A POWERFUL KINEMATIC MODEL FOR PROPORTIONAL NAVIGATION OF GUIDED WEAPONS AGAINST MANEUVERING TARGETS

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ABSTRACT

An accurate solution is presented of the nonlinear differential equations describing the pursuer motion under the PN law in the general case when the target is laterally maneuvering. A quasilinearization (QL) approach is used, followed by a perturbation technique to obtain analytical solutions for the trajectory parameters. Explicit expression for the pursuer lateral acceleration is derived and is shown to contain contributions due to initial heading error and target maneuver, with a coupling between the two effects. The solution is shown to be a substantial and consistent generalization of the classical linear solution for maneuvering targets. The generalized QL solution presented here provides very accurate estimates of pursuer lateral acceleration over a much broader range of engagement geometries and target maneuvers than presently available analytical solutions. The analytical solution is of special value for (i) generating accurate insight into PN behavior for maneuvering targets (ii) aiding rapid design calculation involving tradeoff studies (iii) modelling larger systems in which the PN law appears as a building block, and (iv) performing real-time computations of launch envelope, even on modest airborne computers on launching aircraft, to ensure successful intercept.

INTRODUCTION

Proportional Navigation (PN) has been widely used as guidance law for a wide range of tactical applications in the recent decades. This law is very simple to implement on board and is robust in the sense of ensuring intercept in a wide variety of engagement situations. However, the analysis of PN-based guidance systems has been difficult, because the equations governing motion under the PN law are highly nonlinear.

General analytical solutions to the nonlinear PPN equations have not been available even for the simple case of engagement against non-maneuvering targets. The treatment of the PN problem for non-maneuvering targets, though having great theoretical significance, has relatively little practical value since the overwhelming majority of PN applications involve maneuvering targets. However, for maneuvering targets, the PN equations become highly intractable. Thus, while for non-maneuvering targets at least certain variants of the PN, such as TPN and GTPN have been solved exactly, as also a few special cases of PPN, no general solution to PN motion against maneuvering targets has been reported. Some qualitative treatment [3,4] has been attempted, but such treatment has been confined to the determination only of certain bounds on the pursuer lateral acceleration and has not been able to obtain actual solutions for the pursuer trajectory parameters.

The only practical analytical method of analyzing PN motion has been hitherto based on linearization [1,2,5]. Because of the small-angle approximations necessary for linearization, the linearized solutions are valid only for near-tail-chase geometries and small target maneuvers. Further, the linear superposition of the individual contributions of lateral acceleration due to the target maneuver and the initial heading error does not remain valid for large heading errors.

In this paper, we present a method of accurately solving the PPN equations in explicit form when the target motion involves significant lateral maneuver and the engagement geometry involves large LOS angles and initial heading errors. The method is based on the QL technique and may be considered as a substantial extension of the treatment used in our previous paper [6] in which similar solutions for non-maneuvering targets were presented. Further, the explicit QL solutions presented in this paper are also shown to be a generalization of the classical linear solution for maneuvering targets. This generalization results in a large improvement in the estimation of trajectory parameters and enlarges the validity of the solutions to a much broader range of engagement geometries and maneuver levels than is possible with the currently available classical linear solutions.

PN EQUATIONS FOR MANEUVERING TARGETS

Consider the geometry of Fig. 1. The target is assumed to have a constant forward velocity V_T and a constant lateral acceleration A_T. This implies that the target describes a circular path in a plane with a radius V_T^2/A_T and has an angular velocity (i.e. rate of turn) A_T/V_T.

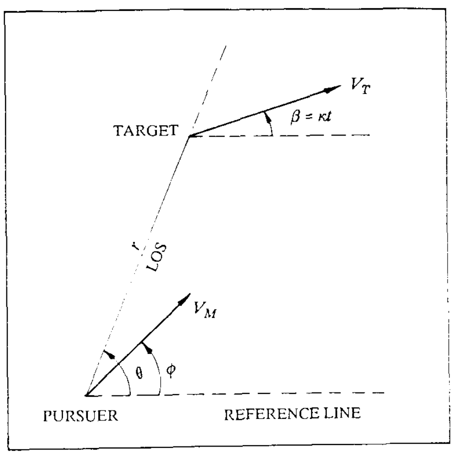

Figure 1. Geometry of Pure Proportional Navigation for a laterally maneuvering target

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By resolving components along and perpendicular to the LOS, the differential equations describing the pursuer motion may be written as

$$\dot{r}=V_T\cos(\theta-\kappa t)-V_M\cos(\theta-\phi) \tag{1}$$

and

$$r\dot{\theta}=-V_T\sin(\theta-\kappa t)+V_M\sin(\theta-\phi) \tag{2}$$

where $\kappa=A_T/V_T$ represents the normal turn rate (angular velocity) of the target.

Using the definition $\dot{\phi}=N\dot{\theta}$ of PPN in eqs (1) and (2)

$$\dot{r}=V_T\cos(\theta-\kappa t)-V_M\cos(b\theta-c) \tag{3}$$

and

$$r\dot{\theta}=-V_T\sin(\theta-\kappa t)+V_M\sin(b\theta-c) \tag{4}$$

where $b=1-N$ and $c=\phi_i-N\theta_i$ (5)

For the normal range of PN parameters, the range-to-go r is a monotonic function of time t. Under the assumption of monotonicity, eqs (3) and (4) can be rewritten with r as the independent variable:

$$r\frac{d\theta}{dr}=\frac{-V_T\sin(\theta-\kappa t)+V_M\sin(b\theta-c)}{V_T\cos(\theta-\kappa t)-V_M\cos(b\theta-c)} \tag{6}$$

and

$$\frac{dt}{dr}=\frac{1}{V_T\cos(\theta-\kappa t)-V_M\cos(b\theta-c)} \tag{7}$$

Normalizing the time t as $\tau=t/T_i$ where $T_i=r_i/V_{riL}$, we obtain

$$\frac{dt}{dr}=T_i\frac{d\tau}{dr} \tag{8a}$$

$$\kappa t=\kappa T_i\tau$$
$$=k\tau \tag{8b}$$

where $k=\kappa T_i=\frac{A_T}{V_T}\frac{r_i}{V_{riL}}$. We note that k is a dimensionless constant.

Substituting eqs (8a) and (8b) in eqs (6) and (7), we obtain

$$r\frac{d\theta}{dr}=\frac{-V_T\sin(\theta-k\tau)+V_M\sin(b\theta-c)}{V_T\cos(\theta-k\tau)-V_M\cos(b\theta-c)}\equiv g(\theta,k) \tag{9}$$

and

$$T_i\frac{d\tau}{dr}=\frac{1}{V_T\cos(\theta-k\tau)-V_M\cos(b\theta-c)}\equiv d(\theta,k) \tag{10}$$

QUASILINEARIZATION OF PN EQUATIONS

The coupled equations (9) and (10) are highly nonlinear and do not admit closed form solution. We derive approximate equivalents of these equations using the QL technique. Expanding the R.H.S. of eqs (9) and (10) respectively in Taylor series in the function space and truncating after the first derivative term,

$$r\frac{d\theta_1}{dr}=G_0+H_0(\theta_1-\theta_0) \tag{11}$$

and

$$T_i\frac{d\tau_1}{dr}=D_0+E_0(\theta_1-\theta_0) \tag{12}$$

where

$$D_0\equiv d(\theta_0) \tag{13a}$$

$$E_0\equiv e(\theta_0)\equiv\left.\frac{\partial d}{\partial\theta}\right|_{\theta=\theta_0} \tag{13b}$$

$$G_0\equiv g(\theta_0) \tag{13c}$$

$$H_0\equiv h(\theta_0)\equiv\left.\frac{\partial g}{\partial\theta}\right|_{\theta=\theta_0} \tag{13d}$$

and the subscript 0 denotes the initial approximation to the solution and 1 denotes the solution after the first iteration. Although more iterations can be performed to improve the accuracy, in this work we stop at the first iteration in order to be able to obtain a closed form solution. Thus θ_1 will be our final solution for the variable θ. To simplify the notation, we replace θ_1 by θ and τ_1 by τ in the following treatment. Thus eqs (11) and (12) can be rewritten as

$$r\frac{d\theta}{dr}=G_0+H_0(\theta-\theta_0) \tag{14}$$

and

$$T_i\frac{d\tau}{dr}=D_0+E_0(\theta-\theta_0) \tag{15}$$

From definition, and after necessary simplification

$$h\equiv\frac{\partial g}{\partial\theta}=\frac{-bV_M^2-V_T^2+(1+b)V_TV_M\cos(N\theta+c-k\tau)}{[V_T\cos(\theta-k\tau)-V_M\cos(b\theta-c)]^2}$$
$$+k\frac{d\tau}{d\theta}\frac{V_T^2-V_TV_M\cos(N\theta+c-k\tau)}{[V_T\cos(\theta-k\tau)-V_M\cos(b\theta-c)]^2} \tag{16}$$

and

$$e\equiv\frac{\partial d}{\partial\theta}=-\frac{-V_T\sin(\theta-k\tau)+bV_M\sin(b\theta-c)}{[V_T\cos(\theta-k\tau)-V_M\cos(b\theta-c)]^2}$$
$$+k\frac{d\tau}{d\theta}\frac{-V_T\sin(\theta-k\tau)}{[V_T\cos(\theta-k\tau)-V_M\cos(b\theta-c)]^2} \tag{17}$$

Equations (14) and (15) along with the definitions in eqs (9), (10), (16) and (17), constitute the QL approximation to the nonlinear PN equations (9) and (10), respectively.

PERTURBATION SOLUTION OF QL EQUATIONS

The QL equations (14) and (15) are still not directly solvable in closed form. A perturbation series method is adopted here to obtain an approximate solution. The variables θ and τ are expressed as truncated perturbation series:

$$\theta(r,k)=\theta_N(r)+k\theta_M(r) \tag{18}$$

and

$$\tau(r,k)=\tau_N(r)+k\tau_M(r) \tag{19}$$

where subscript N denotes the zeroth order solution and M denotes the the second term in the first order solution. It will be shown later that the zeroth order solution corresponds to the non-maneuvering target case and the second term in the first-order solution represents maneuver effects. For the series expansions (18) and (19) to be accurate, k must be small.

Substituting the values of θ and τ from eqs (18) and (19) respectively in eqs (14) and (15), we obtain

$$r\left[\frac{d\theta_N}{dr}+k\frac{d\theta_M}{dr}\right]=G_0+H_0\left[(\theta_N-\theta_{N0})+k(\theta_M-\theta_{M0})\right] \tag{20}$$

and

$$T_i\left[\frac{d\tau_N}{dr}+k\frac{d\tau_M}{dr}\right]=D_0+E_0\left[(\theta_N-\theta_{N0})+k(\theta_M-\theta_{M0})\right] \tag{21}$$

For the QL algorithm, an initial approximation to the solution is necessary. Since at least at the start of the engagement, θ_i is the true solution for θ, we choose the initial LOS angle approximations as

$$\theta_{N0}(r)=\theta_i \tag{22a}$$

$$\theta_{M0}(r)=0 \tag{22b}$$

As an initial approximation for the normalized time τ we use the estimate obtained from the CL solution approach, since time estimates obtained from CL approach are normally quite accurate and yet the form remains simple enough to remain tractable. Thus

$$\tau_{N0}=\frac{t_{N0}}{T_i}=\frac{(r_i-r)/V_{riL}}{r_i/V_{riL}}=\frac{r_i-r}{r_i} \tag{23a}$$

$$\tau_{M0}=0 \tag{23b}$$

We note that in (22b) and (23b) the initial approximation for the maneuver induced components of θ and τ are assumed zero since the entire contribution is assumed to come from the non-maneuvering component which is the dominant one at the start of engagement.

Substituting the values of θ and τ from eqs (18) and (19) respectively in (9), (10), (16) and (17) and using eqs (22) and (23) the following relationships are obtained:

$$G_0\equiv g(\theta_0)=G_{N0}+kG_{M0} \tag{24}$$

$$H_0\equiv h(\theta_0)=H_{N0}+kH_{M0} \tag{25}$$

$$D_0\equiv d(\theta_0)=D_{N0}+kD_{M0} \tag{25}$$

$$E_0\equiv e(\theta_0)=E_{N0}+kE_{M0} \tag{26}$$

where

$$G_{N0}=\frac{-V_T\sin\theta_i+V_M\sin(b\theta_i-c)}{V_T\cos\theta_i-V_M\cos(b\theta_i-c)}=-\frac{r_i\dot{\theta}_i}{V_{riL}} \tag{28}$$

$$G_{M0}=-\frac{V_T}{V_{riL}}\tau_{N0}(\cos\theta_i-G_{N0}\sin\theta_i) \tag{29}$$

$$H_{N0}=\frac{-bV_M^2-V_T^2+(1+b)V_TV_M\cos(N\theta_i+c)}{[V_T\cos\theta_i-V_M\cos(b\theta_i-c)]^2} \tag{30}$$

$$H_{M0}=\frac{-2H_{N0}V_T\tau_{N0}\sin\theta_i}{V_T\cos\theta_i-V_M\cos(b\theta_i-c)}+\frac{(1+b)V_TV_M\tau_{N0}\sin(N\theta_i+c)}{[V_T\cos\theta_i-V_M\cos(b\theta_i-c)]^2} \tag{31}$$

$$D_{N0}=\frac{1}{V_T\cos\theta_i-V_M\cos(b\theta_i-c)}=-\frac{1}{V_{riL}} \tag{32}$$

$$D_{M0}=-D_{N0}^2V_T\tau_{N0}\sin\theta_i \tag{33}$$

$$E_{N0}=-\frac{-V_T\sin\theta_i+bV_M\sin(b\theta_i-c)}{[V_T\cos\theta_i-V_M\cos(b\theta_i-c)]^2} \tag{34}$$

$$E_{M0}=-\frac{2E_{N0}V_T\tau_{N0}\sin\theta_i}{V_T\cos\theta_i-V_M\cos(b\theta_i-c)}-\frac{V_T\tau_{N0}\cos\theta_i}{[V_T\cos\theta_i-V_M\cos(b\theta_i-c)]^2} \tag{35}$$

Substituting the values of G_0, H_0, D_0, and E_0 from (24), (25), (26) and (27) respectively in eqs (20) and (21) and using the initial approximations (22) and (23), we obtain

$$r\left[\frac{d\theta_N}{dr}+k\frac{d\theta_M}{dr}\right]=G_0+H_{N0}(\theta_N-\theta_i)+k\left[G_{M0}+H_{N0}\theta_M+H_{M0}(\theta_N-\theta_i)\right] \tag{36}$$

and

$$T_i\left[\frac{d\tau_N}{dr}+k\frac{d\tau_M}{dr}\right]=D_{N0}+E_{N0}(\theta_N-\theta_i)+k\left[D_{M0}+E_{N0}\theta_M+E_{M0}(\theta_N-\theta_i)\right] \tag{37}$$

Equating the coefficients of k in eqs (36) and (37), we get

$$r\frac{d\theta_N}{dr}=G_{N0}+H_{N0}(\theta_N-\theta_i) \tag{38}$$

$$r\frac{d\theta_M}{dr}=G_{M0}+H_{N0}\theta_M+H_{M0}(\theta_N-\theta_i) \tag{39}$$

$$T_i\frac{d\tau_N}{dr}=\frac{dt_N}{dr}=D_{N0}+E_{N0}(\theta_N-\theta_i) \tag{40}$$

$$T_i\frac{d\tau_M}{dr}=D_{M0}+E_{N0}\theta_M+E_{M0}(\theta_N-\theta_i) \tag{41}$$

The differential equations (37)-(40) are solved subject to the following initial conditions at $r=r_i$

$\theta_N=\theta_i$, $\theta_M=0$

$\tau_N=0$, $\tau_M=0$ (i.e. $\tau=0$)

The zeroth order equations (38) and (40) correspond to the non-maneuvering case and have been solved in closed form in our previous paper [6]. The solution is reproduced below:

$$\theta_N=\theta_i+\frac{G_{N0}}{H_{N0}}\left[\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right] \tag{42}$$

$$T_i\tau_N=t_N=r_i\left[\left(D_{N0}-\frac{E_{N0}G_{N0}}{H_{N0}}\right)\left(\frac{r}{r_i}-1\right)+\frac{E_{N0}G_{N0}}{H_{N0}(H_{N0}+1)}\left(\left(\frac{r}{r_i}\right)^{H_{N0}+1}-1\right)\right] \tag{43}$$

Substituting the value of θ_N from eq (42) in (39)

$$r\frac{d\theta_M}{dr}-H_{N0}\theta_M=V_T\tau_{N0}D_{N0}(\cos\theta_i-G_{N0}\sin\theta_i)+\left[D_{N0}V_T\tau_{N0}\left(-2H_{N0}\sin\theta_i+(1+b)V_MD_{N0}\sin(N\theta_i+c)\right)\right]\times\left[\frac{G_{N0}}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{44}$$

Using the value of $\tau_{N0}=(r_i-r)/r_i$ from eq (23a) in (44)

$$r\frac{d\theta_M}{dr}-H_{N0}\theta_M=V_TD_{N0}\left(1-\frac{r}{r_i}\right)\left[R-S\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{45}$$

where

$$R=\cos\theta_i-G_{N0}\sin\theta_i \tag{46}$$

$$S=-2G_{N0}\sin\theta_i+(1+b)V_MD_{N0}\frac{G_{N0}}{H_{N0}}\sin(N\theta_i+c) \tag{47}$$

Equation (45) is a first order linear differential equation whose solution is given as

$$\theta_M=\frac{V_T}{V_{riL}}\frac{R-S}{H_{N0}-1}\left[\left(1-\frac{r}{r_i}\right)+\frac{1}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{48}$$

Substituting the value of θ_N and θ_M from eqs (42) and (48) respectively in eq (18), we obtain the complete solution for the LOS angle as

$$\theta=\theta_i+\frac{G_{N0}}{H_{N0}}\left[\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right]$$

$$+\frac{r_iA_T}{V_{nL}^2}\frac{R-S}{H_{N0}-1}\left[\left(1-\frac{r}{r_i}\right)+\frac{1}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{49}$$

To solve for the elapsed time t, we substitute the values of θ_N and θ_M from (42) and (48) and D_{M0}, E_{M0} from (33) and (35) in eq (41) and obtain

$$T_i\frac{d\tau_M}{dr}=-D_{N0}^2V_T\tau_{N0}\sin\theta_i-E_{N0}D_{N0}V_T\frac{R-S}{H_{N0}-1}\left[\left(1-\frac{r}{r_i}\right)+\frac{1}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right]$$

$$+\left[-2E_{N0}D_{N0}V_T\tau_{N0}\sin\theta_i-V_T\tau_{N0}D_{N0}^2\cos\theta_i\right]\left[\frac{G_{N0}}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{50}$$

$$=D_{N0}V_T\left[-D_{N0}\left(1-\frac{r}{r_i}\right)\sin\theta_i-E_{N0}\frac{R-S}{H_{N0}-1}\left[\left(1-\frac{r}{r_i}\right)+\frac{1}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right]\right.$$

$$\left.+\left(-2E_{N0}\sin\theta_i-D_{N0}\cos\theta_i\right)\left(1-\frac{r}{r_i}\right)\frac{G_{N0}}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{51}$$

Now,

$$\frac{dt}{dr}=T_i\frac{d\tau}{dr}$$

$$=T_i\left[\frac{d\tau_N}{dr}+k\frac{d\tau_M}{dr}\right]\quad \textit{using eq (19)}$$

$$=\frac{d\tau_N}{dr}+\frac{A_T}{V_T}T_i\frac{d\tau_M}{dr} \tag{52}$$

Substituting the value of $\frac{d\tau_N}{dr}$ from eq (40) and $T_i\frac{d\tau_M}{dr}$ from eq (51) in eq (52), we obtain

$$\frac{dt}{dr}=D_{N0}+E_{N0}\frac{G_{N0}}{H_{N0}}\left[\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right]$$

$$+r_iD_{N0}^2A_T\left[D_{N0}\left(1-\frac{r}{r_i}\right)\sin\theta_i\right.$$

$$+E_{N0}\frac{R-S}{H_{N0}-1}\left[\left(1-\frac{r}{r_i}\right)+\frac{1}{H_{N0}}\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right]$$

$$\left.+\frac{G_{N0}}{H_{N0}}\left(2E_{N0}\sin\theta_i+D_{N0}\cos\theta_i\right)\left(1-\frac{r}{r_i}\right)\left\{\left(\frac{r}{r_i}\right)^{H_{N0}}-1\right\}\right] \tag{53}$$

Equation (53) is now integrated with the initial condition $t=0$ at $r=r_i$ to obtain the general solution for the elapsed time t

$$t=r_i\left[\left(\frac{r}{r_i}-1\right)\left(D_{N0}-\frac{E_{N0}G_{N0}}{H_{N0}}\right)+\frac{E_{N0}G_{N0}}{H_{N0}(HN0+1)}\left[\left(\frac{r}{r_i}\right)^{H_{N0}+1}-1\right]\right.$$

$$+r_iD_{N0}^2A_T\left[\left(\frac{r}{r_i}-1\right)\left(D_{N0}\sin\theta_i+\frac{E_{N0}}{H_{N0}}\cos\theta_i\right)\right.$$

$$-\frac{1}{2}\left[\left(\frac{r}{r_i}\right)^2-1\right]\left(D_{N0}\sin\theta_i+\frac{E_{N0}}{H_{N0}-1}\cos\theta_i\right)$$

$$+\frac{E_{N0}\cos\theta_i}{(H_{N0}-1)H_{N0}(H_{N0}+1)}\left[\left(\frac{r}{r_i}\right)^{H_{N0}+1}-1\right]$$

$$+\frac{G_{N0}r_i}{H_{N0}}\left(2E_{N0}\sin\theta_i+D_{N0}\cos\theta_i\right)$$

$$\times\left\{\frac{1}{H_{N0}+1}\left[\left(\frac{r}{r_i}\right)^{H_{N0}+1}-1\right]-\left(\frac{r}{r_i}-1\right)\right.$$

$$\left.\left.\left.-\frac{1}{H_{N0}+2}\left[\left(\frac{r}{r_i}\right)^{H_{N0}+2}-1\right]+\frac{1}{2}\left[\left(\frac{r}{r_i}\right)^2-1\right]\right\}\right]\right] \tag{54}$$

The final intercept time, t_f, is obtained by putting $r=0$ in eq (54) as

$$t_f=r_i\left[-D_{N0}+\frac{E_{N0}G_{N0}}{H_{N0}+1}-\frac{r_iD_{N0}^2A_T}{2}\left\{D_{N0}\sin\theta_i+\frac{E_{N0}}{H_{N0}+1}\cos\theta_i\right.\right.$$

$$\left.\left.+\frac{G_{N0}}{H_{N0}+1}\left(2E_{N0}\sin\theta_i+D_{N0}\cos\theta_i\right)\right\}\right] \tag{55}$$

The pursuer lateral acceleration A_M is given as

$$A_M=V_MN\frac{d\theta}{dt}$$

$$=NV_M\frac{\frac{d\theta}{dr}}{\frac{dt}{dr}} \tag{56}$$

Differentiating eq (49) and substituting for G_{N0} from eqn (28), we get

$$\frac{d\theta}{dr}=-\frac{\dot\theta_i}{V_{rrL}}\left(\frac{r}{r_i}\right)^{H_{NO}-1}+\frac{A_T}{V_{rrL}^2}\frac{R-S}{H_{NO}-1}\left[\left(\frac{r}{r_i}\right)^{H_{NO}-1}-1\right] \tag{57}$$

Denoting $\frac{dt}{dr}\equiv-\frac{1}{V_{rL}}$ in eq (53), we obtain from eq (56) the final expression for the pursuer lateral acceleration as

$$A_M=NV_M\dot\theta_i\frac{V_{rL}}{V_{rrL}}\left(\frac{r}{r_i}\right)^{H_{NO}-1}$$

$$+\frac{NV_MA_TV_{rL}}{V_{rrL}^2}\frac{R-S}{H_{NO}-1}\left[1-\left(\frac{r}{r_i}\right)^{H_{NO}-1}\right] \tag{58}$$

Equations (49), (54) and (58) are the final explicit expressions for the QL solution for the LOS angle, elapsed time and lateral acceleration for a maneuvering target at a given range-to-go during the engagement. The time till intercept is provided by eq (55).

In the particular case when there is no initial heading error, i.e. $\dot\theta_i=0$, eqs (49), (54), (55) and (58) reduce respectively to

$$\theta=\frac{r_i}{V_{rrL}^2}\frac{\cos\theta_i}{H_{NO}-1}\left[\left(1-\frac{r}{r_i}\right)+\frac{1}{H_{NO}}\left\{\left(\frac{r}{r_i}\right)^{H_{NO}}-1\right\}\right]A_T \tag{59}$$

$$t=r_i\left[\left(\frac{r}{r_i}-1\right)D_{NO}\right.$$

$$+r_iD_{NO}^2A_T\left[\left(\frac{r}{r_i}-1\right)\left(D_{NO}\sin\theta_i+\frac{E_{NO}}{H_{NO}}\cos\theta_i\right)\right.$$

$$-\frac{1}{2}\left(\left(\frac{r}{r_i}\right)^2-1\right)\left(D_{NO}\sin\theta_i+\frac{E_{NO}}{H_{NO}-1}\cos\theta_i\right)$$

$$\left.\left.+\frac{E_{NO}\cos\theta_i}{(H_{NO}-1)H_{NO}(H_{NO}+1)}\left(\left(\frac{r}{r_i}\right)^{H_{NO}+1}-1\right)\right]\right] \tag{60}$$

$$t_f=r_i\left[-D_{NO}-\frac{r_iD_{NO}^2A_T}{2}\left(D_{NO}\sin\theta_i+\frac{E_{NO}}{H_{NO}+1}\cos\theta_i\right)\right]$$

$$=\frac{r_i}{V_{rrL}}\left[1-\frac{r_iA_T}{2V_{rrL}}\left(D_{NO}\sin\theta_i+\frac{E_{NO}}{H_{NO}+1}\cos\theta_i\right)\right] \tag{61}$$

$$A_M=\frac{NV_MV_{rL}}{V_{rrL}^2}\frac{\cos\theta_i}{H_{NO}-1}\left[1-\left(\frac{r}{r_i}\right)^{H_{NO}-1}\right]A_T \tag{62}$$

STRUCTURAL COMPARISON WITH CLASSICAL LINEAR SOLUTION

In this section we show that the general solutions for the trajectory parameters obtained in the last section for the case of maneuvering targets is a true generalization of the more restrictive solutions available earlier. Since the formulation of this paper is essentially a perturbation over the QL solution for non-maneuvering targets , it is easy to see from eq (49),(54), (55) and (58) that forcing A_T to zero (i.e. no maneuver) results in the second term vanishing in all these equations, leaving only the first term which represent the non-maneuvering solution.

To compare the QL solution with the classical linear solution, e.g. [1] we first consider the most important trajectory parameter A_M and recast eq (58) in terms of N' and $\Delta\phi_i$ using the definitions of N' and $\dot\theta_i$ i.e.

$$N'=N\frac{V_M}{V_{rrL}}\cos\phi_c$$

and

$$\dot\theta_i=-\frac{V_M\cos\phi_c}{r_i}\Delta\phi_i$$

We obtain,

$$A_M=-V_M\frac{V_{rL}}{r_i}N'\left(\frac{r}{r_i}\right)^{H_{NO}-1}\Delta\phi_i$$

$$+\frac{N'}{H_{NO}-1}\frac{R-S}{\cos\phi_c}\frac{V_{rL}}{V_{rrL}}\left[1-\left(\frac{r}{r_i}\right)^{H_{NO}-1}\right]A_T \tag{63}$$

The general solution (63) has two distinct parts, the first corresponding to a non-maneuvering target and the second providing the contribution due to target maneuver. However, unlike the classical linear solution [1], where the effects of initial heading error and target maneuver are distinct and uncoupled, here the second term in eq (63) also contains cross-coupling between A_T and $\Delta\phi_i$.

For a near-tail-chase situation, as assumed for the classical linear solution, θ_i and $\Delta\phi_i$ are small and eq (63) reduces to

$$A_M=-V_M\frac{V_{rL}}{r_i}N'\left(\frac{r}{r_i}\right)^{H_{NO}-1}\Delta\phi_i$$

$$+\frac{N'}{H_{NO}-1}\frac{1}{\cos\phi_c}\left[1-\left(\frac{r}{r_i}\right)^{H_{NO}-1}\right]A_T \tag{64}$$

It is readily seen that the coupling between A_T and $\Delta\phi_i$ has disappeared in eq (64) because of the small-angle approximation.

The small-angle QL eq (64) is identical to the classical linear result [1] if H_{NO} in the former is replaced by N'-1 in the latter. Indeed, H_{NO} is a refinement of the classical effective navigation constant N' as has been established earlier in our previous paper [6]. The QL solution for maneuvering target derived in this paper is thus a consistent generalization of the classical linear treatment.

A similar approach can be employed to show that the QL expressions for the LOS angle and elapsed time reduces to classical linear form for small LOS and heading angles and small target maneuvers.

RESULTS AND DISCUSSIONS

Figures 2, 3 and 4 show plots of the lateral acceleration A_M of the pursuer engaging a maneuvering target. The QL estimate of A_M is obtained from eq (58) and (62) for pursuit with and without initial heading error respectively. The results derived from classical linear formulation [1] are shown alongside for comparison. Further, for ascertaining the absolute accuracy of the results, the "exact" estimates of A_M are also plotted in the figures. The exact estimates are obtained by an accurate numerical solution of the original nonlinear equations (3) and (4) for $\dot\theta$ using a 4/5th order Runge-Kutta-Fehlberg algorithm [7] and substituting this $\dot\theta$ in eq (56) to obtain A_M.

In Figs. 2, 3 and 4 three representative values of the effective navigation constant N' are considered. The (a) part in each figure depicts results for small-angles of the geometry, representing a near-tail-chase pursuit, and also relatively low target maneuver. For this case, all the three estimates (QL, linear and "exact") of A_M are found to be close. This is as would have been expected, since linear

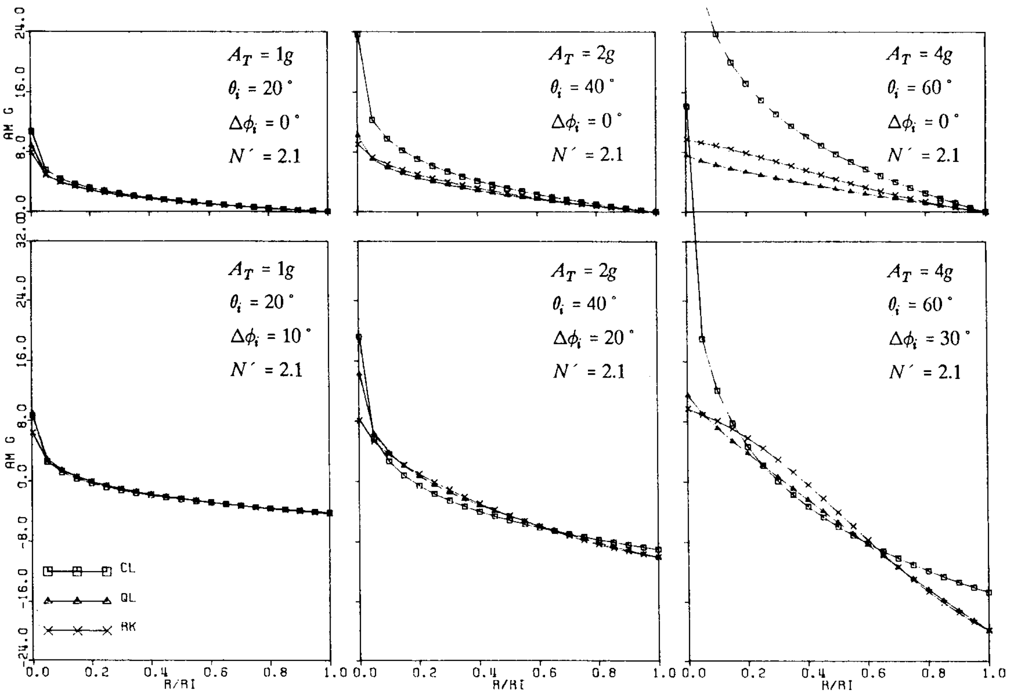

Figure 2. QL solution for A_M against a laterally maneuvering target as a function of r/r_i. CL solution and 'exact' RK estimates are also shown for comparison. N'=2.1

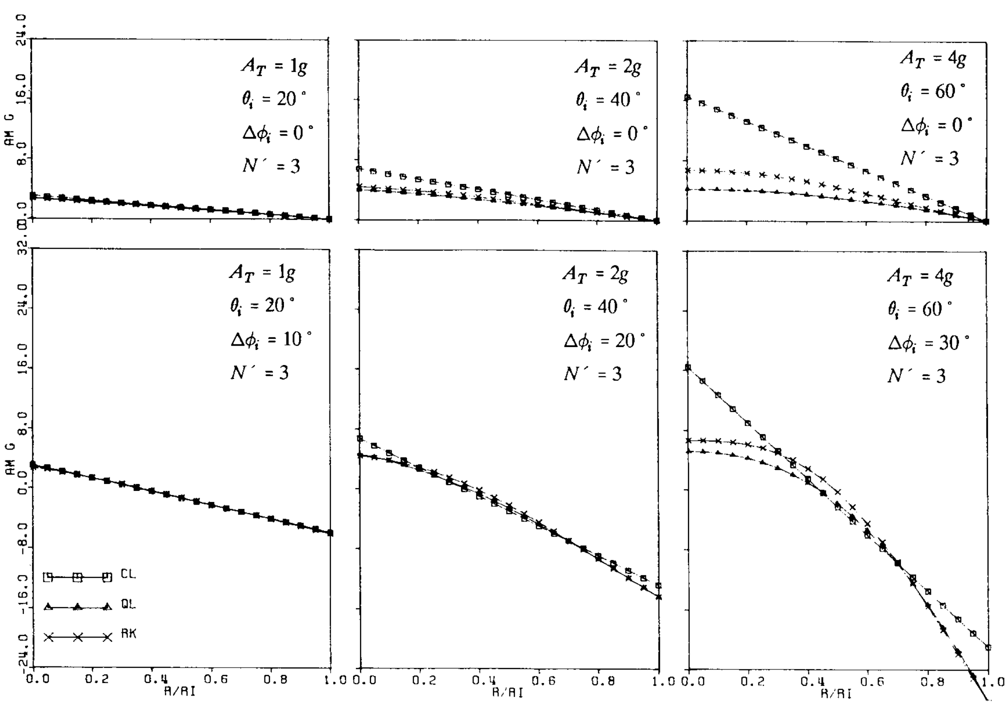

Figure 3. QL solution for A_M against a laterally maneuvering target as a function of r/r_i. CL solution and 'exact' RK estimates are also shown for comparison. N'=3

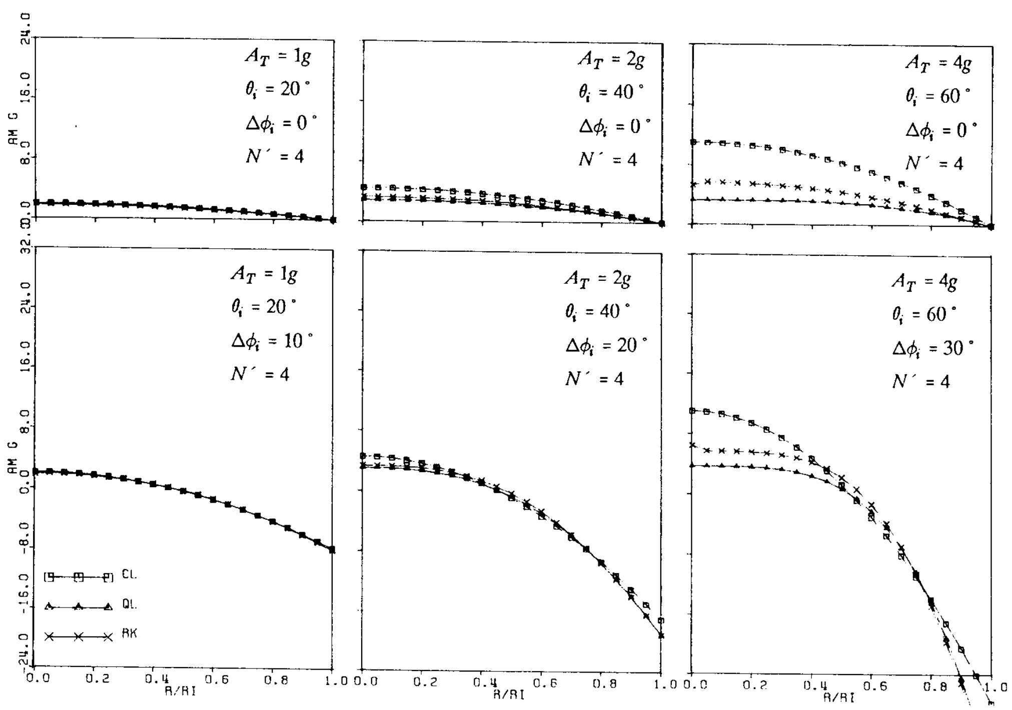

Figure 4. QL solution for A_M against a laterally maneuvering target as a function of r/r_i. CL solution and 'exact' RK estimates are also shown for comparison. N'=4

solutions for maneuvering targets are also based on such assumptions. However, even for this situation, the QL result shows a much closer coincidence with the exact results than the linear treatment.

In (b) and (c) parts of Figs. 2, 3 and 4 the angles and the target maneuver levels are progressively increased to high values. This represents a highly generalized pursuit scenario, with high target maneuvers and engagement geometries that are far from tail-chase and collision- course situations. The linear solutions currently available are not expected to be valid under such conditions, and they indeed are not, as seen from parts (b) and (c) of Figs. 2, 3 and 4. However, the QL technique of this paper continues to yield much more accurate results even under such general conditions.

In addition to the plots for A_M given in Figs. 2 to 4, the behavior of the LOS angle θ and time of flight t_f is depicted in Table 1 for the commonly used value of N'=3. The results obtained from CL formulation and the "exact" R.K. estimates are also tabulated alongside for comparison. The results for both pursuit with and without heading error are shown. The initial conditions correspond to the case of Fig. 3. It is obvious from Table 1 that the QL solution yields much more accurate results, even under highly generalized pursuit scenario, as compared to the classical linear solution for LOS angle and time of flight also.

CONCLUSIONS

In this paper, an attempt has been made to solve the difficult problem of obtaining the trajectory parameters of a projectile pursuing a maneuvering target under the pure proportional guidance strategy. Currently, only linear solutions of the nonlinear equations governing PN motion are available, the validity of which is restricted to low target maneuvers and to nearly tail-chase and collision-course pursuits. A quasilinearization approach has been followed in this paper to obtain closed-form solutions for the pursuer lateral acceleration which are found to yield accurate results for a wide range of engagement geometries and target maneuver levels.

It has been shown analytically that the closed form solution for the maneuvering target is a generalization of quasilinear solution for non-maneuvering targets derived in our previous paper [6], and also that the solutions reduce to the classical linear form for maneuvering targets, under the small-angle approximations employed for the linear solutions. The solution presented in this paper may thus be viewed as the highest member of a family of solutions of increasing generalization in the PN context.

The validity of the formulation, as also the fact of its generalization over the linear approach, is also demonstrated from the actual results derived from the formulation, using the exact numerical solution of the original PN equations as the standard. Even under conditions where the linear solutions are also valid, the generalized quasilinear solution of this paper shows a distinct improvement in lateral acceleration estimates; when the conditions become more severe and the linear solution ceases to be applicable, the quasilinear solution still continues to follow the true solution quite faithfully.

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TABLE 1. Comparison of solutions obtained through CL and QL for θ and t_f for maneuvering target case. The 'exact' RK estimates are also shown for comparison. N'=3

LOS angle θ $\theta_i = 20°$ $\Delta\phi_i = 0°$ $A_T = 1g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
1.000	20.0000	20.0000	20.0000
.900	20.0375	20.0352	20.0354
.800	20.1500	20.1403	20.1422
.700	20.3374	20.3149	20.3210
.600	20.5999	20.5583	20.5718
.500	20.9373	20.8699	20.8943
.400	21.3497	21.2488	21.2876
.300	21.8370	21.6942	21.7504
.200	22.3994	22.2047	22.2806
.100	23.0368	22.7786	22.8750
.001	23.7416	23.4063	23.5142

LOS angle θ $\theta_i = 40°$ $\Delta\phi_i = 0°$ $A_T = 2g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
1.000	40.0000	40.0000	40.0000
.900	40.0667	40.0505	40.0520
.800	40.2667	40.1999	40.2108
.700	40.6000	40.4447	40.4796
.600	41.0667	40.7809	40.8589
.500	41.6667	41.2044	41.3471
.400	42.4000	41.7104	41.9398
.300	43.2667	42.2934	42.6296
.200	44.2667	42.9467	43.4060
.100	45.4000	43.6621	44.2551
.001	46.6533	44.4188	45.1375

LOS angle θ $\theta_i = 60°$ $\Delta\phi_i = 0°$ $A_T = 4g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
1.000	60.0000	60.0000	60.0000
.900	60.1110	60.0537	60.0585
.800	60.4439	60.2080	60.2448
.700	60.9988	60.4522	60.5720
.600	61.7757	60.7757	61.0466
.500	62.7745	61.1678	61.6672
.400	63.9952	61.6177	62.4231
.300	65.4379	62.1144	63.2955
.200	67.1026	62.6469	64.2600
.100	68.9892	63.2036	65.2882
.001	71.0757	63.7673	66.3314

Time of flight t_f $\theta_i = 20°$ $\Delta\phi_i = 0°$ $A_T = 1g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
.001	8.1670	8.2921	8.2505

Time of flight t_f $\theta_i = 40°$ $\Delta\phi_i = 0°$ $A_T = 2g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
.001	7.7008	8.1262	8.0442

Time of flight t_f $\theta_i = 60°$ $\Delta\phi_i = 0°$ $A_T = 4g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
.001	7.0257	7.9990	7.8536

LOS angle θ $\theta_i = 20°$ $\Delta\phi_i = 10°$ $A_T = 1g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
1.000	20.0000	20.0000	20.0000
.900	18.6881	18.6519	18.6524
.800	17.5921	17.5251	17.5283
.700	16.7121	16.6224	16.6319
.600	16.0480	15.9460	15.9655
.500	15.5998	15.4973	15.5286
.400	15.3676	15.2765	15.3187
.300	15.3514	15.2819	15.3301
.200	15.5511	15.5095	15.5539
.100	15.9667	15.9506	15.9764
.001	16.5909	16.5768	16.5606

LOS angle θ $\theta_i = 40°$ $\Delta\phi_i = 20°$ $A_T = 2g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
1.000	40.0000	40.0000	40.0000
.900	37.7763	37.5119	37.5127
.800	35.9215	35.4594	35.4675
.700	34.4358	33.8500	33.8801
.600	33.3191	32.6844	32.7567
.500	32.5713	31.9556	32.0888
.400	32.1926	31.6476	31.8504
.300	32.1828	31.7324	31.9972
.200	32.5421	32.1670	32.4674
.100	33.2703	32.8856	33.1830
.001	34.3548	33.7717	34.0298

LOS angle θ $\theta_i = 60°$ $\Delta\phi_i = 30°$ $A_T = 4g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
1.000	60.0000	60.0000	60.0000
.900	57.5324	56.7248	56.7169
.800	55.5491	54.2389	54.2027
.700	54.0498	52.5052	52.4522
.600	53.0348	51.4569	51.4446
.500	52.5039	51.0020	51.1130
.400	52.4572	51.0288	51.3389
.300	52.8946	51.4115	51.9702
.200	53.8163	52.0199	52.8535
.100	55.2221	52.7319	53.8607
.001	57.0907	53.4492	54.8837

Time of flight t_f $\theta_i = 20°$ $\Delta\phi_i = 10°$ $A_T = 1g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
.001	8.1109	8.3639	8.2546

Time of flight t_f $\theta_i = 40°$ $\Delta\phi_i = 20°$ $A_T = 2g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
.001	7.5504	8.2003	8.0681

Time of flight t_f $\theta_i = 60°$ $\Delta\phi_i = 30°$ $A_T = 4g$ N' = 3			
r/r_i	Classical Linear	Quasilinear	R.K. Estimate
.001	6.8857	7.9248	7.8672

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