourier analysis

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Abstract: A recent work derived expressions for the induced p-norm of a special class of circulant matrices $A(n, a, b) \in \mathbb{R}^{n\times n}$, with the diagonal entries equal to $a \in \mathbb{R}$ and the off-diagonal entries equal to $b \geq 0$. We provide shorter proofs for all the results therein using Fourier analysis. The key observation is that a circulant matrix is diagonalized by a DFT matrix. The results comprise an exact expression for $\|A\|_p$, $1 \leq p \leq \infty$, where $A = A(n, a, b)$, $a \geq 0$ and for $\|A\|_2$ where $A = A(n, -a, b)$, $a \geq 0$; for the other p-norms of $A(n, -a, b)$, $2 < p < \infty$, upper and lower bounds are derived.

Keywords: self-adjoint, unitary invariance, induced norm, Riesz-Thorin interpolation

MSC: 15B05, 47A30

1 Introduction

Circulant matrices arise in many applications ranging from wireless communication [10] to cryptography [7] to solving differential equations [11] (see [2] and the references therein for historical context and more recent theoretical studies on circulant matrices). A real circulant matrix is of the form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{bmatrix},$$

where $a_j \in \mathbb{R}$, $1 \leq j \leq n$. For a matrix $A \in \mathbb{R}^{n\times n}$, A_{jk} denotes its (j, k)-th entry and $A^\star$ denotes its adjoint. Further, we define the induced p-norm of A to be

$$\|A\|_p := \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p},$$

for $1 \leq p \leq \infty$, where $x = [x_1, \dots, x_n]^T$, with $[\cdot]^T$ denoting the transpose, is a vector in $\mathbb{R}^n$,

$$\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\},$$

and for $1 \leq p < \infty$,

$$\|x\|_p := \left(|x_1|^p + \dots + |x_n|^p\right)^{1/p}.$$

It is well-known [4, 6] that the eigen-decomposition of a circulant matrix A is of the form $F^\star \Lambda F$ where F denotes the Discrete Fourier Transform (DFT) matrix with entries $F_{jk} = \frac{1}{\sqrt{n}} \cdot \omega_n^{-jk}$, $0 \leq j, k \leq n-1$, where

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$\omega_n = e^{\frac{2\pi i}{n}}$ (see [3, Section 3.2] for a proof). The eigenvalues, namely the diagonal entries of Λ, are given by

$$\lambda_k := \Lambda_{kk} = \sum_{j=0}^{n-1} a_{j+1}\omega_n^{jk}, \quad 0 \le k \le n-1. \tag{1}$$

We use this property of circulant matrices to study the induced p-norm of a special class of circulant matrices, $A(n,a,b) \in \mathbb{R}^{n\times n},\ a, b \in \mathbb{R}$, where

$$A(n,a,b) := \begin{bmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}.$$

The results proved here were first obtained in the recent work [2]; we provide shorter proofs for all the results using Fourier analysis. The 1-norm and the infinity norm of $A(n,a,b)$ are easily calculated to be $|a|+(n-1)|b|$ by inspection. As observed in [2], it suffices to consider the following two cases: $A(n,a,b)$ and $A(n,-a,b)$ where $a,b \ge 0$. The results comprise an exact expression for $\|A\|_p,\ 1<p<\infty$, where $A = A(n,a,b)$ and for $\|A\|_2$ where $A = A(n,-a,b)$; for $A = A(n,-a,b)$, upper and lower bounds for $\|A\|_p,\ 2<p<\infty$ are provided.

2 Results and Proofs

For a diagonal matrix, all the induced p-norms are equal to the maximum of the absolute value of the entries [5]. We calculate this value for Λ in Lemma 1. This lemma is the key to deriving the other results in this paper – essentially, for $1<p<\infty$, we relate $\|A\|_p$ to $\|A\|_2$, which, by unitary invariance, equals $\|\Lambda\|_2$.

Lemma 1. *For $a, b \ge 0$, for $1 \le p \le \infty$,*

i. for $A = A(n,a,b)$ and $A = F^\star \Lambda F$, we have

$$\|\Lambda\|_p = a + (n-1)b,$$

ii. for $A = A(n,-a,b)$ and $A = F^\star \Lambda F$, we have

$$\|\Lambda\|_p = \begin{cases} -a+(n-1)b & \text{if } 2a \le (n-2)b \\ a+b & \text{otherwise.} \end{cases}$$

Proof. For $A = A(n,a,b),\ a,b \in \mathbb{R}$ and $A = F^\star \Lambda F$, by (1), the diagonal entries of Λ are given by

$$\begin{aligned}\lambda_k &= a\omega_n^{kk} + \sum_{\substack{j=0\\ j\neq k}}^{n-1} b\omega_n^{jk} \\ &= b\sum_{j=0}^{n-1}\omega_n^{jk} + (a-b)\omega_n^{kk},\end{aligned}$$

for $0 \le k \le n-1$. Using the well-known identity (see, for example, [8]),

$$\sum_{j=0}^{n-1}\omega_n^{jk} = \begin{cases} n & \text{if } k = 0 \pmod{n} \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\lambda_0 = bn + (a-b) = a + (n-1)b,$$

and for $0 < k \leq n-1$,

$$|\lambda_k| = |a - b|.$$

The result follows by calculating $\|\Lambda\|_p = \max_{0 \leq k \leq n-1} |\lambda_k|$ for $1 \leq p \leq \infty$, for $A(n, a, b)$ and $A(n, -a, b)$. □

We use Lemma 1 to derive an exact expression for $\|A\|_2$ for $A(n, a, b)$ as well as $A(n, -a, b)$, for $a, b \geq 0$.

Theorem 2. *[2, Theorem 3.2, 4.1] For $a, b \geq 0$,*
i. for $A = A(n, a, b)$, we have

$$\|A\|_2 = a + (n-1)b.$$

ii. for $A = A(n, -a, b)$, we have

$$\|A\|_2 = \begin{cases} -a + (n-1)b & \text{if } 2a \leq (n-2)b \\ a + b & \text{otherwise.} \end{cases}$$

Proof. For $A = F^{\star} \Lambda F$, we have $\|A\|_2 = \|\Lambda\|_2$ since F is unitary and the spectral norm is unitarily invariant. The result now follows by Lemma 1. □

Remark. In [1], similar techniques are employed to calculate the (unitarily invariant) Schatten p-norms of block circulant matrices.

As observed in [2], $A = A(n, a, b)$, $a \in \mathbb{R}$, $b \geq 0$ is self-adjoint and hence $\|A\|_p = \|A\|_q$, for p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ (see [5]). Thus, it suffices to focus on either $p \in (1, 2]$ or $p \in [2, \infty)$. First, we compute the p-norm of $A = A(n, a, b)$ with $a, b \geq 0$ for $p \geq 2$.

Theorem 3. *[2, Theorem 3.2] For $A = A(n, a, b)$, $a, b \geq 0$, and for $p \geq 2$,*

$$\|A\|_p = a + (n-1)b.$$

Proof. Using the vector $x = [1, 1, \ldots, 1]^T$, we have

$$\|A\|_p \geq a + (n-1)b.$$

Next, observe that

$$\|A\|_\infty = a + (n-1)b = \|A\|_2,$$

where the last identity is by Theorem 2. By the Riesz-Thorin interpolation theorem [9, Theorem 2.1] we have, for every $0 < \theta < 1$,

$$\|A\|_{p_\theta} \leq \|A\|_q^{1-\theta} \|A\|_r^{\theta},$$

where p_θ, q, and r satisfy

$$\frac{1}{p_\theta} = \frac{1-\theta}{q} + \frac{\theta}{r}. \tag{2}$$

Setting $p_\theta = p$, $q = 2$, and $r = \infty$ in (2) yields

$$\|A\|_p \leq \|A\|_2 = a + (n-1)b.$$

□

Remark. Using similar arguments as above, one can calculate $\|A\|_p$, $1 \leq p \leq \infty$, of a general real circulant matrix A with non-negative entries $a_1, \ldots, a_n$ to be $a_1 + \cdots + a_n$.

Next, we estimate the p-norm of $A(n, -a, b)$ with $a, b \geq 0$ for $p \geq 2$.

Theorem 4. *[2, Theorem 5.1] For $A = A(n, -a, b)$, $a, b \geq 0$, and for $p \geq 2$, we have*

$$-a + (n-1)b \leq \|A\|_p \leq n^{\frac{1}{2}-\frac{1}{p}} \cdot (-a + (n-1)b) \qquad \text{if } 2a \leq (n-2)b,$$
$$a + b \leq \|A\|_p \leq n^{\frac{1}{2}-\frac{1}{p}} \cdot (a + b) \qquad \text{if } 2a \geq (n-2)b.$$

Proof. For $p \geq 2$ and $x \neq 0$, we have

$$\begin{aligned}\|Ax\|_p &\leq \|Ax\|_2 \\ &\leq \|F^\star\|_2 \cdot \|\Lambda\|_2 \cdot \|\widehat{x}\|_2,\end{aligned}$$

where $\widehat{x} = Fx$ denotes the Fourier transform of x. By Plancherel's relation, we have $\|\widehat{x}\|_2 = \|x\|_2$ and $\|F^\star\|_2 = 1$. Hence

$$\|Ax\|_p \leq \|\Lambda\|_2 \cdot \|x\|_2,$$

whereby

$$\|A\|_p \leq n^{\frac{1}{2}-\frac{1}{p}} \cdot \|\Lambda\|_2,$$

where we have used the inequality $\|x\|_r \leq \|x\|_p \cdot n^{\frac{1}{r}-\frac{1}{p}}$ for $2 = r \leq p$. The upper bounds in the theorem now follow from Lemma 1. To obtain the lower bounds, we exhibit a vector $x \neq 0$ such that $\|Ax\|_p/\|x\|_p$ equals the quantity in the desired lower bound.

i. $2a \leq (n-2)b$: for $x = [1, 1, \ldots, 1]^T$,

$$\frac{\|Ax\|_p}{\|x\|_p} = |-a + (n-1)b| \geq -a + (n-1)b.$$

ii. $2a \geq (n-2)b$: for $x = [-1, 1, 0, \ldots, 0]^T$,

$$\frac{\|Ax\|_p}{\|x\|_p} = \left(\frac{|a+b|^p + |-a-b|^p}{2}\right)^{1/p} = a + b.$$

□

Finally, we provide an improved upper bound for the p-norm of $A(n, -a, b)$ for $p > 2$, using the Riesz-Thorin interpolation theorem (see (2)). In particular, for $p > 2$, we "interpolate between $\|A\|_2$ and $\|A\|_\infty$" to obtain an upper bound for $\|A\|_p$.

Theorem 5. *[2, Theorem 5.2] For $A = A(n, -a, b)$, $a, b \geq 0$, and for $p > 2$, we have,*

$$\|A\|_p \leq \|A\|_2^{\frac{2}{p}} \|A\|_\infty^{1-\frac{2}{p}}.$$

Proof. Setting $q = 2$ and $r = \infty$ in (2) yields

$$\frac{1}{p_\theta} = \frac{1-\theta}{2}.$$

We choose $p_\theta = p$ to get

$$\|A\|_p \leq \|A\|_2^{1-\theta} \|A\|_\infty^{\theta},$$

from which the result follows since $\theta = 1 - 2/p$. □

Using similar arguments we can show the following monotonicity property of the induced p-norm of $A(n, -a, b)$, $a, b \geq 0$. In fact, the result is true for all real self-adjoint matrices.

Theorem 6. *For $A = A(n, -a, b)$, $a, b \geq 0$, and for $p \geq 2$, $\|A\|_p$ is monotonically non-decreasing in p.*

Proof. Fix $\beta > 0$. Setting $q = p - \alpha$, $\alpha > 0$, $r = p + \beta$, and $p_\theta = p$ in (2) yields

$$\|A\|_p \leq \|A\|_{p-\alpha}^{1-\theta} \|A\|_{p+\beta}^{\theta}.$$

Choose α such that

$$\frac{1}{p-\alpha}+\frac{1}{p+\beta}=1.$$

Since A is self-adjoint, we have $\|A\|_{p-\alpha}=\|A\|_{p+\beta}$, and hence

$$\|A\|_p \le \|A\|_{p+\beta}.$$

□

Remark. As a corollary, we get $\|A\|_p \ge \|A\|_2$ for all $p \ge 2$, which is the same as the lower bound in Theorem 4. Further, using the Riesz-Thorin interpolation theorem, it can be shown that for every $\beta > 0$,

$$\|A\|_p \le \|A\|_2^{\frac{\frac{1}{p}-\frac{1}{p+\beta}}{\frac{1}{2}-\frac{1}{p+\beta}}} \cdot \|A\|_{p+\beta}^{\frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p+\beta}}}.$$

3 Summary

Using the observation that a circulant matrix is diagonalized by a DFT matrix, we have computed the p-norm of a special class of circulant matrices $A(n, a, b) \in \mathbb{R}^{n\times n}$, with the diagonal entries equal to $a \in \mathbb{R}$ and the off-diagonal entries equal to $b \ge 0$. The 1-norm and the infinity norm of $A(n, a, b)$ are easily calculated to be $|a| + (n-1)|b|$ by inspection. For $A = A(n, a, b)$ with $a, b \ge 0$, we show that for all $1 \le p \le \infty$,

$$\|A\|_p = a + (n-1)b.$$

For $A = A(n, -a, b)$ with $a, b \ge 0$, we obtain an exact expression for $\|A\|_2$. Since A is self-adjoint, $\|A\|_q = \|A\|_p$ for conjugate pairs, p and q. This, along with the Riesz-Thorin interpolation theorem, implies that for $2 \le p \le \infty$, $\|A\|_p$ is monotonically non-decreasing in p. Further, we show that for $2 \le p \le \infty$,

$$\|A\|_2 \le \|A\|_p \le \|A\|_2^{\frac{2}{p}} \|A\|_\infty^{1-\frac{2}{p}}.$$

An exact expression for $\|A\|_p,\ 2 < p < \infty$ for $A = A(n, -a, b),\ a, b \ge 0$, remains elusive.

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